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Jean-Baptiste Caillau, Bilel Daoud

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Minimum time control of the restricted three-body problem

J.-B. Caillau

B. Daoud

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For $\mu \in (0,1)$, let $Q_{\mu} = \mathbf{C} \setminus \{-\mu, 1-\mu\}$. For $q \in Q_{\mu} \subset \mathbf{C} \simeq \mathbf{R}^2$ and positive ε , define the controlled circular restricted three body problem (planar model) according to¹

$$\ddot{q}(t) - 2\overline{\partial}\Omega_{\mu}(q(t)) + 2i\dot{q}(t) = \varepsilon u(t), \quad |u(t)| \le 1.$$

Here, $(q, \dot{q}) \in X_{\mu} = TQ_{\mu} \simeq Q_{\mu} \times \mathbf{C}$ are Cartesian coordinates in the rotating frame and

$$\Omega_{\mu}(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}$$

Another choice of coordinates consists in letting $X_{\mu} = T^*Q_{\mu}$, taking the cotangent bundle instead to write the drift in Hamiltonian form. Let $p = \dot{q} + iq$ and

$$J_{\mu}(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \Omega_{\mu}(q),$$

= $\frac{1}{2}|p|^2 + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|}$

be the Jacobian integral. Then,

$$\dot{q}(t) = \frac{\partial J_{\mu}}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial J_{\mu}}{\partial q}(q(t), p(t)) + \varepsilon u(t), \quad |u(t)| \leq 1.$$

More compactly,

$$\dot{x}(t) = F_0(x(t)) + \varepsilon u_1(t) F_1(x(t)) + \varepsilon u_2(t) F_2(x(t)), \quad |u(t)| \le 1,$$

with, in (q, p) coordinates for $x \in X_{\mu}$,

$$F_0 = \overrightarrow{J_{\mu}}, \quad F_1 = \frac{\partial}{\partial p_1}, \quad F_2 = \frac{\partial}{\partial p_2},$$

where $\overrightarrow{J_{\mu}} = (\nabla_p J_{\mu}, -\nabla_q J_{\mu})$ is the symplectic gradient. The drift has five equilibrium points $L_1(\mu), \ldots, L_5(\mu)$ known as Lagrange points [3] whose position depends on μ . Let $j_i(\mu) = J_{\mu}(L_i(\mu)), i = 1, \dots, 5$ denote the Jacobian constants of these points. For $\mu \in (0,1)$, $j_2 < j_1 < j_3 < j_4 =$ j_5 . The open subset $\{x \in X_{\mu} \mid J_{\mu}(x) < j_1(\mu)\}\$ has two connected components, and we denote by X_{μ}^{1} the component containing $L_{2}(\mu)$ (see Fig. 1).

¹We use $\overline{\partial} = (1/2)(\partial/\partial q_1 + i\partial/\partial q_2)$ to represent the gradient with respect to q is complex

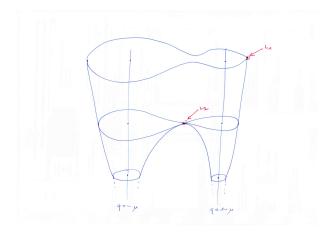


Figure 1: Projection of the open submanifold X^1_{μ} in the (q_1,q_2,J_{μ}) -space. The boundary of the volume is an apparent contour generated by the projection. It is the zero velocityset. Above each interior point there is an \mathbf{S}^1 -fiber corresponding to the argument of \dot{q} . For $\mu \in (0,1), j_2 < j_1$ and X^1_{μ} is connex.

Theorem 1. For any $\mu \in (0,1)$, for any positive ε , the system is controllable on X^1_{μ} .

Proof. The control set is a neighbourhood of the origin and, as the system is a second order controlled system, the family $\{F_0, F_1, F_2\}$ is bracket generating as, for all $x \in X_\mu$, $\mathrm{Vect}_x\{F_1, F_2, [F_0, F_1], [F_0, F_2]\}$ is of rank four. We may so conclude that the system is controllable, provided the drift is recurrent [1]. Let x_0, x_f in X^1_μ , and let j be strictly smaller than the Jacobian constants of both endpoints. Set $\widetilde{X}^1_\mu = X^1_\mu \cap \{J_\mu > j\}$. Outside a subset of zero measure associated with initial conditions generating collisions $(q = -\mu \text{ or } 1 - \mu)$, the drift is a complete Hamiltonian vector field whose exponential is defined for all times and is a volume preserving bijection in (q,p) coordinates. By definition, \widetilde{X}^1_μ which is a union of level sets of the Hamiltonian J_μ is invariant with respect to the exponential. For $x = (q,p) \in \widetilde{X}^1_\mu$,

$$j + \Omega_{\mu}(q) < \frac{1}{2}|p - iq|^2 < j_1(\mu) + \Omega_{\mu}(q).$$

Then, for a fixed q, the volume of the q-section of \widetilde{X}^1_μ is bounded by $2\pi(j_1(\mu)-j)$ as is clear integrating with respect to $\mathrm{d} p_1 \mathrm{d} p_2 = \rho \, \mathrm{d} \rho \, \mathrm{d} \alpha$ (set $p-iq=\rho \exp(i\alpha)$). Since the projection on the (q_1,q_2) -space of \widetilde{X}^1_μ is also bounded, the $\mathrm{d} q \wedge \mathrm{d} p$ -measure of \widetilde{X}^1_μ is finite (Fubini). We conclude as in [2] that almost every point of \widetilde{X}^1_μ is recurrent by Poincaré theorem. Controllability on \widetilde{X}^1_μ implies the existence of a trajectory joining x_0 to x_f , which in turn implies controllability on X^1_μ .

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