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# Minimum time control of the restricted three-body problem

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For  $\mu \in (0, 1)$ , let  $Q_\mu = \mathbf{C} \setminus \{-\mu, 1 - \mu\}$ . For  $q \in Q_\mu \subset \mathbf{C} \simeq \mathbf{R}^2$  and positive  $\varepsilon$ , define the controlled circular restricted three body problem (planar model) according to<sup>1</sup>

$$\ddot{q}(t) - 2\bar{\partial}\Omega_\mu(q(t)) + 2i\dot{q}(t) = \varepsilon u(t), \quad |u(t)| \leq 1.$$

Here,  $(q, \dot{q}) \in X_\mu = TQ_\mu \simeq Q_\mu \times \mathbf{C}$  are Cartesian coordinates in the rotating frame and

$$\Omega_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}.$$

Another choice of coordinates consists in letting  $X_\mu = T^*Q_\mu$ , taking the cotangent bundle instead to write the drift in Hamiltonian form. Let  $p = \dot{q} + iq$  and let

$$\begin{aligned} J_\mu(q, \dot{q}) &= \frac{1}{2}|\dot{q}|^2 - \Omega_\mu(q), \\ &= \frac{1}{2}|p|^2 + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|} \end{aligned}$$

be the Jacobian integral. Then,

$$\dot{q}(t) = \frac{\partial J_\mu}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial J_\mu}{\partial q}(q(t), p(t)) + \varepsilon u(t), \quad |u(t)| \leq 1.$$

More compactly,

$$\dot{x}(t) = F_0(x(t)) + \varepsilon u_1(t)F_1(x(t)) + \varepsilon u_2(t)F_2(x(t)), \quad |u(t)| \leq 1,$$

with, in  $(q, p)$  coordinates for  $x \in X_\mu$ ,

$$F_0 = \vec{J}_\mu, \quad F_1 = \frac{\partial}{\partial p_1}, \quad F_2 = \frac{\partial}{\partial p_2},$$

where  $\vec{J}_\mu = (\nabla_p J_\mu, -\nabla_q J_\mu)$  is the symplectic gradient.

The drift has five equilibrium points  $L_1(\mu), \dots, L_5(\mu)$  known as Lagrange points [3] whose position depends on  $\mu$ . Let  $j_i(\mu) = J_\mu(L_i(\mu))$ ,  $i = 1, \dots, 5$  denote the Jacobian constants of these points. For  $\mu \in (0, 1)$ ,  $j_2 < j_1 < j_3 < j_4 = j_5$ . The open subset  $\{x \in X_\mu \mid J_\mu(x) < j_1(\mu)\}$  has two connected components, and we denote by  $X_\mu^1$  the component containing  $L_2(\mu)$  (see Fig. 1).

<sup>1</sup>We use  $\bar{\partial} = (1/2)(\partial/\partial q_1 + i\partial/\partial q_2)$  to represent the gradient with respect to  $q$  in complex form.

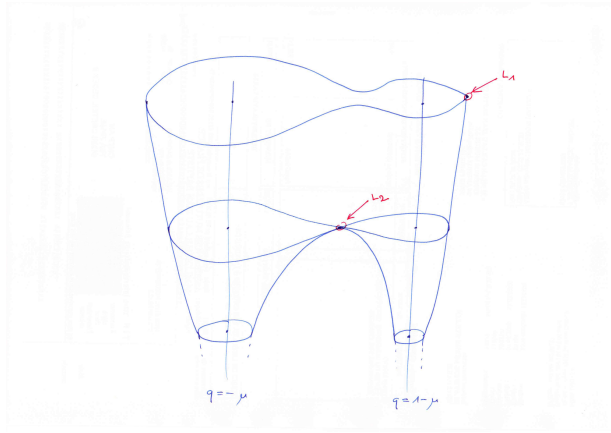


Figure 1: Projection of the open submanifold  $X_\mu^1$  in the  $(q_1, q_2, J_\mu)$ -space. The boundary of the volume is an apparent contour generated by the projection. It is the *zero velocity set*. Above each interior point there is an  $\mathbf{S}^1$ -fiber corresponding to the argument of  $\dot{q}$ . For  $\mu \in (0, 1)$ ,  $j_2 < j_1$  and  $X_\mu^1$  is connex.

**Theorem 1.** *For any  $\mu \in (0, 1)$ , for any positive  $\varepsilon$ , the system is controllable on  $X_\mu^1$ .*

*Proof.* The control set is a neighbourhood of the origin and, as the system is a second order controlled system, the family  $\{F_0, F_1, F_2\}$  is bracket generating as, for all  $x \in X_\mu$ ,  $\text{Vect}_x\{F_1, F_2, [F_0, F_1], [F_0, F_2]\}$  is of rank four. We may so conclude that the system is controllable, provided the drift is recurrent [1]. Let  $x_0, x_f$  in  $X_\mu^1$ , and let  $j$  be strictly smaller than the Jacobian constants of both endpoints. Set  $\tilde{X}_\mu^1 = X_\mu^1 \cap \{J_\mu > j\}$ . Outside a subset of zero measure associated with initial conditions generating collisions ( $q = -\mu$  or  $1 - \mu$ ), the drift is a complete Hamiltonian vector field whose exponential is defined for all times and is a volume preserving bijection in  $(q, p)$  coordinates. By definition,  $\tilde{X}_\mu^1$  which is a union of level sets of the Hamiltonian  $J_\mu$  is invariant with respect to the exponential. For  $x = (q, p) \in \tilde{X}_\mu^1$ ,

$$j + \Omega_\mu(q) < \frac{1}{2}|p - iq|^2 < j_1(\mu) + \Omega_\mu(q).$$

Then, for a fixed  $q$ , the volume of the  $q$ -section of  $\tilde{X}_\mu^1$  is bounded by  $2\pi(j_1(\mu) - j)$  as is clear integrating with respect to  $dp_1 dp_2 = \rho d\rho d\alpha$  (set  $p - iq = \rho \exp(i\alpha)$ ). Since the projection on the  $(q_1, q_2)$ -space of  $\tilde{X}_\mu^1$  is also bounded, the  $dq \wedge dp$ -measure of  $\tilde{X}_\mu^1$  is finite (Fubini). We conclude as in [2] that almost every point of  $\tilde{X}_\mu^1$  is recurrent by Poincaré theorem. Controllability on  $\tilde{X}_\mu^1$  implies the existence of a trajectory joining  $x_0$  to  $x_f$ , which in turn implies controllability on  $X_\mu^1$ .  $\square$

## References

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- [3] Szebehely, V. *Theory of orbits: The restricted problem of three bodies*, Academic Press, 1967.