Complexifiable characteristic classes

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COMPLEXIFIABLE CHARACTERISTIC CLASSES

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Abstract. We examine the topological characteristic cohomology classes of complexified vector bundles. In particular, all the classes coming from the real vector bundles underlying the complexification are determined.

1. Introduction and statement of the results

In the theory of characteristic classes (in the sense of Milnor and Stasheff [4], whom we follow in terminology and notation in this article), it is well-known how the Chern classes are mapped to even Stiefel-Whitney classes when converting complex vector space bundles to real vector space bundles by forgetting the complex structure.

In the other direction, we have the fibre-wise complexification: Given a real vector bundle $F \to B$ with fibre $\mathbb{R}^n$, its complexification is the complex vector bundle $F^{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C} \to B$ obtained by declaring complex multiplication on $F \oplus F$ in each fibre $\mathbb{R}^n \oplus \mathbb{R}^n$ by $i(x, y) := (-y, x)$ for the imaginary unit i. The Pontrjagin classes of a real vector bundle are (up to a sign) constructed as Chern classes of its complexification. Conversely, which classes of a real vector bundle can be attributed to its complexification? These are the *complexifiable* characteristic classes which we determine in this article, under the request that they are characteristic classes in the sense of [4].

Consider a real vector bundle $F \to B$ and a complex vector bundle $E \to B$ over the same paracompact Hausdorff base space B (we keep the latter assumption on B throughout this article).

Definition 1. *A real vector bundle F is called a* real generator bundle *of E, if its complexification $F^{\mathbb{C}}$ is isomorphic to E. In the case that such a bundle F exists, we call E* real-generated.

Not every complex vector bundle is real-generated; it is an easy exercise to show that no complex vector bundle with some nonzero and non-2-torsion odd Chern class can admit a real generator bundle. This makes it seem possible that the subcategory of real-generated vector bundles could admit information additional to its Chern classes, in the complexifiable classes of the real generator bundles. However, we will see that the Chern classes already contain all of the relevant information.

Definition 2. *A characteristic class c of real vector bundles is* complexifiable *if for all pairs (F, G) of real vector bundles with isomorphic complexification $F^{\mathbb{C}} \cong G^{\mathbb{C}}$, the identity $c(F) = c(G)$ holds.*

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We will now give a complete classification of the complexifiable characteristic classes. Denote by $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ the group with two elements.

Theorem 1. *Let c be a polynomial in the Stiefel-Whitney classes w_i. Then the following two conditions are equivalent:*

(i) *The class c is an element of the sub-ring $\mathbb{Z}_2[w_i^2]_{i\in\mathbb{N}\cup\{0\}}$ of the polynomials in the Stiefel-Whitney classes.*
(ii) *The class c is complexifiable.*

The implication (i)⇒(ii) follows easily from the fact that the square of the n-th Stiefel-Whitney class of a real vector bundle is the mod-2-reduction of the n-th Chern class of the complexified vector bundle. The proof of the implication (ii)⇒(i) is prepared with several intermediary steps leading to it. One ingredient, lemma 1, follows essentially from work of Cartan on Hopf fibrations. But this only allows to show that complexifiable characteristic classes in cohomology with $\mathbb{Z}_2$–coefficients are contained in the *ideal* generated by the squares of the Stiefel-Whitney classes. To show that they constitute exactly the *subring* generated by the squares of the Stiefel-Whitney classes, which is much smaller, we need the technical decomposition of lemma 2 that we prove by induction.

By their naturality, characteristic classes are uniquely determined on the universal bundle over the classifying space ($B\mathcal{O}$ for real vector bundles). As the cohomology ring $\mathrm{H}^*(B\mathcal{O},\mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes of the universal bundle, all modulo–2–classes are polynomials in the Stiefel-Whitney classes, and theorem 1 tells us which of them are complexifiable.

We build on this result to investigate which integral cohomology classes are complexifiable. To express our result, we use Feshbach's description [3] of the cohomology ring of the classifying space $B\mathcal{O}$ with $\mathbb{Z}$–coefficients. Consider the Steenrod squaring operation Sq^1 and the mod–2–reduction homomorphism

$$\rho : \mathrm{H}^*(B\mathcal{O},\mathbb{Z}) \to \mathrm{H}^*(B\mathcal{O},\mathbb{Z}_2).$$

Feshbach uses as generators Pontrjagin classes and classes V_I with index sets I that are finite nonempty subsets of $\left\{\frac{1}{2}\right\} \cup \mathbb{N}$, such that

$$\rho(V_I) = Sq^1\left(\bigcup_{i\,\in\, I} \omega_{2i}\right),$$

where ω_i is the i-th Stiefel Whitney class of the universal bundle over $B\mathcal{O}$. We give the details of Feshbach's description in the appendix. We will write v_I for the characteristic class that is V_I on the universal bundle over $B\mathcal{O}$. Our final result now takes the following shape.

Theorem 2. *Let C be a polynomial in v_I^2, with I arbitrary, $v_{\{\frac{1}{2}\}}$ and the Pontrjagin classes p_i. Then C is complexifiable.*

And conversely, we can say the following.

Theorem 3. *Let C be a complexifiable integral characteristic class. Then for any real vector bundle ξ, $C(\xi)$ is completely determined by some Chern classes $c_k(\xi^{\mathbb{C}})$, $k \in \mathbb{N}$.*

2. Classes in cohomology with $\mathbb{Z}_2$–coefficients

In this section, we shall prove theorem 1, after developing all the tools we need to do so. For the whole of this section, we only consider classes in cohomology with $\mathbb{Z}_2$–coefficients. Let $F \to B$ be a real vector bundle over a paracompact Hausdorff base space. Let c be a complexifiable polynomial in the Stiefel-Whitney classes w_i.

Let $\mathcal{O}$ be the direct limit of the orthogonal groups, U the direct limit of the unitary groups and EU the universal total space to the classifying space BU for stable complex vector bundles. Let $B\mathcal{O} := EU/\mathcal{O}$, via the inclusion $\mathcal{O} \subset U$ induced by the canonical inclusion $\mathbb{R} \subset \mathbb{C}$. Let $\gamma(\mathbb{R}^\infty)$ be the universal bundle over $B\mathcal{O}$, and denote its Stiefel-Whitney classes by $\omega_i := w_i(\gamma(\mathbb{R}^\infty))$.

Lemma 1. *Let c be a complexifiable class in cohomology with $\mathbb{Z}_2$–coefficients. Then $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ is contained in the ideal $\langle \omega_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}}$.*

Proof. We use the Hopf spaces fibration of Cartan [2, p. 17-22],

$$U/\mathcal{O} \xrightarrow{\quad f \quad} B\mathcal{O} \xrightarrow{\quad p \quad} BU,$$

where the projection p is the rest class map to dividing the whole group U out of EU; and $f : U/\mathcal{O} \to B\mathcal{O}$ embeds a fibre. The cohomology ring $\mathrm{H}^*(B\mathcal{O}, \mathbb{Z}_2)$ is the polynomial algebra $\mathbb{Z}_2[\omega_1, \omega_2, ...]$ with generators the Stiefel-Whitney classes of the universal bundle. Cartan [2, p. 17-22] has shown that f^* maps these generators ω_i to the generators $\nu_i := w_i(f^*\gamma(\mathbb{R}^\infty))$ of the exterior algebra

$$\mathrm{H}^*(U/\mathcal{O}, \mathbb{Z}_2) = \bigwedge(\mathbb{Z}_2[\nu_1, \nu_2, ...]),$$

which is obtained by dividing the ideal $\langle \nu_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}}$ out of the polynomial algebra $\mathbb{Z}_2[\nu_1, \nu_2, ...]$. Hence, exactly the ideal $\langle \omega_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}}$ is mapped to zero. So to write

$$\langle \omega_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}} = \ker f^*.$$

Composing f with the projection $p : B\mathcal{O} \to BU$, we obtain a constant map (the whole fibre is mapped to its basepoint) and therefore a trivial bundle $(p \circ f)^*\gamma(\mathbb{C}^\infty)$. This pullback of the complex universal bundle is the complexification of $f^*\gamma(\mathbb{R}^\infty)$:

$$(p \circ f)^*\gamma(\mathbb{C}^\infty) = f^*p^*EU \times_U \mathbb{C}^\infty = f^*E\mathcal{O} \times_{\mathcal{O}} \mathbb{C}^\infty = f^*(E\mathcal{O} \times_{\mathcal{O}} \mathbb{R}^\infty)^{\mathbb{C}}$$

$$= f^*\gamma(\mathbb{R}^\infty)^{\mathbb{C}} = (f^*\gamma(\mathbb{R}^\infty))^{\mathbb{C}}.$$

So, $f^*\gamma(\mathbb{R}^\infty)$ admits a trivial complexification, and all of the complexifiable classes c must treat it like the trivial bundle ε: $c(f^*\gamma(\mathbb{R}^\infty)) = c(\varepsilon)$. A pullback of the trivial bundle is trivial too, so

$$0 = c(f^*\gamma(\mathbb{R}^\infty)) - c(f^*\varepsilon) = f^*(c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon))$$

by naturality. Whence, $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ is an element of the kernel of f^*, which we have identified with the ideal $\langle \omega_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}}$. □

The above lemma allows to split off one square of a Stiefel-Whitney class as a factor the characteristic class c under investigation. But we must inductively split off squares until we achieve the decomposition in the following lemma.

Lemma 2. *Any complexifiable characteristic class c admits a decomposition*

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$$
$$= \sum_{j_1=1}^{m} \omega^2_{i_{j_1}} \cup \sum_{j_2=1}^{m_{j_1}} \omega^2_{i_{(j_1,j_2)}} \cup \sum ... \cup \sum_{j_k=1}^{m_{(j_1,...,j_{k-1})}} \omega^2_{i_{(j_1,...,j_k)}} \cup r_{(j_1,...,j_k)}(\gamma(\mathbb{R}^\infty))$$
$$+ \sum_{j_1=1}^{m} \omega^2_{i_{j_1}} \cup r_{j_1}(\varepsilon) + ... + \sum_{j_1=1}^{m} \omega^2_{i_{j_1}} \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,...,j_{k-2})}} \omega^2_{i_{(j_1,...,j_{k-1})}} \cup r_{(j_1,...,j_{k-1})}(\varepsilon)$$

for some $m, m_{j_1}, ..., m_{(j_1,...,j_{k-1})} \in \mathbb{N} \cup \{0\}$, some $i_{j_1}, ..., i_{(j_1,...,j_k)} \in \mathbb{N} \setminus \{0\}$,
some $r_{(j_1,...,j_k)}(\gamma(\mathbb{R}^\infty)) \in \mathrm{H}^*(B\mathcal{O}, \mathbb{Z}_2)$,
and some coefficients $r_{j_1}(\varepsilon), ..., r_{(j_1,...,j_{k-1})}(\varepsilon) \in \{0,1\}$,
such that for all $\vec{j} := (j_1, ..., j_k)$ and $I(\vec{j}) := \{i_{j_1}, ..., i_{(j_1,...,j_k)}\}$, the following inequality holds:

$$2 \sum_{p \,\in\, I(\vec{j})} p > \deg c.$$

Once this lemma is established, we use that the degree must be the same on both sides of its equation, to deduce that the sum over all terms containing a factor $\bigcup_{p \,\in\, I(\vec{j})} \omega_p^2$ of too high degree $\left(2 \sum_{p \,\in\, I(\vec{j})} p\right)$ must vanish.

As the classes $c(\varepsilon)$, $r_{\vec{j}}(\varepsilon)$ of the trivial bundle ε are just coefficients in $\mathrm{H}^0(B\mathcal{O}, \mathbb{Z}_2) \cong \{0,1\}$, a polynomial $c(\gamma(\mathbb{R}^\infty))$ in some squares ω_p^2 , $p \in \mathbb{N} \cup \{0\}$ will remain, so this argument implies theorem 1, (ii)⇒(i).

Before giving the proof of the required lemma 2, we shall introduce two notations just to make that proof more readable.

Definition 3. *An index vector $\vec{j}$* appears *in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$, if this decomposition admits a summand of the form*

$$r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \,\in\, I(\vec{j})} \omega_p^2,$$

and if $\left(2 \sum_{p \,\in\, I(\vec{j})} p\right) \leq \deg c.$

Note that the terms $\left(r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \,\in\, I(\vec{j})} \omega_p^2\right)$ with $\left(2 \sum_{p \,\in\, I(\vec{j})} p > \deg c\right)$ must vanish in any decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. That is why we do not let them contribute in the last definition.

Definition 4. *Set $\ell := \min_{\vec{j} \text{ appears}} \max I(\vec{j})$. Consider an index vector $\vec{j}$ appearing in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$.*
If $\max I(\vec{j}) = \ell$, then we call $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$ a low situated rest term.

As seen so far, $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ lies in $\ker f^* = \langle \omega_i^2 \rangle_{i \in \mathbb{N}\setminus\{0\}}$, so there is a decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\gamma(\mathbb{R}^\infty)),$$

for some $m \in \mathbb{N} \cup \{0\}$, some $i_{j_1} \in \mathbb{N} \setminus \{0\}$, and some $r_{j_1}(\gamma(\mathbb{R}^\infty)) \in \mathrm{H}^*(B\mathcal{O}, \mathbb{Z}_2)$. We will show that there is a low situated rest term $r_{j_1}(\gamma(\mathbb{R}^\infty)) - r_{j_1}(\varepsilon)$ in this decomposition that lies in $\ker f^*$. Then, that low situated rest term admits a decomposition as a linear combination of squares $\omega^2_{i_{(j_1,j_2)}}$ with coefficients $r_{(j_1,j_2)}(\gamma(\mathbb{R}^\infty))$ in $\mathrm{H}^*(B\mathcal{O}, \mathbb{Z}_2)$, leading to a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. So, inductively, we will replace a low situated rest term in any given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ by a linear combination the coefficients of which are rest terms with longer index vectors. That is why after a finite number of these steps, the index vectors $\vec{j}$ will no more appear, because the sums $\left(2 \sum_{p \in I(\vec{j})} p\right)$ will exceed the degree of c. This is the moment when all low situated rest terms are eliminated and the decomposition described in lemma 2 is achieved.

To carry out this strategy, we first need to introduce the following truncation procedure.

Truncated stable invariance. With lemma 3, we shall give a sense to "the truncation of the equation $c(F \oplus G) = c(G)$ at the dimension ℓ". Define the bundles

$$F := pr_1^* f^* \gamma(\mathbb{R}^\infty) \longrightarrow U/\mathcal{O} \times B\mathcal{O}$$

and

$$G := pr_2^* \gamma(\mathbb{R}^\infty) \longrightarrow U/\mathcal{O} \times B\mathcal{O},$$

where pr_i shall be the projection on the i-th factor of the base space $U/\mathcal{O} \times B\mathcal{O}$. Let $\ell \in \mathbb{N}$. Consider the map

$$(id, emb_l) : (U/\mathcal{O} \times B\mathcal{O}_\ell) \hookrightarrow (U/\mathcal{O} \times B\mathcal{O})$$

where $emb_l : B\mathcal{O}_\ell \hookrightarrow B\mathcal{O}$ shall be the natural embedding, recalling that $B\mathcal{O}$ is the direct limit over all $B\mathcal{O}_\ell$, $\ell \in \mathbb{N}$. Then the bundle $G_l := (id, emb_l)^* G$ admits Stiefel-Whitney classes that are in bijective correspondence with those of the ℓ-dimensional universal bundle $\gamma_l(\mathbb{R}^\infty) \to B\mathcal{O}_\ell$.

To be precise, $G_l \cong pr_{B\mathcal{O}_\ell}{}^* \gamma_l(\mathbb{R}^\infty)$ and the situation is

$$\begin{array}{ccccccc}
\gamma_l(\mathbb{R}^\infty) & & G_l \cong pr_{B\mathcal{O}_\ell}{}^*\gamma_l(\mathbb{R}^\infty) & & G := pr_2^*\gamma(\mathbb{R}^\infty) & & \gamma(\mathbb{R}^\infty) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\mathcal{O}_\ell & \xleftarrow{pr_{B\mathcal{O}_\ell}} & (U/\mathcal{O} \times B\mathcal{O}_\ell) & \xhookrightarrow{(id, emb_l)} & (U/\mathcal{O} \times B\mathcal{O}) & \xrightarrow{pr_2} & B\mathcal{O}.
\end{array}$$

Especially, $w_p(G_l)$ vanishes for $p > \ell$.

Lemma 3. *Under the above assumptions, the following equation holds:*

$$\sum_{\substack{\max I(\vec{j}) \leq \ell \\ \vec{j} \text{ appears}}} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l) = \sum_{\substack{\max I(\vec{j}) \leq \ell \\ \vec{j} \text{ appears}}} r_{\vec{j}}(G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l).$$

We will call it the equation $c(F \oplus G) = c(G)$ truncated at dimension ℓ.

Proof. The bundle F inherits from $f^*\gamma(\mathbb{R}^\infty)$ the property to admit a trivial complexification. As c is complexifiable, we have $c(F\oplus G)=c(G)$. Applying the induced cohomology map $(id, emb_l)^*$ to this equation, we obtain

$$c(id^*F \oplus emb_l^* G) = c(emb_l^* G)$$

and hence

$$c(F \oplus G_l) = c(G_l).$$

By the universality of $\gamma(\mathbb{R}^\infty)$, and the naturality of all characteristic classes with respect to the classifying maps of G_l and $F \oplus G_l$, any given decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \bigcup_{p \in I(\vec{j})} \omega_p^2$$

gives analogous decompositions

$$c(G_l) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l)$$

and

$$c(F \oplus G_l) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \in I(\vec{j})} w_p^2(F \oplus G_l).$$

By theorem 1, (i)⇒(ii) the square w_p^2 is complexifiable and hence invariant under adding the bundle F of trivial complexification :

$$w_p^2(F \oplus G_l) = w_p^2(G_l).$$

Thus, the equation $c(F \oplus G_l) = c(G_l)$ can be rewritten using that all summands containing a factor $w_p(G_l)$ with $p > \ell$ vanish:

$$\sum_{\vec{j}}^{\max I(\vec{j}) \leq \ell} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j}}^{\max I(\vec{j}) \leq \ell} r_{\vec{j}}(G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l).$$

For not to exceed the degree of c, also all terms with $2 \sum_{p \in I(\vec{j})} p > \deg c$ must vanish:

$$\sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \leq \ell} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \leq \ell} r_{\vec{j}}(G_l) \bigcup_{p \in I(\vec{j})} w_p^2(G_l).$$

So, this last equation is the equation $c(F\oplus G)=c(G)$ truncated at the dimension ℓ. □

Proof of lemma 2. We carry out the proof by *induction over the index vector pointing at a low situated rest term.*

Base case. Lemma 1 implies $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\gamma(\mathbb{R}^\infty))$,
with r_{j_1} some polynomial in the Stiefel-Whitney classes.
Rename $i_1, ..., i_m$ such that $i_1 < i_2 < ... < i_m$.
We truncate the equation $c(F \oplus G) = c(G)$ at the dimension i_1, and obtain

$$\sum_{j_1 \text{ appears}}^{i_{j_1} \leq i_1} r_{j_1}(F \oplus G_{i_1}) \cup w_{i_{j_1}}^2(G_{i_1}) = \sum_{j_1 \text{ appears}}^{i_{j_1} \leq i_1} r_{j_1}(G_{i_1}) \cup w_{i_{j_1}}^2(G_{i_1}).$$

As $i_1 < i_2 < ... < i_m$, this is just $r_1(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = r_1(G_{i_1}) \cup w_{i_1}^2(G_{i_1})$.

Injectivity of the multiplication map $\cup w_{i_1}^2(G_{i_1})$ in $\mathrm{H}^*(U/\mathcal{O} \times B\mathcal{O}_{i_1}, \mathbb{Z}_2)$ then holds $r_1(F \oplus G_{i_1}) = r_1(G_{i_1})$. Then we pull this back with

$$(id \times const) : U/\mathcal{O} \to (U/\mathcal{O} \times B\mathcal{O}_{i_1}),$$

(where the map *const* takes just one, arbitrary, value), to obtain

$$r_1(f^*\gamma(\mathbb{R}^\infty) \oplus \varepsilon) = r_1(\varepsilon).$$

Due to Whitney sum formula, the Stiefel-Whitney classes in which r_1 is a polynomial are stable under adding a trivial bundle, and the above left hand term equals $r_1(f^*\gamma(\mathbb{R}^\infty))$. Using naturality of characteristic classes with respect to pullbacks, this shows that $r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon)$ lies in $\ker f^*$. So we can replace it with a linear combination of quadratic terms, providing a new decomposition,

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \omega_{i_1}^2 \sum_{j_2=1}^{m_1} \omega_{i_{(1,j_2)}}^2 r_{(1,j_1)}(\gamma(\mathbb{R}^\infty)) + \omega_{i_1}^2 r_1(\varepsilon) + \sum_{j_1=2}^{m} \omega_{i_{j_1}}^2 r_{j_1}(\gamma(\mathbb{R}^\infty)).$$

Induction hypothesis. Consider a given decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \bigcup_{p \,\in\, I(\vec{j})} \omega_p^2$$

$$+ \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + ... + \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,...,j_{k-2})}} \omega_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon).$$

Inductive claim. The decomposition of the induction hypothesis admits a low situated rest term that lies in $\ker f^*$. We show this in the inductive step.

Inductive step. We truncate the equation $c(F \oplus G) = c(G)$ at the dimension

$$\ell := \min_{\vec{j} \text{ appears}} \max I(\vec{j}).$$

Then the remaining terms of $c(G_l) - c(\varepsilon)$ do all have the common factor $w_l^2(G_l)$. This is no zero divisor in $\mathrm{H}^*(U/\mathcal{O} \times B\mathcal{O}_\ell, \mathbb{Z}_2)$ and further its multiplication map $\cup w_l^2(G_l)$ is injective. Now, in $c(F \oplus G_l) = c(G_l)$, this injectivity implies

$$\sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, \ell} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\in\, I(\vec{j})\setminus\{\ell\}} w_p^2(G_l) = \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, \ell} r_{\vec{j}}(G_l) \bigcup_{p \,\in\, I(\vec{j})\setminus\{\ell\}} w_p^2(G_l).$$

$\diamondsuit$ If there is just one low situated rest term $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$, then we use the injectivity of the multiplication map

$$\cup \left(\bigcup_{p \,\in\, I(\vec{j})\setminus\{\ell\}} w_p^2(G_l) \right) \text{ on } \mathrm{H}^*(U/\mathcal{O} \times B\mathcal{O}_\ell, \mathbb{Z}_2)$$

to obtain $r_{\vec{j}}(F \oplus G_l) = r_{\vec{j}}(G_l)$. Then we pull this back with

$$(id \times const) : U/\mathcal{O} \to (U/\mathcal{O} \times B\mathcal{O}_\ell)$$

to obtain $r_{\vec{j}}(f^*\gamma(\mathbb{R}^\infty) \oplus \varepsilon) = r_{\vec{j}}(\varepsilon)$. Using naturality, we see now that the low situated rest term $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$ lies in $\ker f^*$.

$\diamondsuit$ Else we truncate the remaining equation again at the dimension

$$\ell' := \min_{\vec{j} \text{ appears}}^{\max I(\vec{j})=\ell} \max(I(\vec{j}) \setminus \{\ell\}),$$

such as to obtain

$$\sum_{\vec{j} \text{ appears}}^{\max(I(\vec{j})\setminus\{\ell\}) \leq \ell'} r_{\vec{j}}(F \oplus G_{\ell'}) \bigcup_{p \in (I(\vec{j})\setminus\{\ell\})} w_p^2(G_{\ell'})$$

$$= \sum_{\vec{j} \text{ appears}}^{\max(I(\vec{j})\setminus\{\ell\}) \leq \ell'} r_{\vec{j}}(G_{\ell'}) \bigcup_{p \in (I(\vec{j})\setminus\{\ell\})} w_p^2(G_{\ell'}).$$

Now we proceed analogously with the choice marked with the "$\diamondsuit$" signs, and after finitely many steps, find a low situated rest term in $\ker f^*$.
This low situated rest term can be replaced by a linear combination of squares, holding a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$.

This completes the induction. □

Proof of theorem 1, (ii)⇒(i). Let c be a complexifiable characteristic class. By the universality of $\gamma(\mathbb{R}^\infty)$, the decomposition of lemma 2 yields the decomposition

$$c = c(\varepsilon) + \sum_{j_1=1}^{m} w_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + ... + ...$$

$$+ \sum_{j_1=1}^{m} w_{i_{j_1}}^2 \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,...,j_{k-2})}} w_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon).$$

As $c(\varepsilon), r_{j_1}(\varepsilon), ..., r_{(j_1,...,j_{k-1})}(\varepsilon)$ are elements of $\{0, 1 = w_0 = w_0^2\}$, the class c is in the sub-ring $\mathbb{Z}_2[w_i^2]_{i \in \mathbb{N}\cup\{0\}}$ of the polynomial ring of Stiefel-Whitney classes. □

This completes the proof of theorem 1.

3. Classes in cohomology with integral coefficients

We will build on our results obtained for $\mathbb{Z}_2$-coefficients and use the mod–2–reduction homomorphism

$$\rho : \mathrm{H}^*(-, \mathbb{Z}) \to \mathrm{H}^*(-, \mathbb{Z}_2)$$

to prove the theorems with $\mathbb{Z}$–coefficients stated in the introduction. Define the element $V_I \in \mathrm{H}^*(B\mathcal{O}, \mathbb{Z})$ as in appendix A, and let v_I be the characteristic class that is V_I on the universal bundle.

Lemma 4. *For any real bundle ξ, the mod–2–reduced class $\rho(v_I^2(\xi))$ equals*

$$\left(\sum_{i \in I \cap \{\frac{1}{2}\}} w_1^2 \cup \bigcup_{j \in I\setminus\{i\}} w_{4j} + \sum_{i \in I\setminus\{\frac{1}{2}\}} (w_{4i+2} + w_2 \cup w_{4i}) \cup \bigcup_{j \in I\setminus\{i\}} w_{4j} \right) (\xi \oplus \xi).$$

Proof. By Feshbach's description (in the appendix), the mod–2–reduction is

$$\rho\left(v_I^2(\xi)\right) = \left(Sq^1\left(\bigcup_{i\,\in\, I} w_{2i}(\xi)\right)\right)^2.$$

We expand this expression until it is a polynomial in the Stiefel-Whitney classes, and use that $2 = 0$ in $\mathrm{H}^*(B\mathcal{O}, \mathbb{Z}_2)$. Then we rearrange the expression using the Whitney sum formula and the symmetry of the terms. □

Proof of theorem 2. For $v_{\{\frac{1}{2}\}}$ and the Pontrjagin classes p_i, the result is obvious. Now let $F \to B$, $G \to B$ be real bundles with $F^{\mathbb{C}} \cong G^{\mathbb{C}}$. Forgetting the complex structure, this is $F \oplus F \cong G \oplus G$. By naturality of the Stiefel-Whitney classes, for any finite nonempty index set $I \subset (\{\frac{1}{2}\} \cup \mathbb{N} \setminus \{0\})$, the polynomial given in lemma 4 is the same for the arguments $(F \oplus F)$ and $(G \oplus G)$. Applying lemma 4, this means that $\rho(v_I^2(F)) = \rho(v_I^2(G))$.
As V_I^2 is in the torsion of $\mathrm{H}^*(B\mathcal{O}, \mathbb{Z})$, restricted on which ρ is injective [3, p. 513], this proves the theorem: $v_I^2(F) = v_I^2(G)$. □

Proof of theorem 3. Feshbach [3, p. 513] shows that

$$\mathrm{H}^*(B\mathcal{O}, \mathbb{Z}) = \mathbb{Z}[\pi_i]_{i\,\in\,\mathbb{N}} \oplus \{2\text{–torsion}\},$$

where π_i is the Pontrjagin class p_i of the universal bundle. Then $C = P(p_i) + T$ with P a formal polynomial in the Pontrjagin classes and T some 2-torsion class. So for every real bundle ξ,

$$\rho(C)(\xi) = P\left(\rho\left(p_i(\xi)\right)\right) + \rho(T)(\xi).$$

By definition of the Pontrjagin classes, $p_i(\xi) = (-1)^i c_{2i}(\xi^{\mathbb{C}})$; and using the reduction $\rho\left(c_{2i}(\xi^{\mathbb{C}})\right) = w_{4i}(\xi \oplus \xi)$ from Chern classes to Stiefel-Whitney classes, further the Whitney sum formula and the symmetry of the summands, we deduce

$$\rho(C)(\xi) = P(w_{2i}^2(\xi)) + \rho(T)(\xi).$$

It follows from theorem 1 that the mod-2-reduction $\rho(C)(\xi)$ is a polynomial in the squares of Stiefel-Whitney classes; and hence also $\rho(T)(\xi)$ is a polynomial $Q(w_j^2(\xi))$ in the squares of Stiefel-Whitney classes. As according to [3, p. 513], ρ is injective on the torsion elements, there is a local inverse ${\rho|_{\{2-\mathrm{torsion}\}}}^{-1}$ lifting $\rho(T)$ back to T. So,

$$T(\xi) = {\rho|_{\{2-\mathrm{torsion}\}}}^{-1}\left(Q(w_j^2(\xi))\right);$$

and with the Whitney sum formula and the symmetry of the summands, further the reduction from Chern classes to Stiefel-Whitney classes, and finally using the decomposition of C, we obtain

$$C(\xi) = P\left((-1)^i c_{2i}(\xi^{\mathbb{C}})\right) + {\rho|_{\{2-\mathrm{torsion}\}}}^{-1}\left(Q\left(\rho(c_j(\xi^{\mathbb{C}}))\right)\right).$$

□

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Appendix A. The cohomology ring of $B\mathcal{O}$ with $\mathbb{Z}$–coefficients

The cohomology ring of $B\mathcal{O}$ with $\mathbb{Z}$–coefficients is known since Thomas [5], [6] and with all relations between its generators since Brown [1] and Feshbach [3]. It can be obtained as follows. Define the set of generators of $\mathrm{H}^*(B\mathcal{O}_n, \mathbb{Z})$ as in [3, definition 1]:
It consists of the Pontrjagin classes p_i of the universal bundle over $B\mathcal{O}_n$, and classes V_I with I ranging over all finite nonempty subsets of

$$\left\{\frac{1}{2}\right\} \cup \left\{ k \in \mathbb{N} \;\middle|\; 0 < k < \frac{n+1}{2} \right\}$$

with the proviso that I does not contain both $\frac{1}{2}$ and $\frac{n}{2}$, for $n > 1$.
According to [3, theorem 2], $\mathrm{H}^*(B\mathcal{O}_n, \mathbb{Z})$ is for all $n \leq \infty$ isomorphic to the polynomial ring over $\mathbb{Z}$ generated by the above specified elements modulo the ideal generated by the following six types of relations.
In all relations except the first, the cardinality of I is less than or equal to that of J and greater than one. (Most of the restrictions on I and J are to avoid repeating relations). By convention, $p_{\frac{1}{2}}$ where it occurs means $V_{\{\frac{1}{2}\}}$. Also, if $\{\frac{n}{2}, \frac{1}{2}\} \subset I \cup J$, then $V_{I\cup J}$ shall mean $V_{\{\frac{n}{2}\}} V_{(I\cup J)\setminus\{\frac{n}{2},\frac{1}{2}\}}$.

1) $2V_I = 0$.

2) $V_I V_J + V_{I\cup J} V_{I\cap J} + V_{I\setminus J} V_{J\setminus I} \prod\limits_{i \in I\cap J} p_i = 0 \quad$ (for $I \cap J \neq \emptyset,\ I \not\subseteq J$).

3) $V_I V_J + \sum\limits_{i \in I} V_{\{i\}} V_{(J\setminus I)\cup\{i\}} \prod\limits_{j \in I\setminus\{i\}} p_j = 0 \quad$ (for $I \subset J$).

4) $V_I V_J + \sum\limits_{i \in I} V_{\{i\}} V_{(I\cup J)\setminus\{i\}} = 0 \quad$ (for $I \cap J = \emptyset$; if I and J have the same cardinality, then the smallest element of I is less than that of J).

5) $\sum\limits_{i \in I} V_{\{i\}} V_{I\setminus\{i\}} = 0$.

6) $V_{\{\frac{1}{2}\}} p_{\frac{n}{2}} + V_{\{\frac{n}{2}\}}^2 = 0$, if n is even.

Then $\rho(V_I) = Sq^1(\bigcup\limits_{i \in I} w_{2i})$, with the Steenrod squaring operation Sq^1.

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