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► **To cite this version:**

Caroline Kalla. Breathers and solitons of generalized nonlinear Schrödinger equations as degenerations of algebro-geometric solutions. 2011. hal-00597964

HAL Id: hal-00597964

<https://hal.science/hal-00597964v1>

Preprint submitted on 3 Jun 2011

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Breathers and solitons of generalized nonlinear Schrödinger equations as degenerations of algebro-geometric solutions

C. Kalla*

June 1, 2011

Abstract

We present new solutions in terms of elementary functions of the multi-component nonlinear Schrödinger equations and known solutions of the Davey-Stewartson equations such as multi-soliton, breather, dromion and lump solutions. These solutions are given in a simple determinantal form and are obtained as limiting cases in suitable degenerations of previously derived algebro-geometric solutions. In particular we present for the first time breather and rational breather solutions of the multi-component nonlinear Schrödinger equations.

1 Introduction

One of the significant advances in mathematical physics at the end of the 19th century has been the discovery by Gardner, Greene, Kruskal and Miura [18] of the applicability of the Inverse Scattering Transform (IST) to the Korteweg-de Vries equation, and the construction of multi-soliton solutions. The most important physical property of solitons is that they are localized wave packets which survive collisions with other solitons without change of shape. For a guide to the vast literature on solitons, see for instance [31, 10]. Existence of soliton solutions to the nonlinear Schrödinger equation (NLS)

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2\rho |\psi|^2 \psi = 0, \quad (1.1)$$

where $\rho = \pm 1$, was proved by Zakharov and Shabat [42] using a modification of the IST. The NLS equation is a famous nonlinear dispersive partial differential equation with many applications, e.g. in hydrodynamics (deep water waves), plasma physics and nonlinear fiber optics. The N -soliton solutions to both the self-focusing NLS equation ($\rho = 1$), as well as the defocusing NLS equation ($\rho = -1$), can also be computed by Darboux transformations [28], Hirota's bilinear method (see e.g. [20, 34, 9]) or Wronskian techniques (see [17, 30, 16]). Hirota's method relies on a transformation of the underlying equation to a bilinear equation. The resulting multi-soliton solutions are expressed in the form of polynomials in exponential functions. Wronskian techniques formulate the N -soliton solutions in terms of the Wronskian determinant of N functions. This

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method allows a straightforward direct check that the obtained solutions satisfy the equation since differentiation of a Wronskian is simple. On the other hand, multi-soliton solutions of (1.1) can be directly derived from algebro-geometric solutions when the associated hyperelliptic Riemann surface degenerates into a Riemann surface of genus zero, see for instance [7].

In the present paper, we construct solutions in terms of elementary functions of two generalizations of the NLS equation (1.1): the multi-component NLS equation (n-NLS), where the number of dependent variables is increased, and the Davey-Stewartson equation (DS), an integrable generalization to $2 + 1$ dimensions. The solutions of n-NLS and DS presented in this paper are obtained by degenerating algebro-geometric solutions, previously investigated by the author in [22] using Fay's identity [29]. This method for finding solutions in terms of elementary functions has not been applied to n-NLS and DS so far. It provides a unified approach to various solutions of n-NLS and DS expressed in terms of a simple determinantal form, and allows to present new solutions to the multi-component NLS equation in terms of elementary functions.

One way to generalize the NLS equation is to increase the number of dependent variables in (1.1). This leads to the multi-component nonlinear Schrödinger equation

$$i \frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + 2 \left(\sum_{k=1}^n s_k |\psi_k|^2 \right) \psi_j = 0, \quad j = 1, \dots, n, \quad (1.2)$$

denoted by n-NLS^s, where $s = (s_1, \dots, s_n)$, $s_k = \pm 1$. Here $\psi_j(x, t)$ are complex valued functions of the real variables x and t . The case $n = 1$ corresponds to the NLS equation. The two-component NLS equation ($n = 2$) is relevant in the study of electromagnetic waves in optical media in which the electric field has two nontrivial components. Integrability of the two-component NLS equation in the case $s = (1, 1)$ was first established by Manakov [25]. In optical fibers, for arbitrary $n \geq 2$, the components ψ_j in (1.2) correspond to components of the electric field transverse to the direction of wave propagation. These components of the transverse field form a basis of the polarization states. Integrability for the multi-component case with any $n \geq 2$ and $s_k = \pm 1$ was established in [38]. Multi-soliton solutions of (1.2) were considered in a series of papers, see for instance [25, 35, 36, 23, 1].

In this paper, we present a family of dark and bright multi-solitons, breather and rational breather solutions to the multi-component NLS equation. This appears to be the first time that breathers and rational breathers are given for the multi-component case. The notion of a dark soliton refers to the fact that the solution tends asymptotically to a non-zero constant, i.e., it describes a darkening on a bright background, whereas the bright soliton is a localized bright spot being described by a solution that tends asymptotically to zero. The name 'breather' reflects the behavior of the profile which is periodic in time or space and localized in space or time. It is remarkable that degenerations of algebro-geometric solutions to the multi-component NLS equation lead to the breather solutions, well known in the context of the one-component case as the soliton on a finite background [3] (breather periodic in space), the Ma breather [24] (breather periodic in time) and the rational breather [33]. In the NLS framework, these solutions have been suggested as models for a class of extreme, freak or rogue wave events (see e.g. [19, 32, 4]). A family of rational solutions to the focusing NLS equation was constructed in [13] and was rediscovered recently in [11] via Wronskian techniques. Here we give for the first time a family of breather and rational breather solutions of the multi-component NLS equation. For the one component case, our solutions consist of the well known breather and Peregrine breather of the focusing NLS equation. For the multi-component case, we find new profiles of breathers and rational breathers which do not exist in the scalar case.

Another way to generalize the NLS equation is to increase the number of spatial dimensions to two. This leads to the DS equations,

$$\begin{aligned} i\psi_t + \psi_{xx} - \alpha^2 \psi_{yy} + 2(\Phi + \rho|\psi|^2)\psi &= 0, \\ \Phi_{xx} + \alpha^2 \Phi_{yy} + 2\rho|\psi|_{xx}^2 &= 0, \end{aligned} \tag{1.3}$$

where $\alpha = i, 1$ and $\rho = \pm 1$; $\psi(x, y, t)$ and $\Phi(x, y, t)$ are functions of the real variables x, y and t , the latter being real valued and the former being complex valued. In what follows, $DS1^\rho$ corresponds to the case $\alpha = i$, and $DS2^\rho$ to $\alpha = 1$. The DS equation (1.3) was introduced in [12] to describe the evolution of a three-dimensional wave package on water of finite depth. Complete integrability of the equation was shown in [5]. A main feature of equations in $1 + 1$ dimensions is the existence of soliton solutions which are localized in one dimension. Solutions of the $2 + 1$ dimensional integrable equations which are localized only in one dimension (plane solitons) were constructed in [2, 6]. Moreover, various recurrent solutions (the growing-and-decaying mode, breather and rational growing-and-decaying mode solutions) were investigated in [40]. The spectral theory of soliton type solutions to the DS1 equation (called dromions) with exponential fall off in all directions on the plane, and their connection with the initial-boundary value problem, have been studied by different methods in a series of papers [8, 15, 39, 37]. The lump solution (a rational non-singular solution) to the $DS2^-$ equation was discovered in [6].

Here we present a family of dark multi-soliton solutions to the $DS1$ and $DS2^+$ equations, as well as a family of bright multi-solitons for the $DS1$ and $DS2^-$ equations, obtained by degenerating algebro-geometric solutions. Moreover, a class of breather and rational breather solutions of the $DS1$ equation is given. These solutions have a very similar appearance to those in $1 + 1$ dimensions. In this paper it is shown how the simplest solutions, the dromion and the lump solutions can be derived from algebro-geometric solutions.

The paper is organized as follows: Section 2 contains various facts from the theory of theta functions and identities due to Fay. These identities were used to construct algebro-geometric solutions of n-NLS and DS equations in [22], and will be needed for the degeneration of the underlying theta-functional solutions. Section 3 provides technical tools dealing with the degeneration of Riemann surfaces. We present a method which allows to degenerate algebro-geometric solutions associated to an arbitrary Riemann surface that can be applied to general integrable equations. In Section 4 solutions in terms of elementary functions to the complexified n-NLS equation are derived by degenerating algebro-geometric solutions; for an appropriate choice of the parameters one gets multi-solitonic solutions, and for the first time breather and rational breather solutions to the multi-component NLS equation (1.2). In Section 5 a similar program is carried out for the DS equations; well known solutions such as multi-solitons, dromion or lump are rediscovered from an algebro-geometric approach.

2 Theta functions and Fay's identity

Solutions of equations (1.2) and (1.3) in terms of the multi-dimensional theta function were discussed in [22]. In this section we recall some facts from the construction of these solutions which will be used in the following to get particular solutions as limiting cases of algebro-geometric solutions.

2.1 Theta functions

Let \mathcal{R}_g be a compact Riemann surface of genus $g > 0$. Denote by $\{\mathcal{A}_j, \mathcal{B}_j\}_{j=1}^g$ a canonical homology basis, and by $\{\omega_j\}_{j=1}^g$ the dual basis of holomorphic differentials normalized via

$$\int_{\mathcal{A}_k} \omega_j = 2i\pi\delta_{k,j} \quad k, j = 1, \dots, g. \quad (2.1)$$

The matrix \mathbb{B} of \mathcal{B} -periods of the normalized holomorphic differentials with entries $(\mathbb{B})_{kj} = \int_{\mathcal{B}_k} \omega_j$ is symmetric and has a negative definite real part. The theta function with (half integer) characteristic $\delta = [\delta', \delta'']$ is defined by

$$\Theta[\delta](\mathbf{z}|\mathbb{B}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp \left\{ \frac{1}{2} \langle \mathbb{B}(\mathbf{m} + \delta'), \mathbf{m} + \delta' \rangle + \langle \mathbf{m} + \delta', \mathbf{z} + 2i\pi\delta'' \rangle \right\} \quad (2.2)$$

for any $\mathbf{z} \in \mathbb{C}^g$; here $\delta', \delta'' \in \{0, \frac{1}{2}\}^g$ are the vectors of characteristic and $\langle \cdot, \cdot \rangle$ denotes the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i u_i v_i$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^g$. The theta function $\Theta[\delta](\mathbf{z})$ is even if the characteristic δ is even i.e. $4\langle \delta', \delta'' \rangle$ is even, and odd if the characteristic δ is odd, i.e. $4\langle \delta', \delta'' \rangle$ is odd. An even characteristic is called non-singular if $\Theta[\delta](0) \neq 0$, and an odd characteristic is called non-singular if the gradient $\nabla\Theta[\delta](0)$ is non-zero.

2.2 Corollaries of Fay's identity

Let us first introduce some notation. Let k_a denote a local parameter near $a \in \mathcal{R}_g$. Consider the following expansion of the normalized holomorphic differentials ω_j near a ,

$$\omega_j(p) = (V_{a,j} + W_{a,j}k_a(p) + o(k_a(p))) dk_a(p), \quad (2.3)$$

where p lies in a neighbourhood of a , and $V_{a,j}, W_{a,j} \in \mathbb{C}$. Let us denote by D_a the operator of directional derivative along the vector $\mathbf{V}_a = (V_{a,1}, \dots, V_{a,g})^t$:

$$D_a F(\mathbf{z}) = \sum_{j=1}^g \partial_{z_j} F(\mathbf{z}) V_{a,j} = \langle \nabla F(\mathbf{z}), \mathbf{V}_a \rangle, \quad (2.4)$$

where $F : \mathbb{C}^g \rightarrow \mathbb{C}$ is an arbitrary function, and denote by D'_a the operator of directional derivative along the vector $\mathbf{W}_a = (W_{a,1}, \dots, W_{a,g})^t$.

Now let δ be a non-singular odd characteristic. For any $\mathbf{z} \in \mathbb{C}^g$ and any distinct points $a, b \in \mathcal{R}_g$, the following two versions of Fay's identity [14] hold (see [29] and [22])

$$D_a D_b \ln \Theta(\mathbf{z}) = q_1 + q_2 \frac{\Theta(\mathbf{z} + \mathbf{r}) \Theta(\mathbf{z} - \mathbf{r})}{\Theta(\mathbf{z})^2}, \quad (2.5)$$

$$D'_a \ln \frac{\Theta(\mathbf{z} + \mathbf{r})}{\Theta(\mathbf{z})} + D_a^2 \ln \frac{\Theta(\mathbf{z} + \mathbf{r})}{\Theta(\mathbf{z})} + \left(D_a \ln \frac{\Theta(\mathbf{z} + \mathbf{r})}{\Theta(\mathbf{z})} - K_1 \right)^2 + 2 D_a^2 \ln \Theta(\mathbf{z}) + K_2 = 0, \quad (2.6)$$

where the scalars q_i, K_i for $i = 1, 2$ depend on the points a, b and are given by

$$q_1(a, b) = D_a D_b \ln \Theta[\delta](\mathbf{r}), \quad (2.7)$$

$$q_2(a, b) = \frac{D_a \Theta[\delta](0) D_b \Theta[\delta](0)}{\Theta[\delta](\mathbf{r})^2}, \quad (2.8)$$

$$K_1(a, b) = \frac{1}{2} \frac{D'_a \Theta[\delta](0)}{D_a \Theta[\delta](0)} + D_a \ln \Theta[\delta](\mathbf{r}), \quad (2.9)$$

$$K_2(a, b) = -D'_a \ln \Theta(\mathbf{r}) - D_a^2 \ln (\Theta(\mathbf{r}) \Theta(0)) - \left(D_a \ln \Theta(\mathbf{r}) - K_1(a, b) \right)^2. \quad (2.10)$$

Here we used the notation $\mathbf{r} = \int_a^b \omega$ where $\omega = (\omega_1, \dots, \omega_g)^t$ is the vector of the normalized holomorphic differentials.

2.3 Integral representation of $q_2(a, b)$ and $K_1(a, b)$

Quantities $q_2(a, b)$ and $K_1(a, b)$ defined in (2.8) and (2.9) respectively, admit integral representation which will be more convenient for our purposes. These integral representations follow from the fact that meromorphic differentials normalized by the condition of vanishing \mathcal{A} -periods can be expressed in terms of theta functions.

Let $a, b \in \mathcal{R}_g$ be two distinct points connected by a contour which does not intersect \mathcal{A} and \mathcal{B} -cycles. Hence we can define the normalized meromorphic differential of the third kind Ω_{b-a} which has residue 1 at b and residue -1 at a . Now let $a \in \mathcal{R}_g$, and $N \in \mathbb{N}$ with $N > 1$. The normalized meromorphic differential of the second kind $\Omega_a^{(N)}$ has only one singularity at the point a and is of the form

$$\Omega_a^{(N)}(p) = \left(\frac{1}{k_a(p)^N} + O(1) \right) dk_a(p), \quad p \in \mathcal{R}_g, \quad (2.11)$$

where k_a is a local parameter in a neighbourhood of a .

Proposition 2.1. *Let $a, b \in \mathcal{R}_g$ be distinct points. Denote by k_a and k_b local parameters in a neighbourhood of a and b respectively. The quantities $q_2(a, b)$ and $K_1(a, b)$ defined in (2.8) and (2.9) respectively admit the following integral representations:*

$$q_2(a, b) = - \lim_{\substack{\tilde{b} \rightarrow b \\ \tilde{a} \rightarrow a}} \left[\left(k_a(\tilde{a}) k_b(\tilde{b}) \right)^{-1} \exp \left\{ \int_{\tilde{a}}^{\tilde{b}} \Omega_{b-a}(p) \right\} \right], \quad (2.12)$$

where the integration contour does not cross any cycle of canonical basis, and

$$K_1(a, b) = \lim_{\tilde{a} \rightarrow a} \left[\int_c^{\tilde{a}} \Omega_a^{(2)}(p) + \frac{1}{k_a(\tilde{a})} \right] - \int_c^b \Omega_a^{(2)}(p), \quad (2.13)$$

where c is an arbitrary point on \mathcal{R}_g .

Proof of (2.12) can be found in [22], where similar statements lead to (2.13).

3 Uniformization map and degenerate Riemann surfaces

It is well known that solutions in terms of theta functions are almost periodic due to the periodicity properties of the theta functions. In the limit when the Riemann surface degenerates to a surface of genus zero, periods of the surface diverge, and the theta series breaks down to elementary functions. Whereas this procedure is well-known in the case of a hyperelliptic surface, i.e., a two-sheeted branched covering of the Riemann sphere, where such a degeneration consists

in colliding branch points pairwise, it has not been applied so far to theta-functional solutions on non-hyperelliptic surfaces.

We present here a method to treat this case based on the uniformization theorem for Riemann surfaces. In particular, we show that the theta function tends to a finite sum of exponentials in the limit when the arithmetic genus of the associated Riemann surface drops to zero, and give explicitly the constants (2.7)-(2.10) in this limit. As illustrated in Section 4 and 5, particular solutions of n-NLS and DS such as multi-solitons, well known in the theory of soliton equations, arise from such degenerations of algebro-geometric solutions.

3.1 Degeneration to genus zero

Let us first recall some techniques used for degenerating Riemann surfaces (see [14] for more details). There exist basically two ways for degenerating a Riemann surface by pinching a cycle: a cycle homologous to zero in the first case, and a cycle non-homologous to zero in the second case. The first degeneration leads to two Riemann surfaces whose genera add up to the genus of the pinched surface, whereas the limiting situation for the second degeneration is one Riemann surface of genus $g-1$ with two points identified, g being the genus of the non-degenerated surface. In both cases, locally one can identify the pinched region to a hyperboloid

$$y^2 = x^2 - \epsilon, \quad (3.1)$$

where $\epsilon > 0$ is a small parameter, such that the vanishing cycle coincides with the homology class of a closed contour around the cut $[-\sqrt{\epsilon}, \sqrt{\epsilon}]$ in the x -plane. In what follows, we deal with the degeneration of the second type and make consecutive pinches until the surface degenerates to genus zero.

To degenerate the Riemann surface \mathcal{R}_g of genus g into a Riemann surface \mathcal{R}_0 of genus zero, we pinch all \mathcal{A}_i -cycles into double points. After desingularization one gets \mathcal{R}_0 , and each double point corresponds to two different points on \mathcal{R}_0 , denoted by u_i and v_i for $i = 1, \dots, g$. In this limit, holomorphic normalized differentials ω_i become normalized differentials of the third kind with poles at u_i and v_i . Note that the normalized differential of the second kind $\Omega_a^{(N)}$ with a pole of order $N > 1$ at a remains a differential of the second kind with the same order of the pole after degeneration to genus zero. We keep the same notation for the differential of the second kind on the degenerated surface.

The compact Riemann surface \mathcal{R}_0 of genus zero is conformally equivalent to the Riemann sphere with the coordinate w . This mapping between \mathcal{R}_0 and the w -sphere is called the uniformization map and we denote it by $w(p) = w$ for any $p \in \mathcal{R}_0$. Therefore, in what follows we let \mathcal{R}_0 stand also for the Riemann sphere with the coordinate w .

Meromorphic differentials on \mathcal{R}_0 can be constructed using the fact that in genus zero, such differentials are entirely defined by their behaviors near their singularities. This leads to the following third and second kind differentials on \mathcal{R}_0 :

- *Differentials of the third kind:*

$$\Omega_{v_i-u_i} = \left(\frac{1}{w-w_{v_i}} - \frac{1}{w-w_{u_i}} \right) dw. \quad (3.2)$$

- *Differentials of the second kind:*

$$\Omega_a^{(2)} = \frac{1}{k'_a(w_a)} \frac{dw}{(w - w_a)^2}, \quad (3.3)$$

where k_a is a local parameter in a neighbourhood of $w_a \in \mathcal{R}_0$ and the prime denotes the derivative with respect to the argument. This is the differential on \mathcal{R}_0 , obtained from $\Omega_a^{(2)}$ (2.11) defined on \mathcal{R}_g , in the limit as the surface \mathcal{R}_g degenerates to \mathcal{R}_0 . The factor $(k'_a(w_a))^{-1}$ ensures that the biresidue of $\Omega_a^{(2)}$ with respect to the local parameter k_a is 1 as before the degeneration.

3.2 Degenerate theta function

To study the theta function with zero characteristic in the limit when the genus tends to zero, let us first analyse the behavior of the matrix \mathbb{B} of \mathcal{B} -periods of the normalized holomorphic differentials. Since holomorphic normalized differentials ω_i become differentials of the third kind with poles at u_i and v_i , for a small parameter $\epsilon > 0$, elements $(\mathbb{B})_{ik}$ of the matrix \mathbb{B} have the following behavior

$$\begin{aligned} (\mathbb{B})_{ik} &= \int_{u_i}^{v_i} \Omega_{v_k - u_k} + O(\epsilon), \quad i \neq k, \\ (\mathbb{B})_{kk} &= \ln \epsilon + O(1). \end{aligned} \quad (3.4)$$

Therefore, the real parts of diagonal terms of the Riemann matrix tend to $-\infty$ when ϵ tends to zero, that is when the Riemann surface degenerates into the Riemann surface \mathcal{R}_0 . It follows that the theta function (2.2) with zero characteristic tends to one, since only the term corresponding to the vector $\mathbf{m} = 0$ in the series may give a non-zero contribution.

To get non constant solutions of (1.2) and (1.3) after the degeneration of the Riemann surface, let us write the argument of the theta-function in the form $\mathbf{Z} - \mathbf{D}$, where \mathbf{D} is a vector with components $D_k = (1/2) (\mathbb{B})_{kk} + d_k$, for some $d_k \in \mathbb{C}$ independent of ϵ . Hence for any $\mathbf{Z} \in \mathbb{C}^g$ one gets

$$\lim_{\epsilon \rightarrow 0} \Theta(\mathbf{Z} - \mathbf{D}) = \sum_{\mathbf{m} \in \{0,1\}^g} \exp \left\{ \sum_{1 \leq i < k \leq g} (\mathbb{B})_{ik} m_i m_k + \sum_{k=1}^g m_k (Z_k - d_k) \right\}. \quad (3.5)$$

Here we use the same notation for the quantities $(\mathbb{B})_{ik}$ on the degenerated surface. The expression in the right hand side of (3.5) can be put into a determinantal form (see Proposition 3.1) which will be used in the whole paper. This determinantal form can be obtained from the following representation of the components $(\mathbb{B})_{ik}$ after degeneration, obtained from (3.2) and (3.4),

$$(\mathbb{B})_{ik} = \ln \left\{ \frac{w_{v_i} - w_{v_k}}{w_{v_i} - w_{u_k}} \frac{w_{u_i} - w_{u_k}}{w_{u_i} - w_{v_k}} \right\}. \quad (3.6)$$

Hence, following [27] one gets

Proposition 3.1. *For any $\mathbf{z} \in \mathbb{C}^g$ the following holds*

$$\sum_{\mathbf{m} \in \{0,1\}^g} \exp \left\{ \sum_{1 \leq i < k \leq g} (\mathbb{B})_{ik} m_i m_k + \sum_{k=1}^g m_k z_k \right\} = \det(\mathbb{T}), \quad (3.7)$$

where \mathbb{T} is a $g \times g$ matrix with entries

$$(\mathbb{T})_{ik} = \delta_{i,k} + \frac{w_{v_i} - w_{u_i}}{w_{v_i} - w_{u_k}} e^{\frac{1}{2}(z_i + z_k)}. \quad (3.8)$$

3.3 Degenerate constants

The next step is to give explicitly the quantities (independent of the vector \mathbf{z}) appearing in (2.5) and (2.6), i.e., \mathbf{V}_a , \mathbf{W}_a , \mathbf{r} , q_2 , etc, after the degeneration to genus zero. We use the same notation for these quantities on the degenerated surface. For any distinct points $a, b \in \mathcal{R}_0$, it follows from (2.3) and (3.2) that

$$V_{a,k} = \frac{1}{k'_a(w_a)} \left(\frac{1}{w_a - w_{v_k}} - \frac{1}{w_a - w_{u_k}} \right), \quad (3.9)$$

$$W_{a,k} = \frac{1}{k'_a(w_a)^2} \left(-\frac{1}{(w_a - w_{v_k})^2} + \frac{1}{(w_a - w_{u_k})^2} \right) - \frac{k''_a(w_a)}{k'_a(w_a)^2} V_{a,k}, \quad (3.10)$$

$$r_k = \ln \left\{ \frac{w_b - w_{v_k}}{w_b - w_{u_k}} \frac{w_a - w_{u_k}}{w_a - w_{v_k}} \right\}, \quad (3.11)$$

for $k = 1, \dots, g$. Moreover, from the integral representation of $q_2(a, b)$ and $K_1(a, b)$ (see (2.12) and (2.13)), using (3.2) and (3.3) one gets

$$q_2(a, b) = \frac{1}{k'(w_a)k'(w_b)(w_a - w_b)^2}, \quad (3.12)$$

$$K_1(a, b) = \frac{1}{k'(w_a)(w_b - w_a)} - \frac{1}{2} \frac{k''(w_a)}{k'^2(w_a)}. \quad (3.13)$$

Putting $\mathbf{z} = 0$ in (2.5) and taking the limit $\epsilon \rightarrow 0$ leads to

$$q_1(a, b) = -q_2(a, b), \quad (3.14)$$

due to the fact that the theta function tends to one and that its partial derivatives tend to zero. In the same way, taking the limit $\epsilon \rightarrow 0$ in (2.6) one gets

$$K_2(a, b) = -\left(K_1(a, b)\right)^2. \quad (3.15)$$

4 Degenerate algebro-geometric solutions of n-NLS

One way to construct solutions of (1.2) is first to solve its complexified version, a system of $2n$ equations of $2n$ dependent variables $\left\{ \psi_j, \psi_j^* \right\}_{j=1}^n$,

$$\begin{aligned} i \frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + 2 \left(\sum_{k=1}^n \psi_k \psi_k^* \right) \psi_j &= 0, \\ -i \frac{\partial \psi_j^*}{\partial t} + \frac{\partial^2 \psi_j^*}{\partial x^2} + 2 \left(\sum_{k=1}^n \psi_k \psi_k^* \right) \psi_j^* &= 0, \quad j = 1, \dots, n, \end{aligned} \quad (4.1)$$

where $\psi_j(x, t)$ and $\psi_j^*(x, t)$ are complex valued functions of the real variables x and t . This system reduces to the n-NLS^s equation (1.2) under the *reality conditions*

$$\psi_j^* = s_j \overline{\psi_j}, \quad j = 1, \dots, n. \quad (4.2)$$

Algebro-geometric solutions of the system (4.1) were obtained in [22] by the use of the degenerated versions (2.5) and (2.6) of Fay's identity; these solutions are given by:

Theorem 4.1. *Let \mathcal{R}_g be a compact Riemann surface of genus $g > 0$ and let f be a meromorphic function of degree $n + 1$ on \mathcal{R}_g . Let $z_a \in \mathbb{C}$ be a non critical value of f , and consider the fiber $f^{-1}(z_a) = \{a_1, \dots, a_{n+1}\}$ over z_a . Choose the local parameters $k_{a_j}(p) = f(p) - z_a$, for any point $p \in \mathcal{R}_g$ lying in a neighbourhood of a_j . Let $\mathbf{D} \in \mathbb{C}^g$ and $A_j \neq 0$ be arbitrary constants. Then the following functions $\{\psi_j\}_{j=1}^n$ and $\{\psi_j^*\}_{j=1}^n$ are solutions of the system (4.1)*

$$\begin{aligned} \psi_j(x, t) &= A_j \frac{\Theta(\mathbf{Z} - \mathbf{D} + \mathbf{r}_j)}{\Theta(\mathbf{Z} - \mathbf{D})} \exp\{i(-E_j x + F_j t)\}, \\ \psi_j^*(x, t) &= \frac{q_2(a_{n+1}, a_j)}{A_j} \frac{\Theta(\mathbf{Z} - \mathbf{D} - \mathbf{r}_j)}{\Theta(\mathbf{Z} - \mathbf{D})} \exp\{i(E_j x - F_j t)\}. \end{aligned} \quad (4.3)$$

Here Θ denotes the theta function (2.2) with zero characteristic, and $\mathbf{Z} = i\mathbf{V}_{a_{n+1}}x + i\mathbf{W}_{a_{n+1}}t$, where vectors $\mathbf{V}_{a_{n+1}}$ and $\mathbf{W}_{a_{n+1}}$ are defined in (2.3). Moreover, $\mathbf{r}_j = \int_{a_{n+1}}^{a_j} \omega$, where ω is the vector of normalized holomorphic differentials, and the scalars E_j, F_j are given by

$$E_j = K_1(a_{n+1}, a_j), \quad F_j = K_2(a_{n+1}, a_j) - 2 \sum_{k=1}^n q_1(a_{n+1}, a_k). \quad (4.4)$$

The scalars q_i, K_i for $i = 1, 2$ are defined in (2.7)-(2.10).

The proof of this theorem is based on the following identity:

$$\sum_{k=1}^{n+1} \mathbf{V}_{a_k} = 0, \quad (4.5)$$

which is satisfied by the vectors \mathbf{V}_{a_k} associated to the fiber $f^{-1}(z_a) = \{a_1, \dots, a_{n+1}\}$ over z_a . We shall use this relation to construct solutions of (1.2) in terms of elementary functions.

Remark 4.1. The relationship between solutions of the Kadomtsev-Petviashvili (KP1) equation (generalization of the KdV equation to two spatial variables, see, for instance, [7]) and solutions of the multi-component NLS equation was investigated in [22]. This relationship implies that all solutions of equation (1.2) constructed in this paper provide also solutions of the KP1 equation as explained in [22].

In the next section, solutions of (4.1) in terms of elementary functions are derived from solutions (4.3) by degenerating the associated Riemann surface \mathcal{R}_g into a Riemann surface of genus zero. Imposing reality conditions (4.2), by an appropriate choice of the parameters one gets special solutions of (1.2) such as multi-solitons and breathers. To the best of our knowledge, such an approach to multi-solitonic solutions of n-NLS^s has not been studied before. Moreover, breather and rational breather solutions to the multi-component case are derived here for the first time.

4.1 Determinantal solutions of the complexified n-NLS equation

Solutions of the complexified scalar NLS equation in terms of elementary functions were obtained in [7], when the genus of the associated hyperelliptic spectral curve tends to zero. For specific choices of parameters, they get dark and bright multi-solitons of the NLS equation, as well as quasi-periodic modulations of the plane wave solutions previously constructed in [21]. A direct generalization of this approach to the multi-component case is not obvious, due to the complexity of the associated spectral curve. To bypass this problem and to construct spectral data associated to algebro-geometric solutions (4.3) in the limit when the genus tends to zero, we use the uniformization map between the degenerate Riemann surface and the sphere. Details of such a degeneration were presented in Section 3.

Let us discuss solutions of n-NLS in genus zero. Consider the following meromorphic function $f(w)$ on the sphere:

$$f(w) = \alpha \prod_{i=1}^{n+1} \frac{w - w_{a_i}}{w - w_{b_i}} \quad (4.6)$$

where $w_{a_j} \neq w_{b_k}$ for all j, k , $w_{a_j} \neq w_{a_k}$ for $j \neq k$, and $\alpha \in \mathbb{C}$. Without loss of generality, put $\alpha = 1$. This function is of degree $n + 1$ on the sphere, hence it represents a genus zero $(n+1)$ -sheeted branched covering of \mathbb{CP}^1 . Recall that a meromorphic function f on the sphere is called real if its zeros as well as its poles are real or pairwise conjugate.

If not stated otherwise, the local parameter in a neighbourhood of a regular point w_a (i.e. $f'(w_a) \neq 0$) is chosen to be $k_a(w) = f(w) - f(w_a)$ for any w lying in a neighbourhood of w_a . Solutions of the complexified system (4.1) associated to the meromorphic function f (4.6) on the sphere are given by:

Proposition 4.1. *Let $j, k \in \mathbb{N}$ satisfy $1 \leq j \leq n$ and $1 \leq k \leq g$. Let f be a meromorphic function (4.6) of degree $n + 1$ on the sphere, with complex zeros $\{w_{a_i}\}_{i=1}^{n+1}$ and complex poles $\{w_{b_i}\}_{i=1}^{n+1}$. Let $\mathbf{d} \in \mathbb{C}^g$ and $A_j \neq 0$ be arbitrary constants. Moreover, assume that $w_{u_k}, w_{v_k} \in \mathbb{C}$ satisfy*

$$f(w_{u_k}) = f(w_{v_k}). \quad (4.7)$$

Then the following functions are solutions of the complexified system (4.1)

$$\begin{aligned} \psi_j(x, t) &= A_j \frac{\det(\mathbb{T}_{j,1})}{\det(\mathbb{T}_{j,0})} \exp\{i(-E_j x + F_j t)\}, \\ \psi_j^*(x, t) &= \frac{q_2(a_{n+1}, a_j)}{A_j} \frac{\det(\mathbb{T}_{j,-1})}{\det(\mathbb{T}_{j,0})} \exp\{i(E_j x - F_j t)\}. \end{aligned} \quad (4.8)$$

For $\beta = -1, 0, 1$, $\mathbb{T}_{j,\beta}$ denotes the $g \times g$ matrix with entries (3.8) where $z_k^j = Z_k - d_k + \beta r_{j,k}$. Here $Z_k = iV_{a_{n+1},k} x + iW_{a_{n+1},k} t$, where the scalars $V_{a_{n+1},k}$ and $W_{a_{n+1},k}$ are defined in (3.9) and (3.10), and $r_{j,k}$ is defined in (3.11) with $w_a := w_{a_{n+1}}$ and $w_b := w_{a_j}$. The scalars E_j and F_j are given by

$$E_j = K_1(a_{n+1}, a_j), \quad F_j = -(K_1(a_{n+1}, a_j))^2 + 2 \sum_{k=1}^n q_2(a_{n+1}, a_k),$$

where $q_2(a_{n+1}, a_j)$ and $K_1(a_{n+1}, a_j)$ are defined in (3.12) and (3.13).

Proof. Consider solutions (4.3) associated to a Riemann surface \mathcal{R}_g of genus g , and assume $f(a_i) = 0$ for any $1 \leq i \leq n+1$. Pinch all \mathcal{A} -cycles of the associated Riemann surface \mathcal{R}_g into double points, as explained in Section 3. After desingularization, the meromorphic function f of degree $n+1$ on \mathcal{R}_g becomes a meromorphic function of degree $n+1$ on the sphere, given in general form by (4.6). In the limit considered here, the theta function tends to the determinantal form (3.7). Quantities defined on the degenerated surface and independent of the variables x and t were constructed in Section 3.3 and are given in (3.9)-(3.15). Condition (4.7) follows from the fact that double points appearing after degeneration of \mathcal{R}_g are desingularized into two distinct points w_{u_k} and w_{v_k} having the same projection under the meromorphic function f . Note that equation (4.5) holds in the limit, since by (2.3) and (4.6) one has

$$\sum_{i=1}^{n+1} V_{a_i, k} = \frac{1}{f(w_{u_k})} - \frac{1}{f(w_{v_k})} \quad (4.9)$$

which by (4.7) equals zero for $k = 1, \dots, g$. □

Remark 4.2. Functions (4.8) give a family of solutions to the complexified multi-component NLS equation (4.1) depending on $3n + g + 2$ complex parameters: w_{a_i}, w_{b_i} for $1 \leq i \leq n+1$, d_k for $1 \leq k \leq g$, and A_j for $1 \leq j \leq n$.

Remark 4.3. The following transformations leave equation (4.1) invariant

$$\begin{aligned} \psi_j(x, t) &\longrightarrow \psi_j(\beta x + 2\beta\lambda t, \beta^2 t) \exp\{-i(\lambda x + \lambda^2 t)\}, \\ \psi_j^*(x, t) &\longrightarrow \beta^2 \psi_j^*(\beta x + 2\beta\lambda t, \beta^2 t) \exp\{i(\lambda x + \lambda^2 t)\}, \end{aligned} \quad (4.10)$$

where $\lambda = \mu\beta^{-1}$ for any $\mu \in \mathbb{C}$ and any $\beta \neq 0$. Such a transformation may be useful to simplify the expressions in the obtained solutions and thus to facilitate the numerical implementation.

4.2 Multi-solitonic solutions of n-NLS

Imposing reality conditions (4.2) on the degenerate solutions (4.8) of the complexified system, one gets particular solutions of (1.2) such as dark and bright multi-solitons. Dark and bright solitons differ by the fact that the modulus of the first tends to a non zero constant and the modulus of the second tends to zero when the spatial variable tends to infinity. Such solutions were obtained in [7] for the one component case by degenerating algebro-geometric solutions, and describe elastic collisions between solitons. Elastic means that the solitons asymptotically retain their shape and speed after interaction. The interaction of vector solitons is more complex than the one of scalar solitons because inelastic collisions can appear in all components of one solution (see for instance [1]).

In what follows $N \in \mathbb{N}$ with $N \geq 1$.

4.2.1 Dark multi-solitons of n-NLS^s, $s \neq (1, \dots, 1)$.

Dark multi-soliton solutions of 2-NLS^s were investigated in [35]. The dark N -soliton solution derived here corresponds to elastic interactions between N dark solitons. Moreover, it is shown that this type of solutions does not exist for the focusing multi-component nonlinear Schrödinger equation, i.e., in the case where $s = (1, \dots, 1)$.

Proposition 4.2. Let $j, k \in \mathbb{N}$ satisfy $1 \leq j \leq n$ and $1 \leq k \leq N$. Let f be a real meromorphic function (4.6) of degree $n+1$ on the sphere, having $n+1$ real zeros $\{w_{a_i}\}_{i=1}^{n+1}$. Choose $\theta \in \mathbb{R}$ and $\mathbf{d} \in \mathbb{R}^N$. Moreover, assume that $w_{u_k}, w_{v_k} \in \mathbb{C}$ satisfy (4.7) and

$$\overline{w_{u_k}} = w_{v_k}. \quad (4.11)$$

Put $s_j = \text{sign}(f'(w_{a_{n+1}})f'(w_{a_j}))$. Then the following functions define smooth dark N -soliton solutions of n -NLS^s, where $s = (s_1, \dots, s_n)$ with $s \neq (1, \dots, 1)$,

$$\psi_j(x, t) = A_j e^{i\theta} \frac{\det(\mathbb{T}_{j,1})}{\det(\mathbb{T}_{j,0})} \exp\{i(-E_j x + F_j t)\}. \quad (4.12)$$

Here $A_j = |q_2(a_{n+1}, a_j)|^{1/2}$, and the remaining notation is as in Proposition 4.1 with $g = N$.

Proof. Let us check that the functions ψ_j and ψ_j^* defined in (4.8) satisfy reality conditions (4.2) with $s_j = \text{sign}(f'(w_{a_{n+1}})f'(w_{a_j}))$. Put $A_j = |q_2(a_{n+1}, a_j)|^{1/2}$ in (4.8). Then with the above assumptions, it is straightforward to see that $\psi_j^* = s_j \overline{\psi_j}$ where $s_j = \text{sign}(q_2(a_{n+1}, a_j))$, which by (3.12) leads to $s_j = \text{sign}(f'(w_{a_{n+1}})f'(w_{a_j}))$. Moreover, with (4.11) it can be seen that condition (4.5) is equivalent to

$$\frac{1}{|w_{a_{n+1}} - w_{v_k}|^2} + \sum_{j=1}^n \frac{f'(w_{a_{n+1}})}{f'(w_{a_j})} \frac{1}{|w_{a_j} - w_{v_k}|^2} = 0, \quad (4.13)$$

for $k = 1, \dots, N$. Therefore, by (4.13) the quantity $f'(w_{a_{n+1}})f'(w_{a_j})$ cannot be positive for all j , which yields $s \neq (1, \dots, 1)$. The solutions are smooth since the denominator in (4.12) is a finite sum of real exponentials. \square

Remark 4.4. The dark N -soliton solutions (4.12) depend on $N+1$ real parameters d_k, θ and a real meromorphic function f (4.6) defined by $2n+2$ real parameters. The solitons are dark since the modulus of the ψ_j tends to A_j when $x \in \mathbb{R}$ tends to infinity.

Example 4.1. With the notation of Proposition 4.1 and 4.2, functions ψ_j (4.12) are given for $N = 1$ by

$$\psi_j(x, t) = A_j \frac{1 + e^{Z_1 - d_1 + r_{j,1}}}{1 + e^{Z_1 - d_1}} e^{i(-E_j x + F_j t)}.$$

4.2.2 Bright multi-solitons of n -NLS^s.

Bright multi-solitons of the NLS equation presented in [7] were obtained by collapsing all branch cuts of the underlying hyperelliptic curve of the algebro-geometric solutions. This way they get solutions expressed as the quotient of a finite sum of exponentials similar to dark multi-solitons, except that the modulus of the solutions tends to zero instead of a non-zero constant when the spatial variable tends to infinity. Following this approach, a family of bright multi-solitons of n -NLS^s is obtained here by further degeneration of (4.8).

For the multi-component case there exist two sorts of bright soliton interactions: elastic or inelastic. Inelastic collisions between bright solitons were investigated in [36] for the two component case and in [23] for the multi-component case. The family of bright multi-solitons of n -NLS^s obtained here describes the standard elastic collision with phase shift. Notice that there exist various ways to degenerate algebro-geometric solutions. Therefore, it appears possible that bright solitons with inelastic collision can be obtained by different degenerations.

Proposition 4.3. Let $j \in \mathbb{N}$ satisfy $1 \leq j \leq n$. Take $w_{a_j}, \theta \in \mathbb{R}$ and choose $\hat{\mathbf{d}} \in \mathbb{C}^{2N}$ such that $\overline{\hat{d}_{2k-1}} = \hat{d}_{2k}$. Moreover, let $w_{u_{2k}}, w_{v_{2k-1}} \in \mathbb{C}$ satisfy

$$\overline{w_{u_{2k}}} = w_{v_{2k-1}} \quad (4.14)$$

for $1 \leq k \leq N$. Choose $\gamma_j \in \mathbb{R}$ and put $s_j = \text{sign}(\gamma_j)$. Then the following functions give bright N -soliton solutions of n -NLS^s

$$\psi_j(x, t) = A_j e^{i\theta} \frac{\det(\mathbb{K}_j)}{\det(\mathbb{M})}, \quad (4.15)$$

where $A_j = |\gamma_j|^{1/2} |w_{a_j}|^{-1}$. Here \mathbb{K}_j and \mathbb{M} are $2N \times 2N$ matrices with entries $(\mathbb{K}_j)_{ik}$ and $(\mathbb{M})_{ik}$ given by:

$$\begin{aligned} - \text{ for } i \text{ and } k \text{ even: } & (\mathbb{K}_j)_{ik} = \delta_{2,i} \frac{w_{u_i}}{w_{u_k}} e^{\frac{1}{2}(z_2+z_k+\hat{r}_{j,2}+\hat{r}_{j,k})} + \delta_{i,k} - \delta_{2,i} \delta_{2,k} \\ & + \delta_{2,k} (\delta_{2,i} - 1) \frac{w_{u_i}}{w_{u_2}} e^{\frac{1}{2}(z_i-z_2+\hat{r}_{j,i}-\hat{r}_{j,2})} \\ - \text{ for } i \text{ even and } k \text{ odd: } & (\mathbb{K}_j)_{ik} = \alpha_{u_k}^2 \frac{w_{u_i}}{\alpha_{v_i} - \alpha_{u_k}} \frac{\alpha_{v_2} - \alpha_{v_i}}{\alpha_{v_2} - \alpha_{u_k}} e^{\frac{1}{2}(z_i+z_k+\hat{r}_{j,i}+\hat{r}_{j,k})} \\ - \text{ for } i \text{ odd and } k \text{ even: } & (\mathbb{K}_j)_{ik} = \frac{w_{v_i}}{w_{v_i} - w_{u_k}} e^{\frac{1}{2}(z_i+z_k+\hat{r}_{j,i}+\hat{r}_{j,k})} \\ - \text{ for } i \text{ and } k \text{ odd: } & (\mathbb{K}_j)_{ik} = \delta_{i,k} \\ - \text{ for } i, k \text{ even, or } i, k \text{ odd: } & (\mathbb{M})_{ik} = \delta_{i,k} \\ - \text{ for } i \text{ even and } k \text{ odd: } & (\mathbb{M})_{ik} = \alpha_{u_k} \alpha_{v_i} \frac{w_{u_i}}{\alpha_{v_i} - \alpha_{u_k}} e^{\frac{1}{2}(z_i+z_k)} \\ - \text{ for } i \text{ odd and } k \text{ even: } & (\mathbb{M})_{ik} = - \frac{w_{v_i}}{w_{v_i} - w_{u_k}} e^{\frac{1}{2}(z_i+z_k)}. \end{aligned}$$

Here $\overline{\alpha_{v_{2k}}} = \alpha_{u_{2k-1}}$ where

$$\alpha_{u_{2k-1}} = \sum_{j=1}^n \gamma_j \left(\frac{1}{w_{a_j}} - \frac{1}{w_{a_j} - w_{v_{2k-1}}} \right). \quad (4.16)$$

Moreover, z_k is a linear function of the variables x and t satisfying $\overline{z_{2k}} = z_{2k-1}$, given by

$$z_{2k-1} = i \alpha_{u_{2k-1}} x + i \alpha_{u_{2k-1}}^2 t - \hat{d}_{2k-1}.$$

The scalars $\hat{r}_{j,k}$ satisfy $\overline{\hat{r}_{j,2k}} = -\hat{r}_{j,2k-1}$ where

$$\hat{r}_{j,2k-1} = \ln \left\{ \frac{w_{a_j} - w_{v_{2k-1}}}{w_{a_j} w_{v_{2k-1}} \alpha_{u_{2k-1}}} \right\}.$$

Proof. Consider functions (4.8) obtained from (4.3) for the choice of local parameters k_{a_i} :

$$k_{a_i}(w) = (\gamma_i f'(w_{a_i}))^{-1} f(w)$$

for any w lying in a neighbourhood of w_{a_i} , $i = 1, \dots, n+1$, and assume $g = 2N$. Hence condition (4.5) becomes

$$\sum_{i=1}^{n+1} \gamma_i \left(\frac{1}{w_{a_i} - w_{v_k}} - \frac{1}{w_{a_i} - w_{u_k}} \right) = 0 \quad (4.17)$$

for $k = 1, \dots, N$. Now put $A_j = |q_2(a_{n+1}, a_j)|^{1/2}$ in (4.8), where

$$q_2(a_{n+1}, a_j) = \gamma_{n+1} \gamma_j (w_{a_{n+1}} - w_{a_j})^{-2}.$$

Choose a small parameter $\epsilon > 0$ and define $d_k = -\ln \epsilon + \hat{d}_k$, for $k = 1, \dots, 2N$, and

$$w_{u_{2k-1}} = w_{a_{n+1}} + \epsilon^2 \alpha_{u_{2k-1}}^{-1}, \quad w_{v_{2k}} = w_{a_{n+1}} + \epsilon^2 \alpha_{v_{2k}}^{-1}, \quad (4.18)$$

for $k = 1, \dots, N$. Now put $\gamma_{n+1} = \epsilon^2$ and consider in the determinant $\det(\mathbb{T}_{j,1})$ appearing in (4.8) the substitution

$$L_{2i} \longrightarrow L_{2i} - \frac{(\mathbb{T}_{j,1})_{2i,2}}{(\mathbb{T}_{j,1})_{2,2}} L_2,$$

for $i = 2, \dots, N$, where L_k denotes the line number k of the matrix $\mathbb{T}_{j,1}$, and $(\mathbb{T}_{j,1})_{i,k}$ denotes the entries of this matrix. In the limit $\epsilon \rightarrow 0$, it can be seen that the functions ψ_j given in (4.8) converge towards functions (4.15), where the following change of parameters (eliminating the parameter $w_{a_{n+1}}$) has been made:

$$w_{a_j} \rightarrow w_{a_j} + w_{a_{n+1}}, \quad w_{u_{2k}} \rightarrow w_{u_{2k}} + w_{a_{n+1}}, \quad w_{v_{2k-1}} \rightarrow w_{v_{2k-1}} + w_{a_{n+1}},$$

for $j = 1, \dots, n$ and $k = 1, \dots, N$. Analogous statements can be made for the functions ψ_j^* . By assumption, it is straightforward to see that the functions ψ_j and ψ_j^* obtained in the limit considered here satisfy the reality conditions $\psi_j^* = s_j \overline{\psi_j}$ with $s_j = \text{sign}(\gamma_j)$. Moreover, in this limit condition (4.17) yields (4.16). \square

Remark 4.5. The bright N -soliton solutions (4.15) depend on $2N$ complex parameters \hat{d}_{2k-1} , $w_{v_{2k-1}}$, and $2n+1$ real parameters w_{a_j} , γ_j , θ . Moreover, all parameters appearing in (4.15) are free, contrary to the dark multi-solitons (4.12) where parameters w_{u_k} and w_{v_k} have to satisfy the polynomial equation (4.7). The solitons are bright since the modulus of the ψ_j tends to zero when $x \in \mathbb{R}$ tends to infinity, in contrast to the dark solitons.

4.3 Breather and rational breather solutions of n-NLS

Solutions obtained here differ from the dark multi-solitons studied in Section 4.2.1 by the reality condition imposed on parameters w_{u_i} and w_{v_i} in solutions (4.8) of the complexified system for $i = 1, \dots, g$. By an appropriate choice of parameters, one gets periodic solutions (breathers) as well as rational solutions (rational breathers). The name ‘breather’ reflects the behavior of the profile of the solution which is periodic in time (respectively, space) and localized in space (respectively, time). This appears to be the first time that explicit breather and rational breather solutions of n-NLS^s are given.

In what follows $N \in \mathbb{N}$ with $N \geq 1$.

4.3.1 Multi-Breathers of n-NLS^s.

Multi-breather solutions of n-NLS^s are given in the following proposition. The N -breather solution corresponds to an elastic interaction between N breathers.

Proposition 4.4. *Let $j, k \in \mathbb{N}$ satisfy $1 \leq j \leq n$ and $1 \leq k \leq N$. Let f be a real meromorphic function (4.6) of degree $n+1$ on the sphere, having $n+1$ real zeros $\{w_{a_i}\}_{i=1}^{n+1}$. Choose $\theta \in \mathbb{R}$ and take $\hat{\mathbf{d}} \in \mathbb{C}^{2N}$ such that $\overline{\hat{d}_{2k-1}} = \hat{d}_{2k}$. Let $w_{u_{2k}}, w_{u_{2k-1}}, w_{v_{2k}}, w_{v_{2k-1}} \in \mathbb{C}$ satisfy (4.7) and*

$$\overline{w_{u_{2k}}} = w_{v_{2k-1}}, \quad \overline{w_{u_{2k-1}}} = w_{v_{2k}}. \quad (4.19)$$

Put $s_j = \text{sign}(f'(w_{a_{n+1}})f'(w_{a_j}))$. Then the following functions define N -breather solutions of n-NLS^s

$$\psi_j(x, t) = A_j e^{i\theta} \frac{\det(\mathbb{T}_{j,1})}{\det(\mathbb{T}_{j,0})} \exp\{i(-E_j x + F_j t)\}, \quad (4.20)$$

where $A_j = |q_2(a_{n+1}, a_j)|^{1/2}$, and the remaining notation is the same as in Proposition 4.1 for $g = 2N$.

Remark 4.6. Functions (4.20) cover a family of breather solutions of n-NLS^s depending on N complex parameters d_k , a real parameter θ , and a real meromorphic function f (4.6) defined by $2n+2$ real parameters.

To simplify the computation of the solutions, we apply transformation (4.10) to the solutions (4.20), with β and λ given by

$$\beta = 1, \quad \lambda = \frac{1}{2} f''(w_{a_{n+1}}) f'(w_{a_{n+1}})^{-2}. \quad (4.21)$$

Hence, the quantity $f''(w_{a_{n+1}})f'(w_{a_{n+1}})^{-2} V_{a_{n+1},k}$ in the expression (3.10) for the scalar $W_{a_{n+1},k}$ disappears, as well as the quantity $\frac{1}{2} f''(w_{a_{n+1}})f'(w_{a_{n+1}})^{-2}$ in the expression (3.13) for the scalar $K_1(a_{n+1}, a_j)$.

Example 4.2. *Figure 1 shows a breather solution of the 4-NLS^s equation with $s = (-, -, +, -)$. It corresponds to the following choice of parameters: $w_{a_1} = 10$, $w_{a_2} = -5$, $w_{a_3} = -1/3$, $w_{a_4} = 1/4$, $w_{a_5} = 1/2$, and $w_{u_1} \approx 0.55 - 0.11i$ with $f(w_{u_1}) = 2i$, $w_{u_2} \approx -0.35 + 0.07i$ with $f(w_{u_2}) = -2i$.*

Example 4.3. *Figure 2 shows an elastic collision between two breather solutions of the 4-NLS^s equation with $s = (-, +, +, -)$. It corresponds to the following choice of parameters: $w_{a_1} = 1/3$, $w_{a_2} = 3$, $w_{a_3} = 1/7$, $w_{a_4} = 2$, $w_{a_5} = 1$, $w_{b_1} = -1$, $w_{b_2} = 4$, $w_{b_3} = -2$, $w_{b_4} = 0$, and $w_{u_1} \approx 0.55 - 0.11i$ with $f(w_{u_1}) = 2i$, $w_{u_2} \approx -0.35 + 0.07i$ with $f(w_{u_2}) = -2i$, $w_{u_3} \approx -0.91 - 0.52i$ with $f(w_{u_3}) = 10 - 5i$, and $w_{u_4} \approx 14.46 + 5.32i$ with $f(w_{u_4}) = 10 + 5i$.*

4.3.2 N -rational breathers of n-NLS^s, for $1 \leq N \leq n$.

Here we are interested in solutions of n-NLS^s that can be expressed in the form of a ratio of two polynomials (modulo an exponential factor). These solutions, called rational breathers, are neither periodic in time nor in space, but are isolated in time and space. They are obtained from breather solutions (4.20) in the limit when the parameters $w_{v_{2k-1}}$ and $w_{u_{2k-1}}$ tend to each others, as well as the parameters $w_{v_{2k}}$ and $w_{u_{2k}}$, for $k = 1, \dots, N$. An appropriate choice of the parameters d_i in (4.20) for $i = 1, \dots, 2N$, leads to limits of the form $0/0$ in the expression for the breather solutions. Thus, by performing a Taylor expansion of the numerator and denominator in (4.20), one gets a family of N -rational breather solutions of n-NLS^s.

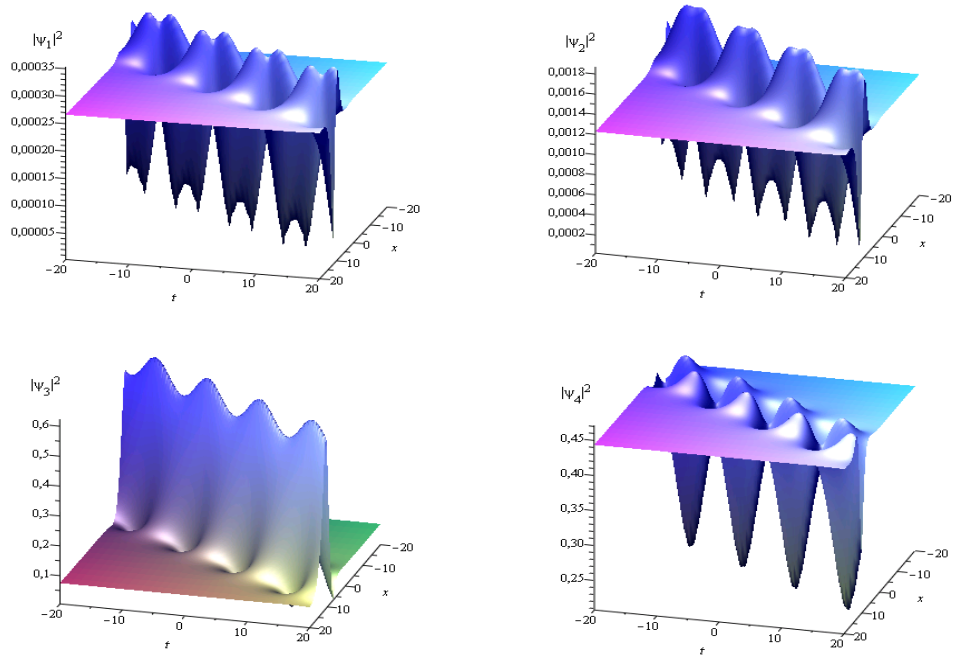


Figure 1: Breather of 4-NLS $^{---+}$.

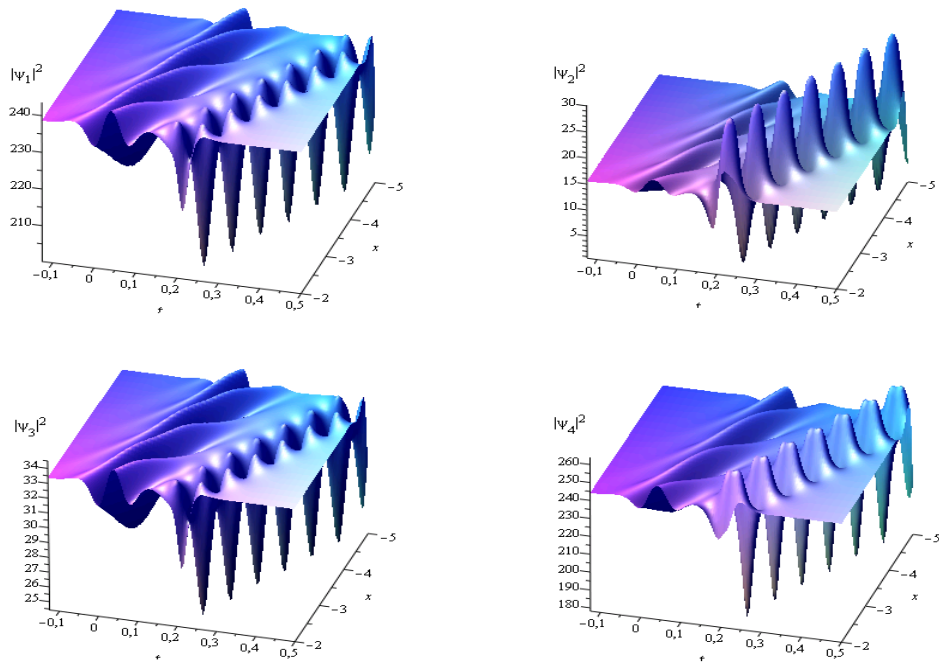


Figure 2: 2-breather of 4-NLS $^{+++}$.

Proposition 4.5. *Let $N, j \in \mathbb{N}$ satisfy $1 \leq N \leq n$ and $1 \leq j \leq n$. Let f be a real meromorphic function (4.6) of degree $n+1$ on the sphere, having $n+1$ real zeros $\{w_{a_i}\}_{i=1}^{n+1}$. Choose $\theta \in \mathbb{R}$ and take $\hat{\mathbf{d}} \in \mathbb{C}^{2N}$ such that $\overline{\hat{d}_{2k}} = \hat{d}_{2k-1}$ for $1 \leq k \leq N$. Moreover, let $w_{u_{2k-1}}, w_{v_{2k}} \in \mathbb{C}$, $1 \leq k \leq N$, be complex conjugate critical points of the meromorphic function f , i.e., they are solutions of $f'(w) = 0$, which is equivalent to*

$$\sum_{i=1}^{n+1} \frac{1}{f'(w_{a_i})} \frac{1}{(w - w_{a_i})^2} = 0. \quad (4.22)$$

Put $s_j = \text{sign}(f'(w_{a_{n+1}})f'(w_{a_j}))$. Then the following functions give N -rational breathers of n -NLS^s

$$\psi_j(x, t) = A_j e^{i\theta} \frac{\det(\mathbb{K}_{j,1})}{\det(\mathbb{K}_{j,0})} \exp\{i(-E_j x + F_j t)\}, \quad (4.23)$$

where $A_j = |q_2(a_{n+1}, a_j)|^{1/2}$. For $\beta = 0, 1$, $\mathbb{K}_{j,\beta}$ denotes a $2N \times 2N$ matrix with entries $(\mathbb{K}_{j,\beta})_{i,k}$ given by:

- for i and k even: $(\mathbb{K}_{j,\beta})_{i,k} = (1 - \delta_{i,k}) \frac{1}{w_{v_i} - w_{v_k}} - \delta_{i,k} (z_k + \beta \hat{r}_{j,k})$
- for i even and k odd: $(\mathbb{K}_{j,\beta})_{i,k} = \frac{1}{w_{v_i} - w_{u_k}}$
- for i odd and k even: $(\mathbb{K}_{j,\beta})_{i,k} = -\frac{1}{w_{u_i} - w_{v_k}}$
- for i and k odd: $(\mathbb{K}_{j,\beta})_{i,k} = -(1 - \delta_{i,k}) \frac{1}{w_{u_i} - w_{u_k}} - \delta_{i,k} (z_k + \beta \hat{r}_{j,k})$.

Here z_k is a linear function of the variables x and t given by

$$z_k = i \hat{V}_{a_{n+1},k} x + i \hat{W}_{a_{n+1},k} t - \hat{d}_k$$

for $k = 1, \dots, 2N$, where $\overline{\hat{V}_{a_{n+1},2k}} = -\hat{V}_{a_{n+1},2k-1}$ and $\overline{\hat{W}_{a_{n+1},2k}} = -\hat{W}_{a_{n+1},2k-1}$ with

$$\hat{V}_{a_{n+1},2k-1} = \frac{1}{f'(w_{a_{n+1}})} \frac{1}{(w_{a_{n+1}} - w_{u_{2k-1}})^2}, \quad \hat{W}_{a_{n+1},2k-1} = -\frac{1}{f'(w_{a_{n+1}})^2} \frac{2}{(w_{a_{n+1}} - w_{u_{2k-1}})^3}$$

for $k = 1, \dots, N$. Scalars $\hat{r}_{j,k}$ satisfy $\overline{\hat{r}_{j,2k}} = -\hat{r}_{j,2k-1}$ and are given by

$$\hat{r}_{j,2k-1} = -\frac{w_{a_{n+1}} - w_{a_j}}{(w_{a_{n+1}} - w_{u_{2k-1}})(w_{a_j} - w_{u_{2k-1}})}.$$

Scalars E_j, F_j are defined by

$$E_j = \frac{1}{f'(w_{a_{n+1}})(w_{a_j} - w_{a_{n+1}})}, \quad F_j = -E_j^2 + 2 \sum_{k=1}^n q_2(a_{n+1}, a_k).$$

Proof. To simplify the expression for the obtained solutions, apply the transformation (4.10) to functions (4.20) with β and λ as in (4.21). Let $\epsilon > 0$ be a small parameter and define $d_k = \epsilon \hat{d}_k + i\pi$, for $k = 1, \dots, 2N$. Moreover, assume

$$w_{v_{2k-1}} = w_{u_{2k-1}} + \epsilon \alpha_{v_{2k-1}}, \quad w_{u_{2k}} = w_{v_{2k}} + \epsilon \alpha_{u_{2k}}, \quad (4.24)$$

for some $\alpha_{v_{2k-1}}, \alpha_{u_{2k}} \in \mathbb{C}$, where $k = 1, \dots, N$. Note that equation number k of system (4.9) can be written as

$$\sum_{j=1}^{n+1} \frac{1}{f'(w_{a_j})} \frac{f(w_{v_k}) f(w_{u_k})}{(w_{a_j} - w_{v_k})(w_{a_j} - w_{u_k})} = - \frac{f(w_{v_k}) - f(w_{u_k})}{w_{v_k} - w_{u_k}}. \quad (4.25)$$

Hence, in the limit $\epsilon \rightarrow 0$, equation (4.25) becomes

$$\sum_{j=1}^{n+1} \frac{1}{f'(w_{a_j})} \frac{f(w_{v_{2k-1}})^2}{(w_{a_j} - w_{v_{2k-1}})^2} = -f'(w_{v_{2k-1}}),$$

and

$$\sum_{j=1}^{n+1} \frac{1}{f'(w_{a_j})} \frac{f(w_{u_{2k}})^2}{(w_{a_j} - w_{u_{2k}})^2} = -f'(w_{u_{2k}}),$$

for $k = 1, \dots, N$. Therefore, choose $w_{v_{2k-1}}$ and $w_{u_{2k}}$ to be distinct critical points of the meromorphic function f for $k = 1, \dots, N$, i.e., they are solutions of $f'(w) = 0$, in such way that equation (4.5) holds in the limit considered here. Since the condition $f'(w) = 0$ is equivalent to solve a polynomial equation of degree $2n$, it follows that $1 \leq N \leq n$. Now take the limit $\epsilon \rightarrow 0$ in (4.20). Note that parameters $\alpha_{v_{2k-1}}, \alpha_{u_{2k}}$ cancel in this limit, and the degenerated functions take the form (4.23). \square

Remark 4.7. Functions (4.23) provide a family of rational breather solutions of n-NLS^s depending on N complex parameters d_k , a real parameter θ , and a real meromorphic function f (4.6) defined by $2n + 2$ real parameters, chosen such that f admits complex conjugate critical points.

Example 4.4. With the notation of Proposition 4.5 the functions ψ_j (4.23) for $N = 1$ are given by

$$\psi_j(x, t) = A_j e^{i\theta} \frac{B + (z_1 + \hat{r}_{j,1})(\bar{z}_1 - \overline{\hat{r}_{j,1}})}{B + |z_1|^2} \exp\{i(-E_j x + N_j t)\},$$

where $B = (2 \operatorname{Im}(w_{u_1}))^{-2}$.

Example 4.5. Figure 3 shows a rational breather solution of the 4-NLS^s equation with $s = (+, +, +, +)$. It corresponds to the following choice of parameters: $k_{a_k}(w) = f'(w_{a_k})f(w)$ for $k = 1, \dots, n + 1$, with $w_{a_1} = 3$, $w_{a_2} = 5$, $w_{a_3} = 7$, $w_{a_4} = 0$, $w_{a_5} = 4$, and $w_{u_1} \approx 4.53 + 0.56i$ being a solution of $\sum_{i=1}^{n+1} (w - w_{a_i})^{-2} = 0$. We observe that functions ψ_2 and ψ_3 coincide with the Peregrine breather well known in the scalar case [33], whereas functions ψ_1, ψ_4 belong to a new class of rational breathers which does not exist in the scalar case. This new type of rational breathers emerges due to the higher degree of the meromorphic function associated to the solutions of n-NLS^s for $n > 1$.

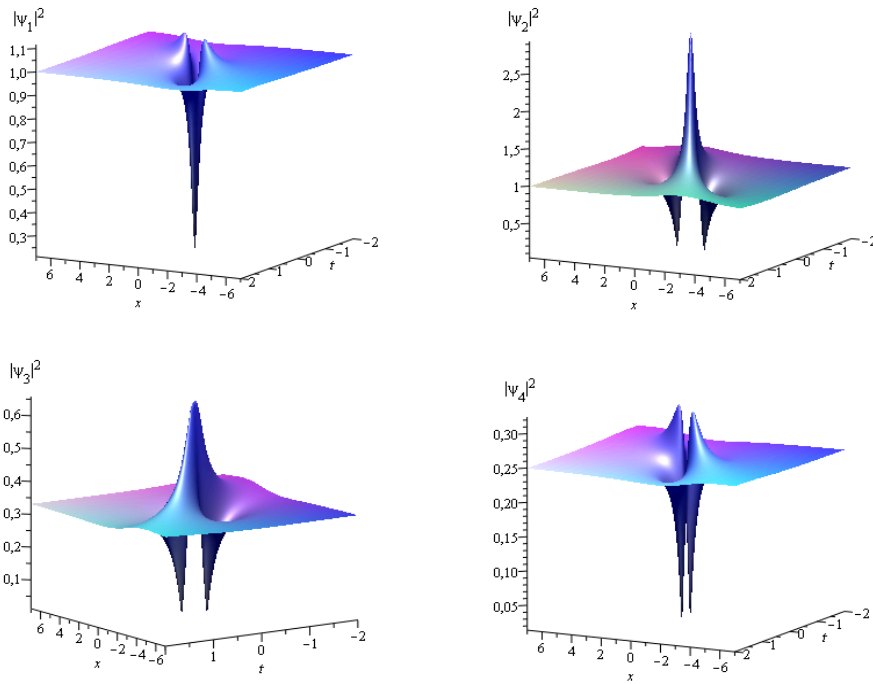


Figure 3: Rational breather of 4-NLS++++.

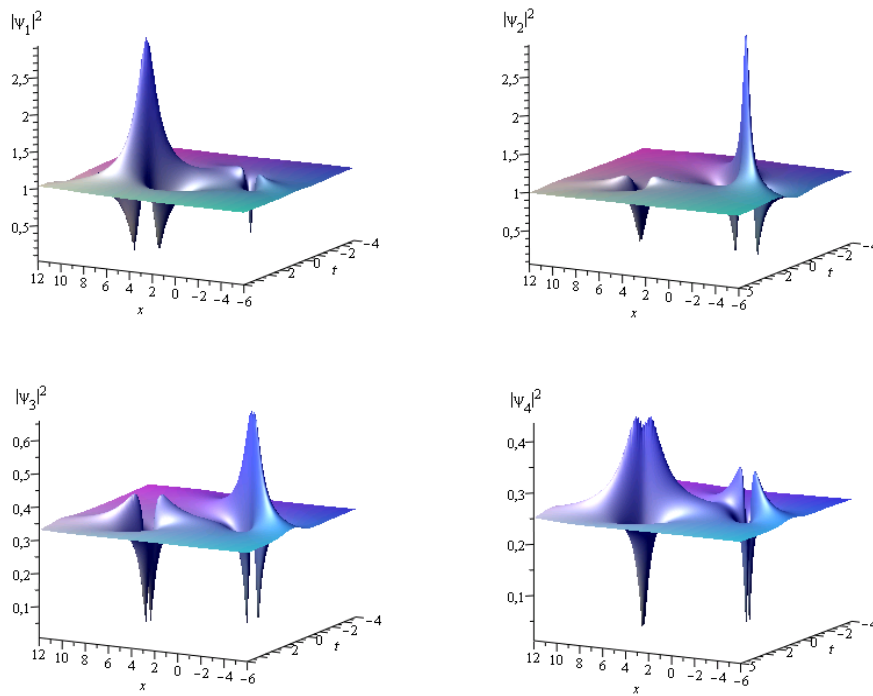


Figure 4: 2-rational breather of 4-NLS++++.

Example 4.6. Figure 4 shows a 2-rational breather solution of the 4-NLS^s equation with $s = (+, +, +, +)$. It corresponds to the following choice of parameters: $k_{a_k}(w) = f'(w_{a_k})f(w)$ for $k = 1, \dots, n+1$, with $w_{a_1} = 3$, $w_{a_2} = 5$, $w_{a_3} = 7$, $w_{a_4} = 0$, $w_{a_5} = 4$, and $w_{u_1} \approx 4.53 + 0.56i$, $w_{u_3} \approx 3.45 + 0.56i$ being solutions of $\sum_{i=1}^{n+1} (w - w_{a_i})^{-2} = 0$, and $d_k = 10$. Variation of the parameters d_k leads to a displacement in the (x, t) -plane of the rational breathers appearing in each of the pictures of Figure 4.

5 Degenerate algebro-geometric solutions of the DS equations

Solutions of the DS equations (1.3) in terms of elementary functions constructed here are obtained analogously to the solutions of the n-NLS equation, therefore some details will be omitted. Let us introduce the function $\phi := \Phi + \rho|\psi|^2$, where $\rho = \pm 1$, and the characteristic coordinates $\xi = \frac{1}{2}(x - i\alpha y)$, $\eta = \frac{1}{2}(x + i\alpha y)$, $\alpha = i, 1$ in (1.3). In these coordinates the DS equations become

$$\begin{aligned} i\psi_t + \frac{1}{2}(\partial_\xi^2 + \partial_\eta^2)\psi + 2\phi\psi &= 0, \\ \partial_\xi\partial_\eta\phi + \frac{\rho}{2}(\partial_\xi^2 + \partial_\eta^2)|\psi|^2 &= 0. \end{aligned} \quad (5.1)$$

Recall that DS1 ^{ρ} denotes the case $\alpha = i$ (here ξ and η are both real), and DS2 ^{ρ} the case $\alpha = 1$ (here ξ and η are pairwise conjugate).

To construct solutions of (5.1) in terms of elementary functions, let us first introduce its complexified version:

$$\begin{aligned} i\psi_t + \frac{1}{2}(\psi_{\xi\xi} + \psi_{\eta\eta}) + 2\varphi\psi &= 0, \\ -i\psi_t^* + \frac{1}{2}(\psi_{\xi\xi}^* + \psi_{\eta\eta}^*) + 2\varphi\psi^* &= 0, \\ \varphi_{\xi\eta} + \frac{1}{2}((\psi\psi^*)_{\xi\xi} + (\psi\psi^*)_{\eta\eta}) &= 0, \end{aligned} \quad (5.2)$$

where $\varphi := \Phi + \psi\psi^*$. This system reduces to (5.1) under the *reality condition*:

$$\psi^* = \rho\bar{\psi}, \quad (5.3)$$

which leads to $\varphi = \phi$. Theta-functional solutions of (5.2) were studied in [22] and can be written in the following form.

Theorem 5.1. *Let \mathcal{R}_g be a compact Riemann surface of genus $g > 0$, and let $a, b \in \mathcal{R}_g$ be distinct points. Take arbitrary constants $\mathbf{D} \in \mathbb{C}^g$ and $A, \kappa_1, \kappa_2 \in \mathbb{C} \setminus \{0\}$, $h \in \mathbb{C}$. Denote by ℓ a contour connecting a and b which does not intersect cycles of the canonical homology basis. Then for any $\xi, \eta, t \in \mathbb{C}$, the following functions ψ , ψ^* and φ are solutions of system (5.2)*

$$\begin{aligned} \psi(\xi, \eta, t) &= A \frac{\Theta(\mathbf{Z} - \mathbf{D} + \mathbf{r})}{\Theta(\mathbf{Z} - \mathbf{D})} \exp\{-i(G_1\xi + G_2\eta - G_3\frac{t}{2})\}, \\ \psi^*(\xi, \eta, t) &= -\frac{\kappa_1\kappa_2 q_2(a, b)}{A} \frac{\Theta(\mathbf{Z} - \mathbf{D} - \mathbf{r})}{\Theta(\mathbf{Z} - \mathbf{D})} \exp\{i(G_1\xi + G_2\eta - G_3\frac{t}{2})\}, \\ \varphi(\xi, \eta, t) &= \frac{1}{2}(\ln \Theta(\mathbf{Z} - \mathbf{D}))_{\xi\xi} + \frac{1}{2}(\ln \Theta(\mathbf{Z} - \mathbf{D}))_{\eta\eta} + \frac{h}{4}. \end{aligned} \quad (5.4)$$

Here $\mathbf{Z} = i\kappa_1 \mathbf{V}_a \xi - i\kappa_2 \mathbf{V}_b \eta + i(\kappa_1^2 \mathbf{W}_a - \kappa_2^2 \mathbf{W}_b) \frac{t}{2}$, where the vectors $\mathbf{V}_a, \mathbf{V}_b$ and $\mathbf{W}_a, \mathbf{W}_b$ were introduced in (2.3). Moreover $\mathbf{r} = \int_{\ell} \omega$, where ω is the vector of normalized holomorphic differentials, and the scalars G_1, G_2, G_3 are given by

$$G_1 = \kappa_1 K_1(a, b), \quad G_2 = \kappa_2 K_1(b, a), \quad (5.5)$$

$$G_3 = \kappa_1^2 K_2(a, b) + \kappa_2^2 K_2(b, a) + h. \quad (5.6)$$

Scalars $q_2(a, b), K_1(a, b), K_2(a, b)$ are defined in (2.8), (2.9), (2.10) respectively.

Remark 5.1. In the case where vectors \mathbf{V}_a and \mathbf{V}_b satisfy $\mathbf{V}_a + \mathbf{V}_b = 0$, as mentioned in [22], solutions of the Davey-Stewartson equation become solutions of the NLS equation (1.1) under an appropriate change of variables.

In this section, we study the behaviour of theta-functional solutions (5.4) of the complexified DS equations when the Riemann surface degenerates into a Riemann surface of genus zero. Imposing the reality condition (5.3), for particular choices of the parameters one gets well-known solutions such as multi-soliton, breather, rational breather, dromion and lump. This appears to be the first time that such solutions of DS are derived from algebro-geometric solutions.

5.1 Determinantal solutions of the complexified DS equations

Here solutions of the complexified system (5.2) are given as a quotient of two determinants. In the next subsections, this particular form will be more convenient to produce special solutions of the DS equations (5.1).

Proposition 5.1. *Let $k \in \mathbb{N}$ satisfy $1 \leq k \leq g$. Let $w_a, w_b, w_{u_k}, w_{v_k}, h \in \mathbb{C}$, and $A, \kappa_1, \kappa_2 \in \mathbb{C} \setminus \{0\}$. Choose $\mathbf{d} \in \mathbb{C}^g$. Then the following functions are solutions of the system (5.2)*

$$\begin{aligned} \psi(\xi, \eta, t) &= A \frac{\det(\mathbb{T}_1)}{\det(\mathbb{T}_0)} \exp \{-i(G_1 \xi + G_2 \eta - G_3 \frac{t}{2})\}, \\ \psi^*(\xi, \eta, t) &= -\frac{\kappa_1 \kappa_2}{A(w_a - w_b)^2} \frac{\det(\mathbb{T}_{-1})}{\det(\mathbb{T}_0)} \exp \{i(G_1 \xi + G_2 \eta - G_3 \frac{t}{2})\}, \\ \varphi(\xi, \eta, t) &= \frac{1}{2} (\ln \det(\mathbb{T}_0))_{\xi\xi} + \frac{1}{2} (\ln \det(\mathbb{T}_0))_{\eta\eta} + \frac{h}{4}. \end{aligned} \quad (5.7)$$

For $\beta = -1, 0, 1$, \mathbb{T}_β denotes the $g \times g$ matrix with entries (3.8) with $z_k = Z_k - d_k + \beta r_k$. Here the scalars r_k are given in (3.11) and

$$\mathbf{Z} = i\kappa_1 \mathbf{V}_a \xi - i\kappa_2 \mathbf{V}_b \eta + i(\kappa_1^2 \mathbf{W}_a - \kappa_2^2 \mathbf{W}_b) \frac{t}{2} \quad (5.8)$$

with

$$V_{c,k} = \frac{1}{w_c - w_{v_k}} - \frac{1}{w_c - w_{u_k}}, \quad W_{c,k} = -\frac{1}{(w_c - w_{v_k})^2} + \frac{1}{(w_c - w_{u_k})^2}, \quad (5.9)$$

where $c \in \{a, b\}$. The scalars G_1, G_2, G_3 are given by

$$G_1 = \frac{\kappa_1}{w_b - w_a}, \quad G_2 = \frac{\kappa_2}{w_a - w_b}, \quad G_3 = -G_1^2 - G_2^2 + h. \quad (5.10)$$

Proof. Consider solutions (5.4) of system (5.2) in the limit when the Riemann surface degenerates to a Riemann surface of genus zero, as explained in Section 3. In this limit, choose the local parameters k_a and k_b near $a \in \mathcal{R}_0$ and $b \in \mathcal{R}_0$ to be the uniformization map between the degenerate Riemann surface \mathcal{R}_0 and the w -sphere. Hence, for any $w \in \mathcal{R}_0$ lying in a neighbourhood of $w_a \in \mathcal{R}_0$, $k_a(w) = w - w_a$. Therefore, quantities independent of variables ξ, η and t are obtained from (3.9)-(3.15). \square

Remark 5.2. Functions (5.7) give a family of solutions of the complexified system, involving elementary functions only. These solutions depend on $3g+6$ complex parameters $w_a, w_b, h, A, \kappa_1, \kappa_2$ and w_{u_k}, w_{v_k}, d_k . Varying these parameters we will obtain different types of physically interesting solutions investigated in the next subsections.

5.2 Multi-solitonic solutions of the DS equations

Soliton solutions of the DS equations were shown to be representable in terms of Wronskian determinants in [5]. Single soliton and multi-soliton solutions corresponding to the known one-dimensional solutions can be obtained from this representation. These solitons are pseudo-one-dimensional in the sense that in the (x, y) -plane, they have the same form as one-dimensional solitons in the (x, t) -plane, but that they move with an angle with respect to the axes. The multi-soliton solution describes the interaction of many such solitons each propagating in different directions.

In what follows $N \in \mathbb{N}$ with $N \geq 1$.

5.2.1 Dark multi-soliton of DS1 $^\rho$ and DS2 $^+$

Here dark multi-solitons of the DS1 $^\rho$ and DS2 $^+$ equations are derived from functions (5.7) for an appropriate choice of the parameters. They were investigated in [41].

Put $g = N$ and $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$ in (5.7). Moreover, assume $h \in \mathbb{R}$ and $\mathbf{d} \in \mathbb{R}^N$.

Reality condition for DS1 $^\rho$. Let us check that with the following choice of parameters,

$$w_a, w_b \in \mathbb{R}, \quad \kappa_1, \kappa_2 \in \mathbb{R} \setminus \{0\}, \quad \overline{w_{v_k}} = w_{u_k}, \quad k = 1, \dots, N, \quad (5.11)$$

functions ψ and ψ^* in (5.7) satisfy the reality condition $\psi^* = \rho \overline{\psi}$ with $\rho = -\text{sign}(\kappa_1 \kappa_2)$. Indeed, this can be deduced from the fact that $G_1, G_2, G_3 \in \mathbb{R}$, and

$$\overline{\det(\mathbb{T}_\beta)} = \det(\overline{\mathbb{T}_\beta}) = \det(\mathbb{T}_{-\beta}), \quad (5.12)$$

since u and v can be interchanged in the proof of (3.7). Therefore, functions ψ and ϕ in (5.7) define dark multi-soliton solutions of DS1 $^\rho$.

Smoothness. The dark multi-soliton solutions obtained here are smooth because the denominator $\det(\mathbb{T}_0)$ of functions ψ and ϕ (5.7) consists of a finite sum of real exponentials (see (3.7)), since ξ, η, t are real.

Remark 5.3. One gets a family of smooth dark multi-soliton of the DS1 $^\rho$ equation, depending on $N+6$ real parameters $w_a, w_b, h, \kappa_1, \kappa_2, d_k$, a phase θ , and N complex parameters w_{u_k} .

Reality condition for $DS2^+$. Let us check that with the following choice of parameters,

$$\overline{w_a} = w_b, \quad \overline{\kappa_1} = \kappa_2, \quad w_{u_k}, w_{v_k} \in \mathbb{R}, \quad k = 1, \dots, N, \quad (5.13)$$

the functions ψ and ψ^* (5.7) satisfy the reality condition $\psi^* = \overline{\psi}$. With (5.13), it is straightforward to see that (5.12) is also satisfied. Moreover, since $\overline{G_1} = G_2$, $G_3 \in \mathbb{R}$ and $(w_a - w_b)^2 < 0$, the functions ψ and ψ^* (5.7) satisfy the reality condition $\psi^* = \overline{\psi}$. Therefore, they define dark multi-soliton solutions of $DS2^+$.

Smoothness. To get smooth solutions, additional conditions are needed to ensure that $\det(\mathbb{T}_0)$ does not vanish for all complex conjugate $\xi = \overline{\eta}$. For instance, if

$$w_{v_1} < w_{u_1} < w_{v_2} < w_{u_2} < \dots < w_{v_N} < w_{u_N},$$

the scalars $(\mathbb{B})_{ik}$ (3.6) are real for any $i, k \in \{1, \dots, N\}$. Therefore, the functions ψ and ϕ (5.7) are smooth, since their denominator does not vanish as a finite sum of real exponentials.

Remark 5.4. One gets a family of smooth dark multi-soliton of the $DS2^+$ equation, depending on $3N + 1$ real parameters h, w_{u_k}, w_{v_k}, d_k , a phase θ , and 2 complex parameters w_a, κ_1 .

5.2.2 Bright multi-soliton of $DS1^\rho$ and $DS2^-$

In this part we construct bright multi-soliton to the $DS1^\rho$ and $DS2^-$ equations. It is well known that such solutions can be written in terms of a quotient of sums of exponentials, for which the modulus tends to zero if the spatial variables tend to infinity.

To get bright multi-soliton solutions, one degenerates once more solutions (5.7) of the complexified system. Put $g = 2N$ and $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$ in (5.7), and take $h \in \mathbb{R}$.

Degeneration. Choose a small parameter $\epsilon > 0$ and define $d_k = -\ln \epsilon + \hat{d}_k$, for $k = 1, \dots, 2N$, and

$$\begin{aligned} w_{u_{2k-1}} &= w_a + \epsilon \alpha_{u_{2k-1}}^{-1} (w_a - w_b), & w_{v_{2k-1}} &= w_b + \epsilon \alpha_{v_{2k-1}}^{-1} (w_a - w_b), \\ w_{u_{2k}} &= w_b + \epsilon \alpha_{u_{2k}}^{-1} (w_a - w_b), & w_{v_{2k}} &= w_a + \epsilon \alpha_{v_{2k}}^{-1} (w_a - w_b), \end{aligned} \quad (5.14)$$

for $k = 1, \dots, N$. Moreover, put $\kappa_1 = \epsilon \hat{\kappa}_1 (w_a - w_b)$, and $\kappa_2 = \epsilon \hat{\kappa}_2 (w_a - w_b)$. Consider in the determinant $\det(\mathbb{T}_1)$ appearing in (5.7) the substitution

$$L_{2i} \longrightarrow L_{2i} - \frac{(\mathbb{T}_1)_{2i,2}}{(\mathbb{T}_1)_{2,2}} L_2$$

for $i = 2, \dots, N$, where L_k denotes the line number k of the matrix \mathbb{T}_1 and $(\mathbb{T}_1)_{i,k}$ the entries of this matrix. An analogous transformation has to be considered for the matrix \mathbb{T}_{-1} appearing in function ψ^* . Now take the limit $\epsilon \rightarrow 0$ in (5.7). The function ψ obtained in this limit has the form (5.17). Notice that in this limit, the dependence on the parameters w_a and w_b disappears.

Reality condition for $DS1^\rho$. It is straightforward to see that, with the following choice of parameters,

$$\hat{\kappa}_1, \hat{\kappa}_2 \in \mathbb{R} \setminus \{0\}, \quad \overline{\hat{d}_{2k-1}} = \hat{d}_{2k}, \quad \overline{\alpha_{u_{2k-1}}} = \alpha_{v_{2k}}, \quad \overline{\alpha_{u_{2k}}} = \alpha_{v_{2k-1}}, \quad k = 1, \dots, N, \quad (5.15)$$

the functions ψ and ψ^* obtained in the limit considered here satisfy the reality condition $\psi^* = \rho \bar{\psi}$ with $\rho = -\text{sign}(\hat{\kappa}_1 \hat{\kappa}_2)$.

Reality condition for $DS2^-$. In the same way, with the following choice of parameters,

$$\overline{\hat{\kappa}_1} = \hat{\kappa}_2, \quad \overline{\hat{d}_{2k-1}} = \hat{d}_{2k}, \quad \overline{\alpha_{u_{2k-1}}} = \alpha_{u_{2k}}, \quad \overline{\alpha_{v_{2k-1}}} = \alpha_{v_{2k}}, \quad k = 1, \dots, N, \quad (5.16)$$

the functions ψ and ψ^* obtained in the considered limit satisfy the reality condition $\psi^* = -\bar{\psi}$.

The solutions. Let $\theta \in \mathbb{R}$. With (5.15), the following functions of the variables ξ, η, t obtained in the considered limit, give bright N -soliton solutions of $DS1^\rho$ where $\rho = -\text{sign}(\hat{\kappa}_1 \hat{\kappa}_2)$ and $\gamma = 0$; because of (5.16) these functions define bright N -soliton solutions of $DS2^-$ where $\gamma = 1$:

$$\begin{aligned} \psi(\xi, \eta, t) &= \hat{A} e^{i\theta} \frac{\det(\mathbb{K})}{\det(\mathbb{M})}, \\ \phi(\xi, \eta, t) &= \frac{1}{2} (\ln \det(\mathbb{M}))_{\xi\xi} + \frac{1}{2} (\ln \det(\mathbb{M}))_{\eta\eta} + \frac{h}{4}, \end{aligned} \quad (5.17)$$

where $\hat{A} = |\hat{\kappa}_1 \hat{\kappa}_2|^{1/2}$. Here \mathbb{K} and \mathbb{M} are $2N \times 2N$ matrices with entries $(\mathbb{K})_{ik}$ and $(\mathbb{M})_{ik}$ given by:

$$\begin{aligned} - \text{ for } i \text{ and } k \text{ even: } & (\mathbb{K})_{ik} = \delta_{i,k} - \delta_{2,i} \delta_{2,k} + \delta_{2,i} e^{\frac{1}{2}(z_2+z_k+\hat{r}_2+\hat{r}_k)} \\ & \quad + \delta_{2,k} (\delta_{2,i} - 1) e^{\frac{1}{2}(z_i-z_2+\hat{r}_i-\hat{r}_2)} \\ - \text{ for } i \text{ even and } k \text{ odd: } & (\mathbb{K})_{ik} = - \frac{\alpha_{u_k}^2}{\alpha_{v_i} - \alpha_{u_k}} \frac{\alpha_{v_2} - \alpha_{v_i}}{\alpha_{v_2} - \alpha_{u_k}} e^{\frac{1}{2}(z_i+z_k+\hat{r}_i+\hat{r}_k)} \\ - \text{ for } i \text{ odd and } k \text{ even: } & (\mathbb{K})_{ik} = - \frac{\alpha_{v_i} \alpha_{u_k}}{\alpha_{v_i} - \alpha_{u_k}} e^{\frac{1}{2}(z_i+z_k+\hat{r}_i+\hat{r}_k)} \\ - \text{ for } i \text{ and } k \text{ odd: } & (\mathbb{K})_{ik} = \delta_{i,k}, \\ - \text{ for } i, k \text{ even, or } i, k \text{ odd: } & (\mathbb{M})_{ik} = \delta_{i,k} \\ - \text{ otherwise: } & (\mathbb{M})_{ik} = (-1)^{i+1} \frac{\alpha_{v_i} \alpha_{u_k}}{\alpha_{v_i} - \alpha_{u_k}} e^{\frac{1}{2}(z_i+z_k)}. \end{aligned}$$

Here z_k is a linear function of the variables ξ, η and t given by

$$\begin{aligned} z_{2k-1} &= i \hat{\kappa}_1 \alpha_{u_{2k-1}} \xi + i \hat{\kappa}_2 \alpha_{v_{2k-1}} \eta + i \left(\hat{\kappa}_1^2 \alpha_{u_{2k-1}}^2 + \hat{\kappa}_2^2 \alpha_{v_{2k-1}}^2 \right) \frac{t}{2} - \hat{d}_{2k-1} + \gamma \frac{i\pi}{2}, \\ z_{2k} &= -i \hat{\kappa}_1 \alpha_{v_{2k}} \xi - i \hat{\kappa}_2 \alpha_{u_{2k}} \eta - i \left(\hat{\kappa}_1^2 \alpha_{v_{2k}}^2 + \hat{\kappa}_2^2 \alpha_{u_{2k}}^2 \right) \frac{t}{2} - \hat{d}_{2k} + \gamma \frac{i\pi}{2}, \end{aligned}$$

for $k = 1, \dots, N$. Moreover, the scalars \hat{r}_k are defined by

$$\hat{r}_k = (-1)^k \ln \{-\alpha_{v_k} \alpha_{u_k}\}, \quad k = 1, \dots, 2N.$$

Remark 5.5. i) With (5.15), functions (5.17) give a family of bright multi-soliton solutions of the $DS1^\rho$ equation depending on $3N$ complex parameters $\hat{d}_{2k-1}, \alpha_{u_{2k-1}}, \alpha_{v_{2k}}$ and 4 real parameters $h, \theta, \hat{\kappa}_1, \hat{\kappa}_2$.

ii) With (5.16), functions (5.17) provide a family of bright multi-soliton solutions of the $DS2^-$ equation depending on $3N+1$ complex parameters $\hat{d}_{2k-1}, \alpha_{u_{2k-1}}, \alpha_{v_{2k-1}}, \hat{\kappa}_1$ and 2 real parameters h, θ .

5.3 Breather and rational breather solutions of the DS equations

The breather solutions of the DS equation were found in [40]. Here a family of breather solutions and rational breather solutions of the DS1 equation are derived from algebro-geometric solutions. These solutions resemble their 1 + 1 dimensional analogues. In particular, the profiles of the corresponding solutions of the DS equation in the (x, y, t) coordinates look as those in the (x, t) coordinates extended along a spatial variable y .

5.3.1 Multi-Breathers of $DS1^\rho$

The N -breather solution obtained here corresponds to an elastic interaction between N breathers. Put $g = 2N$ and $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$ in (5.7). It is straightforward to see that with the following choice of parameters,

$$w_a, w_b, h \in \mathbb{R}, \quad \kappa_1, \kappa_2 \in \mathbb{R} \setminus \{0\}, \quad \overline{d_{2k-1}} = d_{2k}, \quad \overline{w_{v_{2k}}} = w_{u_{2k-1}}, \quad \overline{w_{v_{2k-1}}} = w_{u_{2k}}, \quad (5.18)$$

for $k = 1, \dots, N$, functions ψ and ψ^* (5.7) satisfy the reality condition $\psi^* = \rho \overline{\psi}$ with $\rho = -\text{sign}(\kappa_1 \kappa_2)$. Therefore, analogously to the n-NLS equation, functions ψ and ϕ in (5.7) give N -breather solutions of $DS1^\rho$.

Remark 5.6. One gets a family of breather solutions of $DS1^\rho$ depending on $3N$ complex parameters $d_{2k-1}, w_{u_{2k-1}}, w_{u_{2k}}$ and 6 real parameters $w_a, w_b, h, \kappa_1, \kappa_2$ and a phase θ .

Example 5.1. Figure 5 shows the evolution in time of 2-breather solution of $DS1^-$ with the following choice of parameters: $w_a = 8, w_b = -1, w_{u_1} = 5 - 2i, w_{u_2} = 2 + i, w_{u_3} = 3 - i, w_{u_4} = 1 + 4i, \kappa_1 = \kappa_2 = 1, d_k = h = 0$.

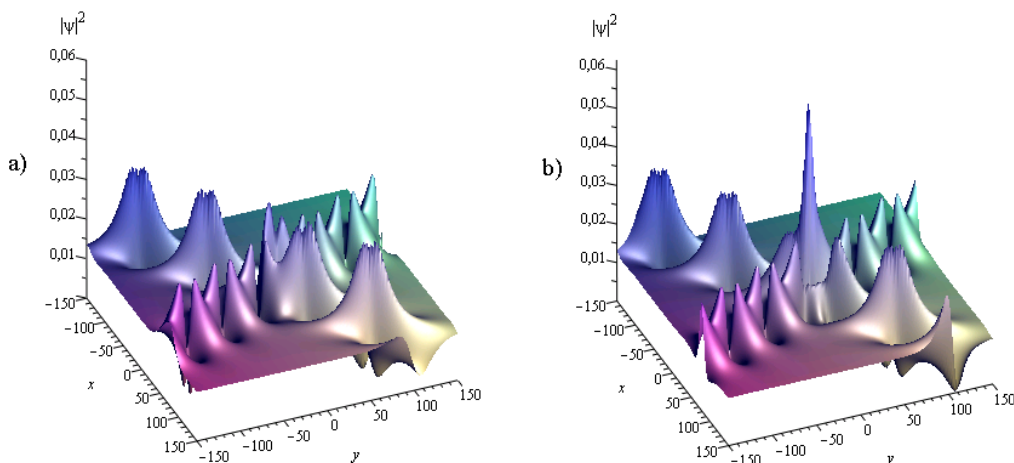


Figure 5: 2-breather of $DS1^-$ at a) $t = 0$, b) $t = 45$.

5.3.2 Multi-rational breathers of DS1 $^\rho$

In this part, we deal with rational solutions (modulo an exponential factor) of the DS1 $^\rho$ equation. These solutions are obtained as limiting cases of the breather solutions. The N -rational solutions describe elastic interaction between N rational breathers, and are expressed as a quotient of two polynomials of degree N in the variables ξ, η, t .

Assume $g = 2N$ and put $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$ in (5.7).

Degeneration. Let $\epsilon > 0$ be a small parameter and define $d_k = \epsilon \hat{d}_k + i\pi$, for $k = 1, \dots, 2N$, and

$$w_{v_{2k-1}} = w_{u_{2k-1}} + \epsilon \alpha_{v_{2k-1}}, \quad w_{u_{2k}} = w_{v_{2k}} + \epsilon \alpha_{u_{2k}} \quad (5.19)$$

for $k = 1, \dots, N$. It is straightforward to see that $\det(\mathbb{T}_\beta) \approx \epsilon^{2N} P_\beta$, where P_β is a polynomial of degree $2N$ with respect to the variables ξ, η and t . Now take the limit $\epsilon \rightarrow 0$ in (5.7). The function ψ obtained in this limit is an N -rational breather solution of DS1 $^\rho$ given by (5.21).

Reality condition. Imposing the following constraints on the parameters:

$$w_a, w_b, h \in \mathbb{R}, \quad \kappa_1, \kappa_2 \in \mathbb{R} \setminus \{0\}, \quad \overline{\hat{d}_{2k}} = \hat{d}_{2k-1}, \quad \overline{w_{u_{2k-1}}} = w_{v_{2k}}, \quad k = 1, \dots, N, \quad (5.20)$$

it can be seen that the functions ψ and ψ^* (5.7) in the considered limit satisfy the reality condition $\psi^* = \rho \bar{\psi}$, with $\rho = -\text{sign}(\kappa_1 \kappa_2)$.

The solutions. Let $\theta \in \mathbb{R}$. Then the following degenerated functions define N -rational breather solutions of DS1 $^\rho$

$$\begin{aligned} \psi(\xi, \eta, t) &= A e^{i\theta} \frac{\det(\mathbb{K}_1)}{\det(\mathbb{K}_0)} \exp \left\{ -i (G_1 \xi + G_2 \eta - G_3 \frac{t}{2}) \right\}, \\ \phi(\xi, \eta, t) &= \frac{1}{2} (\ln \det(\mathbb{K}_0))_{\xi\xi} + \frac{1}{2} (\ln \det(\mathbb{K}_0))_{\eta\eta} + \frac{h}{4}, \end{aligned} \quad (5.21)$$

where \mathbb{K}_β , with $\beta = 0, 1$, is a $2N \times 2N$ matrix with entries $(\mathbb{K}_\beta)_{ik}$ given by

$$\begin{aligned} \text{- for } i \text{ and } k \text{ even: } & (\mathbb{K}_\beta)_{ik} = (1 - \delta_{i,k}) \frac{1}{w_{v_i} - w_{v_k}} - \delta_{i,k} (z_k + \beta \hat{r}_k) \\ \text{- for } i \text{ even and } k \text{ odd: } & (\mathbb{K}_\beta)_{ik} = \frac{1}{w_{v_i} - w_{u_k}} \\ \text{- for } i \text{ odd and } k \text{ even: } & (\mathbb{K}_\beta)_{ik} = -\frac{1}{w_{u_i} - w_{v_k}} \\ \text{- for } i \text{ and } k \text{ odd: } & (\mathbb{K}_\beta)_{ik} = -(1 - \delta_{i,k}) \frac{1}{w_{u_i} - w_{u_k}} - \delta_{i,k} (z_k + \beta \hat{r}_k). \end{aligned}$$

Here z_k is a linear function of the variables ξ, η and t given by

$$z_k = i \kappa_1 \hat{V}_{a,k} \xi - i \kappa_2 \hat{V}_{b,k} \eta + i \left(\kappa_1^2 \hat{W}_{a,k} - \kappa_2^2 \hat{W}_{b,k} \right) \frac{t}{2} - \hat{d}_k.$$

Moreover, for $c \in \{a, b\}$, the scalars $\hat{V}_{c,k}$, $\hat{W}_{c,k}$ and \hat{r}_k satisfy $\overline{\hat{V}_{c,2k}} = \hat{V}_{c,2k-1}$, $\overline{\hat{W}_{c,2k}} = \hat{W}_{c,2k-1}$ and $\overline{\hat{r}_{2k}} = \hat{r}_{2k-1}$, and are given by:

$$\hat{V}_{c,2k-1} = \frac{1}{(w_c - w_{u_{2k-1}})^2}, \quad \hat{W}_{c,2k-1} = -\frac{2}{(w_c - w_{u_{2k-1}})^3},$$

$$\hat{r}_{2k-1} = - \frac{w_a - w_b}{(w_a - w_{u_{2k-1}})(w_b - w_{u_{2k-1}})},$$

for $k = 1, \dots, N$. Constants G_1, G_2, G_3 are given in (5.10).

Remark 5.7. Functions (5.21) give a family of rational solutions of $DS1^\rho$ depending on $2N$ complex parameters $d_{2k-1}, w_{u_{2k-1}}$ and 6 real parameters $w_a, w_b, h, \theta, \kappa_1, \kappa_2$.

Example 5.2. Figure 6 shows the evolution in time of the 2-rational breather solution of $DS1^-$ with the following choice of parameters: $w_a = 2, w_b = 1, w_{u_1} = 2i, w_{u_3} = 2 + i, \kappa_1 = \kappa_2 = 1, d_k = h = 0$.

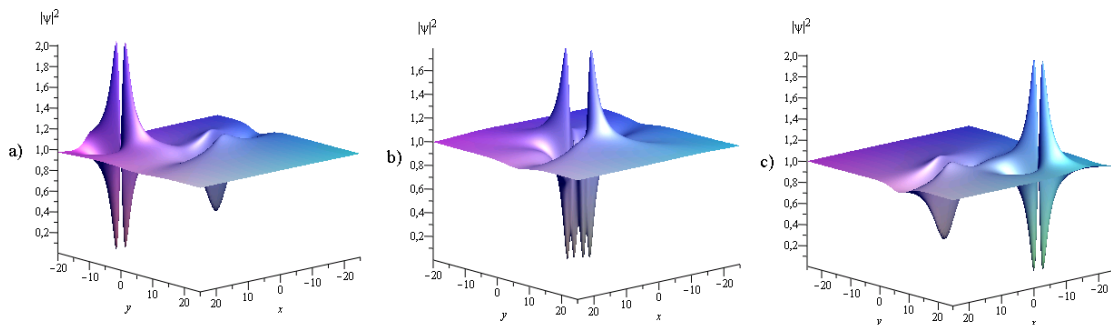


Figure 6: 2-rational breather of $DS1^-$ at a) $t = -5$, b) $t = 0$, c) $t = 5$.

Example 5.3. Figure 7 (resp. Figure 8) shows the interaction between a line rational breather and a rational breather solution of $DS1^-$ with the following choice of parameters: $w_a = 2, w_b = -2, w_{u_1} = 3i$ (resp. $w_{u_1} = 3i + 1$), $w_{u_3} = 2i, \kappa_1 = \kappa_2 = 1, d_k = h = 0$. By line rational breather we denote a growing and decaying mode localized only in one direction.

5.4 Dromion and lump solutions of the DS equations

Here we construct the dromion solution of $DS1^\rho$ and the lump solution of $DS2^-$ which correspond to solutions localized in all directions of the plane. These solutions arise by suitable degenerations of solutions (5.7) to the complexified system, and by imposing the reality condition $\overline{\psi^*} = \rho\psi$. This appears to be the first time that such solutions are obtained as limiting cases of theta-functional solutions.

5.4.1 Dromion of $DS1^\rho$

Boiti et al. [8] have shown that the $DS1$ equation has solutions that decay exponentially in all directions. The solutions they obtained can move along any direction in the plane, and the only effect of their interactions is a shift in their position, independently of their relative initial position in the plane. Later, Fokas and Santini [15, 39] pointed out that by an appropriate choice of the boundary conditions, the localized solitons (called "dromions") of the $DS1$ equation possess properties which are different from the properties of one-dimensional solitons, namely, the

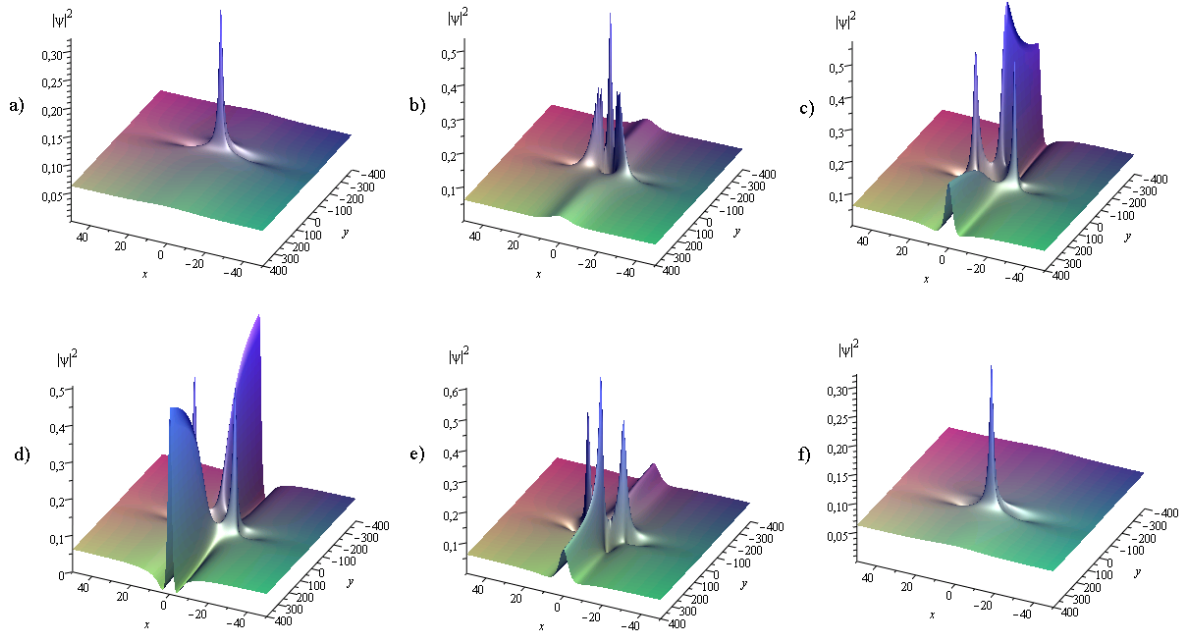


Figure 7: Interaction between a line rational breather and a rational breather of $DS1^-$ at a) $t = -50$, b) $t = -20$, c) $t = -5$, d) $t = 0$, e) $t = 10$, f) $t = 50$. The rational breather propagates in the same direction as the line breather.

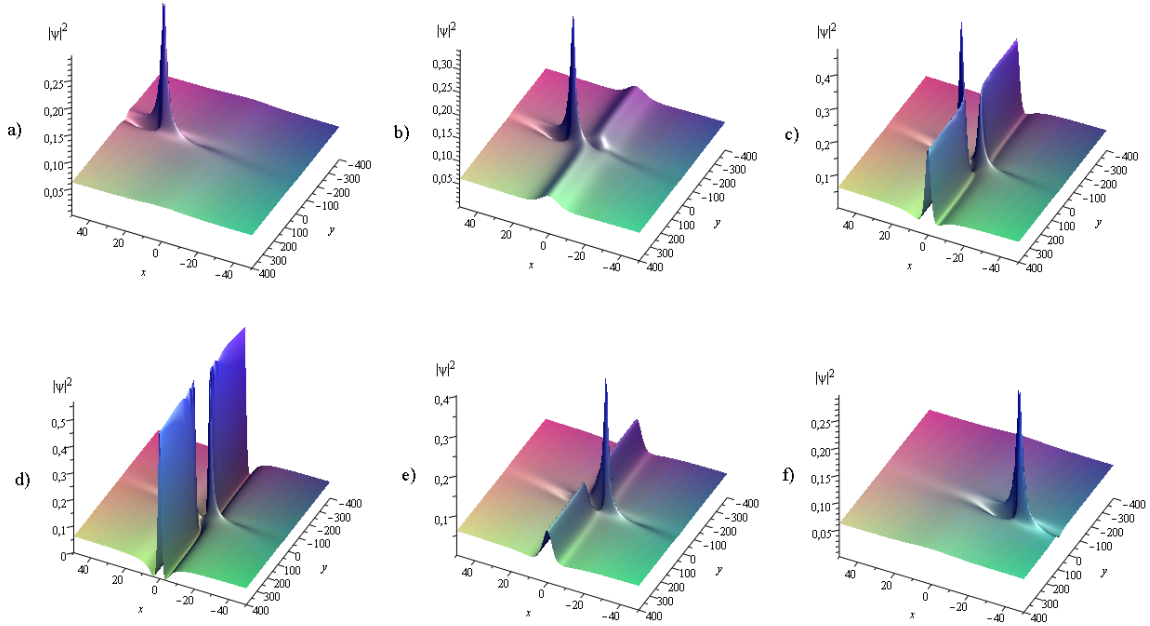


Figure 8: Interaction between a line rational breather and a rational breather of $DS1^-$ at a) $t = -50$, b) $t = -20$, c) $t = -5$, d) $t = 0$, e) $t = 10$, f) $t = 50$. The rational breather propagates transversally to the direction of the line breather.

performed solutions do not preserve their form upon interaction. For a particular choice of their spectral parameters, they recovered solutions previously derived by Boiti et al. For details on the theory of dromion solutions the reader is referred to [37] and references therein. In this section we explore how the simplest dromion solution can be derived from algebro-geometric solutions.

Let us consider solutions of the complexified system obtained in (5.7). Assume $g = 4$ and put $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$.

Degeneration. Choose a small parameter $\epsilon > 0$ and define $d_k = -\ln(\epsilon) + \hat{d}_k$ for $k = 1, \dots, 4$, and

$$\begin{aligned} w_{u_1} &= \epsilon \alpha_{u_1}, & w_{u_2} &= w_a + \epsilon \alpha_{u_2}, & w_{u_3} &= w_b + \epsilon \alpha_{u_3}, & w_{u_4} &= \epsilon \alpha_{u_4}, \\ w_{v_1} &= w_a + \epsilon \alpha_{v_1}, & w_{v_2} &= \epsilon \alpha_{v_2}, & w_{v_3} &= \epsilon \alpha_{v_3}, & w_{v_4} &= w_b + \epsilon \alpha_{v_4}. \end{aligned} \quad (5.22)$$

Moreover, put $\kappa_1 = \epsilon \hat{\kappa}_1 \alpha_{v_1}$ and $\kappa_2 = \epsilon \hat{\kappa}_2 \alpha_{u_3}$. Now consider the limit $\epsilon \rightarrow 0$ in (5.7). The functions ψ and ϕ obtained in this limit are given by (5.24).

Reality condition. Choose $w_a, w_b, h, \theta \in \mathbb{R}$ and $\hat{\kappa}_1, \hat{\kappa}_2 \in \mathbb{R} \setminus \{0\}$. Moreover, assume

$$\overline{\hat{d}_{2k}} = \hat{d}_{2k-1}, \quad \overline{\alpha_{v_{2k-1}}} = \alpha_{u_{2k}}, \quad \overline{\alpha_{v_{2k}}} = \alpha_{u_{2k-1}}, \quad k = 1, 2. \quad (5.23)$$

Put $\rho = -\text{sign}(\hat{\kappa}_1 \hat{\kappa}_2)$. With (5.23), it can be seen that the degenerated functions ψ and ψ^* obtained in the considered limit satisfy the reality condition $\psi^* = \rho \bar{\psi}$. Therefore, the following degenerated functions give the dromion solution of DS1 $^\rho$

$$\begin{aligned} \psi(\xi, \eta, t) &= \hat{A} e^{i\theta} \frac{e^{z_1 + z_3}}{\varphi(\xi, \eta, t)}, \\ \phi(\xi, \eta, t) &= \frac{1}{2} \partial_{\xi\xi} \ln \{\varphi(\xi, \eta, t)\} + \frac{1}{2} \partial_{\eta\eta} \ln \{\varphi(\xi, \eta, t)\} + \frac{h}{4}, \end{aligned} \quad (5.24)$$

where

$$\varphi(\xi, \eta, t) = 1 + A_1 e^{2\text{Re}(z_1)} + A_2 e^{2\text{Re}(z_3)} + A_3 e^{2\text{Re}(z_1) + 2\text{Re}(z_3)}.$$

Here z_k is a linear function of the variables ξ, η, t given by

$$z_1 = -i \frac{\hat{\kappa}_1}{\alpha_{v_1}} \xi - i \frac{\hat{\kappa}_1^2}{\alpha_{v_1}^2} \frac{t}{2} - \hat{d}_1, \quad z_3 = -i \frac{\hat{\kappa}_2}{\alpha_{u_3}} \eta - i \frac{\hat{\kappa}_2^2}{\alpha_{u_3}^2} \frac{t}{2} - \hat{d}_3.$$

Constants \hat{A}, A_1, A_2 and A_3 are given by

$$\begin{aligned} \hat{A} &= |\hat{\kappa}_1 \hat{\kappa}_2|^{1/2} \frac{w_a w_b}{(\alpha_{v_3} - \alpha_{u_1}) \alpha_{v_1} \alpha_{u_3}}, & A_1 &= \frac{w_a}{4 \text{Im}(\alpha_{v_1}) \text{Im}(\alpha_{u_1})}, \\ A_2 &= \frac{w_b}{4 \text{Im}(\alpha_{v_3}) \text{Im}(\alpha_{u_3})}, & A_3 &= A_1 A_2 + \frac{w_a w_b}{4 \text{Im}(\alpha_{v_1}) \text{Im}(\alpha_{u_3})} \frac{1}{|\alpha_{u_1} - \alpha_{v_3}|^2}. \end{aligned}$$

Moreover, in the case where $A_1 > 0$, $A_2 > 0$ and $A_3 > 0$, functions (5.24) are smooth solutions of DS1 $^\rho$.

Remark 5.8. i) Functions (5.24) define a family of dromion solutions of DS1^ρ depending on 6 complex parameters $\hat{d}_1, \hat{d}_3, \alpha_{u_1}, \alpha_{v_1}, \alpha_{u_3}, \alpha_{v_3}$ and 6 real parameters $w_a, w_b, \hat{\kappa}_1, \hat{\kappa}_2, h, \theta$.
ii) In the case where $\alpha_{u_1}, \alpha_{v_3} \in \mathbb{R}$, one gets localized breathers, namely, the solution oscillates with respect to the time variable (modulus of ψ is constant with respect to t).

Different degenerations can be investigated for larger values of g . The performed functions lead to particular solutions such as dromions which move along sets of straight and curved trajectories, as well as oscillating dromion solutions. We do not discuss these solutions here.

5.4.2 Lump of DS2⁻

The lump solutions were discovered in [26] for the KP1 equation, and have been extensively studied. Arkadiev et al. [6] have constructed a family of travelling waves (the lump solutions) of DS2⁻ that we rediscover here.

Let us consider functions ψ, ψ^*, ϕ given in (5.7), assume $g = 2$ and put $A = |\kappa_1 \kappa_2|^{1/2} |w_a - w_b|^{-1}$. Moreover, consider the following transformation which leaves the system (5.2) invariant:

$$\begin{aligned}\psi(\xi, \eta, t) &\rightarrow \psi(\xi + \beta_1 t, \eta + \beta_2 t, t) \exp\left\{-i\left(\beta_1 \xi + \beta_2 \eta + (\beta_1^2 + \beta_2^2) \frac{t}{2}\right)\right\}, \\ \psi^*(\xi, \eta, t) &\rightarrow \psi^*(\xi + \beta_1 t, \eta + \beta_2 t, t) \exp\left\{i\left(\beta_1 \xi + \beta_2 \eta + (\beta_1^2 + \beta_2^2) \frac{t}{2}\right)\right\}, \\ \phi(\xi, \eta, t) &\rightarrow \phi(\xi + \beta_1 t, \eta + \beta_2 t, t),\end{aligned}\tag{5.25}$$

where $\beta_i = \mu_i \kappa_i^{-1}$ for some $\mu_i \in \mathbb{C}$.

Degeneration. Choose a small parameter $\epsilon > 0$ and define $d_k = i\pi + \epsilon \hat{d}_k$, for $k = 1, 2$, and

$$\begin{aligned}w_{v_1} &= w_a + \epsilon \alpha_{v_1}, & w_{u_1} &= w_a + \epsilon \alpha_{u_1}, \\ w_{v_2} &= w_b + \epsilon \alpha_{v_2}, & w_{u_2} &= w_b + \epsilon \alpha_{u_2}.\end{aligned}$$

Moreover, put $\kappa_k = \epsilon^2 \hat{\kappa}_k$, and $\mu_k = \epsilon^2 \hat{\mu}_k$ for $k = 1, 2$. Now take the limit $\epsilon \rightarrow 0$ in (5.25). The functions ψ and ϕ obtained in this limit are given in (5.26).

Reality condition. Choose $w_a, w_b \in \mathbb{C}$ such that $\overline{w_a} = -w_b$ or $w_a, w_b \in \mathbb{R}$. Take $h, \theta \in \mathbb{R}$ and assume

$$\overline{\hat{\kappa}_1} = \hat{\kappa}_2, \quad \overline{\hat{\mu}_1} = \hat{\mu}_2, \quad \overline{\hat{d}_1} = \hat{d}_2, \quad \overline{\alpha_{v_1}} = \alpha_{v_2}, \quad \overline{\alpha_{u_1}} = \alpha_{u_2}.$$

With this choice of parameters, it can be seen that the functions ψ and ψ^* obtained in the limit considered here satisfy the reality condition $\psi^* = -\overline{\psi}$.

The solutions. Therefore, the following degenerated functions provide smooth solutions of DS2⁻

$$\begin{aligned}\psi(x, y, t) &= \frac{\hat{A} e^{i\theta}}{\hat{B} + |z_1|^2} \exp\left\{-i\left(2 \operatorname{Re}(\beta_1 \xi) + \operatorname{Re}(\beta_1^2) t\right)\right\}, \\ \phi(x, y, t) &= \frac{1}{2} \partial_{\xi \xi} \ln \left\{\hat{B} + |z_1|^2\right\} + \frac{1}{2} \partial_{\xi \xi} \ln \left\{\hat{B} + |z_1|^2\right\} + \frac{h}{4},\end{aligned}\tag{5.26}$$

where $\xi = x + iy$ and $\beta_1 = \hat{\mu}_1 \hat{\kappa}_1^{-1}$. Here $z_1 = i \hat{V}_{a,1} (\hat{\kappa}_1 \xi + \hat{\mu}_1 t) - \hat{d}_1$ and

$$\hat{V}_{a,1} = -\frac{\alpha_{u_1} - \alpha_{v_1}}{\alpha_{u_1} \alpha_{v_1}}, \quad \hat{A} = \frac{|\hat{\kappa}_1| |\alpha_{u_1} - \alpha_{v_1}|^2}{|w_a - w_b| \alpha_{v_1} \overline{\alpha_{u_1}}}, \quad \hat{B} = \frac{|\alpha_{u_1} - \alpha_{v_1}|^2}{(w_b - w_a)^2}.$$

Simplifications. To simplify (5.26), put

$$\hat{d}_1 = -\frac{i\mu}{\hat{V}_{a,1} \hat{\kappa}_1}, \quad \nu = \frac{\alpha_{u_1} \overline{\alpha_{v_1}}}{|\hat{\kappa}_1| |w_a - w_b|}, \quad \lambda = \beta_1,$$

for arbitrary $\mu \in \mathbb{C}$. In this way, functions (5.26) become

$$\begin{aligned} \psi(x, y, t) &= \nu \frac{\exp\{-2i \operatorname{Re}(\lambda \xi) - i \operatorname{Re}(\lambda^2) t + i\theta\}}{|\xi + \lambda t + \mu|^2 + |\nu|^2}, \\ \phi(x, y, t) &= \frac{1}{2} \partial_{\xi\xi} \ln \{|\xi + \lambda t + \mu|^2 + |\nu|^2\} + \frac{1}{2} \partial_{\bar{\xi}\bar{\xi}} \ln \{|\xi + \lambda t + \mu|^2 + |\nu|^2\} + \frac{h}{4}, \end{aligned} \quad (5.27)$$

where $\xi = x + iy$. Here λ, ν, μ are arbitrary complex constants, and $\theta, h \in \mathbb{R}$. Solutions (5.27) coincide with the lump solution previously obtained in [6].

6 Outlook

In this paper, various classes of solutions to the multi-component NLS equation and the DS equations in terms of elementary functions have been presented as limiting cases of algebro-geometric solutions discussed in a previous paper [22]. We did not construct all families of solutions present in the literature, but we believe that different degenerations will lead to interesting new or known solutions that are not presented here.

In particular, future investigations might address bright multi-solitons of n-NLS with inelastic collision. This novel type of inelastic collision, which is not observed in 1 + 1 dimensional soliton systems, follows from a family of bright soliton solutions having more parameters than the ones presented here with standard elastic collision. We believe that also this kind of solutions arises from algebro-geometric solutions after suitable degenerations.

I thank C. Klein who interested me in the subject, and V. Shramchenko for carefully reading the manuscript and providing valuable hints. I am grateful to D. Korotkin and V. Matveev for useful discussions and hints. This work has been supported in part by the project From-PDE funded by the European Research Council through the Advanced Investigator Grant Scheme, the Conseil Régional de Bourgogne via a FABER grant and the ANR via the program ANR-09-BLAN-0117-01.

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