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## Minimum time control of the restricted three-body problem

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For  $\mu \in (0,1)$ , let  $Q_{\mu} = \mathbf{C} \setminus \{-\mu, 1-\mu\}$ . For  $q \in Q_{\mu} \subset \mathbf{C} \simeq \mathbf{R}^2$  and positive  $\varepsilon$ , define the controlled circular restricted three body problem (planar model) according to<sup>1</sup>

$$\ddot{q}(t) - 2\overline{\partial}\Omega_{\mu}(q(t)) + 2i\dot{q}(t) = \varepsilon u(t), \quad |u(t)| \le 1.$$

Here,  $(q, \dot{q}) \in X_{\mu} = TQ_{\mu} \simeq Q_{\mu} \times \mathbf{C}$  are Cartesian coordinates in the rotating frame and

$$\Omega_{\mu}(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}.$$

Another choice of coordinates consists in letting  $X_{\mu} = T^*Q_{\mu}$ , taking the cotangent bundle instead to write the drift in Hamiltonian form. Let  $p = \dot{q} + iq$  and let

$$\begin{aligned} J_{\mu}(q,\dot{q}) &= \frac{1}{2}|\dot{q}|^2 - \Omega_{\mu}(q), \\ &= \frac{1}{2}|p|^2 + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|} \end{aligned}$$

be the Jacobian integral. Then,

$$\dot{q}(t) = \frac{\partial J_{\mu}}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial J_{\mu}}{\partial q}(q(t), p(t)) + \varepsilon u(t), \quad |u(t)| \le 1.$$

More compactly,

$$\dot{x}(t) = F_0(x(t)) + \varepsilon u_1(t)F_1(x(t)) + \varepsilon u_2(t)F_2(x(t)), \quad |u(t)| \le 1,$$

with, in (q, p) coordinates for  $x \in X_{\mu}$ ,

$$F_0 = \overrightarrow{J_{\mu}}, \quad F_1 = \frac{\partial}{\partial p_1}, \quad F_2 = \frac{\partial}{\partial p_2},$$

where  $\overrightarrow{J_{\mu}} = (\nabla_p J_{\mu}, -\nabla_q J_{\mu})$  is the symplectic gradient. The drift has five equilibrium points  $L_1(\mu), \ldots, L_5(\mu)$  known as Lagrange points [3] whose position depends on  $\mu$ . Let  $j_i(\mu) = J_{\mu}(L_i(\mu)), i = 1, \dots, 5$  denote the Jacobian constants of these points. For  $\mu \in (0, 1)$ ,  $j_2 < j_1 < j_3 < j_4 =$  $j_5$ . The open subset  $\{x \in X_{\mu} \mid J_{\mu}(x) < j_1(\mu)\}$  has two connected components, and we denote by  $X_{\mu}^1$  the component containing  $L_2(\mu)$  (see Fig. 1).

<sup>&</sup>lt;sup>1</sup>We use  $\overline{\partial} = (1/2)(\partial/\partial q_1 + i\partial/\partial q_2)$  to represent the gradient with respect to q is complex form.



Figure 1: Projection of the open submanifold  $X^1_{\mu}$  in the  $(q_1, q_2, J_{\mu})$ -space. The boundary of the volume is an apparent contour generated by the projection. It is the *zero velocity*set. Above each interior point there is an  $\mathbf{S}^1$ -fiber corresponding to the argument of  $\dot{q}$ . For  $\mu \in (0, 1), j_2 < j_1$  and  $X^1_{\mu}$  is connex.

**Theorem 1.** For any  $\mu \in (0,1)$ , for any positive  $\varepsilon$ , the system is controllable on  $X^1_{\mu}$ .

Proof. The control set is a neighbourhood of the origin and, as the system is a second order controlled system, the family  $\{F_0, F_1, F_2\}$  is bracket generating as, for all  $x \in X_{\mu}$ ,  $\operatorname{Vect}_x\{F_1, F_2, [F_0, F_1], [F_0, F_2]\}$  is of rank four. We may so conclude that the system is controllable, provided the drift is recurrent [1]. Let  $x_0, x_f$  in  $X^1_{\mu}$ , and let j be strictly smaller than the Jacobian constants of both endpoints. Set  $\tilde{X}^1_{\mu} = X^1_{\mu} \cap \{J_{\mu} > j\}$ . Outside a subset of zero measure associated with initial conditions generating collisions  $(q = -\mu \text{ or } 1 - \mu)$ , the drift is a complete Hamiltonian vector field whose exponential is defined for all times and is a volume preserving bijection in (q, p) coordinates. By definition,  $\tilde{X}^1_{\mu}$  which is a union of level sets of the Hamiltonian  $J_{\mu}$  is invariant with respect to the exponential. For  $x = (q, p) \in \tilde{X}^1_{\mu}$ ,

$$j + \Omega_{\mu}(q) < \frac{1}{2}|p - iq|^2 < j_1(\mu) + \Omega_{\mu}(q).$$

Then, for a fixed q, the volume of the q-section of  $\widetilde{X}^1_{\mu}$  is bounded by  $2\pi(j_1(\mu)-j)$ as is clear integrating with respect to  $dp_1dp_2 = \rho d\rho d\alpha$  (set  $p - iq = \rho \exp(i\alpha)$ ). Since the projection on the  $(q_1, q_2)$ -space of  $\widetilde{X}^1_{\mu}$  is also bounded, the  $dq \wedge dp$ measure of  $\widetilde{X}^1_{\mu}$  is finite (Fubini). We conclude as in [2] that almost every point of  $\widetilde{X}^1_{\mu}$  is recurrent by Poincaré theorem. Controllability on  $\widetilde{X}^1_{\mu}$  implies the existence of a trajectory joining  $x_0$  to  $x_f$ , which in turn implies controllability on  $X^1_{\mu}$ .

#### References

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