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A REILLY INEQUALITY FOR THE FIRST STEKLOV EIGENVALUE

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ABSTRACT. Let M be a compact submanifold with boundary of a Euclidean space or a Sphere. In this paper, we derive an upper bound for the first non zero eigenvalue p_1 of Steklov problem on M in terms of the r-th mean curvatures of its boundary ∂M. The upper bound obtained is sharp.

1. INTRODUCTION

Let (M, g) be an m-dimensional Riemannian manifold which admits an isometric immersion ϕ in the Euclidean space $(\mathbb{R}^n, \text{can})$ and denote by H its mean curvature. In the case where M is compact and without boundary, the well-known Reilly inequality (see [29]) gives an extrinsic upper bound for the first non zero eigenvalue $\lambda_1(M)$ of the Laplacian $\Delta = -\text{tr} \circ \text{Hess}$ of (M, g) in terms of the mean curvature of its immersion ϕ:

$$(1) \qquad \lambda_1 \leq \frac{m}{\text{Vol}(M)} \int_M |H|^2 dv_g$$

where dv_g, $\text{Vol}(M)$ denote respectively the Riemannian volume element and the volume of (M, g), and $|H|^2$ denotes the square of the length of the mean curvature H of the immersion ϕ.

More precisely, if we denote by H_r the r-th mean curvatures of the immersion (see the section below for the definitions), Reilly [29] proved the following more general result

Theorem 1.1.   1) *If $n > m + 1$ and r is an odd integer, $1 \leq r \leq m$, then*

$$(2) \qquad \lambda_1 \Big(\int_M H_{r-1} dv_g \Big)^2 \leq m \, \text{Vol}(M) \Big(\int_M |H_r|^2 dv_g \Big).$$

If for some such r, we have equality in this last inequality and if H_r does not vanish identically, then ϕ immerses M minimally into some hypersphere of $\mathbb{R}^n$ and H_r is parallel in the normal bundle of M in $\mathbb{R}^n$. In particular, if $r = 1$ and we have equality, then ϕ immerses M as a minimal submanifold of some hypersphere of $\mathbb{R}^n$ (here we don't

> *need to assume that the mean curvature vector $H_1 = H$ does not vanish identically).*
>
> 2) *If $n = m + 1$ and r is any integer, $0 \leq r \leq m$, then*

$$(3) \qquad \lambda_1 \Big(\int_M H_{r-1} dv_g \Big)^2 \leq m \operatorname{Vol}(M) \Big(\int_M H_r^2 dv_g \Big).$$

> *We get equality in this last inequality for some r, $0 \leq r \leq m$, if and only if ϕ immerses M as a hypersphere in $\mathbb{R}^n$.*

For $r = 1$, the Reilly inequality above can easily be extended to immersed submanifolds of spheres and after a partial result of Heintze [21], it has been generalized to hyperbolic submanifolds by El Soufi and Ilias ([6] and [8]). The generalization to all r for spherical and hyperbolic submanifolds was obtained by Grosjean (see [16], [15] and [17]). Note that until now, there is no similar inequalities when M has boundary.

Let M be a compact Riemannian manifold with boundary ∂M. In the present article, we will be interested in the first non zero eigenvalue p_1 of the Steklov problem

$$(4) \qquad \begin{cases} \Delta f = 0 \ \text{ in } \ M \\ \dfrac{\partial f}{\partial \nu} = p \, f \ \text{ on } \ \partial M, \end{cases}$$

where ν is the outward unit normal of ∂M in M.

We observe that one can consider more generally the Steklov problem with a density ρ and the modified boundary condition $\dfrac{\partial f}{\partial \nu} = p \, \rho \, f$. But for simplicity, we consider only the case $\rho \equiv 1$. The general case can be treated easily with minor changes.

It is known that the Steklov boundary problem (4) has a discrete spectrum

$$0 = p_0 < p_1 \leq p_2 \leq \ldots \to \infty.$$

The boundary value problem (4) was first introduced by Steklov [31], in 1902, for a bounded planar domain M. His motivation came from physics. In fact, the function f represents the steady state temperature on M such that the flux on the boundary is proportional to the temperature (for other physical motivations see for instance [2]). Steklov problem is also important in harmonic analysis and inverse problems (see for instance [4]), this is because it has the same sets of eigenvalues and eigenfunctions as the well-known Dirichlet-Neumann map which associates to each function defined on the boundary ∂M, the normal derivative of its harmonic extension to M.

The first isoperimetric inequality, for the first non zero eigenvalue p_1, to appear in the literature was obtained by Weinstock [33]. He considered Steklov problem on a simply-connected plane domain M and proved that

$$(5) \qquad p_1 \leq \frac{2\pi}{L(\partial M)},$$

where $L(\partial M)$ is the length of ∂M. Since then, many authors have discussed this problem and have derived various eigenvalues estimates (see for instance [27], [1], [22], [23], [25], [30], [10], [11], [12], [13], [3], [5], [18], [19],[20] and the recent survey [19]). Unfortunately, we note that the majority of the results on this subject concerns simply connected planar domains.

Recently, Fraser and Schoen [14] extended the result of Weinstock (5) to arbitrary bounded Riemannian surfaces M of genus γ with k boundary components and obtained

$$p_1 \, L(\partial M) \leq 2(\gamma + k)\pi.$$

These last authors also considered Steklov problem on a compact m-dimensional Riemannian manifold with non-empty boundary admitting proper conformal immersions into the unit ball B^n and obtained, in the spirit of the work of Li and Yau [26] (extended by El-Soufi and Ilias [7]) concerning the first eigenvalue of Laplacian (or the Neumann Laplacian if the manifold has boundary), an upper bound for p_1 in terms of the relative n-conformal volume $V_{rc}(M, n)$ (see [14] for details) :

$$p_1 \, Vol(\partial M) \, Vol(M)^{\frac{2-m}{m}} \leq m \, V_{rc}(M, n)^{\frac{2}{m}}.$$

In this article, we will consider Steklov problem on a compact Riemannian manifold M isometrically immersed in $\mathbb{R}^n$ or in $\mathbb{S}^n$ and will derive upper bounds for p_1 in terms of the r-th mean curvatures of ∂M in $\mathbb{R}^n$ or in $\mathbb{S}^n$, similar to those for λ_1 obtained by Reilly and Grosjean. In fact, for Euclidean submanifolds, we prove the following

Theorem 1.2. *Let (M^m, g) be a compact Riemannian manifold with boundary ∂M and $\phi : (M^m, g) \longrightarrow (\mathbb{R}^n, \mathrm{can})$ an isometric immersion. Denote by H_r the r-th mean curvatures of ∂M in $\mathbb{R}^n$. We have*

    1) If $n > m$ and r is an odd integer, $1 \leq r \leq m - 1$, then

$$(6) \qquad p_1 \Big(\int_{\partial M} H_{r-1} dv_g \Big)^2 \leq m \, \mathrm{Vol}(M) \Big(\int_{\partial M} |H_r|^2 dv_g \Big).$$

    If in addition H_r does not vanish identically, then we have equality in (6) for such r if and only if ϕ immerses M minimally in $B^n(\frac{1}{p_1})$, such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ orthogonally and H_r is proportional to ϕ.

    In particular, if $r = 1$ then, we have equality in (6) if and only if ϕ is a minimal immersion of M in $B^n(\frac{1}{p_1})$ such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ minimally and orthogonally (here we don't need to assume that the mean curvature vector $H_1 = H$ of ∂M does not vanish identically).

    2) If $n = m$, let M be a bounded domain of $\mathbb{R}^m$. In this case, if r is any integer, $0 \leq r \leq m - 1$, then

$$(7) \qquad p_1 \Big(\int_{\partial M} H_{r-1} dv_g \Big)^2 \leq m \, \mathrm{Vol}(M) \Big(\int_{\partial M} H_r^2 dv_g \Big).$$

And for any r, $0 \leq r \leq m-1$, equality holds in (7) if and only if M is a ball.

We also obtain similar estimates for compact spherical submanifolds with boundary (see theorem 3.2). The case of submanifolds of a hyperbolic space, which is more difficult, remains open.

2. Prerequisites

To begin we shall briefly recall some definitions and results needed in all the sequel (for the details see [28], [29], [16] and [15]). Let (M, g) be an orientable m-dimensional Riemannian manifold isometrically immersed by ϕ in an n-dimensional Riemannian manifold (N, h) of constant sectional curvature. Before defining some important extrinsic invariants, we recall the definition of the generalized Kronecker symbols. If $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ are integers between 1 and m, then $\delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ is $+1$ or -1 according as the i's are distinct and the j's are an even or an odd permutation of the i's and is 0 in all the other cases. Let B be the second fundamental form of the immersion ϕ, which is normal-vector valued and let $(B_{ij} = B(e_i, e_j))_{i,j \leq m}$ be its matrix at a point $x \in M$ with respect to an orthonormal frame $(e_i)_{1 \leq i \leq m}$ at x. For any $r \in \{1, \ldots, m\}$, we define the r-th mean curvature of the immersion ϕ as follows (following [29])

- If r is even, the r-th mean curvature is a function H_r defined at x by,

$$H_r = \binom{m}{r}^{-1} \frac{1}{r!} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} \delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} \, h(B_{i_1j_1}, B_{i_2j_2}) \ldots h(B_{i_{r-1}j_{r-1}}, B_{i_rj_r})$$

and

- If r is odd, the r-th mean curvature is a normal vector field defined at x by,

$$H_r = \binom{m}{r}^{-1} \frac{1}{r!} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} \delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} \, h(B_{i_1j_1}, B_{i_2j_2}) \ldots h(B_{i_{r-2}j_{r-2}}, B_{i_{r-1}j_{r-1}}) \, B_{i_rj_r}$$

And by convention, we set $H_0 = 1$ and $H_{m+1} = 0$. We also observe that $H_1 = H$ is the usual mean curvature vector. If the codimension of the immersion ϕ is 1 and if M is oriented by a unit normal vector field ν, it is convenient to work with the real valued second fundamental form b (i.e $b(X,Y) = h(B(X,Y),\nu)$). Therefore, the r-th mean curvature of odd order can be defined as real valued (replace the vector field H_r by the scalar $h(H_r,\nu)$) and if we choose an orthonormal frame at x which diagonalizes b (i.e $b_x(e_i, e_j) = \mu_i \, \delta_{ij}$), we get the following expression for any

$r \in \{1, \ldots, m\}$,

$$H_r = \binom{m}{r}^{-1} \sum_{i_1 < \cdots < i_r} \mu_{i_1} \ldots \mu_{i_r}$$

where (μ_i) are the principal curvatures of the immersion ϕ at x.

For convenience, when $N = \mathbb{R}^{m+1}$, we extend the definition of H_r for $r = -1$ by setting $H_{-1} = - <\phi, \nu>$.

Remark 2.1. *All our definitions work for a general Riemannian manifold N, without any assumption about its sectional curvature.*

Let $(e_i)_{1 \leq i \leq m}$ be an orthonormal frame at $x \in M$, $(e_i^*)_{1 \leq i \leq m}$ its dual coframe and as before, (B_{ij}) the matrix of B at x with respect to (e_i). We define the following $(0,2)$-tensors T_r for $r \in \{1, \ldots, m\}$:

- If r is even, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i,i_1,\ldots,i_r \\ j,j_1,\ldots,j_r}} \delta_{j,j_1,\ldots,j_r}^{i,i_1,\ldots,i_r} \, h(B_{i_1 j_1}, B_{i_2 j_2}) \ldots h(B_{i_{r-1} j_{r-1}}, B_{i_r j_r}) \, e_i^* \otimes e_j^*$$

  and

- If r is odd, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i,i_1,\ldots,i_r \\ j,j_1,\ldots,j_r}} \delta_{j,j_1,\ldots,j_r}^{i,i_1,\ldots,i_r} \, h(B_{i_1 j_1}, B_{i_2 j_2}) \ldots h(B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}) \, B_{i_r j_r} \otimes e_i^* \otimes e_j^*.$$

By convention $T_0 = g$. As for the r-th mean curvatures, if the codimension of M is one (i.e $n = m + 1$), then we can unify these two formulas. In fact, if ν is the unit normal vector field (determined by the orientation of M) and (e_i) is an orthonormal frame at x which diagonalizes the scalar valued second fundamental form b, then the tensors T_r can be viewed as scalar valued $(0, 2)$ tensors (by replacing T_r by $h(T_r(.,.), \nu)$ if r is odd) and we have at x

$$T_r = \sum_{\substack{i_1 < \cdots < i_r \\ i_j \neq i}} \mu_{i_1} \ldots \mu_{i_r} \, e_i^* \otimes e_i^*.$$

Using Codazzi equation and the fact that the sectional curvature of N is constant, one can easily derive $\mathrm{div}_M T_r = 0$, for $r \in \{1, \ldots, m\}$ and r even if the codimension is not 1. The properties of the r-th mean curvatures and the tensors T_r can be summarized as follows (see for instance [15])

Lemma 2.1. *For any integer $r \in \{1, \ldots, m\}$, we have*

$$\mathrm{tr}(T_r) = k(r) H_r.$$

Moreover, if r is even

$$\sum_{i,j} T_r(e_i, e_j) \, B(e_i, e_j) = k(r) H_{r+1}$$

and, if r is odd

$$\sum_{i,j} h(T_r(e_i, e_j), B(e_i, e_j)) = k(r)H_{r+1}$$

where $k(r) = (m-r)\binom{m}{r}$.

As pointed out before, if the codimension is 1, we use the real valued r-th mean curvatures of odd order and if $N = \mathbb{R}^{m+1}$, we set $H_{-1} = - <\phi, \nu>$. Let us now recall the Hsiung-Minkowski formulas (the case of codimension 1 is due to Hsiung [24] and the generalization for arbitrary codimension is due to Reilly [28], [29])

Lemma 2.2. *(Hsiung-Minkowski formulas) If M is compact without boundary, then we have*

  1) *If $n > m+1$ and r is an odd integer, $1 \le r \le m$, then*

$$\int_M \left(h(\phi, H_r) + H_{r-1} \right) dv_g = 0.$$

  2) *If $n = m+1$ and r is any integer, $0 \le r \le m$, then*

$$\int_M \left(h(\phi, H_r \, \nu) + H_{r-1} \right) dv_g = 0.$$

3. THE REILLY INEQUALITY

3.1. **The Euclidean case.** We first assume that $N = \mathbb{R}^n$ endowed with its standard metric "*can*" and use the same notations as in the preceding paragraph. We consider the Steklov eigenvalue problem on a compact Riemannian manifold M immersed in the Euclidean space $\mathbb{R}^n$,

$$(8) \qquad \begin{cases} \Delta f = 0 \ \text{ in } \ M, \\ \dfrac{\partial f}{\partial \nu} = p \, f \ \text{ on } \ \partial M. \end{cases}$$

The first eigenvalue is 0 with constant eigenfunctions and the second eigenvalue p_1 has the following variational characterization

$$(9) \qquad p_1 = min_{\left\{ f \in \mathcal{C}^1(\bar{M}) \, , \, \int_{\partial M} f \, dv_g = 0 \right\}} \frac{\int_M |\nabla f|^2 dv_g}{\int_{\partial M} f^2 dv_g}$$

where dv_g is the volume element of g on M and on ∂M. In addition, for a function $f \in \mathcal{C}^1(\bar{M})$ such that $\int_{\partial M} f \, dv_g = 0$, we have $p_1 = \dfrac{\int_M |\nabla f|^2 dv_g}{\int_{\partial M} f^2 dv_g}$ if and only if the function f is an eigenfunction of the Steklov problem (4) associated to the eigenvalue p_1.
In the sequel, the n-dimensional Euclidean ball of radius r will be denoted by $B^n(r)$.
Our main result is the following

Theorem 3.1. *Let (M^m, g) be a compact Riemannian manifold with boundary, immersed isometrically by ϕ in $(\mathbb{R}^n, \mathrm{can})$. Denote by H_r the r-th mean curvatures of the boundary ∂M of M in $\mathbb{R}^n$. We have*

1) *If $n > m$ and r is an odd integer, $1 \le r \le m - 1$, then*

$$(10) \qquad p_1 \Big(\int_{\partial M} H_{r-1} dv_g \Big)^2 \le m \, \mathrm{Vol}(M) \Big(\int_{\partial M} |H_r|^2 dv_g \Big).$$

If in addition H_r does not vanish identically, then we have equality in (10) for such r if and only if ϕ immerses M minimally in $B^n(\frac{1}{p_1})$, such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ orthogonally and H_r is proportional to ϕ.

In particular, if $r = 1$ then, we have equality in (10) if and only if ϕ is a minimal immersion of M in $B^n(\frac{1}{p_1})$ such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ minimally and orthogonally (here we don't need to assume that the mean curvature vector $H_1 = H$ of ∂M does not vanish identically).

2) *If $n = m$, we limit ourselves to the case where M is a bounded domain of $\mathbb{R}^m$. In this case, if r is any integer, $0 \le r \le m - 1$, then*

$$(11) \qquad p_1 \Big(\int_{\partial M} H_{r-1} dv_g \Big)^2 \le m \, \mathrm{Vol}(M) \Big(\int_{\partial M} H_r^2 dv_g \Big).$$

And for any r, $0 \le r \le m - 1$, equality holds in (11) if and only if M is a ball.

Proof. Translating the immersion ϕ if necessary, we can suppose that $\int_{\partial M} \phi \, dv_g = 0$ (in other words, $\forall i \in \{1, \dots, n\}$, $\int_{\partial M} \phi_i \, dv_g = 0$). Applying the variational characterization (9) to the components ϕ_i of ϕ and summing on i give

$$p_1 \sum_{i=1}^n \Big\{ \int_{\partial M} \phi_i^2 \, dv_g \Big\} \le \sum_{i=1}^n \int_M |\nabla \phi_i|^2 \, dv_g = m(\mathrm{Vol(M)})$$

or

$$(12) \qquad p_1 \int_{\partial M} |\phi|^2 dv_g \le m \, (\mathrm{Vol(M)}).$$

To prove inequality (10) of the first assertion 1) , we multiply both sides of (12) by the quantity $\int_{\partial M} |H_r|^2 \, dv_g$ and use the Cauchy-Schwarz inequality to obtain

$$p_1 \Big(\int_{\partial M} < \phi, H_r > \, dv_g \Big)^2 \le p_1 \Big(\int_{\partial M} |\phi||H_r| \, dv_g \Big)^2$$

$$\le p_1 \Big(\int_{\partial M} |\phi|^2 \, dv_g \Big) \Big(\int_{\partial M} |H_r|^2 \, dv_g \Big)$$

$$(13) \qquad \le m \, \mathrm{Vol}(M) \Big(\int_{\partial M} |H_r|^2 \, dv_g \Big)$$

and we get the desired inequality by applying the Hsiung-Minkowski formula (Lemma 2.2) to ∂M considered as a compact submanifold of $\mathbb{R}^n$ (without boundary).

If $H_r = 0$ (which can not be the case for $r = 1$) , the Hsiung-Minkowski formula gives $\int_{\partial M} H_{r-1} dv_g = 0$ and equality holds trivially in inequality (10).

Now suppose that H_r does not vanish identically (this is always the case for $r = 1$). If we have equality in (10), then all the inequalities in (13) are in fact equalities. This implies that on ∂M, we have $H_r = \lambda \phi$ for some constant $\lambda \neq 0$ and all the components ϕ_i of ϕ are eigenfunctions associated to p_1. Since H_r is normal, ϕ must be also normal and thus, for any tangent vector field X on ∂M, we have $X(|\phi|^2) = 2 < \nabla_X \phi, \phi >= 0$. Therefore, $|\phi|$ and $|H_r|$ are constant on any connected component of ∂M. Thus ϕ maps the connected components of ∂M into hyperspheres of $\mathbb{R}^n$ (note that, since $\nabla_X \phi$ is tangent to M, the r-th mean curvature vector $H_r = \lambda \phi$ is in fact parallel in the normal bundle). On the other hand, since the components ϕ_i of ϕ are eigenfunctions associated to p_1, the immersion ϕ is in fact minimal and $\nu = p_1 \phi$ on ∂M. Hence, the connected components of the boundary ∂M meets orthogonally the same Euclidean hypersphere, the one of radius $\dfrac{1}{p_1}$.

Now, let $f = \sum_{i=1}^n \phi_i^2$. Since M is minimal in $\mathbb{R}^n$, a direct calculation gives $\Delta f = 2m$. Thus, we have

$$
(14) \qquad \begin{cases} \Delta f = 2m \ \text{ in } \ M, \\ f = \left(\dfrac{1}{p_1}\right)^2 \ \text{ on } \ \partial M. \end{cases}
$$

Therefore, using the maximum principle, we deduce that $f \leq \left(\frac{1}{p_1}\right)^2$ on M, which means

$$
\phi(M) \subset B^n\left(\dfrac{1}{p_1}\right).
$$

Reciprocally, the minimality of M in $\mathbb{R}^n$ and the fact that ∂M meets orthogonally the hypersphere $\partial B^n\left(\frac{1}{p_1}\right)$ means that the components ϕ_i of ϕ are eigenfunctions of the Steklov problem associated to p_1. Using, in addition, the hypothesis of proportionality of H_r and ϕ, we obtain equality in all the inequalities of (13). To be complete, we observe that for $r = 1$, by Takahashi theorem (see [32]) the proportionality of H and ϕ means that ϕ is a minimal immersion in an Euclidean hypersphere. This concludes the proof of assertion 1).

The proof of the second assertion 2) of theorem 3.1 is analogous to that of the first assertion, except for the case $r = 0$ and the equality case. For the

special case $r = 0$, it suffices to use instead of inequality (13) the following,

$$p_1 \left(\int_{\partial M} < \phi, \nu > dv_g \right)^2 \leq p_1 \left(\int_{\partial M} |\phi||\nu| dv_g \right)^2$$

$$\leq p_1 \left(\int_{\partial M} |\phi|^2 dv_g \right) \left(\int_{\partial M} |\nu|^2 dv_g \right)$$

$$(15) \qquad \leq m \, \mathrm{Vol}(M) \mathrm{Vol}(\partial M) = m \, \mathrm{Vol}(M) \left(\int_{\partial M} |H_0|^2 dv_g \right).$$

Now if the equality holds for some $r \in \{0, \ldots, m-1\}$, then $\nu = p_1 \phi$ and thus ∂M is a hypersphere of $\mathbb{R}^m$. In conclusion M is an Euclidean ball (recall that for a ball $B^m(r)$ of radius r, $p_1 = \frac{1}{r}$). Reciprocally, to verify the equality in (11) for an Euclidean ball, by homogeneity it suffices to verify it for the unit ball. For $r = 0$, it is obvious and for $1 \leq r \leq m-1$, this follows from the fact that for the unit Sphere $\mathbb{S}^{m-1}(1)$ we have $H_r = 1$, $p_1 = 1$ on the unit Ball $B^m(1)$ and $\mathrm{Vol}(\mathbb{S}^{m-1}(1)) = m \, \mathrm{Vol}(B^m(1))$. $\qquad \square$

Theorem 3.1 has two consequences that should be reported. The first one is the following classical isoperimetric upper bound for p_1

Corollary 3.1. *Let M be a bounded domain of $\mathbb{R}^m$. We have*

$$(16) \qquad p_1 \leq \frac{\mathrm{Vol}(\partial M)}{m \, \mathrm{Vol}(M)}.$$

Moreover, we have equality if and only if M is a ball.

Proof. It suffices to use the second assertion of theorem 3.1 with $r = 0$ after observing that $|\int_{\partial M} H_{-1} dv_g| = m \, \mathrm{Vol}(M)$, which follows easily from Gauss divergence formula applied to the vector field $X = \phi$ on M. $\qquad \square$

Remark 3.1. *In dimension 2 and when the domain has connected boundary, inequality (5) of Weinstock is better than inequality (16). In fact, the isoperimetric inequality gives,*

$$\frac{\mathrm{Vol}(\partial \mathrm{M})^2}{\mathrm{Vol}(\mathrm{M})} \geq 4\pi,$$

and therefore, using the Weinstock inequality we get

$$p_1 \leq \frac{2\pi}{\mathrm{Vol}(\partial \mathrm{M})} = \frac{4\pi}{2\,\mathrm{Vol}(\partial \mathrm{M})} \leq \frac{\mathrm{Vol}(\partial \mathrm{M})}{2\,\mathrm{Vol}(\mathrm{M})}$$

which is inequality (16).

For the second consequence, we will just rewrite theorem 3.1 for $r = 1$ (i.e the Reilly inequality in terms of the mean curvature)

Corollary 3.2. *Let $\phi : (M, g) \longrightarrow (\mathbb{R}^n, \mathrm{can})$ be an isometric immersion of a compact Riemannian manifold M with boundary and denote by H the mean curvature of its this boundary ∂M in $\mathbb{R}^n$. We have*

$$(17) \qquad p_1 \leq m \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\partial M)^2} \int_{\partial M} |H|^2 dv_g.$$

Furthermore,

  i) *If $n > m$, equality holds in (17) if and only if ϕ is a minimal immersion of M in $B^n(\frac{1}{p_1})$ such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ minimally and orthogonally .*

  ii) *If $n = m$ and M is a domain of $\mathbb{R}^m$, then equality holds in (17) if and only if M is a ball.*

Remark 3.2. • *A similar inequality for submanifolds of a sphere or a projective space can be easily deduced from inequality (17). In fact, using the standard embeddings of these spaces in an Euclidean space (see for instance [9] and the references therein), a straightforward calculation gives the following :*

If we denote by (N, h) the sphere $\mathbb{S}^n$, the real projective space $\mathbb{R}P^n$, the complex projective space $\mathbb{C}P^n$ or the quaternionic projective space $\mathbb{Q}P^n$ endowed with their respective standard metrics and consider an m-dimensional immersed compact Riemannian submanifold (M, g) of (N, h) having a boundary, the Steklov eigenvalue p_1 satisfies the following inequality

$$(18) \qquad p_1 \leq m \, \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\partial M)^2} \int_{\partial M} \big(|H|^2 + c(m)\big) dv_g$$

where H is the mean curvature vector of ∂M in N and

$$c(m) = \begin{cases} 1, & \text{if } N = \mathbb{S}^n \\ \dfrac{2(m+1)}{m}, & \text{if } N = \mathbb{R}P^n \\ \dfrac{2(m+2)}{m}, & \text{if } N = \mathbb{C}P^n \\ \dfrac{2(m+4)}{m}, & \text{if } N = \mathbb{Q}P^n. \end{cases}$$

• *Even for spherical submanifolds, one can not adapt easily the inequalities involving the r-th mean curvatures for $r \geq 2$.*

• *One can express inequality (17) in terms of the mean curvature $H^{\partial M}$ of ∂M in M and the mean curvature H^M of M in $\mathbb{R}^n$ by observing that the mean curvature H of ∂M in $\mathbb{R}^n$ satisfies*

$$|H|^2 = |H^{\partial M}|^2 + |H^M|^2.$$

• *The inequality (11), for $r = 2$ and $m \geq 3$ gives an estimate in terms of the scalar curvature $\mathrm{Scal}_{\partial M}$ of ∂M. Indeed, using Gauss equation we have $\mathrm{Scal}_{\partial M} = (m-1)(m-2)H_2$. Therefore, the inequality (11) becomes*

$$p_1 \Big(\int_{\partial M} H dv_g \Big)^2 \leq \frac{m}{((m-1)(m-2))^2} \, \mathrm{Vol}(M) \Big(\int_{\partial M} \mathrm{Scal}_{\partial M}^2 \, dv_g \Big).$$

3.2. **The spherical case.** In this paragraph we will show how, despite some difficulties, we succeed in extending our main theorem 3.1 obtained for Euclidean submanifolds to the spherical ones. In fact, we obtain the following

Theorem 3.2. *Let (M^m, g) be a compact Riemannian manifold with boundary immersed isometrically by ϕ in $(\mathbb{S}^n, \text{can})$. Denote by H_r the r-th mean curvatures of its boundary ∂M in $\mathbb{S}^n$, we have*

1) *If $n > m$ and r is an odd integer, $1 \le r \le m - 1$, then*

$$(19) \qquad p_1 \left(\int_{\partial M} H_{r-1} dv_g \right)^2 \le m \, \text{Vol}(M) \left(\int_{\partial M} (|H_{r-1}|^2 + |H_r|^2) dv_g \right).$$

2) *If $n = m$, we can limit ourselves to the case where M is a bounded domain of $\mathbb{S}^m$. In this case, if r is any integer, $1 \le r \le m - 1$, then*

$$(20) \qquad p_1 \left(\int_{\partial M} H_{r-1} dv_g \right)^2 \le m \, \text{Vol}(M) \left(\int_{\partial M} (H_{r-1}^2 + H_r^2) dv_g \right).$$

Moreover, equality in (19) or in (20) can never occur unless $H_r = H_{r-1} = 0$.

As observed in Remark 3.2, for $r = 1$ it is easy to deduce the Reilly inequality for spherical submanifolds from that of Euclidean ones. Unfortunately, for general r, this is not the case. However, following an idea of Grosjean [15], we will show how one can derive Theorem 3.2 from a more general result, a Reilly inequality for a general second order operator in a divergence form.

We first introduce some definitions and notations. Consider an m-dimensional Riemannian manifold (M, g) (this will be ∂M in the sequel) admitting an isometric immersion in $(\mathbb{R}^n, \text{can})$ and denote, as before, by B its second fundamental form. We endow (M, g) with a divergence-free symmetric $(0, 2)$-tensor T and define a normal vector field H_T by

$$H_T(x) = \sum_{1 \le i, j \le m} T(e_i, e_j) \, B(e_i, e_j)$$

where $(e_i)_{1 \le i \le m}$ is an orthonormal basis of $T_x M$. Let us now introduce the following second order differential operator L_T acting on $\mathcal{C}^\infty(M)$ by

$$L_T(f) = -\text{div}_M(T^\sharp \nabla^M f)$$

where ∇^M is the gradient of (M, g) and $T^\sharp$ is the symmetric $(1, 1)$-tensor associated to T with respect to g. If $(\epsilon_i)_{1 \le i \le n}$ and $(\phi_i)_{i \le n}$ denote respectively the canonical basis of $\mathbb{R}^n$ and the components of the immersion ϕ, we set $L_T \phi = \sum_{i=1}^n (L_T \phi_i) \epsilon_i$. A straightforward computation (see [15]) gives $L_T \phi = -H_T$ and $\frac{1}{2} L_T |\phi|^2 = - <\phi, H_T> -\text{tr}(T)$. This last identity, gives the following generalization of the Hsiung-Minkowski formulas for a closed submanifold M of $\mathbb{R}^n$ (see [16], [15])

$$(21) \qquad 0 = \frac{1}{2} \int_M L_T |\phi|^2 dv_g = - \int_M <\phi, H_T> dv_g - \int_M \text{tr}(T) dv_g.$$

In fact, the Hsiung-Minkowski formulas (in Lemma 2.2) can be deduced from this one by taking $T = T_r$ after using the properties of T_r given in Lemma 2.1.

Applying (21) to ∂M instead of the Hsiung-Minkowski formulas (Lemma 2.2), we obtain the following generalization of theorem 3.1

Theorem 3.3. *Let $\phi : (M^m, g) \longrightarrow (\mathbb{R}^n, \mathrm{can})$ be an isometric immersion of a compact Riemannian manifold (M, g) with boundary. If T is a divergence-free symmetric $(0, 2)$-tensor of ∂M and H_T is the normal vector field associated to T, we have*

$$(22) \qquad p_1 \Big(\int_{\partial M} (\mathrm{tr}\, T) dv_g \Big)^2 \leq m\, \mathrm{Vol}(M) \Big(\int_{\partial M} |H_T|^2 dv_g \Big).$$

Moreover, if H_T does not vanish identically, then we have

1) *If $n > m$, then equality holds in (22) if and only if ϕ immerses M minimally in $B^n(\frac{1}{p_1})$, such that $\phi(\partial M) \subset \partial B^n(\frac{1}{p_1})$ orthogonally and H_T is proportional to ϕ.*

2) *If $n = m$ and M is a bounded domain of $\mathbb{R}^n$, then the equality holds if and only if M is a ball and $\mathrm{tr}(T)$ is constant.*

Proof. To prove inequality (22), we multiply both sides of (12) by the quantity $\int_{\partial M} |H_T|^2$ and use the Cauchy-Schwarz inequality to obtain

$$p_1 \left(\int_{\partial M} <\phi, H_T> dv_g \right)^2 \leq p_1 \left(\int_{\partial M} |\phi||H_T| dv_g \right)^2$$

$$\leq p_1 \left(\int_{\partial M} |\phi|^2 dv_g \right) \left(\int_{\partial M} |H_T|^2 dv_g \right)$$

$$(23) \qquad\qquad \leq m\, \mathrm{Vol}(M) \Big(\int_{\partial M} |H_T|^2 dv_g \Big)$$

but by (21), we have

$$\int_{\partial M} <\phi, H_T> dv_g = - \int_{\partial M} \mathrm{tr}(T) dv_g.$$

Reporting this last equality in (23) we obtain the desired inequality.

As in the proof of theorem 3.1, if $H_T = 0$, then equality holds trivially in (22). Now suppose that H_T does not vanish identically, equality holds in (22) if and only if H_T is proportional to ϕ and the components ϕ_i of ϕ are eigenfunctions associated to p_1. If $n > m$, we may argue as in theorem 3.1 to conclude. In the case where M is a bounded domain of $\mathbb{R}^m$, the condition $\nu = p_1 \phi$ implies that ∂M is in fact the hypersphere of radius $\frac{1}{p_1}$ and thus $M = B^m(\frac{1}{p_1})$. On the other hand, $H_T = \mathrm{tr}(T) \nu$ and $\mathrm{tr}(T)$ is constant. Reciprocally, it suffices to verify the equality in (22) for the Euclidean unit Ball when $\mathrm{tr}(T)$ is constant. This follows from the total umbilicity of the unit hypersphere $\mathbb{S}^{m-1}(1)$ and the fact that $\mathrm{Vol}(\mathbb{S}^{m-1}(1)) = m\, \mathrm{Vol}(B^m(1))$. $\quad\square$

Now we are able to prove theorem 3.2,

Proof of theorem 3.2. We denote by B the second fundamental form of ϕ. Let i be the standard embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and denote by B' the second fundamental form of $i \circ \phi$. As above, we introduce the normal vector field H'_{T_r} associated to B', which is given at $x \in \partial M$ by

$$H'_{T_r} = \sum_{1 \leq i,j \leq m} T_r(e_i, e_j) \, B'(e_i, e_j)$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal frame at x. Therefore, if we apply theorem 3.3 with $i \circ \phi$ we obtain

$$(24) \qquad p_1 \Big(\int_{\partial M} \mathrm{tr}(T_{r-1}) dv_g \Big)^2 \leq m \, \mathrm{Vol}(M) \Big(\int_{\partial M} |H'_{T_{r-1}}|^2 dv_g \Big).$$

But, $B' = B - g \otimes \phi$ and then $H'_{T_{r-1}} = H_{T_{r-1}} - \mathrm{tr}(T_{r-1}) \, \phi$. Hence,

$$|H'_{T_{r-1}}|^2 = |H_{T_{r-1}}|^2 + \mathrm{tr}(T_{r-1})^2$$

reporting this last equality in (24), we get

$$(25) \quad p_1 \Big(\int_{\partial M} \mathrm{tr}(T_{r-1}) dv_g \Big)^2 \leq m \, \mathrm{Vol}(M) \Big(\int_{\partial M} (|H_{T_{r-1}}|^2 + \mathrm{tr}(T_{r-1})^2) dv_g \Big)$$

and finally inequality (19) follows by using Lemma 2.1 which gives $|H_{T_{r-1}}| = k(r-1) |H_r|$ and $\mathrm{tr}(T_{r-1}) = k(r-1) H_{r-1}$.

If we assume that $H_r \neq 0$ or $H_{r-1} \neq 0$ (which is equivalent to $H'_{T_{r-1}} \neq 0$), we can infer from the previous proof that if equality holds, then $i \circ \phi$ immerses M minimally in $\mathbb{R}^{n+1}$. This is impossible, in fact if we denote by H'_M and by H_M respectively the mean curvatures of $i \circ \phi$ and of ϕ, we must have $0 = |H'_M|^2 = |H_M|^2 + 1$. $\qquad \square$

Remark 3.3. *For $r = 1$, Theorem 3.2 gives inequality*

$$(26) \qquad p_1 \leq m \, \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\partial M)^2} \int_{\partial M} (|H|^2 + 1) dv_g$$

which was already observed in Remark 3.2.

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