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► **To cite this version:**

Dimitrios V. Vougas. Generalized least squares transformation and estimation with autoregressive error. *Statistics and Probability Letters*, 2009, 78 (4), pp.402. 10.1016/j.spl.2007.07.025 . hal-00598962

HAL Id: hal-00598962

<https://hal.science/hal-00598962>

Submitted on 8 Jun 2011

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Author's Accepted Manuscript

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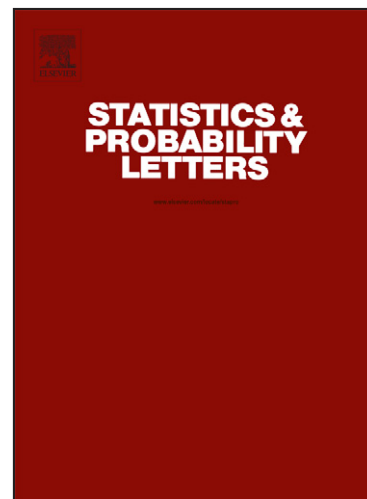
PII: S0167-7152(07)00254-4
DOI: doi:10.1016/j.spl.2007.07.025
Reference: STAPRO 4720

To appear in: *Statistics & Probability Letters*

Received date: 6 December 2005
Revised date: 6 March 2007
Accepted date: 25 July 2007

Cite this article as: Dimitrios V. Vougas, Generalized least squares transformation and estimation with autoregressive error, *Statistics & Probability Letters* (2007), doi:10.1016/j.spl.2007.07.025

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**Generalized Least Squares Transformation and Estimation
with Autoregressive Error**

by

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Brief Title: **Estimation with Autoregressive Error**

Accepted manuscript

Abstract

Approximations of the usual GLS transformation matrices are proposed for estimation with AR error that remove boundary discontinuities. The proposed method avoids constrained optimization or rules of thumb that unnecessarily enforce estimated parameters to be in the interior.

JEL Classification Numbers: C12, C13, C22.

Key Words and Phrases: Gaussian AR Model; GLS Transformation; NLS and QML Estimation; LR Test.

1 Introduction

Nonlinear least squares (NLS) and quasi maximum likelihood (QML) estimation of models with autoregressive (AR) error requires implementing restrictions via (numerically expensive) constrained optimization. For stationary AR(1) error with coefficient ρ , $|\rho| < 1$, a starting error with variance proportional to $(1 - \rho^2)^{-1}$ is assumed. This introduces the Prais and Winsten (1954) transformation, see also Gurland (1954) and Kadiyala (1968). The restriction $|\rho| < 1$ must be implemented to avoid numerical problems, see Hamilton (1994, pp. 118-119), and $|\rho| \geq 1$ is prohibited. Hence, unit root inference is prevented in this context. This paper suitably modifies the usual GLS transformation matrix and removes the discontinuity at $|\rho| \geq 1$, avoiding restrictions and allowing for unit root inference. Note that the approach of Andrews (1999) may not be as successful as the proposed approach. The paper also applies a similar approach for estimation with AR(2) error. The paper is organized as follows: Section 2 presents the new GLS transformations and resulting NLS and QML estimation methods for a regression with AR(1) or AR(2) error, also exemplifying LR (unit root) testing. Finally, Section 3 concludes.

2 Transformations and NLS and QML Estimation

Focusing on the linear regression (generating model)

$$y = X\beta + u, \quad (1)$$

where y and u are $T \times 1$, X is $T \times k$ ($k < T$) and β is $k \times 1$ parameter vector. The error u is AR(1), generated via

$$Pu = \varepsilon, \quad (2)$$

where $\varepsilon \sim N(0, \sigma^2 I_T)$ (I_T is the $T \times T$ identity matrix). The GLS transformation matrix P is defined by $P[1, 1] = \kappa$, $P[j, j] = 1$ and $P[j, j - 1] = -\rho$ for $j = 2, 3, \dots, T$ and 0 elsewhere. A specific value for κ implies a corresponding assumption for u_1 , see Dufour (1990, p. 475) and Dufour and King (1991, p. 116). Setting $\kappa = 0$ ignores the first observation, while $\kappa = 1$ is the assumption of Berenblut and Webb (1973). In general, $u_1 \sim N(0, \sigma^2 \kappa^{-2})$ and independent from $\varepsilon_2, \dots, \varepsilon_T$. Under stationarity, $|\rho| < 1$, and for identification, it is assumed that $\kappa = (1 - \rho^2)^{1/2}$. In this case, $u \sim N(0, \sigma^2 \Sigma(\rho))$ with $\Sigma(\rho) = (P'P)^{-1}$. Still under stationarity, NLS minimizes

$$S(\beta, \rho) = u'P'Pu, \quad (3)$$

while QML maximizes the (concentrated, with a constant ignored) log-likelihood

$$l(\beta, \rho) = -\frac{T}{2} \ln S(\beta, \rho) + \frac{1}{2} \ln(1 - \rho^2), \quad (4)$$

see Judge *et al.* (1985, Chapter 8). Beach and MacKinnon (1978 a) introduce an iterative algorithm to derive the QML estimators. From Eq. (4), $l(\beta, \rho)$ is not defined for $|\rho| \geq 1$. Constrained maximization is required, that is imposing $|\rho| < 1$. Also, NLS and QML estimators differ in general.

To avoid constrained optimization and incorporate stationarity asymptotically, one defines and employs κ as a function of T and ρ ,

$$\kappa \equiv \kappa(T, \rho) := \left\{ \sum_{s=0}^{T-1} \rho^{2s} \right\}^{-1/2} \equiv \left\{ \frac{1 - \rho^{2T}}{1 - \rho^2} \right\}^{-1/2} > 0, \quad \forall \rho, \quad (5)$$

in line with Gurland (1954, p. 221). For $|\rho| < 1$ as $T \rightarrow \infty$, $\kappa(T, \rho) \rightarrow \sqrt{1 - \rho^2}$.

Define P^* equal to P for all elements except $P^*[1, 1] = \kappa(T, \rho)$. It is assumed that $P^*u = \varepsilon$ and $u \sim N(0, \sigma^2 \Sigma(\rho)^*)$ with $\Sigma(\rho)^* = (P^*P^*)^{-1}$. The proposed NLS now minimizes

$$S^*(\beta, \rho) = \varepsilon' P^* P^* \varepsilon, \quad (6)$$

while the new QML maximizes

$$l^*(\beta, \rho) = -\frac{T}{2} \ln S^*(\beta, \rho) + \frac{1}{2} \ln \kappa(T, \rho)^2. \quad (7)$$

Note that any value for ρ is allowed, including $|\rho| \geq 1$. For $|\rho| < 1$ as $T \rightarrow \infty$, $u_1 \xrightarrow{d} N(0, \sigma^2/(1 - \rho^2))$ (the original assumption under stationarity). For $|\rho| = 1$, it holds $\kappa(T, \rho) = T^{-1/2}$. Resulting NLS and QML estimators also differ. Equivalence of S^* and l^* does not apply and LR tests based on S^* differ from LR tests based

on l^* . Let $\check{\beta}$, $\check{\rho}$ and $\bar{\beta}$, $\bar{\rho}$ denote restricted (under a null) and unrestricted estimates, respectively, from NLS estimation, while $\hat{\beta}$, $\hat{\rho}$ and $\tilde{\beta}$, $\tilde{\rho}$ denote restricted (under a null) and unrestricted estimates, respectively, from QML estimation. The corresponding LR tests are

$$LR_{NLS} = T \ln(S^*(\check{\beta}, \check{\rho})/S^*(\bar{\beta}, \bar{\rho})) \approx T(S^*(\check{\beta}, \check{\rho}) - S^*(\bar{\beta}, \bar{\rho}))/S^*(\bar{\beta}, \bar{\rho}), \quad (8)$$

and

$$LR_{QML} = 2(l^*(\tilde{\beta}, \tilde{\rho}) - l^*(\hat{\beta}, \hat{\rho})). \quad (9)$$

Estimation of Eq. (1) with AR(2) error follows similar lines. In this case, u_t is generated by

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t \quad \text{or} \quad Qu = \varepsilon. \quad (10)$$

The first two autocorrelations, θ_1 and θ_2 , are $\theta_1 = \rho_1/(1 - \rho_2)$ and $\theta_2 = \rho_2 + \rho_1\theta_1$, respectively. Under stationarity $|\rho_2| < 1$ and $|\theta_1| < 1$; the last part comes from $\rho_1 < 1 - \rho_2$ and $\rho_2 < 1$. The transformation matrix Q is defined as $Q[1, 1] = \sqrt{1 - \rho_2^2}\sqrt{1 - \theta_1^2}$ (derived from Judge *et al.* (1985, p. 294)), $Q[2, 1] = -\theta_1\sqrt{1 - \rho_2^2}$, $Q[2, 2] = \sqrt{1 - \rho_2^2}$, $Q[j, j] = 1$, $Q[j, j - 1] = -\rho_1$, $Q[j, j - 2] = -\rho_2$ for $j = 3, \dots, T$ and 0 elsewhere. Beach and MacKinnon (1978 b) provide a specific algorithm for QML estimation with AR(2) error. Similarly to the analysis above, one defines

$$\kappa_1 \equiv \kappa_1(T, \theta_1) := \left\{ \sum_{s=0}^{T-1} \theta_1^{2s} \right\}^{-1/2} \equiv \left\{ \frac{1 - \theta_1^{2T}}{1 - \theta_1^2} \right\}^{-1/2} > 0, \quad \forall \rho_1, \rho_2, \quad (11)$$

and

$$\kappa_2 \equiv \kappa_2(T, \rho_2) := \left\{ \sum_{s=0}^{T-1} \rho_2^{2s} \right\}^{-1/2} \equiv \left\{ \frac{1 - \rho_2^{2T}}{1 - \rho_2^2} \right\}^{-1/2} > 0, \quad \forall \rho_1, \rho_2. \quad (12)$$

Under stationarity as $T \rightarrow \infty$, $\kappa_1(T, \theta_1) \rightarrow \sqrt{1 - \theta_1^2}$ while $\kappa_2(T, \rho_2) \rightarrow \sqrt{1 - \rho_2^2}$. The employed approximate transformation matrix Q^* is equal to Q for all elements except $Q^*[1, 1] = \kappa_1 \kappa_2$, $Q^*[2, 1] = -\theta_1 \kappa_2$ and $Q^*[2, 2] = \kappa_2$. The proposed NLS estimation minimizes

$$S_2^*(\beta, \rho_1, \rho_2) = \varepsilon' Q^{*'} Q^* \varepsilon, \quad (13)$$

while the new QML estimation maximizes

$$l_2^*(\beta, \rho_1, \rho_2) = -\frac{T}{2} \ln S_2^*(\beta, \rho_1, \rho_2) + \frac{1}{2} \ln \kappa_1(T, \theta_1)^2 + \ln \kappa_2(T, \rho_2)^2. \quad (14)$$

It is assumed that $Q^*u = \varepsilon$ and $u \sim N(0, \sigma^2 \Sigma(\rho_1, \rho_2)^*)$ with $\Sigma(\rho_1, \rho_2)^* = (Q^{*'} Q^*)^{-1}$.

Constrained optimization is not required. Note that no equivalence between S_2^* and l_2^* holds, and LR tests based on S_2^* differ from LR tests based on l_2^* .

3 Conclusions

This paper proposes NLS and QML estimation of a linear regression with AR(1) or AR(2) error, which is based on modification of the usual GLS transformation matrices. Unconstrained optimization is only required. Generalization to any AR order is intractable and unnecessary, since the methods for models with AR(1) or AR(2) error can handle a number of interesting empirical applications. Extension to

nonlinear models is straightforward. Future research of interest includes simulation of the proposed estimators and LR (especially unit root) testing.

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