HAL
open science

Minimum time control of the restricted three-body problem

Jean-Baptiste Caillau, Bilel Daoud

To cite this version:

Jean-Baptiste Caillau, Bilel Daoud. Minimum time control of the restricted three-body problem. 2011. hal-00599216v3

HAL Id: hal-00599216
https://hal.science/hal-00599216v3

Preprint submitted on 28 Sep 2011 (v3), last revised 10 Apr 2012 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Minimum time control of the restricted three-body problem

J.-B. Caillau* B. Daoud†

July 2010

Abstract

The minimum time control of the circular restricted three-body problem is considered. Controllability is proved on an adequate submanifold. Singularities of the extremal flow are studied by means of a stratification of the switching surface. Properties of homotopy maps in optimal control are framed in a simple case. The analysis is used to perform continuations on the two parameters of the problem: The ratio of the masses, and the magnitude of the control.

Keywords. three-body problem, minimum time control, control-affine systems, homotopy, conjugate points

MSC classification. 49K15, 70Q05

1 Problem statement

The control of two bodies was addressed in [14]. The first body exerted a central force on the second, which was an artificial satellite of negligible mass whose thrust was the control. The resulting controlled Kepler equation was shown to be controllable, and minimization of time was studied (see also the subsequent papers [13, 7]). The present paper is the continuation of this work. Now under the influence of two primary bodies, the artificial satellite is still endowed with a thrust. The motion of the two primaries, not influenced by the third negligible mass, is supposed to be circular. Among the numerous previous studies on space missions in the three-body framework, one has to mention the pioneering work of [17], and more recently of [20]. These approaches are purely celestial mechanical ones and rely on a fine knowledge of the dynamical system with three bodies or more. More on the control side see, *e.g.*, [3] for numerical results using direct methods, [5] for a preliminary study on stabilization, and [23, 22] for a combination of control and dynamical system techniques. The model we consider is the following [28].

*Math. Institute, Bourgogne Univ. & CNRS, 9 avenue Savary, F-21078. Supported by Conseil Régional de Bourgogne (contract no. 2009-160E-160-CE-160T) and ANR Geometric Control Methods (project no. NT09_504490).

†Same address. Supported by the French Ministry for Higher Education & Research (grant no. 31716-2008).

Let $\mu \in (0,1)$ be the ratio of the primaries masses, and let $Q_\mu = \mathbf{C}\backslash\{-\mu, 1-\mu\}$. For $q \in Q_\mu \subset \mathbf{C} \simeq \mathbf{R}^2$ and positive thrust magnitude ε, define the controlled circular restricted three-body problem (planar model) according to[1]

$$\ddot{q}(t) - 2\overline{\partial}\Omega_\mu(q(t)) + 2i\dot{q}(t) = \varepsilon u(t), \quad |u(t)| = \sqrt{u_1^2(t) + u_2^2(t)} \leq 1.$$

Here, $(q, \dot{q}) \in X_\mu = TQ_\mu \simeq Q_\mu \times \mathbf{C}$ are Cartesian coordinates in a rotating frame ($q = e^{-it}\xi$ where ξ is the position vector in a fixed frame) and

$$\Omega_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}.$$

Another choice of coordinates consists in letting $X_\mu = T^*Q_\mu$, taking the cotangent bundle instead to write the drift in Hamiltonian form. Let $p = \dot{q} + iq$ and let

$$\begin{aligned} J_\mu(q, \dot{q}) &= \frac{1}{2}|\dot{q}|^2 - \Omega_\mu(q), \\ &= \frac{1}{2}|p|^2 + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|} \end{aligned}$$

be the *Jacobian integral*. Then,

$$\dot{q}(t) = \frac{\partial J_\mu}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial J_\mu}{\partial q}(q(t), p(t)) + \varepsilon u(t), \quad |u(t)| \leq 1.$$

More compactly,

$$\dot{x}(t) = F_0(x(t)) + \varepsilon u_1(t) F_1(x(t)) + \varepsilon u_2(t) F_2(x(t)), \quad |u(t)| \leq 1,$$

with, in (q, p) coordinates for $x \in X_\mu$,

$$F_0(q,p) = \overrightarrow{J_\mu}, \quad F_1(q,p) = \frac{\partial}{\partial p_1}, \quad F_2(q,p) = \frac{\partial}{\partial p_2},$$

where $\overrightarrow{J_\mu} = (\nabla_p J_\mu, -\nabla_q J_\mu)$ is the symplectic gradient. When $\mu = 0$, we get a two-body problem: $J_0 = E - C$ with energy and momentum respectively equal to

$$E = \frac{1}{2}|\dot{\xi}|^2 - \frac{1}{|\xi|} = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - q \wedge \dot{q} - \frac{1}{|q|},$$

$$C = \xi \wedge \dot{\xi} = q \wedge \dot{q} + |q|^2.$$

Restricting to the *elliptic domain*, $X_0 \cap \{E < 0,\ C > 0\}$, another system of coordinates tailored for the analysis is obtained using orbital elements describing the geometry of the osculating ellipse. Let $n > 0$ be the *mean motion* ($a^3 n^2 = 1$ if a the semi-major axis), $(e_x, e_y) \in \mathbf{D}$ be the *eccentricity vector* (where $\mathbf{D}$ is the open unit ball of $\mathbf{R}^2$), and $l \in \mathbf{R}$ the *longitude* (the class modulo 2π of l is just the polar angle in the fixed (ξ_1, ξ_2)-frame). Alternatively, one can use polar coordinates $(e, \theta) \in (0,1) \times \mathbf{S}^1$ for the eccentricity on the (pointed) Poincaré disk $\mathbf{D}$ (θ is called the *argument of pericenter*). In this system, $x = (n, e, \theta, l)$,

$$F_0(x)_{|\mu=0} = \frac{nW^2}{(1-e^2)^{3/2}} \frac{\partial}{\partial l}, \quad W = 1 + e\cos\tau,$$

[1] We use $\overline{\partial} = (1/2)(\partial/\partial q_1 + i\partial/\partial q_2)$ to represent the gradient with respect to q in complex form.

$$\widetilde{F}_1(x) = \frac{\sqrt{1-e^2}}{n^{1/3}}\left(-\frac{3ne\sin\theta}{1-e^2}\frac{\partial}{\partial n} + \sin\tau\frac{\partial}{\partial e} - \cos\tau\frac{1}{e}\frac{\partial}{\partial\theta}\right),$$

$$\widetilde{F}_2(x) = \frac{\sqrt{1-e^2}}{n^{1/3}}\left(-\frac{3nW}{1-e^2}\frac{\partial}{\partial n} + (\cos\tau + \frac{e+\cos\tau}{W})\frac{\partial}{\partial e} + (\sin\tau + \frac{\sin\tau}{W})\frac{1}{e}\frac{\partial}{\partial\theta}\right)$$

where $\tau = l - \theta$. We have also used a feedback on the control to express the control not in the $\{\partial/\partial\dot{\xi}_1, \partial/\partial\dot{\xi}_2\}$ frame but in the *radial-orthoradial* one, so

$$\widetilde{F}_1(\xi,\dot{\xi}) = \frac{\xi_1}{|\xi|}\frac{\partial}{\partial\dot{\xi}_1} + \frac{\xi_2}{|\xi|}\frac{\partial}{\partial\dot{\xi}_2}, \quad \widetilde{F}_2(\xi,\dot{\xi}) = -\frac{\xi_2}{|\xi|}\frac{\partial}{\partial\dot{\xi}_1} + \frac{\xi_1}{|\xi|}\frac{\partial}{\partial\dot{\xi}_2}.$$

The criterion under consideration is the final time, and the paper is organized as follows.

Section 2 is devoted to controllability. Whatever the bound on the control, it is proven that admissible trajectories between arbitrary points exist provided the Jacobian constant is smaller than the one given for some equilibrium point of the uncontrolled system. The structure of optimal controls is addressed in Section 3, refining the results of [14]. In particular the role of peri and apocenters with respect to global bounds on the number of switchings of the control is emphasized, in connection with averaging of the system. The system has indeed two parameters: The bound of the control, ε, which can be taken very small with low-thrust applications in mind [24] thus leading to averaging; and the ratio of the masses, μ, on which a continuation *à la Poincaré* may be performed to embed the two-body problem into a three-body one. This idea is the key to solving the problem as explained in Section 4 where continuations both with respect to μ and ε are considered. The peculiarity of homotopy maps in optimal control is then illustrated in a simple framework, in relation to second order optimality conditions.

2 Controllability

The drift has five equilibrium points, $L_1(\mu), \ldots, L_5(\mu)$, known as *Lagrange points* [28], whose position depends on μ. The points L_1, L_2 and L_3 are the *colinear* or *Euler points*, the points L_4 and L_5 form two equilateral triangles with $-\mu$ and $1-\mu$. When $\mu = 0$, $L_1 = L_2 = 1$, $L_3 = -1$, and $L_4 = \exp(i\pi/3)$, $L_5 = -\exp(i\pi/3)$ (all belong to $\mathbf{S}^1$ which is a continuum of equilibrium points in this particular case). The Jacobian J_μ is the only first integral of the non-integrable uncontrolled system. Every level set $\{J_\mu = j\}$ projects onto $\Omega_\mu(q) + j = |\dot{q}|^2/2 \geq 0$ in the (q_1, q_2)-space, defining the Hill regions where the free motion has to take place (see Fig. 1). Let $j_i(\mu) = J_\mu(L_i(\mu))$, $i = 1, \ldots, 5$ denote the Jacobian constants of these points. For $\mu \in (0,1)$, $j_2 < j_1 < j_3 < j_4 = j_5$ (these values all degenerate to $-3/2$ when μ goes to zero). The open subset $\{x \in X_\mu \mid J_\mu(x) < j_1(\mu)\}$ has two connected components, and we denote by X_μ^1 the component containing $L_2(\mu)$ (see Fig. 2). Below we recall the basic facts needed to prove controllability on this subset.

Consider a smooth[2] control-affine system on a manifold X,

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=1}^{m} u_i F_i(x(t)), \quad u(t) \in U \subset \mathbf{R}^m,$$

[2]That is $\mathscr{C}^\infty$-smooth.

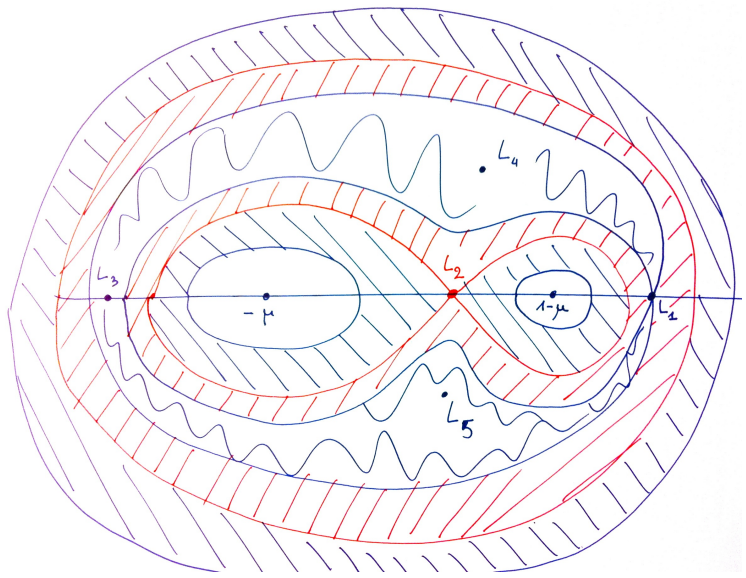

Figure 1: Lagrange points and associated Hill regions, $\mu \in (0,1)$. The (dashed) forbidden regions of motion (complementary to the Hill regions) monotonically decrease as the Jacobian constant tends to $J_\mu(L_4) = J_\mu(L_5)$ (then disappear past this value) and opens up past $J_\mu(L_1)$.

such that U is a neighbourhood of the origin. The attainable set (by piecewise constant controls) from $x_0 \in X$ depends only on the drift, F_0, and on the distribution $\mathscr{D}$ spanned by the vector fields $F_1, \dots, F_m$. It is the set of points obtained by compositions of flows,

$$e^{t_p G_p} \circ \cdots \circ e^{t_1 G_1}(x_0), \quad G_i \in F_0 + \mathscr{D}, \quad t_i \geq 0,$$

with t_i small enough for the composition to be defined. Now, if $\mathscr{F}$ is an arbitrary subsheaf of $\mathscr{C}^\infty$ vector fields on X, assuming for simplicity all $F \in \mathscr{F}$ complete, define the subgroup $\mathscr{G}$ of the diffeomorphisms of X generated by the one parameter subgroups $\exp tF$, $t \in \mathbf{R}$, $F \in \mathscr{F}$. According to the orbit theorem [1, 26], the orbit of $\mathscr{G}$ through x_0 is an immersed submanifold of X whose tangent space is

$$T_x\mathscr{G}(x_0) = \mathrm{Span}_x\{(\mathrm{Ad}\,\varphi)F,\ \varphi \in \mathscr{G},\ F \in \mathscr{F}\}.$$

In the previous expression, the adjoint action $\mathrm{Ad}\,\varphi$ maps F to its conjugate, $\varphi \cdot F \cdot \varphi^{-1}$ $(= (\varphi^{-1})_* F)$ with diffeomorphisms and vector fields acting linearly on $\mathscr{C}^\infty(X, \mathbf{R})$ (defining automorphisms and derivations, respectively). In coordinates,

$$(\mathrm{Ad}\,\varphi)F_{|x} = [\varphi'(x)]^{-1} F(\varphi(x)), \quad x \in X.$$

Restricting to the $\mathscr{C}^\omega$-category, the adjoint action and the exponential commute in the sense that, for arbitrary vector fields F, G,

$$(\mathrm{Ad}\, e^{tF})G_{|x} = (e^{t\,\mathrm{ad}F})G_{|x} = \sum_{n \geq 0} \frac{t^n}{n!}(\mathrm{ad}^n F)G_{|x}, \quad x \in X,$$

where $(\mathrm{ad}F)G$ is the Lie bracket $[F,G] = F\cdot G - G\cdot F$, and recursively for ad^n. In this case, the orbit theorem then simply reads

$$T_x\mathscr{G}(x_0) = \mathrm{Lie}_x\mathscr{F}.$$

If the vector fields are not complete, $\mathscr{G}$ is just a pseudo-group [19] but the conclusion of the orbit theorem—which is local—is preserved. Coming back to control-affine systems, assuming that

$$\mathrm{Lie}_x\{F_0, F_1, \ldots, F_m\} = T_xX, \quad x \in X, \tag{1}$$

and that the drift F_0 is recurrent[3] it can be proven that compositions with the flow of $-F_0$ can be added when computing the attainable set (see [18]). So the attainable set is equal to the orbit of the pseudo-group associated with $\{F_0, F_1, \ldots, F_m\}$. Because of the orbit theorem and of (1), this orbit has to be the whole manifold (supposed to be connected). In the control-affine three-body case, the rank condition holds as is clear from

Lemma 1. *A second order controlled system on* $\mathbf{R}^m$*,*

$$\ddot{q}(t) + g(q(t), \dot{q}(t)) = u(t),$$

is a control-affine system on $\mathbf{R}^{2m}$ *with an involutive distribution* $\mathscr{D}$ *and a drift* F_0 *such that* $\{F_1, \ldots, F_m, [F_0, F_1], \ldots, [F_0, F_m]\}$ *has maximum rank.*

Proof. As a first order system,

$$F_0(q,\dot{q}) = \dot{q}_1\frac{\partial}{\partial q_1} + \cdots + \dot{q}_m\frac{\partial}{\partial q_m} - g_1(q,\dot{q})\frac{\partial}{\partial \dot{q}_1} - \cdots - g_m(q,\dot{q})\frac{\partial}{\partial \dot{q}_m}$$

and $F_i = \partial/\partial\dot{q}_i$ (so $\mathscr{D}$ is clearly involutive). Then

$$[F_0, F_i] = -\partial/\partial q_i \mod \mathscr{D}$$

so the rank is maximum. □

Applying the lemma to the planar three-body problem ($m = 2$), one obtains controllability in the following way.

Theorem 1. *For any* $\mu \in (0,1)$*, for any positive* ε*, the circular restricted three-body problem is controllable on* X^1_μ*.*

Proof. Let x_0, x_f in X^1_μ, and let j be strictly smaller than the Jacobian constants of both endpoints. Set $\widetilde{X}^1_\mu = X^1_\mu \cap \{J_\mu > j\}$. Outside a subset of zero measure associated with initial conditions generating collisions ($q = -\mu$ or $1-\mu$), the drift is a complete Hamiltonian vector field whose exponential is defined for all times and is a volume preserving bijection in (q,p) coordinates. By definition, $\widetilde{X}^1_\mu$ which is a union of level sets of the Hamiltonian J_μ is invariant with respect to the exponential. For $x = (q,p) \in \widetilde{X}^1_\mu$,

$$j + \Omega_\mu(q) < \frac{1}{2}|p - iq|^2 < j_1(\mu) + \Omega_\mu(q).$$

[3]Given a vector field F, a point $x \in X$ is recurrent for F if, for any neighbourhood V of x, for any positive T, there is $t \geq T$ such that $\exp tF(x)$ is defined and belongs to V. The vector field itself is said recurrent when it has a dense subset of recurrent points.

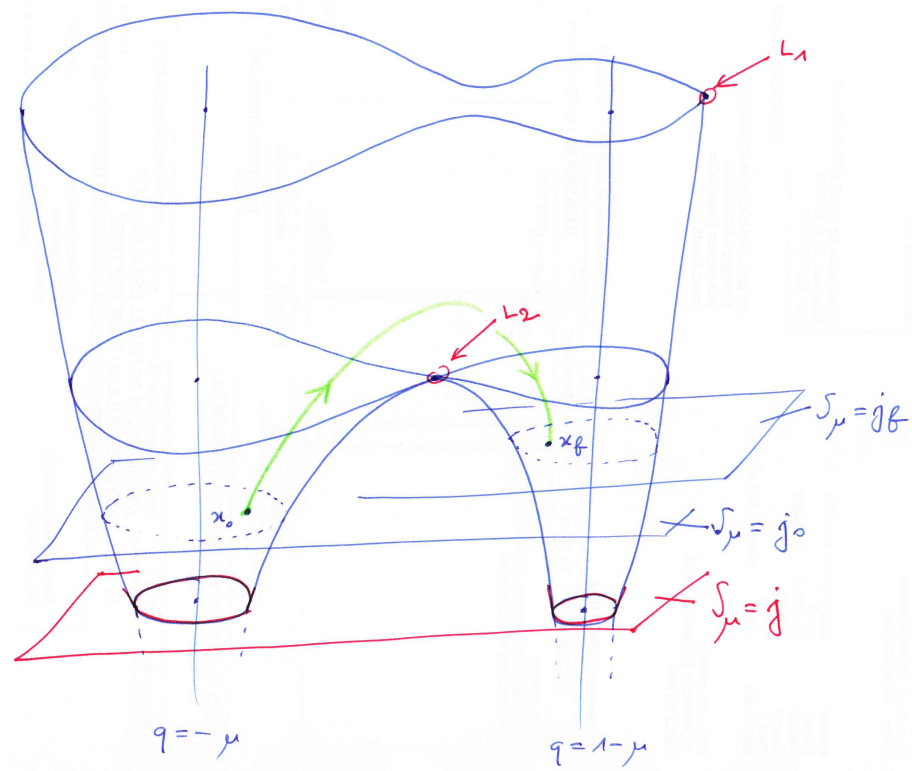

Figure 2: Projection of the open submanifold X^1_μ in the (q_1, q_2, J_μ)-space. The boundary of the volume is an apparent contour generated by the projection. It is the *zero velocity* set. Above each interior point there is an $\mathbf{S}^1$-fibre corresponding to the argument of $\dot{q}$. For $\mu \in (0,1)$, $j_2 < j_1$ and X^1_μ is connex.

Then, for a fixed q, the volume of the q-section of $\widetilde{X}^1_\mu$ is bounded by $2\pi(j_1(\mu) - j)$ as is clear integrating with respect to $\mathrm{d}p_1 \mathrm{d}p_2 = \rho\, \mathrm{d}\rho\, \mathrm{d}\alpha$ (set $p - iq = \rho \exp(i\alpha)$). Since the projection on the (q_1, q_2)-space of $\widetilde{X}^1_\mu$ is also bounded, the $\mathrm{d}q \wedge \mathrm{d}p$-measure of $\widetilde{X}^1_\mu$ is finite (Fubini). We conclude as in [25] that almost every point of $\widetilde{X}^1_\mu$ is recurrent by Poincaré theorem. Controllability on $\widetilde{X}^1_\mu$ follows and implies the existence of a trajectory joining x_0 to x_f, which in turn implies controllability on X^1_μ. □

Remark 1. In the two-body case, $\mu = 0$, controllability still holds on X^1_0, the bounded component of $\{J_0 < j_1(0) = -3/2\}$. Each section of X^1_0 by a level set $\{J_0 = j\}$ is a pointed disk containing bounded, hence periodic, trajectories of of the uncontrolled system. The energy is negative but, as $J_0 = E - C$, X^1_0 contains both direct ($C > 0$), retrograde ($C < 0$) and collision orbits ($C = 0$). In contrast, the controllability result in [14] was obtained on the elliptic domain, $X_0 \cap \{E < 0,\ C > 0\}$, using the periodicity of the drift and excluding collisions (a sign had then to be imposed on the momentum so that the manifold be connex).

Remark 2. In order to obtain an existence result for minimum time, one has to

prove that minimizing trajectories remain into a fixed compact (which depends on the prescribed boundary conditions). Then, from any minimizing sequence one can extract a converging subsequence whose limit is an admissible trajectory by virtue of the convexity of the velocity field $\{F_0(x)+\varepsilon u_1 F_1(x)+u_2 F_2(x),\ |u| \leq 1\}$ for all $x \in X$ (Filippov theorem). One of the difficulties due to collisions is so to give bounds on the distance to the singularities, $-\mu$ and $1-\mu$.

3 Singularities of the extremal flow

Let $u : [0, t_f] \to \mathbf{R}^2$ be a measurable time-minimizing control of the control-affine system

$$\dot{x}(t) = F_0(x(t)) + u_1 F_1(x(t)) + u_2 F_2(x(t)), \quad u_1^2(t) + u_2^2(t) \leq 1,$$

defined on a manifold X of dimension four. Let x denote the associated Lipschitz trajectory. Pontryagin maximum principle asserts that x is the projection of a Lipschitzian curve, the *extremal* $z = (x, p) : [0, t_f] \to T^*X \backslash \{0\}$, valued in the cotangent bundle minus the null section. In coordinates, there is a nonpositive scalar p^0 such that, a.e. on $[0, t_f]$,

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), u(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), u(t), p(t)),$$

with Hamiltonian $H(x, u, p) = p^0 + H_0(x, p) + u_1 H_1(x, p) + u_2 H_2(x, p)$ and Hamiltonian lifts $H_i(x, p) = \langle p, F_i(x)\rangle$, $i = 0, 1, 2$. Moreover, the maximization condition holds a.e.,

$$H(x(t), u(t), p(t)) = \max_{|v|\leq 1} H(x(t), v, p(t)).$$

As a result, H is a.e. equal to a constant (zero, here, because the final time is free) along (x, u, p), and

$$u(t) = \frac{\psi(t)}{|\psi(t)|}$$

whenever the *switching function* $\psi(t) = (H_1, H_2)(x(t), p(t))$ does not vanish. The *switching surface* is

$$\Sigma = \{(x, p) \in T^*X \mid H_1(x, p) = H_2(x, p) = 0\},$$

and we make the following assumption on the distribution $\mathscr{D}$ spanned by F_1 and F_2. Let

$$\begin{aligned} D(x) &= \det(F_1(x), F_2(x), [F_0, F_1](x), [F_0, F_2](x)), \\ D_1(x) &= \det(F_1(x), F_2(x), [F_1, F_2](x), [F_0, F_2](x)), \\ D_2(x) &= \det(F_1(x), F_2(x), [F_0, F_1](x), [F_1, F_2](x)), \end{aligned}$$

we assume that

(i) $D_1^2(x) + D_2^2(x) < D^2(x)$, $x \in X$.

The following results refine those in [14] (see also [7]).

Proposition 1. *Let u be a piecewise continuous optimal control. Then u is bang-bang and, at a switching point $\bar{t}$,*

$$\alpha(\bar{t}+) - \alpha(\bar{t}-) = 2\beta + \pi, \quad \beta = \arcsin \frac{\{H_1, H_2\}(\overline{z})}{\sqrt{\{H_0, H_1\}^2(\overline{z}) + \{H_0, H_2\}^2(\overline{z})}},$$

where α is the angle of the control and $\overline{z} = (x(\bar{t}), p(\bar{t}))$ the associated value of the extremal.

Lemma 2. *For any $u \in \mathbf{R}^2$ such that $|u| \leq 1$, vector fields F_1, F_2, $[F_0, F_1] - u_2[F_1, F_2]$ and $[F_0, F_2] + u_1[F_1, F_2]$ form a frame. For $z = (x, p) \in \Sigma \backslash \{0\}$,*

$$\{H_1, H_2\}^2(z) < \{H_0, H_1\}^2(z) + \{H_0, H_2\}^2(z).$$

Proof. Computing,

$$\det(F_1(x), F_2(x), [F_0, F_1](x) - u_2[F_1, F_2](x), [F_0, F_2](x) + u_1[F_1, F_2](x)) =$$

$$D(x) - u_2 D_1(x) + u_1 D_2(x), \quad x \in X, \quad u \in \mathbf{R}^2.$$

Hence, for $|u| \leq 1$, $\{F_1, F_2, [F_0, F_1] - u_2[F_1, F_2], [F_0, F_2] + u_1[F_1, F_2]\}$ form a frame by virtue of (i). Let $z = (x, p)$ belong to Σ, $p \neq 0$. Then for an arbitrary $u \in \mathbf{R}^2$, $|u| \leq 1$,

$$\begin{bmatrix} \langle p, [F_0, F_1](x) \rangle \\ \langle p, [F_0, F_2](x) \rangle \end{bmatrix} + \langle p, [F_1, F_2](x) \rangle \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since p would otherwise be zero. Whence the conclusion on the Poisson brackets as $\langle p, [F_0, F_1](x) \rangle = \{H_0, H_1\}(z)$ and so on. □

Proof (of Proposition 1). The switching function is almost everywhere differentiable and, at a Lebesgue point,

$$\begin{aligned} \dot{\psi}_1(t) &= \{H_0, H_1\}(z(t)) - u_2(t)\{H_1, H_2\}(z(t)), \\ \dot{\psi}_2(t) &= \{H_0, H_2\}(z(t)) + u_1(t)\{H_1, H_2\}(z(t)). \end{aligned}$$

So $\dot{\psi}$ is as u piecewise continous and has left and right limits everywhere. If $\overline{z} = z(\bar{t})$ belongs to $\Sigma \backslash \{0\}$, neither $\dot{\psi}(\bar{t}-)$ nor $\dot{\psi}(\bar{t}+)$ vanish according to the previous lemma. The time $\bar{t}$ is so an isolated zero and one has $u = \psi/|\psi|$ outside finitely many switching points. Left and right limits of the control angle at $\bar{t}$ verify

$$\{H_1, H_2\}(\overline{z}) - \sin\alpha\{H_0, H_1\}(\overline{z}) + \cos\alpha\{H_0, H_2\}(\overline{z}) = 0$$

which, thanks to the lemma, has exactly two solutions with the predicted discontinuity. □

Assuming moreover that

(ii) $\mathscr{D}$ is involutive,

we get

Corollary 1 ([14])**.** *Any optimal control is piecewise continuous with finitely many switches all of angle π ("π-singularities").*

Proof. When $\mathscr{D}$ is involutive, the switching function is $\mathscr{C}^1$ since the bracket $\{H_1, H_2\}$ vanish at switching points (one has $[F_1, F_2] = 0 \mod \mathscr{D}$). All zeros are isolated by the previous argument and $\beta = 0$, whence the conclusion. □

We finally add the assumption that

(iii) $F_0 \notin \mathrm{Span}\{F_1, F_2, [F_0, F_2]\}$,

and define

$$\Sigma_0 = \Sigma \cap \{(x, p) \in T^*X \mid F_0(x) \in \mathrm{Span}_x\{F_1, F_2, [F_0, F_2]\}\}.$$

Remark 3. Assumptions (i) and (ii) put together are equivalent to saying that $\mathscr{D}$ is involutive and that F_1, F_2, $[F_0, F_1]$ and $[F_0, F_2]$ form a frame. In contrast with (iii), these properties depend only on the distribution (and the drift), not on a particular choice of basis of $\mathscr{D}$.

Theorem 2 ([14])**.** *In the normal case, there cannot be consecutive switchings in Σ_0. In particular, if $\Sigma = \Sigma_0$, any normal optimal control has at most one switching which is a π-singularity.*

Proof. According to (i), there are continous scalar functions λ_1, λ_2 on X such that

$$F_0 = \lambda_1[F_0, F_1] + \lambda_2[F_0, F_2] \mod \mathscr{D}.$$

At a switching point $z(\bar{t}) = (x(\bar{t}), p(\bar{t})) \in \Sigma_0$,

$$H(x(\bar{t}), u(\bar{t}), p(\bar{t})) = -p_0 = \lambda_2(x(\bar{t}))\{H_0, H_2\}(z(\bar{t})) = \lambda_2(x(\bar{t}))\dot{\psi}_2(\bar{t}).$$

In the normal case, $p_0 < 0$, so

$$\lambda_2(x(\bar{t}_1))\dot{\psi}_2(\bar{t}_1)\lambda_2(x(\bar{t}_2))\dot{\psi}_2(\bar{t}_2) = p_0^2 > 0.$$

if we assume that $\bar{t}_1 < \bar{t}_2$ are consecutive such switching times. Because of (iii), λ_2 never vanishes; by continuity its sign is constant so $\dot{\psi}_2(\bar{t}_1)\dot{\psi}_2(\bar{t}_2) > 0$, whence the contradiction. □

Thanks to Lemma 1, these results apply to the problem under consideration: The circular restricted three-body has bang-bang time-minimizing controls with finitely many π-singularities. A remarkably simple geometric interpretation of Σ_0 is obtained when restricting to a two-body system, $\mu = 0$. This case is not only important in itself as it also corresponds to the initial and final phases of a typical three-body low-thrust transfer. Such a trajectory resembles a heteroclinic trajectory connecting periodic orbits around each one of the primaries (see Fig. 4). We use for the analysis the geometric coordinates and the radial-orthoradial frame introduced in Section 1. We assume the eccentricity positive, $0 < e < 1$, to obviate the singularity of these coordinates at circular orbits. With $x = (n, e, \theta, l) \in \mathbf{R}_+^* \times (0, 1) \times \mathbf{S}^1 \times \mathbf{R}$ (and $\tau = l - \theta$), $p = (p_n, p_e, p_\theta, p_l) \in (\mathbf{R}^4)^*$, we set

$$\alpha = -\frac{3n}{1 - e^2}\, p_n, \quad \beta = p_e, \quad \gamma = \frac{p_\theta}{e}, \quad c = \cos\tau, \quad s = \sin\tau,$$

so Σ is defined by the following algebraic system:

$$\alpha e s + \beta s - \gamma c = 0,$$

$$\alpha(1+ec)+\beta(c+\frac{e+c}{1+ec})+\gamma(s+\frac{s}{1+ec})=0,$$
$$c^2+s^2=1.$$

Proposition 2. *The switching surface Σ for $\mu=0$ is stratified as follows:*

(i) If $\gamma=0$, either $(\alpha,\beta)=(0,0)$, or $s=0$ and α,β belong to the union of the two distinct lines

$$(1+e)\alpha+2\beta=0, \quad (1-e)\alpha-2\beta=0.$$

(ii) If $\gamma\neq 0$, s is not zero and α,β are uniquely determined.

Proof. The algebraic system in α,β

$$\alpha es+\beta s=\gamma c,$$
$$\alpha(1+ec)+\beta(c+\frac{e+c}{1+ec})=-\gamma(s+\frac{s}{1+ec}),$$

has determinant

$$\begin{vmatrix} es & s \\ 1+ec & c+\dfrac{e+c}{1+ec} \end{vmatrix} = -\frac{s(1-e^2)}{1+ec},$$

whence the result. □

Proposition 3. *In the two-body case $\mu=0$, the subset Σ_0 is the stratum $\{s=0\}$ of Σ. Accordingly,*

$$\Sigma\cap\{p_\theta=0\}=\Sigma_0\cup\{(p_n,p_e,p_\theta)=(0,0,0)\}.$$

Proof. Using the fact that $F_0\in\mathrm{Span}\{\partial/\partial l\}$ and that, for smooth functions f,g

$$[fF_0,g\widetilde{F}_2]=fg[F_0,\widetilde{F}_2] \mod \mathrm{Span}\{F_0,\widetilde{F}_2\},$$

the condition $\det(F_0,\widetilde{F}_1,\widetilde{F}_2,[F_0,\widetilde{F}_2])=0$ is equivalent to

$$\begin{vmatrix} 0 & es & 1+ec & -es \\ 0 & s & c+\dfrac{e+c}{1+ec} & -s-\dfrac{s(1-e^2)}{(1+ec)^2} \\ 0 & -c & s+\dfrac{s}{1+ec} & c+\dfrac{e+c}{(1+ec)^2} \\ 1 & 0 & 0 & 0 \end{vmatrix}=0,$$

that is to

$$s=0 \quad \text{or} \quad e(1+ec)=0,$$

so the conclusion follows. □

Geometrically, $s=\sin\tau=0$ (that is, $l-\theta=0 \mod \pi$) corresponds to peri and apocenters. The importance of the stratum $\Sigma\cap\{p_\theta=0\}$ comes from the analysis of the system when the control magnitude ε goes to zero (low-thrust case).

When the control is small, not only the time but also the angular length (longitude) needed to connect two Keplerian orbits in the two-body model becomes large. It thus makes sense to use averaging to analyze the behaviour of extremal trajectories. In the case of the minimization of the L^2-norm of the control (*energy* minimization), this approach goes back to [16] (see also [11] for a recent treatment of this question). One first uses the fact that, in the planar model, the drift only acts on the longitude,

$$\dot{l}(t) = \omega(x(t)), \quad \omega(x) = \frac{nW^2}{(1-e^2)^{3/2}}, \quad W = 1 + e\cos\tau,$$

to set l as the new time. In this new parameterization, minimum time extremals are integral curves of the maximized Hamiltonian

$$\hat{H}(l,\hat{x},\hat{p}) = \frac{p^0}{\omega(l,\hat{x})} + \varepsilon\sqrt{\hat{H}_1^2(l,\hat{x},\hat{p}) + \hat{H}_2^2(l,\hat{x},\hat{p})}\,,$$

with $\hat{x} = (n, e, \theta)$, $\hat{p} = (p_n, p_e, p_\theta)$, and

$$\begin{aligned}\hat{H}_1^2(l,\hat{x},\hat{p}) + \hat{H}_2^2(l,\hat{x},\hat{p}) = \frac{(1-e^2)^4}{n^{8/3}W^4}\Bigg\{&\frac{9n^2}{(1-e^2)^2}(1+2e\cos\tau+e^2)p_n^2\\ &-\frac{12n}{1-e^2}(e+\cos\tau)p_np_e\\ &+\left[1+\frac{2(e+\cos\tau)}{W}\cos\tau+\frac{(e+\cos\tau)^2}{W^2}\right]p_e^2\Bigg\}+\cdots\end{aligned}$$

where the dots indicate terms in p_θ^2, p_np_θ and p_ep_θ. The averaged Hamiltonian is

$$\overline{\hat{H}}(\hat{x},\hat{p}) = \frac{1}{2\pi}\int_0^{2\pi}\hat{H}(l,\hat{x},\hat{p})\,\mathrm{d}l. \tag{2}$$

Proposition 4. *The argument of pericenter θ is a cyclic variable of the averaged system. On the stratum $\{p_\theta = 0\}$, the integral (2) is hyperelliptic of genus 2.*

Proof. As is clear from the whole dynamics in geometric coordinates (see Section 1), only the difference $\tau = l - \theta$ appear in $\hat{H}$. Averaging with respect to l—or equivalently to τ—kills terms in θ and p_θ becomes a linear first integral. On $\{p_\theta = 0\}$ then,

$$\sqrt{\hat{H}_1^2 + \hat{H}_2^2} = \frac{1}{W^3}\sqrt{R(\cos\tau)}$$

where R is a degree 3 polynomial with coefficients depending nonlinearly on $\hat{x}$ and quadratically on $\hat{p}$. Setting for instance $u = \cos\tau$ leads to

$$\int\frac{\mathrm{d}u}{1-u^2}\sqrt{(1-u^2)R(u)}$$

which is an integral parameterized by a genus 2 hyperelliptic curve. □

Remark 4. For circular targets (which are of great practical importance in two or three-body control), the transversality condition is $p_\theta = 0$. The previous analysis then suggests that, for small control magnitudes, p_θ should also remain small so that the switching structure be close to the one on $\Sigma \cap \{p_\theta = 0\}$:

Assuming there are no trivial switchings $(p_n, p_e, p_\theta) = 0$, one would get a global bound on the number of π-singularities (at most one). Nevertheless, one should notice that, if p_θ is small but not zero, the switchings does not belong to Σ_0 according to Propositions 2 and 3. Moreover, when studying the convergence of the system towards the averaged one, one should take into accout the lack of regularity due to the radicand vanishing (see also [6] for jn alternative approach to averaging). For instance,

$$f(z) = \int_0^{\pi/2} \sqrt{\sin^2 l + z^2}\,\mathrm{d}l = |z| E(-iz^{-1})$$

where E is the complete elliptic integral of second kind. Such a function has a $z^2 \log|z|$ singularity at the origin and is not $\mathscr{C}^2$ (logarithmic branch on the second derivative).

4 Homotopy

Consider a control problem with smooth data on an n-manifold X,

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

with cost

$$\int_0^{t_f} f^0(x(t), u(t))\mathrm{d}t \to \min$$

and prescribed boundary conditions

$$x(0) = x_0, \quad x(t_f) = x_f.$$

For the sake of simplicity t_f is supposed to be fixed, but the analyis below can be drawn with appropriate changes for free final time as well. We also suppose that U is a manifold without boundary. In coordinates, this is assuming that u belongs to some open subset of $\mathbf{R}^m$, where m is the dimension of U. Let $\overline{u} : [0, t_f] \to U$ be a measurable and essentially bounded control, and let $\overline{x} : [0, t_f] \to X$ be the resulting trajectory. Pontryagin maximum principle implies that there exists a nonpositive constant $\overline{p}^0$ and a Lipschitz covector function $\overline{p} : [0, t_f] \to (\mathbf{R}^n)^*$, not both zero, so that in coordinates on T^*X,

$$\dot{\overline{x}}(t) = \frac{\partial H}{\partial p}(\overline{x}(t), \overline{u}(t), \overline{p}(t)), \quad \dot{\overline{p}}(t) = -\frac{\partial H}{\partial x}(\overline{x}(t), \overline{u}(t), \overline{p}(t)),$$

and

$$H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) = \max_{v \in U} H(\overline{x}(t), v, \overline{p}(t))$$

a.e. on $[0, t_f]$. Here,

$$H(x, u, p) = p^0 f^0(x, u) + \langle p, f(x, u)\rangle.$$

We assume that, on a neighbourhood of the extremal $(\overline{x}, \overline{p})$, the maximized Hamiltonian

$$(x, p) \mapsto \max_{v \in U} H(x, u, p)$$

is well defined and smooth; then $(\overline{x}, \overline{p})$ is an integral curve of the maximized function (see [1]), still denoted H (but now depending only on x and p). We finally make the Legendre regularity assumption that, uniformly on $[0, t_f]$,

$$\nabla^2_{uu} H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) \leq -\alpha I_m$$

for some $\alpha > 0$. As a consequence, there must exist in a neighbourhoud of the extremal a smooth implicit function $u(x, p)$ solving the first order necessary condition $\nabla_u H = 0$ such that $\overline{u}(t) = u(\overline{x}(t), \overline{p}(t))$ and $H(x, p) = H(x, u(x, p), p)$. Summarizing, $\overline{p}(0) \in (\mathbf{R}^n)^*$ is a zero of the *shooting function*[4]

$$p_0 \mapsto x(t_f, x_0, p_0) - x_f,$$

where the *exponential mapping*

$$\exp_{x_0} : (t, p_0) \mapsto x(t, x_0, p_0)$$

sends a given p_0 to the x-projection of the integral curve at t of the maximized Hamiltonian. Both functions are well defined and smooth on neighbourhoods of $\overline{p}(0)$ and $(t_f, \overline{p}(0))$, respectively. A time t_c is *conjugate* to 0 along $(\overline{x}, \overline{p})$ whenever $\overline{p}(0)$ is a critical point of $p_0 \mapsto \exp(t_c, p_0)$. The critical value $x_c = \exp(t_c, \overline{p}(0))$ is the corresponding *conjugate point*. These notions are related to necessary or sufficient second order conditions of optimality [1, 8].

The *endpoint mapping*

$$F_{t_f, x_0} : u \mapsto \hat{x}(t_f, x_0, u)$$

is well defined and smooth on a neighbourhood in $\mathrm{L}^\infty([0, t_f], U)$ of the (class of) $\overline{u}$ and maps a control to the solution at t_f of the *augmented system* ($\hat{x} = (x^0, x)$)

$$\begin{aligned} \dot{x}^0(t) &= f^0(x(t), u(t)), \quad \text{a.e. on } [0, t_f], \\ \dot{x}(t) &= f(x(t), u(t)), \quad x^0(0) = 0, \quad x(0) = x_0. \end{aligned}$$

Pontryagin maximum principle implies that $\overline{u}$ is a critical point[5] of the endpoint mapping: $\operatorname{Im} F'_{t_f, x_0}(\overline{u})$ has codimension in $\mathbf{R}^n$. If we assume that $\overline{u}$ is a corank one critical point, and moreover that it is analytical (with analytical data for the problem as well), the absence of conjugate time in $(0, t_f)$ is necessary for L^∞-local optimality.[6] Conversely, replacing the corank one and analyticity conditions by the assumption that the extremal is normal ($p_0 < 0$), the absence of conjugate time on $(0, t_f]$ is sufficient for $\mathscr{C}^0$-local optimality.[7]

Having in mind these connections with second order conditions in optimal control (see also [9]) we consider a one-parameter smooth Hamiltonian,

$$H : \mathbf{R}^{2n} \times \mathbf{R} \ni (x, p, \lambda) \mapsto H(x, p, \lambda) \in \mathbf{R}.$$

[4] A chart (O, φ) in the neighbourhood of x_f has to be chosen and the definition should read $p_0 \mapsto \varphi(x(t_f, x_0, p_0)) - \varphi(x_f)$. We may actually suppose that $\varphi(x_f) = 0$.

[5] This weaker statement is obvious: Were the function a submersion at $\overline{u}$, it would be locally open and would send neighbourhoods of $\overline{u}$ onto neighbourhoods of the augmented state, $(\overline{x}^0(t_f), \overline{x}(t_f))$. This would contradict L^∞-local optimality.

[6] That is optimality on a neighbourhood of $\overline{u}$ in $\mathrm{L}^\infty([0, t_f], U)$.

[7] Optimality of the trajectory among all admissible trajectories belonging to some neighbourhood of $\overline{x}$.

Given a positive final time t_f and an initial condition x_0, we define the shooting-like *homotopy function*[8]

$$h(p_0, \lambda) = x(t_f, x_0, p_0, \lambda)$$

that maps (p_0, λ) to the coordinate x of the solution at t_f of

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), \lambda), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), \lambda)$$

with initial conditions $x(0) = x_0$, $p(0) = p_0$. By restricting it if necessary, we may assume that its domain of definition, $\Omega \subset \mathbf{R}^{n+1}$, is open and made only of regular points of h so that

$$\operatorname{rank} h'(p_0, \lambda) = n, \quad (p_0, \lambda) \in \Omega.$$

As a consequence, the level set $\{h = 0\}$ is a one-dimensional submanifold of $\mathbf{R}^{n+1}$ called the *path of zeros*. Typically, one knows a zero of $h(., \lambda)$ for, say, $\lambda = 0$, and wants to follow this path to reach if possible a zero for a target value of the parameter, $\lambda = 1$. For any $c = (p_0, \lambda) \in \Omega$, $\dim \operatorname{Ker} h'(c) = 1$ so one can define the (tangent) vector $T(c)$ as being the unique—up to orientation—unit vector in the kernel. The orientation is chosen so that the nonvanishing determinant

$$\det \begin{bmatrix} h'(c) \\ {}^tT(c) \end{bmatrix}$$

has constant sign on each connected component of Ω. This provides a parameterization by arc length of the connected components of $\{h = 0\}$ which can be practically computed by integrating the following differential equation [2] (with $' = \mathrm{d}/\mathrm{d}s$):

$$c'(s) = T(c(s)), \quad c(0) = c_0 \in \{h = 0\}.$$

The aim is to classify each component up to diffeomorphisms, knowing that there are only two possibilities [21]: It is diffeomorphic either to $\mathbf{R}$ or to $\mathbf{S}^1$.

In such a parameterization, a point $c(\overline{s}) = (p_0(\overline{s}), \lambda(\overline{s}))$ in $\{h = 0\}$ is a *turning point* when $\lambda'(\overline{s}) = 0$. This is equivalent to saying that

$$\operatorname{rank} \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) = n - 1,$$

that is to saying that t_f is a conjugate time for $\lambda = \lambda(\overline{s})$ (and that $p_0(\overline{s})$, $x(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s}))$ are the corresponding critical and conjugate point, respectively). A turning point of order one, that is such that $\lambda''(\overline{s}) \neq 0$, actually results in a change of variation on λ, hence the name. We define $\overline{c} = c(\overline{s}) \in \{h = 0\}$ to be a *first turning point* (along the path starting at $c(0)$) if, for all $s \in [0, \overline{s})$, the curve $t \mapsto x(t, x_0, p_0(s), \lambda(s))$ has no conjugate time on $(0, t_f]$.

[8] For the use of homotopy in optimal control (in particular for motion planning), see also [15, 27] where the point of view is slightly different; the idea is to devise a *path lifting equation* to construct a path of zeros in the infinite dimensional set of controls. In this setting, the obstructions described at the end of the current section are translated as non degeneracy and non-explosion issues on the so-called Wazewski equation. Assuming more structure than we do on the dynamics (driftless affine control systems are considered), the authors are able to provide conditions in terms of the Lie algebra of controlled vector fields that overcome these difficulties. The emphasis is also on Galerkin procedures to solve the problem. To some extent, the situation is simpler in our case as considering the shooting function instead of the endpoint mapping restricts the problem to finite dimension.

Theorem 3. *Let $c(\overline{s}) \in \{h = 0\}$ be a first turning point of order one; then for $s > \overline{s}$, $|s - \overline{s}|$ small enough, there exist conjugate times in $(0, t_f)$.*

We recall that, at a corank one critical point $\overline{x}$ of a smooth function $g : \mathbf{R}^n \to \mathbf{R}^n$, one can define (up to a scalar) the *intrinsic second order derivative* [4] as

$$\overline{\mu} g''(\overline{x})_{|\operatorname{Ker} g'(\overline{x}) \times \operatorname{Ker} g'(\overline{x})} \in \mathscr{L}_2(\operatorname{Ker} g'(\overline{x}), \operatorname{Ker} g'(\overline{x}); \mathbf{R}) \simeq \mathbf{R}$$

where $\overline{\mu} \in (\mathbf{R}^n)^*$ is any nonzero covector with kernel $\operatorname{Im} g'(\overline{x})$. The critical point is said nondegenerate provided this quantity is not zero.

Lemma 3. *The turning point $c(\overline{s})$ is of order one if and only if $p_0(\overline{s})$ is a nondegenerate corank one critical point of $p_0 \mapsto h(p_0, \lambda(\overline{s}))$.*

Proof. Differentiating twice $h(c(s)) = 0$, one gets

$$h''(c(\overline{s})) \cdot (c'(\overline{s}), c'(\overline{s})) + h'(c(\overline{s})) \cdot c''(\overline{s}) = 0.$$

As $c'(\overline{s}) = (p_0'(\overline{s}), 0)$, $p_0'(\overline{s})$ generates the kernel of $\partial h / \partial p_0(p_0(\overline{s}), \lambda(\overline{s}))$ and

$$\frac{\partial h}{\partial p_0}(c(\overline{s})) \cdot p_0''(\overline{s}) + \frac{\partial h}{\partial \lambda}(c(\overline{s})) \cdot \lambda''(\overline{s}) = -\frac{\partial^2 h}{\partial p_0^2}(c(\overline{s})) \cdot (p_0'(\overline{s}), p_0'(\overline{s})).$$

Multiplying both sides by any nonzero $\overline{\mu}$ whose kernel coincides with the image of $\partial h / \partial p_0(p_0(\overline{s}), \lambda(\overline{s}))$, one gets

$$\overline{\mu} \frac{\partial h}{\partial \lambda}(c(\overline{s})) \cdot \lambda''(\overline{s}) = -\overline{\mu} \frac{\partial^2 h}{\partial p_0^2}(c(\overline{s})) \cdot (p_0'(\overline{s}), p_0'(\overline{s})).$$

At the turning point $c(\overline{s}) \in \Omega$, $\partial h / \partial \lambda$ is transverse to the image of $\partial h / \partial p_0$ (regularity). So $\overline{\mu}\, \partial h / \partial \lambda(c(\overline{s})) \neq 0$, and $\lambda''(\overline{s}) = 0$ if and only if the intrinsic second derivative vanishes. □

Remark 5. The order one assumption thus puts some restriction on

$$\frac{\partial^2 x}{\partial p_0^2}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})).$$

It was previously mentioned that the first order derivative with respect to p_0 of this function is connected with second order optimality conditions. Here we have a condition of order three.

Lemma 4. *Let $\overline{x}$ be a corank one critical point of a smooth function $g : \mathbf{R}^n \to \mathbf{R}^n$. Then $\overline{x}$ is degenerate if and only if*

$$(\det g')'(\overline{x}) = 0 \quad \textit{on} \quad \operatorname{Ker} g'(\overline{x}).$$

Proof. Let $h \in \mathbf{R}^n$; one has

$$(\det g')'(\overline{x}) \cdot h = \operatorname{tr}(\widetilde{g'(\overline{x})} \cdot g''(\overline{x}) \cdot h).$$

Since $g'(\overline{x})$ is of rank $n - 1$, one can find a nonzero vector $\overline{\xi} \in \operatorname{Ker} g'(\overline{x})$ (*resp.* covector $\overline{\mu}$ with kernel $\operatorname{Im} g'(\overline{x})$) such that the adjugate matrix

$$\widetilde{g'(\overline{x})} = \overline{\xi}\overline{\mu}.$$

Thus,

$$\begin{aligned}(\det g')'(\overline{x})\cdot h &= \sum_{j=1}^{n}\operatorname{tr}(\overline{\xi}\overline{\mu}\frac{\partial g'}{\partial x_j}(\overline{x}))h_j\\ &= \sum_{j=1}^{n}\overline{\mu}\frac{\partial g'}{\partial x_j}(\overline{x})\overline{\xi}h_j\\ &= \overline{\mu}g''(\overline{x})(\overline{\xi},h).\end{aligned}$$

In particular, for $h=\overline{\xi}\in\operatorname{Ker}g'(\overline{x})$,

$$(\det g')'(\overline{x})\cdot\overline{\xi}=\overline{\mu}g''(\overline{x})(\overline{\xi},\overline{\xi}),$$

whence the conclusion. □

Remark 6. Under the assumptions of the lemma, $\chi(x,\mu)=\det(\mu I_n-g'(x))$ has root $\mu=0$ for $x=\overline{x}$, with algebraic multiplicity $k\le n$ (while the geometric multiplicity of 0, as an eigenvalue of $g'(\overline{x})$, is one). By Malgrange preparation theorem, there are smooth scalar functions $a_0,\dots,a_{k-1}$ and b such that, in the neighbourhood of $(\overline{x},0)$,

$$\chi(x,\mu)=b(x,\mu)(\mu^k+a_{k-1}(x)\mu^{k-1}+\cdots+a_0(x)),$$

and $a_0(\overline{x})=\cdots=a_{k-1}(\overline{x})=0$, $b(\overline{x},0)\neq 0$. Accordingly,

$$(\det g')'(\overline{x})=b(\overline{x},0)a_0'(\overline{x}).$$

The nondegeneracy at $\overline{x}$ is then equivalent to the statement that $\overline{x}$ is not a critical point of a_0, plus that f and a_0 are transverse at $\overline{x}$. The quantity a_0 can be interpretated as a smooth (and signed) singular value of f' when x is varied in the neighbourhood of $\overline{x}$, as is clear from the following example. Take $g(x_1,x_2)=(x_2,x_1^2/2)$; $\overline{x}=(0,0)$ is a nondegenerate corank one critical point. In a small enough neighbourhood of $\overline{x}$, the smallest singular value of

$$g'(x)=\begin{bmatrix}0 & 1\\ x_1 & 0\end{bmatrix}$$

is $\sigma(x)=|x_1|$, which is not differentiable at the critical point. In contrast, $a_0(x)=-x_1$ is smooth and provides the information needed to check nondegeneracy.

Proof (of Theorem 3). Define the extended homotopy

$$\widetilde{h}(p_0,\lambda,t_c)=(x(t_f,x_0,p_0,\lambda),\det\frac{\partial x}{\partial p_0}(t_c,x_0,p_0,\lambda)).$$

By assumption, the point $(p_0(\overline{s}),\lambda(\overline{s}),t_f)$ belongs to $\{\widetilde{h}=0\}$, and is regular. Indeed, vectors $(\delta p_0,\delta\lambda,\delta t_c)$ in the kernel of $\widetilde{h}'$ at this point verify $\delta\lambda=0$ and

$$\frac{\partial h}{\partial p_0}(p_0(\overline{s}),\lambda(\overline{s}))\cdot\delta p_0=0,$$

$$\frac{\partial}{\partial p_0}\det\frac{\partial x}{\partial p_0}(t_f,x_0,p_0(\overline{s}),\lambda(\overline{s}))\cdot\delta p_0+\frac{\partial}{\partial t_c}\det\frac{\partial x}{\partial p_0}(t_f,x_0,p_0(\overline{s}),\lambda(\overline{s}))\cdot\delta t_c=0.$$

One has two cases depending on whether the partial derivative with respect to t_c vanishes or not in the second equation. If it does not, the kernel is one-dimensional and parameterized by

$$\delta p_0 \in \operatorname{Ker} \frac{\partial h}{\partial p_0}(p_0(\overline{s}), \lambda(\overline{s})).$$

If it does, δp_0 has to be zero since otherwise the previous lemmas would imply that

$$\frac{\partial}{\partial p_0} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) \cdot \delta p_0 \neq 0$$

because of the order one assumption on the turning point. So the kernel of $\widetilde{h}'$ is parameterized by $\delta t_c \in \mathbf{R}$ and is also of dimension one. The extended homotopy is therefore well defined and regular in a neighbourhood of $(p_0(\overline{s}), \lambda(\overline{s}), t_f)$. Parameterizing by arc length, σ, on $\{\widetilde{h} = 0\}$, one has $t_c(\overline{\sigma}) = t_f$. Since the point is a first turning point, it is enough to prove that $t_c'(\overline{\sigma}) \neq 0$ (here $' = d/d\sigma$); then necessarily $t_c'(\overline{\sigma}) < 0$, as there would otherwise be conjugate times on $(0, t_f)$ for $\sigma < \overline{\sigma}$ in $\{\widetilde{h} = 0\}$, that is for $s < \overline{s}$ in $\{h = 0\}$. According to the description of the kernel,

$$\frac{\partial}{\partial p_0} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{\sigma}), \lambda(\overline{\sigma})) \cdot p_0'(\overline{\sigma}) + \frac{\partial}{\partial t_c} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{\sigma}), \lambda(\overline{\sigma})) \cdot t_c'(\overline{\sigma}) = 0.$$

The two previous alternatives result either in $p_0'(\overline{\sigma})$ being non-zero, in which case neither the first term in the sum nor $t_c'(\overline{\sigma})$ can vanish; or in $p_0'(\overline{\sigma}) = 0$, so $|t_c'(\overline{\sigma})| = 1$ (unit tangent vector). In both situations we conclude that $t_c'(\overline{\sigma})$ cannot be zero. □

In addition to turning points, there are two other issues on homotopy. First, when the connected component of the path considered is diffeomorphic to $\mathbf{R}$, boundary points (if any) are critical points of h. The classification of points in $\partial\Omega$ starts with the following result which is a simple consequence of Morse lemma.

Proposition 5 ([2]). *Let $\overline{c} \in \{h = 0\}$ be a nondegenerate hyperbolic corank one critical point of h. Then, there are coordinates $d_1, \ldots, d_{n+1}$ such that, in the neighbourhood of $\overline{c}$, $\{h = 0\}$ is equal to*

$$d_1^2 - d_2^2 = 0, \quad d_3 = \cdots = d_{n+1} = 0.$$

In this case, we have a critical point jointly in p_0, λ and the intrinsic second order derivative is, up to a scalar,

$$\overline{\mu} h''(\overline{c})_{|\operatorname{Ker} h'(\overline{c}) \times \operatorname{Ker} h'(\overline{c})} \in \operatorname{Sym}(2, \mathbf{R}) \subset \mathbf{M}(2, \mathbf{R}) \simeq \mathscr{L}_2(\operatorname{Ker} h'(\overline{c}), \operatorname{Ker} h'(\overline{c}); \mathbf{R})$$

where $\overline{\mu} \in (\mathbf{R}^n)^*$ is any nonzero covector with kernel $\operatorname{Im} h'(\overline{c})$. Hyperbolicity means that this order 2 symmetric matrix is nondegenerate with eigenvalues of opposite signs. As a consequence, the path of zeros is locally made of two smooth curves intersecting transversally, resulting in a bifurcation. The last issue is due to global features in parametric optimal control. For a given value λ_0 of the parameter, one has to compare the costs associated to zeros in each connected component of $\{h = 0\} \cap \{\lambda = \lambda_0\}$. Each zero of the shooting homotopy function

defines an extremal and global solutions, if any, are those giving the infimum of the cost among them. In the three-body problem, the topology of the state manifold, $X_\mu = T^*Q_\mu$, comes into play; Q_μ has the topology of the eight curve, $\pi_1(Q_\mu) = \mathbf{Z} * \mathbf{Z}$, and a heuristic classification of extremals based on homology is proposed in [12].

Two homotopies are used to compute numerically minimum time trajectories of the restricted three-body problem. A continuation on the ratio of the masses, μ, is first considered. In practice, the isolated contacts with the codimension two switching surface are neglected, and we restrict the computation to smooth extremals without π-singularities. This yields regularity of the μ-parameterized minimum time problem in view of

Lemma 5. *In the absence of π-singularities, Legendre condition holds.*

Proof. If the switching function never vanishes, one has $|u| = 1$ everywhere. So we can restrict the control set U to $\mathbf{S}^1$. The manifold is without boundary and, in the chart $u = (\cos\alpha, \sin\alpha)$, $\alpha \in \mathbf{R}$, one has[9]

$$H(x, \alpha, p) = p^0 + H_0(x, p) + \varepsilon(\cos\alpha H_1(x, p) + \sin\alpha H_2(x, p)).$$

Accordingly,

$$\nabla^2_{uu} H(x, \alpha, p) = -\varepsilon(\cos\alpha H_1(x, p) + \sin\alpha H_2(x, p)).$$

Along an extremal, $\nabla^2_{uu} H(x(t), \alpha(t), p(t)) = -\varepsilon|\psi(t)|$ which is bounded over by some negative constant on $[0, t_f]$ as ψ is smooth and nonvanishing. □

Using previous knowledge on the two-body minimum time trajectories [14, 13], we are able to compute transfers from a circular orbit around the first primary towards the L_2 point when $\mu = 0$, then to follow the path until any value $\mu \in (0, 1)$. The absence of conjugate points—ensuring local optimality—is checked along the path using the `hampath` code [10], see Fig. 3. A continuation on the target eventually allows to obtain solutions, for instance in the Earth-Moon system ($\mu \simeq 1.21e - 2$) from a geostationary orbit to a circular lunar one for average values of the control magnitude ε, see Fig. 4. To reach lower values of ε, a continuation on this parameter is finally employed as in [13], which emphasizes the role of the topology of the state space previously mentioned. Many local minima exist, yielding as many zeros of the shooting function. When decreasing ε, at some point on the resulting path the number of revolutions around the first primary has to be increased to retain global optimality, which means jumping to another connected component (*branch*) of the zero set (see Fig. 5). Schematically, the situation is analogous to the one in Riemannian geometry with cut and conjugate points: Up to some point, the path provides minimizers; then global optimality is lost (this is typically a global phenomenon, for instance due to the topology of the manifold), but local optimality persists up to another point; past this second point (a turning point, in the simple case we framed in the beginning of the Section), even local optimality is lost (see Fig. 6).

The computations combining shooting and homotopy presented here are meant to initialize the solution of more complicated problems. A time minimum

[9] The Hamiltonian lift H_0 implicitly depends on μ, since $F_0 = \overrightarrow{J_\mu}$ (compare Section 1).

trajectory of the three-dimensional model of the SMART1 mission [24] is given Fig. 7. Work in progress include the treatment of the maximization of the final mass.

Acknowledgements

The authors thank Joseph Gergaud for introducing them to homotopy in optimal control, and Bernard Bonnard for his geometric influence on this work.

References

[1] Agrachev, A. A.; Sachkov, Y. L. *Control Theory from the Geometric Viewpoint*, Springer, 2004.

[2] Allgower, E. L.; Georg, K. *Introduction to numerical continuation methods*, SIAM, 2003.

[3] Betts, J. T.; Erb, S. O. Optimal Low Thrust Trajectories to the Moon. *SIAM J. Appl. Dyn. Syst.* **2** (2003), no. 2, 144–170.

[4] Boardman, J. M. Singularities of differentiable maps. *Publ. Math. IHES*, **33** (1967), 383–419.

[5] Bombrun, A.; Chetboun, J.; Pomet, J.-B. Transfert Terre-Lune en poussée faible par contrôle feedback. La mission SMART-1. *INRIA Research report* (2006), no. 5955, 1–27.

[6] Bombrun, A.; Pomet, J.-B. Asymptotic behavior of time optimal orbital transfer for low thrust 2-body control system. *Discrete Contin. Dyn. Syst.* **suppl.** (2007), 122–129.

[7] Bonnard, B.; Caillau, J.-B.; Trélat, E. Geometric optimal control of elliptic Keplerian orbits. *Discrete Contin. Dyn. Syst. Ser. B* **5** (2005), no. 4, 929–956.

[8] Bonnard, B.; Chyba, M. *Singular trajectories and their role in control theory*, Springer, 2003.

[9] Bonnard, B.; Shcherbakova, N.; Sugny, D. The smooth continuation method in optimal control with an application to quantum systems. *ESAIM Control Optim. and Calc. Var.* **17** (2010), no. 1, 267–292.

[10] Caillau, J.-B.; Cots, O.; Gergaud, J. Differential continuation for regular optimal control problems, *Optim. Methods Softw.*, to appear. `apo.enseeiht.fr/hampath`

[11] Caillau, J.-B.; Daoud, B.; Gergaud, J. On some Riemannian aspects of two and three-body controlled problems. *Recent Advances in Optimization and its Applications in Engineering*, 205–224, Springer, 2010. Proceedings of the 14th Belgium-Franco-German conference on Optimization, Leuven, September 2009.

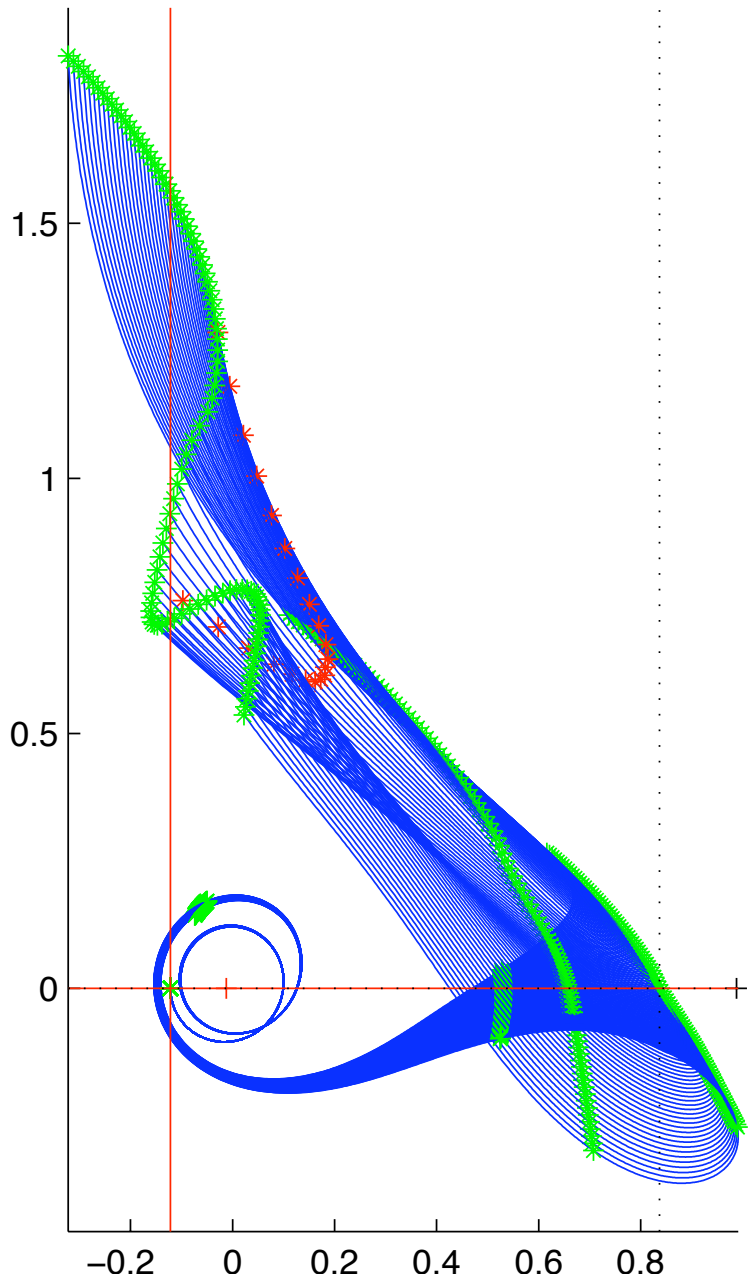

Figure 3: Conjugate point computation (rotating frame). Extremals (here projected on the (q_1, q_2)-space) from a circular orbit around the first primary towards the L_2 Lagrange point are extended beyond the target. Conjugate points, in red, appear after t_f, ensuring local optimality. Green dots indicate isocost (time) lines.

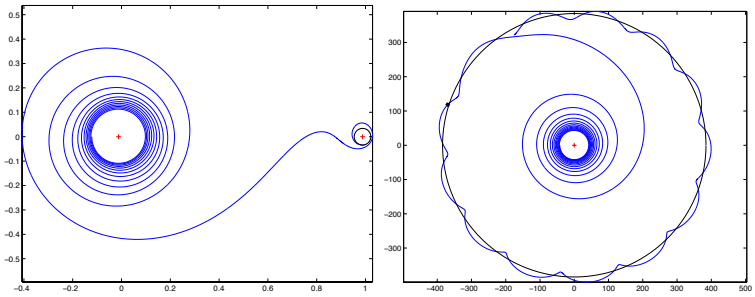

Figure 4: Minimum time trajectory in the Earth-Moon system ($\mu \simeq 1.21e-2$, $\varepsilon = 2.44e-1$). Left, in the rotating frame; right, in the fixed frame to emphasize capture by the second primary at the end of the transfer.

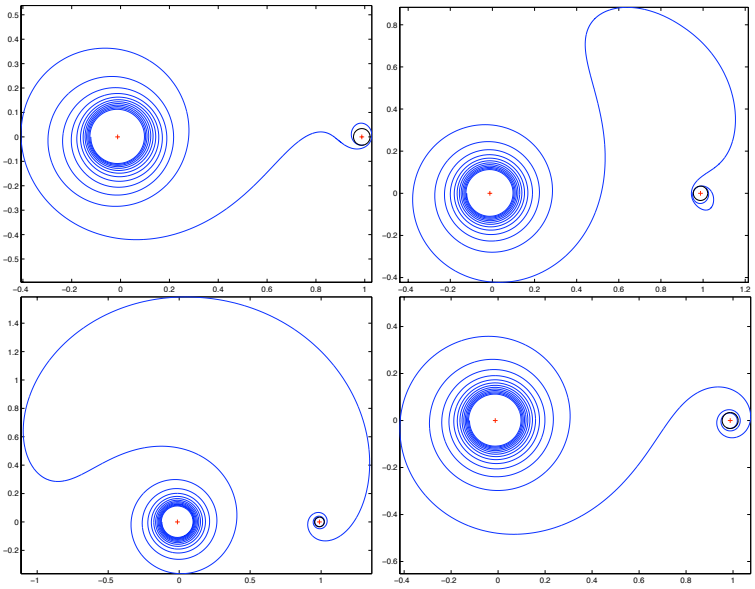

Figure 5: Minimum time trajectory for ε between $2.44e-1$ and $2.196e-1$ ($\mu \simeq 1.21e-2$, rotating frame). As the control magnitude is decreased, strategies are evolved. In the upper graphs, the first two extremals have the same rotation number around the first primary and both wind around the second one positively. Conversely, the third extremal (bottom left) winds negatively around the second primary, while the fourth (bottom right) makes an additional revolution around the first one.

[12] Caillau, J.-B.; Daoud, B.; Gergaud, J. Discrete and differential homotopy in circular restricted three-body control. *AIMS Proceedings*, to appear. Proceedings of 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Dresden, May 2010.

[13] Caillau, J.-B.; Gergaud, J.; Noailles, J. 3D Geosynchronous Transfer of a Satellite: Continuation on the Thrust. *J. Optim. Theory Appl.* **118** (2003), no. 3, 541–565.

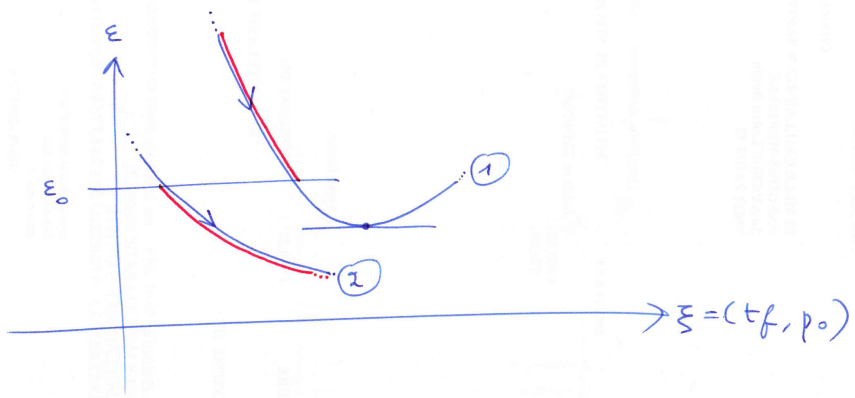

Figure 6: Continuation on ε. Down to ε_0, branch 1 yields minimizers. In the case of free final time, the shooting unknown is $\xi = (t_f, p_0)$, p_0 belonging to the zero level set of the Hamiltonian. Past ε_0, global optimality is lost on branch 1 and a switch to branch 2 has to be made (resulting in a loss of regularity of the value function). Past the turning point on branch 1 (conjugacy of the target point), even local optimality is lost.

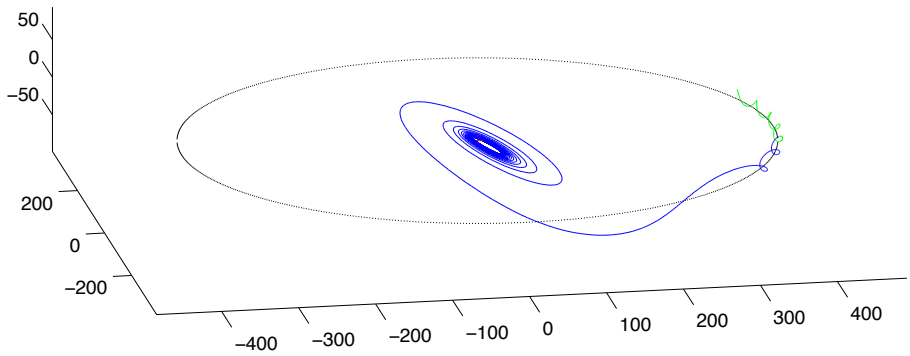

Figure 7: Three-dimensional minimum time transfer, SMART1 boundary conditions (fixed frame). The control magnitude ε is 0.7 Newtons for an initial mass of 350 Kilograms and a specific impulse of 1640 seconds (the variation of mass has been taken into account for this simulation, see [24]). The final time is 26.2 days.

[14] Caillau, J.-B.; Noailles, J. Coplanar control of a satellite around the Earth. *ESAIM Control Optim. and Calc. Var.* **6** (2001), 239–258.

[15] Chitour, Y. A continuation method for motion-planning problems. {em ESAIM Control Optim. Calc. Var.} **12** (2006), no. 1, 139-168.

[16] Edelbaum, T. N. Optimal low-thrust rendez-vous and station keeping. *AIAA J.* **2** (1964), no. 7, 1196–1201.

[17] Gómez, G.; Jorba, A.; Madesmont, J.; Simó, C. *Dynamics and mission design near libration points Vol. III. Advanced methods on collinear points.* World Scientific Publishing, 2001.

[18] Jurdjevic, V. *Geometric control theory*, Cambridge University Press, 1996.

[19] Kupka, I.; Sallet, G. A sufficient condition for the transitivity of pseudo-semigroups: Application to system theory. *J. Differential. Eq.* **47** (1983), 462–470.

[20] Marsden, J. E.; Ross, S. D. New methods in celestial mechanics and mission design. *Bull. Amer. Math. Soc. (N.S.)* **43** (2006), no. 1, 43–73.

[21] Milnor, J. W. *Topology from the Differentiable Viewpoint*, Princeton University Press, 1997.

[22] Mingotti, G.; Topputo, F.; Bernelli-Zazzera, F. Low-energy, low-thrust transfers to the Moon. *Celest. Mech. Dyn. Astr.* **105** (2009), 61–74.

[23] Ozimek, M. T.; Howell, K. C. Low-thrust transfers in the Earth-Moon system, including applications to libration point orbits. *J. Guidance Control Dynam.* **33** (2010), no. 2, 533–549.

[24] Racca, G.; *et al.* SMART-1 mission description and development status. *Planetary and space science* **50** (2002), 1323–1337.

[25] Siegel, C. L.; Moser, J. K. *Lectures on celestial mechanics.* Berlin, 1971.

[26] Sussmann, H. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.* **180** (1973), 171–188.

[27] Sussmann, H. A continuation method for nonholonomic path-finding problems. *IEEE Publications* (1993), 2718–2723. Proceedings of the 32nd IEEE CDC, San Antonio, December 1993.

[28] Szebehely, V. *Theory of orbits: The restricted problem of three bodies*, Academic Press, 1967.