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# HARDY INEQUALITY AND WEYL ASYMPTOTIC FOR DISCRETE LAPLACIANS

SYLVAIN GOLÉRIA

ABSTRACT. In this paper we study in detail some spectral properties of the magnetic discrete Laplacian. We identify its form-domain, characterize the absence of essential spectrum and provide the Weyl asymptotic for the eigenvalues.

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## 1. INTRODUCTION

The uncertainty principle is a central point in quantum physics. It can be expressed by the following Hardy inequality:

$$(1.1) \quad \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \left| \frac{1}{|x|} f(x) \right|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 dx = \langle f, -\Delta_{\mathbb{R}^n} f \rangle, \text{ where } n \geq 3,$$

and  $f \in C_c^\infty(\mathbb{R}^n)$ . Roughly speaking, the Laplacian controls some local singularities of a potential. In this paper, we investigate which potentials a discrete Laplacian is able to control. Obviously, we will not focus on local singularities as the value of a potential on a vertex has to be finite. However, unlike in the continuous case, we will control potentials that explodes at infinity.

We start with some definitions and fix notation for graphs. We refer to [CdV, Chu, MW] for surveys on the matter. Let  $\mathcal{V}$  be a countable set. Let  $\mathcal{E} := \mathcal{V} \times \mathcal{V} \rightarrow [0, \infty)$  and assume that  $\mathcal{E}(x, y) = \mathcal{E}(y, x)$ , for all  $x, y \in \mathcal{V}$ . We say that  $G := (\mathcal{E}, \mathcal{V})$  is an unoriented weighted graph with *vertices*  $\mathcal{V}$  and *weighted edges*  $\mathcal{E}$ . In the setting of electrical networks, the weights correspond to the conductances. We say that  $x, y \in \mathcal{V}$  are *neighbors* if  $\mathcal{E}(x, y) \neq 0$  and denote it by  $x \sim y$ . We say there is a *loop* in  $x \in \mathcal{V}$  if  $\mathcal{E}(x, x) \neq 0$ . The set of *neighbors* of  $x \in E$  is denoted by  $\mathcal{N}_G(x) := \{y \in E \mid x \sim y\}$ . The *degree* of  $x \in V$  is by definition  $|\mathcal{N}_G(x)|$ , the number of neighbors of  $x$ . The graph is of *bounded degree*, if  $\sup_{x \in \mathcal{V}} |\mathcal{N}_G(x)|$  is finite. A graph is *locally finite* if  $|\mathcal{N}_G(x)|$  is finite for all  $x \in V$ . Finally, as we are dealing with magnetic fields, we fix a phase  $\theta : \mathcal{V} \times \mathcal{V} \rightarrow [-\pi, \pi]$ , such that  $\theta(x, y) = -\theta(y, x)$ . We denote by  $\theta_{x,y} := \theta(x, y)$ . A graph is *connected*, if for all  $x, y \in V$ , there exists an *x-y-path*, i.e., there is a finite sequence  $(x_1, \dots, x_{N+1}) \in \mathcal{V}^{N+1}$  such that  $x_1 = x$ ,  $x_{N+1} = y$  and  $x_n \sim x_{n+1}$ , for all  $n \in \{1, \dots, N\}$ . In

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the sequel, we shall always consider (magnetic) graphs  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$ , which are unoriented, locally finite, connected and have no loop. A graph  $G$  is *simple* if  $\mathcal{E}$  has values in  $\{0, 1\}$  and  $\theta = 1$ . Finally, we recall that a *bi-partite* graph is a graph whose vertex set can be partitioned into two subsets in such a way that no two points in the same subset are neighbors. Trees are bi-partite graphs.

We now associate a certain Hilbert space and some operators on it to a given graph  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$ . Let  $\ell^2(G, m^2) := \ell^2(\mathcal{V}, m^2; \mathbb{C})$  be the set of functions  $f : \mathcal{V} \rightarrow \mathbb{C}$ , such that  $\|f\|^2 := \sum_{x \in \mathcal{V}} m(x)^2 |f(x)|^2$  is finite. The associated scalar product is given by  $\langle f, g \rangle = \sum_{x \in \mathcal{V}} m^2(x) \overline{f(x)} g(x)$ , for  $f, g \in \ell^2(\mathcal{V}, m^2)$ . We also denote by  $\mathcal{C}_c(\mathcal{V})$  the set of functions  $f : \mathcal{V} \rightarrow \mathbb{C}$ , which have finite support. We define the quadratic form:

$$(1.2) \quad \mathcal{Q}(f, f) := \mathcal{Q}_{\mathcal{E}, \theta}(f, f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - e^{i\theta_{x,y}} f(y)|^2 \geq 0, \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

It is closable and there exist a unique self-adjoint operator  $\Delta_{\mathcal{E}, \theta}$ , such that  $\mathcal{Q}_{\mathcal{E}, \theta}(f, f) = \langle f, \Delta_{\mathcal{E}, \theta} f \rangle$ , for  $f \in \mathcal{C}_c(\mathcal{V})$  and  $\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}) = \mathcal{D}(\mathcal{Q}_{\mathcal{E}, \theta})$ , where the latter is the completion of  $\mathcal{C}_c(\mathcal{V})$  under  $\|\cdot\|^2 + \mathcal{Q}_{\mathcal{E}, \theta}(\cdot, \cdot)$ . This operator is the *Friedrichs extension* associated to the form  $\mathcal{Q}_{\mathcal{E}, \theta}$ . It acts as follows:

$$(1.3) \quad \Delta_{\mathcal{E}, \theta} f(x) := \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) (f(x) - e^{i\theta_{x,y}} f(y)), \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

When  $m = 1$ , it is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$  (see Section 2.1 for further discussions). If  $G$  is simple, we shall simply write  $\Delta_G$ . There exist other definitions for the discrete Laplacian, e.g., [CdV, Chu, MW], the one we study here is sometimes call the ‘‘physical Laplacian’’.

In  $\ell^2(G, m^2)$ , we define the *weighted degree* by

$$d_G(x) := \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y).$$

Given a function  $V : \mathcal{V} \rightarrow \mathbb{C}$ , we denote by  $V(Q)$  the operator of multiplication by  $V$ . It is elementary that  $\mathcal{D}(d_G^{1/2}(Q)) \subset \mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2})$ . Indeed, one has:

$$(1.4) \quad \langle f, \Delta_{\mathcal{E}, \theta} f \rangle = \frac{1}{2} \sum_{x \in \mathcal{V}} \sum_{y \sim x} \mathcal{E}(x, y) |f(x) - f(y)|^2 \leq \sum_{x \in \mathcal{V}} \sum_{y \sim x} \mathcal{E}(x, y) (|f(x)|^2 + |f(y)|^2) = 2 \langle f, d_G(Q) f \rangle,$$

for  $f \in \mathcal{C}_c(\mathcal{V})$ . This inequality also gives a necessary condition for the absence of essential spectrum for  $\Delta_{\mathcal{E}, \theta}$  (see Corollary 2.1). In Proposition 4.4, we also prove that the constant 2 cannot in general be improved. It is also easy to see that  $\Delta_{\mathcal{E}, \theta}$  is bounded if and only if  $d_G(Q)$  is (see Proposition 2.1).

In this paper we are interested in minorating the Laplacian with the help of the weighted degree. As soon as  $\mathcal{E}$  is non-trivial, one has to face some troubles as Proposition 3.4 ensures that there is no  $a > 0$ , satisfying

$$(1.5) \quad \langle f, d_G(Q) f \rangle \leq a \langle f, \Delta_{\mathcal{E}, \theta} f \rangle, \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

This can be roughly interpreted as a non-violation of the uncertainty principle. Therefore, the hope relies on minorating it with the help of  $d_G(Q) - C$  for some positive constant  $C$ .

On the other hand the Laplacian acting in  $\ell^2(\mathcal{V}, m^2)$  is unitarily equivalent to a Schrödinger operator acting in  $\ell^2(\mathcal{V}, m_0^2)$  (see Proposition 3.2). This has already been noticed before, e.g., [CTT]. By extracting some positivity, we obtain our analog of the Hardy inequality:

**Proposition 1.1.** *Take a locally finite graph  $G = (\mathcal{V}, \mathcal{E}, m_0, \theta)$ . Given  $m : \mathcal{V} \rightarrow (0, \infty)$ , one has*

$$(1.6) \quad \langle f, V_m(Q) f \rangle \leq \langle f, \Delta_{\mathcal{E}, \theta} f \rangle, \text{ for } f \in \mathcal{C}_c(\mathcal{V}),$$

where

$$(1.7) \quad V_m(x) := d_G(x) - W_m(x), \text{ with } W_m(x) := \frac{1}{m_0^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) \frac{m(y)}{m(x)} \frac{m_0(x)}{m_0(y)}.$$

Moreover, if  $G$  is bi-partite, we also get:

$$\langle f, \Delta_{\mathcal{E}, \theta} f \rangle \leq \langle f, (d_G(Q) + W_m(Q)) f \rangle, \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

Note that by choosing  $m = m_0$ , we recover that  $\Delta_{\mathcal{E},\theta} \geq 0$ . Moreover,  $V_m$  is independent of the magnetic field. It is also different from the Kato's inequality of [DM]. Besides, unlike in e.g., [DK, Fuj, Kel, KL, KLW], our approach does not rely on the isoperimetric estimates. We stress that the inequality (1.6) is in some cases trivial, e.g., Proposition 3.3. One has to find a favorable situation in order to exploit it. This is the case for some perturbations of weighted trees. We present our main result:

**Theorem 1.1.** *Let  $G_\circ = (\mathcal{V}, \mathcal{E}_\circ, m, \theta_\circ)$  be a weighted tree. Assume that there is  $\varepsilon_0 \in (0, 1)$ , so that*

$$(1.8) \quad C_0 := \sup_{x \in \mathcal{V}} d_{G_\circ}^{\varepsilon_0 - 1}(x) \mathcal{E}_\circ(\overleftarrow{x}, x) m^{-2}(x) < \infty.$$

*Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a perturbed graph and  $V : \mathcal{V} \rightarrow \mathbb{R}$  be a potential, satisfying:*

$$(1.9) \quad |V(x)| + \Lambda(x) = o(1 + d_{G_\circ}(x)), \text{ as } |x| \rightarrow \infty, \text{ where } \Lambda(x) := \frac{1}{m^2(x)} \sum_{y \sim x} |\mathcal{E}(x, y) - \tilde{\mathcal{E}}(x, y)|.$$

*Then, one has that:*

(a) *The quadratic form associated to  $\Delta_{\mathcal{E},\theta} + V$  bounded is from below by some constant  $C$ . We denote by  $H$  the associated Friedrichs extension.*

(b) *For all  $\varepsilon > 0$ , there is  $c_\varepsilon \geq 0$ , so that*

$$(1.10) \quad (1 - \varepsilon) \langle f, d_{G_\circ}(Q)f \rangle - c_\varepsilon \|f\|^2 \leq \langle f, Hf \rangle \leq (1 + \varepsilon) \langle f, d_{G_\circ}(Q)f \rangle + c_\varepsilon \|f\|^2,$$

*for  $f \in \mathcal{C}_c(\mathcal{V})$ . We have  $\mathcal{D}((H + C)^{1/2}) = \mathcal{D}((d_{G_\circ}(Q))^{1/2})$ .*

(c) *The essential spectrum of  $H$  is equal to the that of  $\Delta_{\mathcal{E}_\circ, \theta_\circ}$ .*

(d) *The essential spectrum of  $H$  is empty if and only if  $\lim_{|x| \rightarrow \infty} d_{G_\circ}(x) = +\infty$ . In this case we obtain:*

$$(1.11) \quad \lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(H)}{\mathcal{N}_\lambda(d_{G_\circ}(Q))} = 1,$$

*where  $\mathcal{N}_\lambda(A) := \dim \text{Ran } \mathbf{1}_{(-\infty, \lambda]}(A)$ , for a self-adjoint operator  $A$ .*

Note that we improve on the bound (1.4). There is no hypothesis on  $\theta$  or on  $\theta_\circ$ . We point out that Hypothesis (1.8) is fulfilled by simple trees, i.e., when  $\mathcal{E} = m_0 = 1$ , and that every simple graph has a maximal subtree. The inequality (1.10) is valid for a larger class of perturbations (see Proposition 5.1).

We point out that the first part of d), namely the absence of the essential spectrum, has been studied in many works, e.g., [Kel, KL, KLW]. They generalize some ideas of [DK, Fuj]. Their approach is based on some isoperimetric estimates and on the Persson's Lemma. The latter characterizes the infimum of the essential spectrum.

The asymptotic of eigenvalues is a novelty and was not considered in the literature before. We stress that one can prescribe any asymptotic of eigenvalues by choosing a proper tree  $G$  (and hence  $d_G$ ). We mention that the spectral asymptotic estimates obtained in [DM] are for some operators with non-empty essential spectrum. They study graphs which are equipped with a free action of a discrete group and establish a bound on the  $\text{tr } e^{-t\Delta_{\mathcal{E},\theta}}$ , where the trace is adapted to a fundamental domain.

To our knowledge, this is the first time that the form-domain of the unbounded discrete Laplacian is identified. It is remarkable that it coincides with that of a multiplication operator. A useful consequence is the stability of the essential spectrum, obtained in c). This is also new. We stress that there are simple bi-partite graphs, such that the form-domain of the Laplacian is different from that of  $d_G(Q)$  (see Proposition 4.4). In this case, (1.10) is not fulfilled.

Having the same form-domain does not necessarily ensure that the domain are also equal. In Proposition 4.2, we construct a simple tree which is such an example. However, under some further hypothesis on the graph, Theorem 4.1 ensures that the domain of the magnetic Laplacian is also the one of  $d_G(Q)$ .

Finally we present the organization of the paper. In section 2.1 we provide a new criterion of essential self-adjointness. Next, in section 2.2, we recall some well-known facts about the min-max principle, its relation to the bottom of the essential spectrum and compactness. Then, in Section 3.1 we prove the Hardy inequality and discuss its triviality. In Section 3.2, we prove Theorem 1.1 in the context of trees. Besides, in Section 4.1 we discuss the question of the domain of the Laplacian on a general graph and that of form-domain on bi-partite graphs. Perturbation theory is developed in Section 5. Finally we provide two appendices, one concerning the  $\mathcal{C}^1$  regularity and another one concerning the Helffer-Sjöstrand's formula.

**Notation:** We denote by  $\mathbb{N}$  the non-negative integer. In particular,  $0 \in \mathbb{N}$ . We set  $\langle x \rangle := (1 + x^2)^{1/2}$ . Given a set  $X$  and  $Y \subseteq X$  let  $\mathbf{1}_Y : X \rightarrow \{0, 1\}$  be the characteristic function of  $Y$ . We denote also by  $Y^c$  the complement set of  $Y$  in  $X$ . We consider only separable complex Hilbert space. We denote by

$\mathcal{B}(\mathcal{H}, \mathcal{H})$ , the space of bounded operator from the Hilbert spaces  $\mathcal{H}$  to  $\mathcal{H}$ . Given a function  $V : \mathcal{V} \rightarrow \mathbb{C}$ , we denote by  $V(Q)$  the operator of multiplication by  $V$ .

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## 2. GENERAL PROPERTIES

**2.1. Essential self-adjointness.** Before talking about essential self-adjointness, we deal with the trivial case, the boundedness of the Laplacian and refer to [KL, KLW] for more discussions in the setting of Dirichlet forms and  $\ell^p$  spaces:

**Proposition 2.1.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph. One has that  $\Delta_{\mathcal{E}, \theta}$  is bounded if and only if  $d_G(Q)$  is bounded.*

*Proof.* First, (1.4) gives one direction. On the other hand,  $\langle \mathbf{1}_{\{x\}}, \Delta_{\mathcal{E}, \theta} \mathbf{1}_{\{x\}} \rangle = d_G(x)$ , for all  $x \in \mathcal{V}$ .  $\square$

Essential self-adjointness of the discrete Laplacian was proved in many situations by Jørgensen (see [Jor] and references therein). In [Jor], he says that every discrete Laplacian is essentially self-adjoint on simple graphs. The proof of [Jor] was incomplete (see [JP]). It was fixed later in [JP]. In the meantime, [Woj] proves this fact. An alternative proof can be found in [Web] where one uses the maximum principle. Similar ideas are found in [KL], where one generalizes this fact to some weighted graphs by studying Dirichlet forms. Then come the works of [Tor, Ma] for weighted graphs which are metrically complete, see also [CTT] for the non-metrically complete case. Some other criteria, based on commutators, are given in [GS] (see also [Gol]). Finally for the magnetic case, we mention the works [CTT2, Mil, Mil2]. We point out that in the older work of [Aom] one gives some characterization of possible self-adjoint extensions of a weighted discrete Laplacian in the limit point/circle spirit in the case of trees. More recently, in [GS], the question of the deficiency indices is discussed. We point out that, in the latter, one can consider potentials that tend to  $-\infty$ .

We now improve a self-adjointness criteria given in [KL] and extend it in two directions: we allow magnetic operators and potentials that are unbounded from below.

**Proposition 2.2.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph and  $V : \mathcal{V} \rightarrow \mathbb{R}$ . Take  $\lambda \in \mathbb{R}$  so that*

$$(2.1) \quad \{\lambda + d_G(x) + V(x), x \in \mathcal{V}\} = \emptyset$$

*Suppose that, for any  $(x_n)_{n \in \mathbb{N}} \in \mathcal{V}^{\mathbb{N}}$ , such that the weight  $\mathcal{E}(x_n, x_{n+1}) > 0$  for all  $n \in \mathbb{N}$ , the property*

$$(2.2) \quad \sum_{n \in \mathbb{N}} m^2(x_n) a_n^2 = \infty, \text{ where } a_n := \prod_{i=0}^{n-1} \left(1 + \frac{\lambda + V(x_i)}{d_G(x_i)}\right)$$

*holds true. Then, the operator  $H := \Delta_{\mathcal{E}, \theta} + V(Q)$  is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ .*

First, note it is always possible to find a  $\lambda$  fulfilling (2.1), as  $\#\{d_G(x) + V(x), x \in \mathcal{V}\} \leq \aleph$ . In [KL], the hypothesis is stronger, i.e., they take  $a_n = 1$ , do not consider magnetic fields, and consider potentials that are bounded from below. Our technique relies on an improvement of [Woj, Theorem 1.3.1].

*Proof.* Let  $f \in \mathcal{D}(H^*) \setminus \{0\}$  such that  $H^*f + (i + \lambda)f = 0$  or  $H^*f + (-i + \lambda)f = 0$ . We get easily:

$$|f(x)| \leq \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \frac{\mathcal{E}(x, y)}{|\lambda + d_G(x) + V(x)|} |f(y)|.$$

We derive:

$$(2.3) \quad |f(y)| \geq \left(1 + \frac{\lambda + V(x)}{d_G(x)}\right) |f(x)|, \text{ for all } x, y \in \mathcal{V}, \text{ so that } \mathcal{E}(x, y) \neq 0.$$

Now, since  $f \neq 0$ , there is  $x_0 \in \mathcal{V}$  so that  $f(x_0) \neq 0$ . Therefore, inductively, we obtain a sequence  $(x_n)_{n \in \mathbb{N}} \in \mathcal{V}^{\mathbb{N}}$  such that  $\mathcal{E}(x_n, x_{n+1}) > 0$ , for all  $n \in \mathbb{N}$ , and so that (2.3) holds for  $y = x_{n+1}$  and  $x = x_n$ . Hence, we get

$$\sum_{n=0}^N m^2(x_n) |f(x_n)|^2 \geq \sum_{i=0}^N \prod_{j=0}^{i-1} \left(1 + \frac{\lambda + V(x_j)}{d_G(x_j)}\right)^2 |f(x_0)|^2.$$

By letting  $N$  goes to infinity and remembering (2.2), we obtain a contradiction to the fact that  $f \in \ell^2(\mathcal{V}, m^2)$ . We conclude with the help of [RS, Theorem X.1].  $\square$

**2.2. Min-max principle.** We recall some well-known results. We refer to [RS][Chapter XIII.1] for more details and to [RS][Chapter XIII.15] for more applications. We start with the form-version of the standard variational characterization of the  $n$ -th eigenvalue.

**Theorem 2.1.** *Let  $A$  be a non-negative self-adjoint operator with form-domain  $\mathcal{Q}(A)$ . For all  $n \geq 1$ , we define:*

$$\mu_n(A) := \sup_{\varphi_1, \dots, \varphi_n} \inf_{\psi \in [\varphi_1, \dots, \varphi_n]^\perp} \langle \psi, A\psi \rangle,$$

where  $[\varphi_1, \dots, \varphi_n]^\perp = \{\psi \in \mathcal{Q}(A), \text{ so that } \|\psi\| = 1 \text{ and } \langle \psi, \varphi_i \rangle = 0, \text{ with } i = 1, \dots, n\}$ . Note that  $\varphi_i$  are not required to be linearly independent.

If  $\mu_n$  is (strictly) below the essential spectrum of  $A$ , it is the  $n$ -th eigenvalue, counted with multiplicity, and we have that:

$$\dim \text{Ran } \mathbf{1}_{[0, \mu_n(A)]}(A) = n.$$

Otherwise,  $\mu_n$  is the infimum of the essential spectrum. Moreover,  $\mu_j = \mu_n$ , for all  $j \geq n$  and there are at most  $n - 1$  eigenvalues, counted with multiplicity, below the essential spectrum. In that case,

$$\dim \text{Ran } \mathbf{1}_{[0, \mu_n(A) + \varepsilon]}(A) = +\infty, \text{ for all } \varepsilon > 0.$$

**Remark 2.1.** *One has a priori no control on the multiplicity of the (possible) eigenvalue which is at the bottom of the essential spectrum.*

This ensures the useful following criteria:

**Proposition 2.3.** *Let  $A, B$  be two self-adjoint operators, with form-domains  $\mathcal{Q}(A)$  and  $\mathcal{Q}(B)$ , respectively. Suppose that*

$$\mathcal{Q}(A) \supset \mathcal{Q}(B) \text{ and } 0 \leq \langle \psi, A\psi \rangle \leq \langle \psi, B\psi \rangle,$$

for all  $\psi \in \mathcal{Q}(B)$ . Then one has  $\inf \sigma_{\text{ess}}(A) \leq \inf \sigma_{\text{ess}}(B)$  and

$$(2.4) \quad \mathcal{N}_\lambda(A) \geq \mathcal{N}_\lambda(B), \text{ for } \lambda \in [0, \infty) \setminus \{\inf \sigma_{\text{ess}}(B)\},$$

where  $\mathcal{N}_\lambda(A) := \dim \text{Ran } \mathbf{1}_{[0, \lambda]}(A)$ .

In particular, if  $A$  and  $B$  have the same form-domain, then  $\sigma_{\text{ess}}(A) = \emptyset$  if and only if  $\sigma_{\text{ess}}(B) = \emptyset$ .

*Proof.* It is enough to notice that  $\mu_n(A) \leq \mu_n(B)$ , for all  $n \geq 0$ . Theorem 2.1 permits us to conclude for the first part. Supposing now they have the same form-domain, by uniform boundedness principle, there is  $a, b > 0$  such that:

$$\langle \psi, |A|\psi \rangle \leq a \langle \psi, |B|\psi \rangle + b \|\psi\|^2 \text{ and } \langle \psi, |B|\psi \rangle \leq a \langle \psi, |A|\psi \rangle + b \|\psi\|^2$$

for all  $\psi \in \mathcal{Q}(|A|) = \mathcal{Q}(|B|)$ . By using twice the previous statement we get the result.  $\square$

Because of Remark 2.1, we stress that we have to remove “ $\inf \sigma_{\text{ess}}(B)$ ” in (2.4), as the min-max principle does not decide whether or not the  $B$  has an eigenvalue at this energy.

We now turn to a criteria of compactness. We recall that a compact and self-adjoint operator  $A$  is in the  $p$ -Schatten class, for  $p \in [1, \infty)$ , if  $\text{tr}(|A|^p) < \infty$ .

**Proposition 2.4.** *Let  $\mathcal{H}$  be a Hilbert space and  $A, B$  be two bounded self-adjoint operators. Suppose that  $B$  is compact and*

$$|\langle f, Af \rangle| \leq \langle f, Bf \rangle, \text{ for all } f \in \mathcal{H}.$$

Then  $A$  is also compact. Moreover, given  $p \in [1, \infty)$ , if  $B$  is in the  $p$ -Schatten class, so is  $A$ .

*Proof.* First note that  $-B \leq A$  and that  $-B \leq -A$  in the form sense. By Proposition 2.3, this implies that the essential spectrum of  $A$  is  $\{0\}$ . Finally, as  $A$  is self-adjoint, we infer that  $A$  is compact. Now assume that  $B$  is in the  $p$ -Schatten class. The min-max principle, applied to  $A$  and  $-A$ , guarantees that  $\text{tr}(|A|^p) \leq 2^p \text{tr}(B^p)$ .  $\square$

Using (1.4), we give a straightforward consequence of Propositions 2.3 and 2.4:

**Corollary 2.1.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph. One has:*

$$\inf \sigma_{\text{ess}}(\Delta_{\mathcal{E}, \theta}) \leq 2 \inf \sigma_{\text{ess}}(d_G(Q)) \text{ and } \mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta}) \geq \mathcal{N}_\lambda(2d_G(Q)),$$

for all  $\lambda \in [0, \infty) \setminus \{\inf \sigma_{\text{ess}}(d_G(Q))\}$ . In particular, if  $0 \in \sigma_{\text{ess}}(d_G(Q))$ , then  $0 \in \sigma_{\text{ess}}(\Delta_{\mathcal{E}, \theta})$  and if  $\Delta_{\mathcal{E}, \theta}$  has compact resolvent then,  $d_G(Q)$  has also compact resolvent. In other words, one has that  $\sigma_{\text{ess}}(d_G(Q)) \neq \emptyset$ , then  $\sigma_{\text{ess}}(\Delta_{\mathcal{E}, \theta}) \neq \emptyset$ .

Moreover, if  $d_G(Q)$  is compact, then  $\Delta_{\mathcal{E}, \theta}$  is also compact.

We mention, that for a simple graph,  $d_G(Q) \geq 1$  and it is not a compact operator.

### 3. SURROUNDING THE LAPLACIAN

**3.1. A Hardy inequality.** We start with a remark about bi-partite graphs:

**Proposition 3.1.** *Given a bi-partite graph  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  and a function  $V : \mathcal{V} \rightarrow [0, \infty)$ , the following assertions are equivalent:*

$$(3.1) \quad \langle f, (d_G(Q) - V(Q))f \rangle \leq \langle f, \Delta_{\mathcal{E}, \theta} f \rangle, \quad \text{for } f \in \mathcal{C}_c(\mathcal{V}),$$

$$(3.2) \quad \langle f, \Delta_{\mathcal{E}, \theta} f \rangle \leq \langle f, (d_G(Q) + V(Q))f \rangle, \quad \text{for } f \in \mathcal{C}_c(\mathcal{V}),$$

$$(3.3) \quad |\langle f, \mathcal{A}_{\mathcal{E}, \theta} f \rangle| \leq \langle f, Vf \rangle, \quad \text{for } f \in \mathcal{C}_c(\mathcal{V}),$$

where  $\mathcal{A}_{\mathcal{E}, \theta}$  is the magnetic adjacency matrix defined by

$$(3.4) \quad (\mathcal{A}_{\mathcal{E}, \theta} f)(x) := \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) e^{i\theta_{x, y}} f(y),$$

for  $f \in \mathcal{C}_c(\mathcal{V})$  and  $x \in \mathcal{V}$ .

*Proof.* Let  $x_0 \in \mathcal{V}$ . Given  $x \in \mathcal{V}$ , by definition of connectedness, there is a sequence  $x_1, \dots, x_n$  of elements of  $\mathcal{V}$ , so that  $\mathcal{E}(x_i, x_{i+1}) \neq 0$ , for  $i = 0, \dots, n-1$  and  $x_n = x$ . The minimal possible  $n$  is the *unweighted length* between  $x_0$  and  $x$ . We denote it by  $\ell(x)$ . Set  $Uf(x) := (-1)^{\ell(x)} f(x)$ . Note that  $U^2 = \text{Id}$  and  $U^{-1} = U^* = U$ . Notice now that

$$U^{-1} \mathcal{A}_{\mathcal{E}, \theta} U = -\mathcal{A}_{\mathcal{E}, \theta} \quad \text{and} \quad \Delta_{\mathcal{E}, \theta} = d_G(Q) - \mathcal{A}_{\mathcal{E}, \theta}.$$

We start with (3.1) and rewrite it as follows:  $\langle f, \mathcal{A}_{\mathcal{E}, \theta} f \rangle \leq \langle f, Vf \rangle$ , for  $f \in \mathcal{C}_c(\mathcal{V})$ . Applying this to  $Uf$ , we infer immediately (3.3). We start now from (3.3). We get:

$$\langle f, \Delta_{\mathcal{E}, \theta} f \rangle = \langle f, (d_G(Q) - \mathcal{A}_{\mathcal{E}, \theta})f \rangle \geq \langle f, (d_G(Q) - V(Q))f \rangle$$

for  $f \in \mathcal{C}_c(\mathcal{V})$ . In the same way, we have that (3.2) is equivalent to (3.3).  $\square$

We now turn to the key estimate of the paper and prove Proposition 1.1. First we mention that a Laplacian in a certain  $\ell^2(\mathcal{V}, m^2)$  is unitarily equivalent to a Schrödinger operator in any other  $\ell^2(\mathcal{V}, m_0^2)$ . This has already been noticed before, e.g., [CTT].

**Proposition 3.2.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph and  $m_0 : \mathcal{V} \rightarrow (0, \infty)$  be a weight. Then the Friedrichs extension of  $\Delta_{\mathcal{E}_0, \theta}$ , acting in  $\ell^2(\mathcal{V}, m_0^2)$ , is unitarily equivalent to that of  $\Delta_{\mathcal{E}, \theta} + V(Q)$ , in  $\ell^2(\mathcal{V}, m^2)$ , where*

$$\mathcal{E}(x, y) := \mathcal{E}_0(x, y) \frac{m(x)m(y)}{m_0(x)m_0(y)} \quad \text{and} \quad V(x) := \frac{1}{m_0^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) \left( 1 - \frac{m(y)m_0(x)}{m(x)m_0(y)} \right).$$

*Proof.* Consider the unitary map  $U : \ell^2(G, m^2) \rightarrow \ell^2(G, m_0^2)$  given by  $(Uf)(x) = (m(x)/m_0(x))f(x)$ . Straightforwardly, using (1.3), one has

$$(3.5) \quad U^{-1} \Delta_{\mathcal{E}_0, \theta} Uf := (\Delta_{\mathcal{E}, \theta} + V(Q))f$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Then it holds true for the closures of  $\Delta_{\mathcal{E}_0, \theta}|_{\mathcal{C}_c}$  and  $(\Delta_{\mathcal{E}, \theta} + V(Q))|_{\mathcal{C}_c}$ . Note now that (3.5) also holds for the adjoints of the last two operators. Therefore, the Friedrichs extensions  $\Delta_{\mathcal{E}_0, \theta}$  and  $\Delta_{\mathcal{E}, \theta} + V(Q)$  are unitarily equivalent.  $\square$

The novelty in the Hardy inequality (1.6) is more in the point of view. Rather than studying  $\Delta_{\mathcal{E}, \theta}$  in  $\ell^2(\mathcal{V}, m^2)$  with the help of a simpler  $\ell^2(\mathcal{V}, m_0^2)$  (where typically  $m_0 = 1$ ), we study it with the help of all other weighted spaces. The applications we consider in the paper are also new.

*Proof of Proposition 1.1.* On  $\ell^2(G, m^2)$ , we consider the quadratic form:

$$\mathcal{Q}_{\mathcal{E}, \theta}(f, f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \tilde{\mathcal{E}}(x, y) |f(x) - e^{i\theta_{x, y}} f(y)|^2 \geq 0, \quad \text{for } f \in \mathcal{C}_c(\mathcal{V}),$$

where

$$\tilde{\mathcal{E}}(x, y) := \mathcal{E}(x, y) \frac{m(x)m(y)}{m_0(x)m_0(y)}.$$

Note that  $\mathcal{Q}(f, f) = \langle f, H_{m^2} f \rangle_{\ell^2(\mathcal{V}, m^2)}$ , where

$$(H_{m^2} f)(x) = \frac{1}{m^2(x)} \sum_{y \sim x} \tilde{\mathcal{E}}(x, y) (f(x) - e^{i\theta_{x,y}} f(y)), \text{ for } f \in \mathcal{C}_c(V).$$

Consider now the unitary map  $U : \ell^2(G, m^2) \rightarrow \ell^2(G, m_0^2)$  given by  $(Uf)(x) = (m(x)/m_0(x))f(x)$ . Set  $H := UH_{m^2}U^{-1}$ . We have:

$$\begin{aligned} (Hf)(x) &= \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \tilde{\mathcal{E}}(x, y) f(x) - \frac{1}{m_0^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) e^{i\theta_{x,y}} f(y) \\ &= \frac{1}{m_0^2(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) (f(x) - e^{i\theta_{x,y}} f(y)) - V_m(x) f(x). \end{aligned}$$

Since  $H \geq 0$ , we obtain (1.6). The rest of the statement is given by Proposition 3.1.  $\square$

In the paper, we are interested in minorating the Laplacian with the help of the weighted degree. Before proving the main result for simple trees in the next section, we start by giving some negative answers. First, it is obvious that one cannot find a non-negative  $V_m$  in (1.6) if the graph is finite. This continues to hold true for some infinite graphs.

**Proposition 3.3.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph and take  $V : \mathcal{V} \rightarrow \mathbb{R}$  so that  $0 \leq V \leq \Delta_{\mathcal{E}, \theta}$ , in the form sense on  $\mathcal{C}_c(\mathcal{V})$ . If the constant function 1 is in  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2})$  then  $V = 0$ .*

If one supposes that  $1 \in \ell^2(\mathcal{V}, m^2)$  and  $\Delta_{\mathcal{E}, \theta}$  is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ , then 1 is in  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2})$ , as  $1 \in \mathcal{D}((\Delta_{\mathcal{E}, \theta}|_{\mathcal{C}_c})^*)$ .

*Proof.* By construction of the Friedrichs extension, there is  $f_n \in \mathcal{C}_c(\mathcal{V})$ , so that  $f_n$  tends to 1 in the graph norm of  $\Delta_{\mathcal{E}, \theta}^{1/2}$ . Moreover,

$$0 \leq \sum_{x \in \mathcal{V}} m^2(x) V(x) |f_n(x)|^2 = \langle f_n, V f_n \rangle \leq \langle f_n, \Delta_{\mathcal{E}, \theta} f_n \rangle \rightarrow 0,$$

as  $n$  goes to  $\infty$ . Now since  $V^{1/2}(Q)$  is closed, the closed graph theorem gives that the l.h.s. tends to  $\sum_{x \in \mathcal{V}} m^2(x) V(x)$ . Therefore  $V = 0$ .  $\square$

We now treat the impossibility of minorating by using the weighted degree. We say that  $\{K_n\}_{n \in \mathbb{N}}$  is a *filtration* of a graph  $(\mathcal{V}, \mathcal{E}, m, \theta)$  if  $K_n \subset \mathcal{V}$  is finite,  $K_n \subset K_{n+1}$ , for all  $n \in \mathbb{N}$ , and such that  $\cup_{n \in \mathbb{N}} K_n = \mathcal{V}$ . Given a subset  $K \in \mathcal{V}$ , let  $\partial K$  be the set  $\{x \in K, \text{ there is } y \sim x \text{ such that } y \notin K\}$ .

**Proposition 3.4.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph, such that  $\mathcal{E} \neq \emptyset$ . Then there is no  $a > 0$  such that (1.5) holds true. Moreover, if there is a filtration  $\{K_n\}_{n \in \mathbb{N}}$  associated with  $G$ , such that:*

$$(3.6) \quad \limsup_{n \rightarrow +\infty} \frac{\sum_{x \in K_n \setminus \partial K_n} m^2(x) d_G(x)}{\sum_{x \in K_n} m^2(x)} = +\infty,$$

then there is no  $a \in [0, 1]$  and  $b \geq 0$  such that

$$(3.7) \quad \langle f, d_G(Q)f \rangle \leq a \langle f, \Delta_{\mathcal{E}, \theta} f \rangle + b \|f\|^2, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}).$$

In particular, (3.6) is fulfilled, if  $1 \in \ell^1(\mathcal{V}, m^2)$  and  $\sup_{x \in \mathcal{V}} m^2(x) d_G(x) = +\infty$ .

*Proof.* Take  $K \subset \mathcal{V}$  finite and set  $f = \mathbf{1}_K$ . Using (1.2), note that

$$(3.8) \quad \langle f, d_G(Q)f \rangle = \sum_{x \in K} m^2(x) d_G(x) \text{ and } \langle f, \Delta_{\mathcal{E}, \theta} f \rangle \leq \sum_{x \in \partial K} m^2(x) d_G(x).$$

We consider the first assertion and suppose that (1.5) holds true for some  $a > 0$ . First, by taking  $f = \mathbf{1}_{\{x\}}$ , for some  $x$  such that  $d_G(x) \neq 0$ , one sees that  $a \leq 1$ . Next, consider  $K$  large enough so that  $\sup_{x \in K} d_G(x) > 0$ . Then, (3.8) implies that  $\sum_{x \in K \setminus \partial K} m^2(x) d_G(x) \leq 0$  and therefore that  $d_G(x) = 0$  for all  $x \in \mathcal{V}$ . For the second assertion, consider  $K = K_n$  and note that (3.7) implies that

$$\sum_{x \in K_n \setminus \partial K_n} m^2(x) d_G(x) \leq b \sum_{x \in K_n} m^2(x).$$

By letting  $n$  goes to infinity, we obtain a contradiction with (3.6).  $\square$



**3.2. The case of trees.** We now turn to a minoration of the magnetic Laplacian and present it in the context of weighted trees. Perturbation theory will be considered in Section 5. We fix some notation.

We first recall that a *tree* is a connected graph  $G = (\mathcal{E}, \mathcal{V}, m, \theta)$  such that for each edge  $e \in \mathcal{V} \times \mathcal{V}$  with  $\mathcal{E}(e) \neq 0$  the graph  $(\tilde{\mathcal{E}}, \mathcal{V}, m, \theta|_{\tilde{\mathcal{E}}})$ , with  $\tilde{\mathcal{E}} := \mathcal{E} \times 1_{\{e\}^c}$ , i.e., with the edge  $e$  removed, is disconnected. It is convenient to choose a root in the tree. Due to its structure, one can take any point of  $\mathcal{V}$ . We denote it by  $\epsilon$ . We define inductively the *spheres*  $S_n$  by  $S_{-1} = \emptyset$ ,  $S_0 := \{\epsilon\}$ , and  $S_{n+1} := \mathcal{N}_G(S_n) \setminus S_{n-1}$ . Given  $n \in \mathbb{N}$ ,  $x \in S_n$ , and  $y \in \mathcal{N}_G(x)$ , one sees that  $y \in S_{n-1} \cup S_{n+1}$ . We write  $x \sim > y$  and say that  $x$  is a *son* of  $y$ , if  $y \in S_{n-1}$ , while we write  $x \sim < y$  and say that  $x$  is a *father* of  $y$ , if  $y \in S_{n+1}$ . Notice that  $\epsilon$  has no father. Given  $x \neq \epsilon$ , note that there is a unique  $y \in V$  with  $x \sim > y$ , i.e., everyone apart from  $\epsilon$  has one and only one father. We denote the father of  $x$  by  $\overset{\leftarrow}{x}$ .

*Proof of Theorem 1.1.* We focus on the left hand side of (1.10) for  $G_\circ$ , i.e., for  $H = \Delta_{\mathcal{E}_\circ, \theta_\circ}$ . Set  $K := \{x \in \mathcal{V}, d_{G_\circ}(x) \leq 1\}$  and  $\tilde{\mathcal{V}} := \mathcal{V} \setminus K$ . Note that induced graph  $\tilde{G}_\circ$  of  $G_\circ$  w.r.t.  $\tilde{\mathcal{V}} = \cup_{n \in \mathbb{N}} \tilde{\mathcal{V}}_n$  is a forest of disjoint and maximal subtrees  $\tilde{G}_n := (\mathcal{E}_n, \tilde{\mathcal{V}}_n, m|_{\tilde{\mathcal{V}}_n}, \theta_\circ|_{\tilde{\mathcal{E}}_n})$ , where  $\tilde{\mathcal{E}}_n := \mathcal{E}_\circ|_{\tilde{\mathcal{V}}_n \times \tilde{\mathcal{V}}_n}$ . For each  $\tilde{G}_n$  of  $\tilde{G}_\circ$ , we define:

$$\tilde{m}(\epsilon_{\tilde{\mathcal{V}}_n}) := 1 \text{ and } \tilde{m}(x) := \eta \tilde{m}(\overset{\leftarrow}{x}) \frac{m(x)}{m(\overset{\leftarrow}{x})} d_{G_\circ}^{-\varepsilon_0/2}(x), \text{ for all } x \in \tilde{\mathcal{V}}_n.$$

With this definition and  $V_{\tilde{m}}$  as in (1.7), we obtain:

$$\begin{aligned} \frac{V_{\tilde{m}}(x)}{d_{\tilde{G}_\circ}(x)} &= 1 - \frac{1}{d_G(x)m^2(x)} \left( \mathcal{E}(\overset{\leftarrow}{x}, x) \frac{\tilde{m}(\overset{\leftarrow}{x})}{\tilde{m}(x)} \frac{m(x)}{m(\overset{\leftarrow}{x})} + \sum_{y \sim > x} \mathcal{E}(y, x) \frac{\tilde{m}(y)}{\tilde{m}(x)} \frac{m(x)}{m(y)} \right) \\ &= 1 - \frac{1}{\eta d_G^{(1-\varepsilon_0/2)}(x)m^2(x)} \mathcal{E}(\overset{\leftarrow}{x}, x) - \frac{\eta}{d_G(x)m^2(x)} \sum_{y \sim > x} \mathcal{E}(y, x) d_G^{-\varepsilon_0/2}(y) \\ (3.9) \quad &\geq 1 - \eta - \frac{1}{\eta} d_G^{-\varepsilon_0/2}(x) C_0, \end{aligned}$$

for all  $x \in \tilde{\mathcal{V}}_n$ . Consider  $f \in \mathcal{C}_c(\mathcal{V})$ . We apply Proposition 1.1 to  $\tilde{G}$ . Since  $\mathcal{E} \geq 0$  we derive that there is a constant  $C(\eta) > 0$ , so that:

$$\begin{aligned} \langle f, \Delta_{\mathcal{E}_\circ, \theta_\circ} f \rangle &\geq \langle f|_{\tilde{\mathcal{V}}}, \Delta_{\mathcal{E}_\circ, \theta_\circ} (f|_{\tilde{\mathcal{V}}}) \rangle \geq \langle f|_{\tilde{\mathcal{V}}}, V_{\tilde{m}}(Q) (f|_{\tilde{\mathcal{V}}}) \rangle \\ &\geq \langle d_G^{1/2}(Q) f|_{\tilde{\mathcal{V}}}, (1 - \eta - C_0/d_G^{\varepsilon_0/2}(Q)\eta) d_G^{1/2}(Q) f|_{\tilde{\mathcal{V}}} \rangle \\ &\geq (1 - 2\eta) \langle f|_{\tilde{\mathcal{V}}}, d_{\tilde{G}}(Q) f|_{\tilde{\mathcal{V}}} \rangle - C(\eta) \|f|_{\tilde{\mathcal{V}}}\|^2 \\ (3.10) \quad &\geq (1 - 2\eta) \langle f, d_{G_\circ}(Q) f \rangle - (C_\eta + 1) \|f\|^2. \end{aligned}$$

This gives (1.10). The reverse inequality is obtained by applying Proposition 3.1.

Next, the equality of the domains of the forms follow immediately and so the essential spectrum of  $\Delta_{\mathcal{E}_\circ, \theta_\circ}$  is empty if and only if  $\lim_{|x| \rightarrow \infty} d_G(x) = +\infty$ . Finally, we use twice Proposition 2.3 with the double inequality (1.10). This yields:

$$1 - \varepsilon \leq \liminf_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta})}{\mathcal{N}_\lambda(d_G(Q))} \leq \limsup_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta})}{\mathcal{N}_\lambda(d_G(Q))} \leq 1 + \varepsilon.$$

By letting  $\varepsilon$  goes to zero we obtain the Weyl asymptotic (1.11) for  $\Delta_{\mathcal{E}_\circ, \theta_\circ}$ . It remains to use Proposition 5.1 to obtain the result for  $\Delta_{\mathcal{E}, \theta}$ .  $\square$

**Remark 3.1.** *It is important to note that  $\varepsilon$  has to be positive in (1.10). For instance, on a simple tree, if  $\varepsilon$  was equal to 0, this would imply that the magnetic adjacency matrix was bounded (see Proposition 3.1). Finally note that, by considering delta functions as test functions, the magnetic adjacency matrix is bounded if and only if the weighted degree is bounded.*

**Remark 3.2.** *We could not provide an example of Schrödinger operator which have the same form-domain as  $d_G(Q)$  and which is not essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ .*

## 4. COMPARISON OF DOMAINS

**4.1. From form-domain to domain.** Once one knows that the form-domains are equal, the next question is to guarantee that the domains are equal. We present an approach by commutators. Some subtleties arise since we have to deal with the square root of the Laplacian.

We first prove the invariance of the domain under the  $C_0$ -group  $\{e^{itd_G(Q)}\}_{t \in \mathbb{R}}$  and the  $\mathcal{C}^1$  regularity (see Appendix A for a definition).

**Proposition 4.1.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph, such that  $\Delta_{\mathcal{E}, \theta}$  is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ ,  $\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ , and*

$$(4.1) \quad \sup_{x \in \mathcal{V}} \left( \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \frac{\mathcal{E}(x, y)}{\min(\langle d_G(x) \rangle^{1/2}, \langle d_G(y) \rangle^{1/2})} |d_G(x) - d_G(y)| \right) < \infty.$$

*Then, one has that the invariance  $e^{itd_G(Q)}\mathcal{D}(\Delta_{\mathcal{E}, \theta}) \subset \mathcal{D}(\Delta_{\mathcal{E}, \theta})$ , for all  $t \in \mathbb{R}$ . Moreover, we have that  $\Delta_{\mathcal{E}, \theta} \in \mathcal{C}^1(d_G(Q))$  and that  $[\Delta_{\mathcal{E}, \theta}, d_G(Q)]|_{\mathcal{C}_c(\mathcal{V})}$  extends to  $[\Delta_{\mathcal{E}, \theta}, d_G(Q)]_0 \in \mathcal{B}(\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}), \ell^2(\mathcal{V}, m^2))$ .*

*Proof.* We denote by  $[\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}]_0$  the closure of the commutator  $[\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}]|_{\mathcal{C}_c(\mathcal{V})}$  and by  $c_0$  the constant appearing in (4.1). We have that:

$$(4.2) \quad \begin{aligned} |\langle f, [\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}]_0 g \rangle| &= \sum_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} \overline{f(x)} \mathcal{E}(x, y) \left( e^{itd_G(x)} - e^{itd_G(y)} \right) g(y) \\ &\leq \frac{|t|}{2} \sum_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} \frac{\mathcal{E}(x, y)}{\langle d_G(y) \rangle^{1/2}} |d_G(x) - d_G(y)| \left( |f(x)|^2 + |\langle d_G(Q) \rangle^{1/2} g(y)|^2 \right) \\ &\leq c_0 |t| \left( \|f\|^2 + \left\| \langle d_G(Q) \rangle^{1/2} g \right\|^2 \right), \end{aligned}$$

for all  $f, g \in \mathcal{C}_c(\mathcal{V})$ . Therefore,  $[\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}]_0$  is bounded from  $\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2})$  to  $\ell^2(\mathcal{V}, m^2)$ . On the other hand, we get:

$$\begin{aligned} \|\Delta_{\mathcal{E}, \theta} e^{itd_G(Q)} f\| &\leq \|e^{itd_G(Q)} \Delta_{\mathcal{E}, \theta} f\| + \|[\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}] \langle d_G(Q) \rangle^{-1/2} \langle d_G(Q) \rangle^{1/2} f\| \\ &\leq C (\|f\| + \|\Delta_{\mathcal{E}, \theta} f\|), \end{aligned}$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . By density we obtain the invariance of the domain.

Next, thanks to (4.2), we obtain that

$$\liminf_{t \rightarrow 0^+} \left\| [\Delta_{\mathcal{E}, \theta}, e^{itd_G(Q)}] / t \right\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}), \ell^2(\mathcal{V}, m^2))} < \infty.$$

Therefore, Theorem A.2 yields that  $\Delta_{\mathcal{E}, \theta} \in \mathcal{C}^1(d_G(Q))$ . Finally, by estimating as in (4.2), we obtain that  $[\Delta_{\mathcal{E}, \theta}, d_G(Q)]_0$ , belongs  $\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}), \ell^2(\mathcal{V}, m^2))$ .  $\square$

We turn to the central result of this section.

**Theorem 4.1.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted graph, such that  $\mathcal{D}(\Delta_{\mathcal{E}, \theta}^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ ,  $\Delta_{\mathcal{E}, \theta}$  is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ , and there is  $\varepsilon > 0$  such that*

$$(4.3) \quad \sup_{x \in \mathcal{V}} \left( \frac{1}{m^2(x)} \sum_{y \in \mathcal{V}} \frac{\mathcal{E}(x, y)}{\min(\langle d_G(x) \rangle^{1/2-\varepsilon}, \langle d_G(y) \rangle^{1/2-\varepsilon})} |d_G(x) - d_G(y)| \right) < \infty.$$

*Take also a potential  $V : \mathcal{V} \rightarrow \mathbb{R}$ , such that:*

$$(4.4) \quad \lim_{|x| \rightarrow +\infty} \frac{V(x)}{d_G(x) + 1} = 0.$$

*Then,  $\mathcal{D}(\Delta_{\mathcal{E}, \theta} + V(Q)) = \mathcal{D}(d_G(Q))$ .*

**Remark 4.1.** *We point out that the hypothesis on the essential self-adjointness is also necessary, as  $d_G(Q)$  is essentially self-adjoint on  $\mathcal{C}_c(\mathcal{V})$ .*

**Remark 4.2.** *By taking  $\varepsilon = 0$  in (4.3), i.e., under the same hypothesis as Proposition 4.1, the proof ensures only that  $\mathcal{D}(d_G(Q)) \subset \mathcal{D}(\Delta_{\mathcal{E}, \theta})$ .*

*Proof.* By Kato-Rellich's Theorem, e.g. [RS, Theorem X.12], it is enough to consider  $V = 0$ . We assume for the moment that

$$M := (\Delta_{\mathcal{E}, \theta} + 1)^{1/2} [d_G(Q), (\Delta_{\mathcal{E}, \theta} + 1)^{1/2}]_0 + [\Delta_{\mathcal{E}, \theta}, (d_G(Q) + 1)^{1/2}]_0 (d_G(Q) + 1)^{1/2}.$$

is a bounded operator from  $\mathcal{D}(d_G^{1-\varepsilon}(Q))$  to  $\mathcal{D}(d_G^{1-\varepsilon}(Q))^*$ . The 1 is here to make the square root smooth over the spectrum. Since the form-domain are equal of  $\Delta_{\mathcal{E}, \theta}$  and  $d_G(Q)$ , we have that there is  $a, b > 0$  so that (3.7) holds true. By using twice (3.7) and working in the form sense on  $\mathcal{C}_c(\mathcal{V})$ , we infer:

$$\begin{aligned}
(\Delta_{\varepsilon, \theta} + 1)^2 &\leq a(\Delta_{\varepsilon, \theta} + 1)^{1/2}(d_G(Q) + 1)(\Delta_{\varepsilon, \theta} + 1)^{1/2} + (b + 1)(\Delta_{\varepsilon, \theta} + 1) \\
&= a(d_G(Q) + 1)^{1/2}\Delta_{\varepsilon, \theta}(d_G(Q) + 1)^{1/2} + aM + (b + 1)(\Delta_{\varepsilon, \theta} + 1) \\
(4.5) \quad &\leq a'd_G^2(Q) + b',
\end{aligned}$$

for some  $a', b' > 0$ . By using the fact that  $\varepsilon > 0$ , the reverse inequality holds true for the same reasons for some  $a' \in (0, 1)$  and  $b' > 0$ . Therefore, the domains are equal. It remains to prove the boundedness of  $M$ . We start with the r.h.s. term. Let  $c_1$  be the constant in (4.3). We estimate as above:

$$\begin{aligned}
&|\langle f, [\Delta_{\varepsilon, \theta}, (d_G(Q) + 1)^{1/2}] \circ \langle d_G(Q) \rangle^{1/2} g \rangle| \\
&= \sum_{x \in \mathcal{Y}} \sum_{y \in \mathcal{Y}} \overline{f(x)} \mathcal{E}(x, y) ((d_G(x) + 1)^{1/2} - (d_G(y) + 1)^{1/2}) \langle d_G(y) \rangle^{1/2} g(y) \\
&\leq \frac{1}{2} \sum_{x \in \mathcal{Y}} \sum_{y \in \mathcal{Y}} \frac{\mathcal{E}(x, y)}{(d_G(y))^{1/2 - \varepsilon}} |d_G(x) - d_G(y)| (|f(x)|^2 + |\langle d_G(Q) \rangle^{1 - \varepsilon} g(y)|^2) \\
&\leq c_1 \left( \|f\|^2 + \|\langle d_G(Q) \rangle^{1 - \varepsilon} g\|^2 \right),
\end{aligned}$$

for all  $f, g \in \mathcal{C}_c(\mathcal{Y})$ . In particular,  $[\Delta_{\varepsilon, \theta}, (d_G(Q) + 1)^{1/2}] \circ \langle d_G(Q) \rangle^{1/2}$  is bounded from  $\mathcal{D}(d_G^{1 - \varepsilon}(Q))$  to  $\ell^2(\mathcal{Y}, m^2)$ .

We turn to the second part of  $M$  and use the Appendix B. Let  $\varphi$  be in  $\mathcal{S}^{1/2}$  such that  $\varphi(x) = \sqrt{x}$ , for all  $x \geq 1$ . Since  $\Delta_{\varepsilon, \theta}$  is non-negative,  $\varphi(\Delta_{\varepsilon, \theta} + 1) = (\Delta_{\varepsilon, \theta} + 1)^{1/2}$ . We cannot use the (B.5) directly with  $\varphi$  as the integral does not seem to exist. We proceed as in [GJ]. Take  $\chi_1 \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$  with values in  $[0, 1]$  and being 1 on  $[-1, 1]$ . Set  $\chi_R := \chi(\cdot/R)$ . As  $R$  goes to infinity,  $\chi_R$  converges pointwise to 1. Moreover,  $\{\chi_R\}_{R \in [1, \infty)}$  is bounded in  $\mathcal{S}^0$ . We infer  $\varphi_R := \varphi \chi_R$  tends pointwise to  $\varphi$  and that  $\{\varphi_R\}_{R \in [1, \infty)}$  is bounded in  $\mathcal{S}^{1/2}$ . Now, recalling  $\Delta_{\varepsilon, \theta} \in \mathcal{C}^1(d_G(Q))$  and (A.3), we obtain

$$\begin{aligned}
(4.6) \quad &[\varphi_R(\Delta_{\varepsilon, \theta}), d_G(Q)] \circ \langle \Delta_{\varepsilon, \theta} \rangle^{-1/2} = \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \varphi_R^{\mathbb{C}}}{\partial \bar{z}} (z - \Delta_{\varepsilon, \theta})^{-1} [d_G(Q), \Delta_{\varepsilon, \theta}] \circ (z - \Delta_{\varepsilon, \theta})^{-1} \langle \Delta_{\varepsilon, \theta} \rangle^{-1/2} dz \wedge d\bar{z}. \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \varphi_R^{\mathbb{C}}}{\partial \bar{z}} (z - \Delta_{\varepsilon, \theta})^{-1} [d_G(Q), \Delta_{\varepsilon, \theta}] \circ \langle \Delta_{\varepsilon, \theta} \rangle^{-1/2} (z - \Delta_{\varepsilon, \theta})^{-1} dz \wedge d\bar{z}.
\end{aligned}$$

By Proposition 4.1,  $[d_G(Q), \Delta_{\varepsilon, \theta}] \circ \langle \Delta_{\varepsilon, \theta} \rangle^{-1/2}$  is bounded. Moreover, using (B.2) with  $l = 2$ , we bound the integrand, uniformly in  $R$ , by  $C \langle x \rangle^{-1+1/2-2} |y|^2 |y|^{-1} |y|^{-1}$ , for some constant  $C$ . It is integrable on the domain given by (B.4). By Lebesgue domination, the r.h.s. of (4.6) has a limit in norm. Note now that the l.h.s., as form on  $\mathcal{C}_c(\mathcal{Y})$ , tends to the operator  $[(\Delta_{\varepsilon, \theta} + 1)^{1/2}, d_G(Q)] \langle \Delta_{\varepsilon, \theta} \rangle^{-1/2}$ . This gives that

$$\langle \Delta_{\varepsilon, \theta} \rangle^{1/2} [(\Delta_{\varepsilon, \theta} + 1)^{1/2}, d_G(Q)] \circ \leq C(d_G(Q) + 1)$$

in the form sense on  $\mathcal{C}_c(\mathcal{Y})$  for some constant  $C$ . This ensures the announced boundedness of  $M$ .  $\square$

**Remark 4.3.** Assuming also that for all  $\varepsilon > 0$ , there is  $c_\varepsilon \geq 0$  so that (1.10) holds true, one observes that  $a'$  can be arbitrary close to 1 in (4.5). Therefore, using again the Kato-Rellich's Theorem, one can weaken (4.4) in Theorem 4.1 and replace it by

$$(4.7) \quad \limsup_{|x| \rightarrow +\infty} \frac{|V(x)|}{d_G(x) + 1} < 1.$$

to obtain  $\mathcal{D}(\Delta_{\varepsilon, \theta} + V(Q)) = \mathcal{D}(d_G(Q))$ .

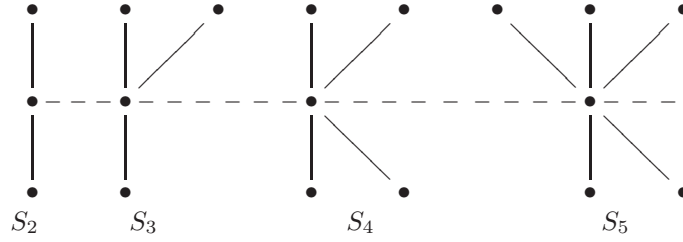
Finally, we provide an example of tree for which the situation is different.

**Proposition 4.2.** *There is a simple tree  $T$ , such that  $\mathcal{D}(\Delta_T) \neq \mathcal{D}(d_T(Q))$  and such that the form-domains  $\mathcal{D}(\Delta_T^{1/2}) = \mathcal{D}(d_T^{1/2}(Q))$ .*

*Proof.* The second point follows from Theorem 1.1. We start by constructing the star graph  $S_n$ . Let  $S_n := (\mathcal{E}_n, \mathcal{Y}_n)$  be defined as follows:  $\mathcal{Y}_n := \{1, \dots, n+1\}$  and so that  $\mathcal{E}_n((1, j)) := 1, \forall j \in \{2, \dots, n+1\}$  and  $\mathcal{E}_n((j, k)) := 0, \forall j, k \in \{2, \dots, n+1\}$ . Consider now  $f_n(x) := 1$  on  $\mathcal{Y}_n$ . One has:

$$(4.8) \quad \|\Delta_{S_n} f_n\|^2 = 0, \quad \|f_n\|^2 = n + 1, \quad \text{and} \quad \|d_{S_n}(Q) f_n\|^2 = n(n + 1).$$

Now take  $T := (\mathcal{E}, \mathcal{V})$ , where  $\mathcal{V} := \cup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{V}_n$  and  $\mathcal{E}(x, y) := \mathcal{E}_n(x, y)$ , if  $x, y \in \mathcal{V}_n$ ,  $\mathcal{E}(x, y) := 1$ , if  $x = 1 \in \mathcal{V}_n$  and  $y = 1 \in \mathcal{V}_{n+1}$ , for all  $n \geq 1$ , and  $\mathcal{E}(x, y) := 0$  otherwise.



Graph of  $T$

Now if  $\mathcal{D}(d_T(Q)) \subset \mathcal{D}(\Delta_T)$ , by the uniform boundedness principle, there are constants  $a, b > 0$ , so that

$$\|d_T(Q)f\|^2 \leq a\|\Delta_T f\|^2 + b\|f\|^2, \quad \text{for all } f \in \mathcal{C}_c(T).$$

This leads to a contradiction with (4.8), as  $\|\Delta_T f_n\|^2 = 4$  and  $\|d_T(Q)f_n\|^2 = (n+2)^2 + n$  for  $n \geq 3$ .  $\square$

**4.2. The form-domain for bi-partite graphs.** As (1.10) holds true for trees, it is natural to ask the question for bi-partite graphs. The answer is no. We start by relating the form-domain of the magnetic Laplacian with the inequality (4.9).

**Proposition 4.3.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  be a weighted bi-partite graph. Then there is  $a \in (0, 1]$  and  $C_a > 0$  so that:*

$$(4.9) \quad (1-a)\langle f, d_G(Q)f \rangle - C_a\|f\|^2 \leq \langle f, \Delta_{\mathcal{E}, \theta} f \rangle \leq (1+a)\langle f, d_G(Q)f \rangle + C_a\|f\|^2,$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Moreover, one can take some  $a < 1$  in (4.9) if and only if  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ .

Suppose also that  $\Delta_{\mathcal{E}, \theta}$  has compact resolvent. Then,  $d_G(Q)$  has also compact resolvent and, with the same  $a$  as in (4.9), one has:

$$(4.10) \quad 1-a \leq \liminf_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta})}{\mathcal{N}_\lambda(d_G(Q))} \leq \limsup_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta})}{\mathcal{N}_\lambda(d_G(Q))} \leq 1+a.$$

We recall that (1.4) ensures that  $\mathcal{D}(d_G^{1/2}(Q)) \subset \mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2})$ .

*Proof.* First, note that (4.9) follows from (1.4) and Proposition 3.1. Suppose that (4.9) holds true for some  $a < 1$ , this gives immediately that  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ . Reciprocally, suppose now that  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ . By the uniform boundedness principle, there is  $a_0, b_0 > 0$  so that

$$a_0\langle f, d_G(Q)f \rangle - b_0\|f\|^2 \leq \langle f, \Delta_{\mathcal{E}, \theta} f \rangle,$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Using again Proposition 3.1, (4.9) holds true with  $a = 1 - a_0 < 1$ .

We turn to the second part and work under the hypothesis that  $\Delta_{\mathcal{E}, \theta}$  has compact resolvent. Corollary 2.1 ensures that  $d_G(Q)$  has also compact resolvent. We conclude by using Proposition 2.3 twice.  $\square$

**Remark 4.4.** *For a bi-partite graph, the constant 2 in (1.4) can be improved, in the sense of (4.9), if and only if  $\mathcal{D}((\Delta_{\mathcal{E}, \theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$ .*

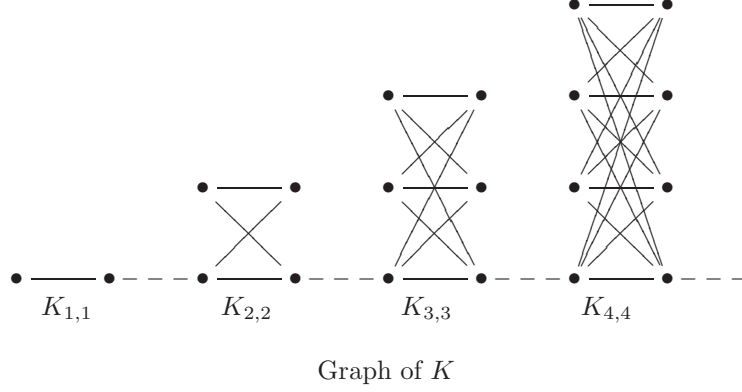
We finally provide an example.

**Proposition 4.4.** *There is a simple bi-partite graph  $K$  such that  $\mathcal{D}(d_K^{1/2}) \subsetneq \mathcal{D}(\Delta_K^{1/2})$ . In particular, the constant 2 in (1.4) is optimal in the sense of (4.9).*

*Proof.* We start by constructing a complete bi-partite graph. Let  $K_{n,n} := (\mathcal{E}_n, \mathcal{V}_n)$  be defined as follows:  $\mathcal{V}_n = \{1, \dots, n\} \times \{1, \dots, n\}$  and such that  $\mathcal{E}_n((k, i), (j, i)) = 0, \forall j, k \in \{1, \dots, n\}$  and  $i = 1, 2$  and  $\mathcal{E}_n((k, 1), (j, 2)) = 0, \forall j, k \in \{1, \dots, n\}$ . Consider now  $f_n(x) = 1$  on  $K_{n,n}$ . One has:

$$(4.11) \quad \langle f_n, \Delta_K f_n \rangle = 0, \quad \|f_n\|^2 = 2n, \quad \text{and} \quad \langle f_n, d_K(Q)f_n \rangle = 2n^2.$$

Now take  $K := (\mathcal{E}, \mathcal{V})$ , where  $\mathcal{V} = \cup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{V}_n$  and  $\mathcal{E}(x, y) := \mathcal{E}_n(x, y)$ , if  $x, y \in \mathcal{V}_n$ ,  $\mathcal{E}(x, y) = 1$ , if  $x = (1, 2) \in \mathcal{V}_n$  and  $y = (1, 1) \in \mathcal{V}_{n+1}$ , for all  $n \geq 1$ , and  $\mathcal{E}(x, y) = 0$  otherwise.



On the other hand, if  $\mathcal{D}(d_K^{1/2}(Q)) = \mathcal{D}(\Delta_K^{1/2})$ , the uniform boundedness principle ensures that there are constants  $a, b > 0$ , so that

$$\langle f, d_K(Q)f \rangle \leq a \langle f, \Delta_K f \rangle + b \|f\|^2, \quad \text{for all } f \in \mathcal{C}_c(K).$$

This leads to a contradiction with (4.11), as  $\langle f_n, \Delta_K f_n \rangle = 2$ , for  $n \geq 2$ . Optimality follows by Proposition and 4.3 and Remark 4.4.  $\square$

## 5. PERTURBATION THEORY

We finally go into perturbation theory in order to obtain the stability of the essential spectrum, of the inequality (5.1), and of the asymptotic of eigenvalues.

**Proposition 5.1.** *Let  $G = (\mathcal{V}, \mathcal{E}, m, \theta)$  and  $G_o = (\mathcal{V}, \mathcal{E}_o, m, \theta_o)$  be weighted graphs and  $V : \mathcal{V} \rightarrow \mathbb{R}$ . Suppose that (1.10) holds true for  $\Delta_{\mathcal{E}_o, \theta_o}$  and suppose that there is  $\eta \in (0, 1)$  and  $\kappa_\eta \geq 0$ , so that*

$$(5.1) \quad |\langle f, V(Q)f \rangle| + 2 \langle f, \Lambda(Q)f \rangle \leq \eta \langle f, d_{G_o}(Q)f \rangle + \kappa_\eta \|f\|^2,$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ , where

$$\Lambda(x) := \frac{1}{m^2(x)} \sum_{y \sim x} |\mathcal{E}(x, y) - \tilde{\mathcal{E}}(x, y)|.$$

Then, one has that:

- (a) *The operator  $\Delta_{\mathcal{E}, \theta} + V(Q)$  is bounded from below by some constant  $C$ , in the form sense, and also that  $\mathcal{D}((\Delta_{\mathcal{E}, \theta} + V(Q) + C)^{1/2}) = \mathcal{D}((\Delta_{\mathcal{E}_o, \theta_o})^{1/2})$ .*
- (b) *The essential spectrum of  $\Delta_{\mathcal{E}, \theta} + V(Q)$  is empty if and only if the essential spectrum of  $\Delta_{\mathcal{E}_o, \theta_o}$  is empty and if and only if  $\lim_{|x| \rightarrow \infty} d_{G_o}(x) = +\infty$ .*
- (c) *Supposing that the essential spectrum of  $\Delta_{\mathcal{E}, \theta} + V(Q)$  is empty and that for all  $\eta \in (0, 1)$  there is  $\kappa_\eta \geq 0$ , so that (5.1) holds true, then:*

$$\lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{E}, \theta} + V(Q))}{\mathcal{N}_\lambda(d_{G_o}(Q))} = 1.$$

- (d) *Supposing that*

$$(5.2) \quad |V(x)| + \Lambda(x) = o(1 + d_{G_o}(x)), \quad \text{as } |x| \rightarrow \infty,$$

we obtain that  $\sigma_{\text{ess}}(\Delta_{\mathcal{E}_o, \theta_o}) = \sigma_{\text{ess}}(\Delta_{\mathcal{E}, \theta} + V(Q))$ .

*Proof.* We start by noticing that, as in (1.4), (5.1) ensures that:

$$(5.3) \quad |\langle f, (\Delta_{\mathcal{E}, \theta} + V(Q) - \Delta_{\mathcal{E}_o, \theta_o})f \rangle| \leq |\langle f, V(Q)f \rangle| + 2 \langle f, \Lambda(G)f \rangle \leq \eta \langle f, d_{G_o}(Q)f \rangle + \kappa_\eta \|f\|^2$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Therefore, for all  $\varepsilon > 0$  and all  $\eta \in (0, 1)$ , satisfying (5.1), there is  $c_{\varepsilon, \eta}$  so that:

$$(1 - \eta - \varepsilon) \langle f, d_{G_o}(Q)f \rangle - c_{\varepsilon, \eta} \|f\|^2 \leq \langle f, (\Delta_{\mathcal{E}, \theta} + V(Q))f \rangle \leq (1 + \eta + \varepsilon) \langle f, d_{G_o}(Q)f \rangle + c_{\varepsilon, \eta} \|f\|^2,$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . This gives, directly, the first point. Moreover, as above, the second and third points follow by Proposition 2.3.

We now turn to the stability of the essential spectrum. By Weyl's Theorem, e.g., [RS, Theorem XII.1], it is enough to show that the difference of the resolvents is compact. As we have to work with forms, one should be careful with the resolvent equation. We give a complete proof and refer to [GG0] for more discussions of this matter. To lighten notation, we set  $H_0 := \Delta_{\varepsilon_0, \theta_0}$  and  $H := \Delta_{\varepsilon, \theta} + V$ . To start off, we give a rigorous meaning to

$$(5.4) \quad (H + i)^{-1} - (H_0 + i)^{-1} = (H + i)^{-1}(H_0 - H)(H_0 + i)^{-1}$$

Since  $\mathcal{G} := \mathcal{D}((H + C)^{1/2}) = \mathcal{D}((H_0)^{1/2})$ , both operators extend to an element of  $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$ . Here we use the Riesz lemma to identify  $\mathcal{H}$  with its anti-dual  $\mathcal{H}^*$ . We denote these extensions with a tilde.

We have  $(H_0 + i)^{-1*} \mathcal{H} \subset \mathcal{G}$ . This allows one to deduce that  $(H_0 + i)^{-1}$  extends to a unique continuous operator  $\mathcal{G}^* \rightarrow \mathcal{H}$ . We denote it for the moment by  $R$ . From  $R(H_0 + i)u = u$  for  $u \in \mathcal{D}(H_0)$  we get, by density of  $\mathcal{D}(H_0)$  in  $\mathcal{G}$  and continuity,  $R(\widetilde{H}_0 + i)u = u$  for  $u \in \mathcal{G}$ . In particular

$$(H + i)^{-1} = R(\widetilde{H}_0 + i)(H + i)^{-1}.$$

Clearly,

$$(H_0 + i)^{-1} = (H_0 + i)^{-1}(H + i)(H + i)^{-1} = R(\widetilde{H} + i)(H + i)^{-1}.$$

We subtract the last two relations to obtain that

$$(H_0 + i)^{-1} - (H + i)^{-1} = R(\widetilde{H} - \widetilde{H}_0)(H + i)^{-1}$$

Since  $R$  is uniquely determined as the extension of  $(H_0 + i)^{-1}$  to a continuous map  $\mathcal{G}^* \rightarrow \mathcal{H}$ , one may keep the notation  $(H_0 + i)^{-1}$  for it. With this convention, the rigorous version of (5.4) that we use is:

$$(5.5) \quad (H_0 + i)^{-1} - (H + i)^{-1} = (H_0 + i)^{-1}(\widetilde{H} - \widetilde{H}_0)(H + i)^{-1}.$$

Therefore, to prove c), it is enough to show that  $\widetilde{H} - \widetilde{H}_0$  is a compact operator from  $\mathcal{G}$  to  $\mathcal{G}^*$ . By (5.3), one get:

$$\begin{aligned} & | \langle (1 + d_{G_0})^{-1/2}(Q)f, (\Delta_{\varepsilon, \theta} + V(Q) - \Delta_{\varepsilon_0, \theta_0})(1 + d_{G_0})^{-1/2}(Q)f \rangle | \leq \\ & \quad \langle (1 + d_{G_0})^{-1/2}(Q)f, (V + 2\Lambda)(Q)(1 + d_{G_0})^{-1/2}(Q)f \rangle, \end{aligned}$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Now, by hypothesis (5.2),  $(V + 2\Lambda)(1 + d_{G_0})^{-1}(Q)$  is compact in  $\ell^2(\mathcal{V}, m^2)$ . We conclude by using Proposition 2.4.  $\square$

**Remark 5.1.** Note that (5.1) allows us to consider some potential  $V(Q)$  that are unbounded from below, whereas  $\Delta_{\varepsilon, \theta} + V$  is bounded from below. On the other hand, for a discrete Schrödinger operator, the associated quadratic form is a Dirichlet form if and only if  $V \geq 0$ , e.g., [KL]. The perturbation theory that we consider here is then more efficient. This is due to the fact that we know the form-domain explicitly.

## APPENDIX A. THE $\mathcal{C}^1$ -REGULARITY

We start with some generalities. Given a bounded operator  $B$  and a self-adjoint operator  $A$  acting in a Hilbert space  $\mathcal{H}$ , one says that  $B \in \mathcal{C}^k(A)$  if  $t \mapsto e^{-itA} B e^{itA}$  is strongly  $\mathcal{C}^k$ . Given a closed and densely defined operator  $B$ , one says that  $B \in \mathcal{C}^k(A)$  if for some (hence any)  $z \notin \sigma(B)$ ,  $t \mapsto e^{-itA} (B - z)^{-1} e^{itA}$  is strongly  $\mathcal{C}^k$ . The two definitions coincide in the case of a bounded self-adjoint operator. We recall a result following from Lemma 6.2.9 and Theorem 6.2.10 of [ABG].

**Theorem A.1.** Let  $A$  and  $B$  be two self-adjoint operators in the Hilbert space  $\mathcal{H}$ . For  $z \notin \sigma(A)$ , set  $R(z) := (B - z)^{-1}$ . The following points are equivalent to  $B \in \mathcal{C}^1(A)$ :

(a) For one (then for all)  $z \notin \sigma(B)$ , there is a finite  $c$  such that

$$(A.1) \quad | \langle Af, R(z)f \rangle - \langle R(\bar{z})f, Af \rangle | \leq c \|f\|^2, \text{ for all } f \in \mathcal{D}(A).$$

(b) a. There is a finite  $c$  such that for all  $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$ :

$$(A.2) \quad | \langle Af, Bf \rangle - \langle Bf, Af \rangle | \leq c (\|Bf\|^2 + \|f\|^2).$$

b. For some (then for all)  $z \notin \sigma(B)$ , the set  $\{f \in \mathcal{D}(A) \mid R(z)f \in \mathcal{D}(A) \text{ and } R(\bar{z})f \in \mathcal{D}(A)\}$  is a core for  $A$ .

Note that the condition (2.b) could be uneasy to check (see [GGé]). We mention [GM][Lemma A.2] and [GL][Lemma 3.2.2] to overcome this subtlety.

Note that (A.1) yields the commutator  $[A, R(z)]$  extends to a bounded operator, in the form sense. We shall denote the extension by  $[A, R(z)]_\circ$ . In the same way, since  $\mathcal{D}(B) \cap \mathcal{D}(A)$  is dense in  $\mathcal{D}(B)$ , (A.2) ensures that the commutator  $[B, A]$  extends to a unique element of  $\mathcal{B}(\mathcal{D}(B), \mathcal{D}(B)^*)$  denoted by  $[B, A]_\circ$ . Moreover, when  $B \in \mathcal{C}^1(A)$ , one has:

$$(A.3) \quad [A, (B-z)^{-1}]_\circ = \underbrace{(B-z)^{-1}}_{\mathcal{H} \leftarrow \mathcal{D}(B)^*} \underbrace{[B, A]_\circ}_{\mathcal{D}(B)^* \leftarrow \mathcal{D}(B)} \underbrace{(B-z)^{-1}}_{\mathcal{D}(B) \leftarrow \mathcal{H}}.$$

Here we use the Riesz lemma to identify  $\mathcal{H}$  with its anti-dual  $\mathcal{H}^*$ .

It turns out that an easier characterization is available if the domain of  $B$  is conserved under the action of the  $C_0$ -group generated by  $A$ .

**Theorem A.2.** ([ABG, p. 258]) *Let  $A$  and  $B$  be two self-adjoint operators in the Hilbert space  $\mathcal{H}$  such that  $e^{itA}\mathcal{D}(B) \subset \mathcal{D}(B)$ , for all  $t \in \mathbb{R}$ . Then,  $B \in \mathcal{C}^1(A)$  if and only if*

$$\liminf_{t \rightarrow 0^+} \|[B, e^{itB}/t]\|_{\mathcal{B}(\mathcal{D}(B), \mathcal{D}(B)^*)} < \infty.$$

Note that  $e^{itA}\mathcal{D}(B)^* \subset \mathcal{D}(B)^*$  by duality.

#### APPENDIX B. THE HELFFER-SJÖSTRAND'S FORMULA

We present briefly the Helffer-Sjöstrand's formula. We refer to [GJ][Appendix B] and [BG][Appendix A] (see also [DG, HSS]) for commutator expansion. We first recall some well-known facts about almost analytic extensions. For  $\rho \in \mathbb{R}$ , let  $\mathcal{S}^\rho$  be the class of function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{C})$  such that

$$(B.1) \quad \forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t \in \mathbb{R}} \langle t \rangle^{-\rho+k} |\varphi^{(k)}(t)| < \infty.$$

Equipped with the semi-norms defined by (B.1),  $\mathcal{S}^\rho$  is a Fréchet space. Leibniz' formula implies the continuous embedding:  $\mathcal{S}^\rho \cdot \mathcal{S}^{\rho'} \subset \mathcal{S}^{\rho+\rho'}$ . We shall use the following result, e.g., [DG, Appendix C.2].

**Lemma B.1.** *Let  $\varphi \in \mathcal{S}^\rho$  with  $\rho \in \mathbb{R}$ . For all  $l \in \mathbb{N}$ , there is a smooth function  $\varphi^{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ , such that:*

$$(B.2) \quad \varphi^{\mathbb{C}}|_{\mathbb{R}} = \varphi, \quad \left| \frac{\partial \varphi^{\mathbb{C}}}{\partial \bar{z}}(z) \right| \leq c_1 \langle \Re(z) \rangle^{\rho-1-l} |\Im(z)|^l$$

$$(B.3) \quad \text{supp } \varphi^{\mathbb{C}} \subset \{x + iy \mid |y| \leq c_2 \langle x \rangle\},$$

$$(B.4) \quad \varphi^{\mathbb{C}}(x + iy) = 0, \text{ if } x \notin \text{supp } \varphi.$$

for some constants  $c_1, c_2$  depending on the semi-norms (B.1) of  $\varphi$  in  $\mathcal{S}^\rho$  and not on  $\varphi$ .

One calls  $\varphi^{\mathbb{C}}$  an *almost analytic extension* of  $\varphi$ . Let  $A$  be a self-adjoint operator,  $\rho < 0$  and  $\varphi \in \mathcal{S}^\rho$ . By functional calculus, one has  $\varphi(A)$  bounded. The Helffer-Sjöstrand's formula, e.g., [HS] and [DG], gives that for all almost analytic extension of  $\varphi \in \mathcal{S}^\rho$ , with  $\rho < 0$ , one has:

$$(B.5) \quad \varphi(A) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \varphi^{\mathbb{C}}}{\partial \bar{z}}(z - A)^{-1} dz \wedge d\bar{z}.$$

Note the integral exists in the norm topology, by (B.2) with  $l = 1$  and by taking in account the domain of integration given in (B.3).

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