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Maxwell's far-field methodology predicting elastic properties of multi-phase composites reinforced with aligned transversely isotropic spheroids

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ABSTRACT

Maxwell's methodology, developed to estimate the effective electrical conductivity of isotropic particulate composites, is used with a far-field elasticity result of Eshelby to derive closed-form formulae for effective transversely isotropic elastic properties of multi-phase composites comprising aligned transversely isotropic spheroidal inclusions embedded in an isotropic matrix. Very simple expressions are derived for the effective shear moduli. Closed-form analytical results for all elastic constants are shown, using exact numerical methods, to be identical to more complex results derived by Qiu and Weng on applying Mori-Tanaka theory to spheroidal reinforcements. This is a contradictory result as Maxwell's approach neglects inclusion interactions while Mori-Tanaka theory is designed, to some extent, to take such interactions into account. The rational conclusion is that inclusion interaction effects for volume fractions of practical relevance do not affect the far-field to any significant degree so that Maxwell's methodology, when combined with Eshelby's analysis, has much wider applicability than expected.

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Results for isotropic composites having distributions of spherical particles, and transversely isotropic composites having distributions of aligned fibres, correspond with known expressions, and can coincide with, or lie between, variational bounds for all volume fractions. A new simple expression having a 'mixtures' structure is obtained for the axial modulus of multi-phase fibre reinforced composites that reduces to concentric cylinders estimates when there are just two phases. To demonstrate accuracy, property results for a variety of composites are compared with accurate numerical results in the literature for two-phase composites having reinforcement volume fractions in the range 0 to 0.7.

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Keywords: Maxwell methodology, Eshelby analysis, elastic properties, particulate composites, spheroidal inclusions, multi-phase.

1. Introduction

This paper, which is a tribute to Professor Anthony Kelly, CBE, FRS, marking his 80th birthday, revisits a classical problem of composite science by presenting a new analysis that indicates how effective elastic properties can be estimated approximately for isotropic matrices that are reinforced by aligned multi-phase transversely isotropic spheroids which all have the same aspect ratio. Discussion of key issues and formulae will, in effect, review many results that have already been given in the literature.

The development of methods to estimate the effective properties of multi-phase composites reinforced by spheroidal reinforcements has a long history. For example, Mori-Tanaka theory [1], estimating the average stress and elastic energy for composites reinforced with mis-fitting inclusions, has been applied by Weng [2] to develop an approximate method of estimating effective elastic properties of general multi-phase anisotropic composites reinforced with arbitrarily oriented anisotropic inclusions. Explicit formulae for effective elastic properties were derived for suspensions of uniformly distributed, multi-phase isotropic spherical particles in an isotropic matrix. Norris [3] emphasised the relationship of effective properties, estimated using Mori-Tanaka theory, to general bounds, showing that predicted properties for two-phase composites always satisfy the Hashin-Shtrikman bounds [4], a result that does not generalise to multi-phase composites so that caution should be used in this case. Weng [5] reformulated Mori-Tanaka theory so that it is recast into a form that has an identical structure to that used when deriving the Hashin-Shtrikman bounds, and he uses a notation based on the treatment of fourth order tensors for elastic constants developed and used by Walpole [6]-[9]. Qiu and Weng [10] apply the modified Mori-Tanaka theory to composites having transversely isotropic spheroidal inclusions and derive explicit but complex formulae that can be used to estimate all elastic properties. Benveniste et al [11] investigate the diagonal and elastic symmetry of the fourth order effective elastic property tensor for heterogeneous media and show that the Mori-Tanaka and self-consistent methods lead to diagonal and symmetric property tensors for all two-phase composites. If, however, all the inclusions have a similar shape and are aligned, then they show that the symmetry properties apply also to multi-phase composites. Chen et al [12] have applied the Mori-Tanaka method to estimate explicit formulae for the effective elastic properties of composites reinforced with aligned or randomly oriented, transversely isotropic fibres or platelets, and for fibrous composites reinforced with cylindrical orthotropic fibres. For the general case of transversely isotropic spheroidal inclusions embedded in an isotropic matrix, the effective elastic properties presented as formulae for fourth order tensors are very difficult to interpret in terms of the five independent elastic constants, as shown by Qiu and Weng [10]. This has meant that the practical application in an engineering context of the valuable results has been difficult.

As a result of collaborating with Professor Anthony Kelly in recent years, regarding methods of estimating the effective thermo-elastic constants and conductivities of composite materials, the author was introduced to the pioneering work of James Clerk Maxwell [13] who provided an ingenious method of estimating the effective electrical conductivity of a cluster of spherical particles, having the same size and embedded in an infinite medium, by considering the effect of the cluster on the far-field, when the system is subject to a uniform electrical field. Maxwell modestly asserted that the sizes and distribution of the particles must be such that particle interaction effects may be neglected, and he infers that his result will be valid only for small volume fractions of reinforcing particles. A result for effective permittivity that is analogous to Maxwell's result for electrical conductivity is known as the 'Maxwell-

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Garnett mixing formula', and it has a microscopic analogue that is known as the 'Clausius-Mossotti' (or 'Lorentz-Lorenz') formula, which has been related to effective elastic property estimation by Felderhof and Iske [14] and Cohen and Bergman [15], [16].

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More recently McCartney and Kelly [17], studied the method used by Maxwell and demonstrated that his methodology, focusing only on the far-field, can also be applied to the estimation of other properties of composite materials. The principal objective was to show how Maxwell's methodology could be used to estimate explicit formulae for the effective bulk modulus, shear modulus and thermal expansion coefficient of multi-phase isotropic composites reinforced with homogeneous spherical particles. The methodology of Maxwell was naturally extended so that assemblies of multi-phase spherical particles having a range of radii and/or properties could be considered. A second objective was to show that Maxwell's methodology is one reliable technique that provides closed-form estimates of effective properties and is not necessarily restricted to low volume fractions of particulate reinforcement as has often been claimed in the literature.

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More recent unpublished work has shown that, when Maxwell's methodology is applied to clusters of aligned transversely isotropic cylindrical fibres of different types, embedded in an infinite isotropic matrix material, a similar situation arises to that described in [17]. For the fibre case, the effective thermo-elastic properties of the composite are transversely isotropic but it is not known how to estimate the axial Young's modulus and axial thermal expansion coefficient for multi-phase fibre reinforced composites. This leads on to the idea of considering clusters of aligned spheroidal inclusions having various sizes and properties that can represent aligned short fibres and particulate composites of various types, and of making use of a classical analysis due to Eshelby [18], [19]. He considers the elastic field for isolated ellipsoidal inclusions (both isotropic and anisotropic) that are embedded in an infinite isotropic matrix and subjected to loading that would in a homogenous material lead to uniform stress and strain fields. Of particular relevance is an expression for the far-field displacement field that provides a very convenient method of extending Maxwell's methodology for spheres to the case of aligned transversely isotropic spheroids and fibres embedded in an isotropic matrix, the investigation of which is the principal objective here. It is noted that Torquato [20] has observed: i) this correspondence for the case of isotropic ellipsoidal inclusions when using a formulation based on the fourth order elastic constants described above, and ii) that the Mori and Tanaka results, and hence those in this paper based on Maxwell's methodology, coincide with one of the bounds due to Willis [21] depending on whether the matrix is stiffer or more compliant than all of the inclusions, as shown by Weng [22].

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This paper first steps back in time combining the pioneering work of both Maxwell [13] and Eshelby [18], to develop a new method of estimating explicit closed-form formulae for the effective elastic properties of composites reinforced by aligned transversely isotropic spheroidal inclusions embedded in an isotropic matrix. The new approach will be shown to generate very simple expressions for the shear moduli, and lead to a new result for the effective axial Young's modulus for a multi-phase composite reinforced with aligned fibres. The approach will be shown capable of generating many of the well-known results for effective properties that have been derived in the literature using a variety of other methods. It is thought that the new method of estimating effective properties will enable engineers to understand more readily, and calculate more efficiently, the transversely isotropic effective properties of composites reinforced with aligned spheroidal inclusions.

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The paper is organised so that a general description is first given in Section 2 of the application of Maxwell's methodology to a uniform distribution of aligned spheroidal inclusions, and in Section 3 relations are given defining the transformation tensor for a transversely isotropic inclusion. The far-field relations for the displacement field derived by Eshelby [18] are then given in Section 4. The effective shear properties of the composite are derived in Section 5, and the corresponding non-shear effective properties are derived in Sections 6-8. The resulting effective properties for spherical and fibre reinforcements are given in Sections 9 and 10 respectively. A comparison is made in Section 10 between the predicted effective properties of two-phase composites and various accurate numerical results that are available in the literature. Results obtained are discussed in Section 11 and conclusions are drawn in Section 12.

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2. General description of Maxwell's methodology applied to spheroidal inclusions

The following description is based on Maxwell's [13] far-field approach, when estimating the electrical conductivity of a cluster of isotropic spherical particles embedded in an infinite isotropic matrix. The methodology is generalised here so that multi-phase composites having aligned transversely isotropic spheroidal inclusions of various types can be considered.

2.1 Description of geometry

In a well-mixed cluster of N types of aligned spheroidal reinforcement embedded in and perfectly bonded to an infinite isotropic matrix, there are n_i spheroidal inclusions having major axes a_i and minor axes b_i , $i = 1 \dots N$. The centres of the spheroids representing the inclusions are assumed to be homogeneously and isotropically distributed within the cluster. Inclusion properties of type i , which may differ from those of other types, are denoted by a subscript or superscript i , and they are assumed to be transversely isotropic with the principal direction aligned with the major axes of the spheroids. The cluster of all inclusion types may be just enclosed by a spheroid of major axis \bar{a} and minor axes \bar{b} having the same alignment as the inclusions. The homogeneous inclusion distribution leads to transversely isotropic effective properties of the composite formed by the cluster of aligned transversely isotropic spheroidal inclusions and isotropic matrix lying within this enclosing spheroid. The volume fractions of inclusions of type i within the enclosing spheroid of radii \bar{a} and \bar{b} are given by

$$V_p^i = \frac{n_i a_i b_i^2}{\bar{a} \bar{b}^2}, \quad i = 1, \dots, N, \quad \text{such that} \quad V_m + \sum_{i=1}^N V_p^i = 1, \quad (2.1)$$

where V_m is the volume fraction of matrix. For just one type of inclusion, as shown in Figure 1, with n inclusions having major axis a and minor axis b , the particulate volume fraction V_p is such that

$$V_p = \frac{n a b^2}{\bar{a} \bar{b}^2} = 1 - V_m. \quad (2.2)$$

Whatever the nature and arrangement of the aligned spheroidal transversely isotropic inclusions in the cluster, Maxwell's methodology considers the far-field when replacing the discrete particulate composite, that can be enclosed by a spheroid having axes \bar{a} and \bar{b} , by a homogeneous effective composite spheroid having the same axes \bar{a} and \bar{b} embedded in the matrix. There is no restriction on sizes, properties and locations of inclusions provided that

the equivalent effective medium is homogeneous and transversely isotropic. Composites having statistical distributions of both inclusion size and properties can clearly be analysed, but they must all be aligned in the same direction.

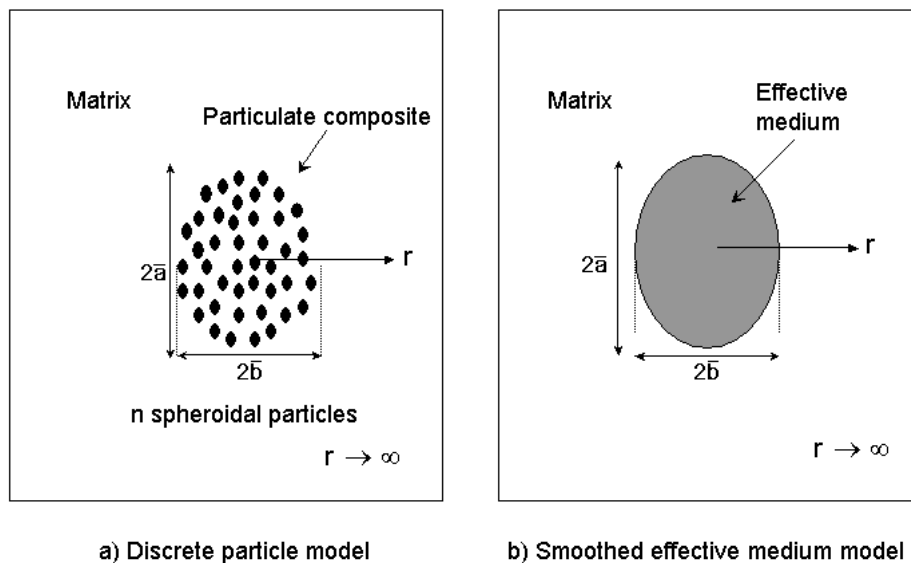


Figure 1: Diagrams illustrating (a) discrete model and (b) effective medium model of a particulate composite having aligned spheroidal reinforcements embedded in an infinite isotropic matrix material.

2.2 Maxwell's methodology for estimating elastic constants

The first step considers the effect of embedding in the infinite matrix, an isolated cluster of aligned spheroidal transversely isotropic inclusions of different types that can be just contained within the spheroid having the same alignment. At infinity a stress is applied that would lead in matrix material alone to uniform stress and strain fields. For a single isolated inclusion embedded in an infinite matrix, the matrix displacement distribution is perturbed by the presence of the inclusion, and the perturbation from the uniform strain field depends on inclusion geometry and properties. The analysis of Eshelby [18], [19] enables the perturbations of the matrix displacement field to be determined at large distances from the inclusion. According to Maxwell's methodology (see [17]), the perturbing effect in the matrix at large distances from all the inclusions in the cluster is estimated by superimposing the perturbations caused by each inclusion, regarded as being isolated. The second step recognises that, at very large distances from the cluster, all the inclusions can be considered to be located at the origin that is chosen to be situated at the centre of one of the inclusions in the cluster. The third step replaces the composite having discrete inclusions lying within the bounding spheroid by the homogeneous spheroidal effective medium having axes \bar{a} and \bar{b} , and having the transversely isotropic effective elastic properties of the composite.

Maxwell's methodology [13] will now be combined with that of Eshelby [18], [19] so that a method can be developed that enables the estimation of the transversely isotropic effective elastic properties of a multi-phase particulate composite reinforced with aligned spheroidal transversely isotropic inclusions having different sizes. An additional assumption

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is now made where it is assumed that the aspect ratios a_i/b_i , $i = 1 \dots N$, of the spheroidal inclusions, and that of the enclosing spheroid \bar{a}/\bar{b} , all have the same value. The principal reasons for this assumption are that it is useful to simplify the analysis that will be rather complex, and as it is one objective of this paper to apply the methodology to aligned cylindrical fibres, these can be enclosed only by a surface having a similar geometry.

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3. Isolated spheroidal inclusion

Eshelby [18] considered the elastic field outside an isolated ellipsoidal inclusion embedded in and perfectly bonded to an infinite matrix for the case where the inclusion and the matrix are isotropic materials. Eshelby [18] stated how the case of an anisotropic inclusion in an isotropic matrix could be solved. The application of Maxwell's methodology to a system of aligned spheroidal inclusions embedded in an infinite isotropic matrix will be considered in this paper, which requires a knowledge of the stress or displacement distribution at large distances from a transversely isotropic spheroidal inclusion.

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From Eshelby [18] the relation determining the transformation strain tensor ϵ_{ij}^T within the inclusion in terms of the strain tensor ϵ_{ij}^A applied to the matrix at infinity is

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$$C_{ijkl}(\epsilon_{kl}^C + \epsilon_{kl}^A) = \lambda(\epsilon_{kk}^C + \epsilon_{kk}^A - \epsilon_{kk}^T) \delta_{ij} + 2\mu(\epsilon_{ij}^C + \epsilon_{ij}^A - \epsilon_{ij}^T), \quad (3.1)$$

where C_{ijkl} are the anisotropic elastic constants for the inclusion, λ and μ (the shear modulus) are Lamé's constants for the isotropic matrix, and where ϵ_{ij}^C is the 'constrained strain' within the inclusion when it transforms while embedded in the matrix. From Eshelby [18],

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$$\epsilon_{ij}^C = S_{ijkl} \epsilon_{kl}^T, \quad (3.2)$$

where the Eshelby tensor S_{ijkl} has dimensionless components depending only on Poisson's ratio of the isotropic matrix and the aspect ratio of the ellipsoid. The elastic constants C_{ijkl} are such that $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$. The substitution of (3.2) in (3.1) leads to

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$$C_{ijkl}(S_{klmn} \epsilon_{mn}^T + \epsilon_{kl}^A) = \lambda(S_{kkmn} \epsilon_{mn}^T - \epsilon_{kk}^T + \epsilon_{kk}^A) \delta_{ij} + 2\mu(S_{ijkl} \epsilon_{kl}^T - \epsilon_{ij}^T + \epsilon_{ij}^A). \quad (3.3)$$

For a transversely isotropic solid, where the axial direction corresponds to the direction of the x_1 -axis, the stress-strain relations defining the elastic coefficients C_{ijkl} for the inclusion have the explicit form

$$\sigma_{11} = (E_A + 4k_T v_A^2) \epsilon_{11} + 2v_A k_T \epsilon_{22} + 2v_A k_T \epsilon_{33}, \quad (3.4)$$

$$\sigma_{22} = 2v_A k_T \epsilon_{11} + (k_T + \mu_T) \epsilon_{22} + (k_T - \mu_T) \epsilon_{33}, \quad (3.5)$$

$$\sigma_{33} = 2v_A k_T \epsilon_{11} + (k_T - \mu_T) \epsilon_{22} + (k_T + \mu_T) \epsilon_{33}, \quad (3.6)$$

$$\sigma_{12} = 2\mu_A \epsilon_{12}, \quad \sigma_{13} = 2\mu_A \epsilon_{13}, \quad \sigma_{23} = 2\mu_T \epsilon_{23}, \quad (3.7)$$

where E_A is the axial Young's modulus, ν_A is the axial Poisson's ratio, k_T is the plane strain bulk modulus, and where μ_A and μ_T are respectively the axial and transverse shear moduli. The corresponding transverse Young's modulus E_T and transverse Poisson's ratio ν_T are obtained using the following relations

$$\frac{4}{E_T} = \frac{1}{k_T} + \frac{1}{\mu_T} + \frac{4\nu_A^2}{E_A}, \quad \nu_T = \frac{E_T}{2\mu_T} - 1. \quad (3.8)$$

The stress-strain relations (3.4)-(3.7) are used to characterise both the reinforcing inclusions and the effective medium representing the composite. For the isotropic matrix the elastic constants are the Young's modulus E , Poisson's ratio ν , the shear modulus μ and the bulk modulus k , which satisfy the relations

$$E = 2\mu(1+\nu), \quad k = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}. \quad (3.9)$$

It is assumed that the major axes of the various spheroids in the composite are aligned with the x_1 axis. It should be noted that for spheroidal inclusions the tensor S_{ijkl} are such that

$$\begin{aligned} S_{2222} = S_{3333}, \quad S_{2211} = S_{3311}, \quad S_{1212} = S_{2121} = S_{1313} = S_{3131}, \quad S_{2323} = S_{3232}, \\ S_{2222} - S_{2233} = 2S_{2323}, \quad S_{2211} - S_{1122} = \nu(S_{2222} + S_{2233} - S_{1111} - S_{1122}). \end{aligned} \quad (3.10)$$

Expanding (3.3) using (3.4)-(3.7) leads to the following six linear equations that determine the components of the strain tensor ε_{ij}^T

$$\begin{aligned} & \left[(E_A + 4k_T\nu_A^2 - \lambda - 2\mu)S_{1111} + (2\nu_A k_T - \lambda)S_{2211} + (2\nu_A k_T - \lambda)S_{3311} \right] \varepsilon_{11}^T \\ & + \left[(E_A + 4k_T\nu_A^2 - \lambda - 2\mu)S_{1122} + (2\nu_A k_T - \lambda)S_{2222} + (2\nu_A k_T - \lambda)S_{3322} \right] \varepsilon_{22}^T \\ & + \left[(E_A + 4k_T\nu_A^2 - \lambda - 2\mu)S_{1133} + (2\nu_A k_T - \lambda)S_{2233} + (2\nu_A k_T - \lambda)S_{3333} \right] \varepsilon_{33}^T \\ & + \lambda(\varepsilon_{11}^T + \varepsilon_{22}^T + \varepsilon_{33}^T) + 2\mu\varepsilon_{11}^T \\ & = - (E_A + 4k_T\nu_A^2 - \lambda - 2\mu)\varepsilon_{11}^A - (2\nu_A k_T - \lambda)\varepsilon_{22}^A - (2\nu_A k_T - \lambda)\varepsilon_{33}^A, \end{aligned} \quad (3.11)$$

$$\begin{aligned} & \left[(2\nu_A k_T - \lambda)S_{1111} + (k_T + \mu_T - \lambda - 2\mu)S_{2211} + (k_T - \mu_T - \lambda)S_{3311} \right] \varepsilon_{11}^T \\ & + \left[(2\nu_A k_T - \lambda)S_{1122} + (k_T + \mu_T - \lambda - 2\mu)S_{2222} + (k_T - \mu_T - \lambda)S_{3322} \right] \varepsilon_{22}^T \\ & + \left[(2\nu_A k_T - \lambda)S_{1133} + (k_T + \mu_T - \lambda - 2\mu)S_{2233} + (k_T - \mu_T - \lambda)S_{3333} \right] \varepsilon_{33}^T \\ & + \lambda(\varepsilon_{11}^T + \varepsilon_{22}^T + \varepsilon_{33}^T) + 2\mu\varepsilon_{22}^T \\ & = - (2\nu_A k_T - \lambda)\varepsilon_{11}^A - (k_T + \mu_T - \lambda - 2\mu)\varepsilon_{22}^A - (k_T - \mu_T - \lambda)\varepsilon_{33}^A, \end{aligned} \quad (3.12)$$

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$$\begin{aligned}
& \left[(2\nu_A k_T - \lambda) S_{1111} + (k_T - \mu_T - \lambda) S_{2211} + (k_T + \mu_T - \lambda - 2\mu) S_{3311} \right] \varepsilon_{11}^T \\
& + \left[(2\nu_A k_T - \lambda) S_{1122} + (k_T - \mu_T - \lambda) S_{2222} + (k_T + \mu_T - \lambda - 2\mu) S_{3322} \right] \varepsilon_{22}^T \\
& + \left[(2\nu_A k_T - \lambda) S_{1133} + (k_T - \mu_T - \lambda) S_{2233} + (k_T + \mu_T - \lambda - 2\mu) S_{3333} \right] \varepsilon_{33}^T \\
& + \lambda (\varepsilon_{11}^T + \varepsilon_{22}^T + \varepsilon_{33}^T) + 2\mu \varepsilon_{33}^T \\
& = - (2\nu_A k_T - \lambda) \varepsilon_{11}^A - (k_T - \mu_T - \lambda) \varepsilon_{22}^A - (k_T + \mu_T - \lambda - 2\mu) \varepsilon_{33}^A,
\end{aligned} \tag{3.13}$$

$$\varepsilon_{12}^T = \frac{\varepsilon_{12}^A}{\frac{\mu}{\mu - \mu_A} - 2S_{1212}}, \quad \varepsilon_{13}^T = \frac{\varepsilon_{13}^A}{\frac{\mu}{\mu - \mu_A} - 2S_{1313}}, \quad \varepsilon_{23}^T = \frac{\varepsilon_{23}^A}{\frac{\mu}{\mu - \mu_T} - 2S_{2323}}. \tag{3.14}$$

4. Far-field displacement distribution

Consider an isolated ellipsoidal inclusion having axes a, b, c perfectly bonded to an infinite isotropic matrix having elastic properties λ, μ . Eshelby [18] derived the following expression for the perturbation displacement field at large distances from the ellipsoidal inclusion

$$u_i^C = \frac{abc}{6(1-\nu)} \left[(1-2\nu) (\varepsilon_{ik}^T l_k + \varepsilon_{ki}^T l_k - \varepsilon_{kk}^T l_i) + 3\varepsilon_{jk}^T l_i l_j l_k \right] \frac{1}{r^2}, \tag{4.1}$$

where $\nu = \frac{1}{2} \lambda / (\lambda + \mu)$ is Poisson's ratio for the matrix and where l_i are direction cosines of the point $\mathbf{r} = (x_1, x_2, x_3)$ relative to the origin of spherical coordinates (r, θ, ϕ) defined by

$$l_1 = \sin \theta \cos \phi, \quad l_2 = \sin \theta \sin \phi, \quad l_3 = \cos \theta. \tag{4.2}$$

Eshelby [18] justified the use of (4.1) for the case of anisotropic inclusions embedded in an isotropic matrix. For the case where $\varepsilon_{11}^T, \varepsilon_{22}^T, \varepsilon_{33}^T$ are the only non-zero strains, (4.1) may be written, for $k = 1, 2, 3$, (no summation over repeated suffices)

$$u_k^C = \frac{abc}{6(1-\nu)} \left[(1-2\nu) (2\varepsilon_{kk}^T - \varepsilon_{11}^T - \varepsilon_{22}^T - \varepsilon_{33}^T) + 3\varepsilon_{11}^T l_1^2 + 3\varepsilon_{22}^T l_2^2 + 3\varepsilon_{33}^T l_3^2 \right] \frac{l_k}{r^2}. \tag{4.3}$$

For the case when ε_{12}^T is the only non-zero strain, (4.1) may be written

$$u_1^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_1^2] \varepsilon_{12}^T \frac{l_2}{r^2}, \quad u_2^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_2^2] \varepsilon_{12}^T \frac{l_1}{r^2}, \quad u_3^C = \frac{abc}{1-\nu} \varepsilon_{12}^T \frac{l_1 l_2 l_3}{r^2}. \tag{4.4}$$

For the case when ε_{13}^T is the only non-zero strain

$$u_1^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_1^2] \varepsilon_{13}^T \frac{l_3}{r^2}, \quad u_2^C = \frac{abc}{1-\nu} \varepsilon_{13}^T \frac{l_1 l_2 l_3}{r^2}, \quad u_3^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_3^2] \varepsilon_{13}^T \frac{l_1}{r^2}, \tag{4.5}$$

and for the case when ε_{23}^T is the only non-zero strain

$$u_1^C = \frac{abc}{1-\nu} \varepsilon_{23}^T \frac{l_1 l_2 l_3}{r^2}, u_2^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_1^2] \varepsilon_{23}^T \frac{l_3}{r^2}, u_3^C = \frac{abc}{3(1-\nu)} [1-2\nu+3l_3^2] \varepsilon_{23}^T \frac{l_2}{r^2}. \quad (4.6)$$

The subsequent analysis in this paper assumes that $b = c$ so that the inclusions are spheroidal. Also, matrix properties will be denoted by a suffix m as there will not now be confusion with the tensor notation.

5. Estimating shear properties

On combining (3.14)₁ with (4.4), it follows that, for a strain field ε_{12}^A applied to a single transversely isotropic spheroid of type i embedded in an infinite isotropic matrix, the far-field displacement distribution has the form

$$u_1^C = \frac{a_i b_i^2}{3(1-\nu_m)} [1-2\nu_m+3l_1^2] \frac{\varepsilon_{12}^A}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}} \frac{l_2}{r^2}, \quad (5.1)$$

$$u_2^C = \frac{a_i b_i^2}{3(1-\nu_m)} [1-2\nu_m+3l_2^2] \frac{\varepsilon_{12}^A}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}} \frac{l_1}{r^2}, \quad (5.2)$$

$$u_3^C = \frac{a_i b_i^2}{1-\nu_m} \frac{\varepsilon_{12}^A}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}} \frac{l_1 l_2 l_3}{r^2}. \quad (5.3)$$

As it is assumed that the aspect ratios of all types of spheroid in the cluster are identical and equal to that for the enclosing spheroid, Maxwell's methodology asserts that the far-field displacement distribution for a cluster of N spheroidal inclusions is given by

$$u_1^C = \frac{1-2\nu_m+3l_1^2}{3(1-\nu_m)} \frac{l_2}{r^2} \varepsilon_{12}^A \sum_{i=1}^N \frac{n_i a_i b_i^2}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}}, \quad (5.4)$$

$$u_2^C = \frac{1-2\nu_m+3l_2^2}{3(1-\nu_m)} \frac{l_1}{r^2} \varepsilon_{12}^A \sum_{i=1}^N \frac{n_i a_i b_i^2}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}}, \quad (5.5)$$

$$u_3^C = \frac{\varepsilon_{12}^A}{1-\nu_m} \frac{l_1 l_2 l_3}{r^2} \sum_{i=1}^N \frac{n_i a_i b_i^2}{\frac{\mu_m}{\mu_m - \mu_A^i} - 2S_{1212}}. \quad (5.6)$$

When the equivalent single spheroidal transversely isotropic inclusion representing the cluster of aligned spheroids in the matrix is subject to the same applied strain field, the far-field displacement distribution will have the form

$$u_1^C = \frac{1-2\nu_m+3l_1^2}{3(1-\nu_m)} \frac{l_2}{r^2} \varepsilon_{12}^A \frac{\bar{a} \bar{b}^2}{\frac{\mu_m}{\mu_m - \mu_A^{\text{eff}}} - 2S_{1212}}, \quad (5.7)$$

$$u_2^C = \frac{1-2\nu_m + 3l_2^2}{3(1-\nu_m)} \frac{l_1}{r^2} \varepsilon_{12}^A \frac{\bar{a} \bar{b}^2}{\frac{\mu_m}{\mu_m - \mu_A^{\text{eff}}} - 2S_{1212}}, \quad (5.8)$$

$$u_3^C = \frac{\varepsilon_{12}^A}{1-\nu_m} \frac{l_1 l_2 l_3}{r^2} \frac{\bar{a} \bar{b}^2}{\frac{\mu_m}{\mu_m - \mu_A^{\text{eff}}} - 2S_{1212}}. \quad (5.9)$$

The application of Maxwell's methodology demands that the far-field displacements defined by (5.4)-(5.6) and (5.7)-(5.9) are identical, and this leads to the following simple relationship that can be used to estimate the effective *axial* modulus μ_A^{eff} of the composite

$$\frac{1}{\frac{\mu_m}{\mu_A^{\text{eff}}} - \mu_m + 2S_{1212}} = \sum_{i=1}^N \frac{V_p^i}{\frac{\mu_m}{\mu_A^i} + 2S_{1212}}, \quad (5.10)$$

where the inclusion volume fractions V_p^i have been introduced using (2.1). As $S_{1212} = S_{1313}$, it follows that (5.10) can be obtained also from the relations (3.14)₂ and (4.5) which are relevant when the applied strain field is given by ε_{13}^A . However, on using (3.14)₃ in conjunction with (4.6), it can be shown that, when the applied strain field is given by ε_{23}^A , the effective *transverse* shear modulus μ_T^{eff} may be found from the simple relation

$$\frac{1}{\frac{\mu_m}{\mu_T^{\text{eff}}} - \mu_m + 2S_{2323}} = \sum_{i=1}^N \frac{V_p^i}{\frac{\mu_m}{\mu_T^i} + 2S_{2323}}. \quad (5.11)$$

It can be shown, on using (2.1), that the results (5.10) and (5.11) may be also be expressed as the following simple 'mixtures' relationships

$$\frac{1}{\mu_A^{\text{eff}} + \mu_A^*} = \sum_{i=1}^N \frac{V_p^i}{\mu_A^i + \mu_A^*} + \frac{V_m}{\mu_m + \mu_A^*}, \quad \text{where } \mu_A^* = \left(\frac{1}{2S_{1212}} - 1 \right) \mu_m, \quad (5.12)$$

$$\frac{1}{\mu_T^{\text{eff}} + \mu_T^*} = \sum_{i=1}^N \frac{V_p^i}{\mu_T^i + \mu_T^*} + \frac{V_m}{\mu_m + \mu_T^*}, \quad \text{where } \mu_T^* = \left(\frac{1}{2S_{2323}} - 1 \right) \mu_m. \quad (5.13)$$

6. Far-field solution for non-shear case

For a strain field ε_{11}^A , ε_{22}^A , ε_{33}^A applied to a single transversely isotropic spheroidal inclusion of type i embedded in an infinite matrix, it follows from (4.1) that the far-field displacement distribution has the form, for $k = 1, 2, 3$, (no summation over repeated suffices)

$$u_k^C = \frac{a_i b_i^2}{6(1-\nu_m)} \left[(1-2\nu_m) \left(2\varepsilon_{kk}^{\text{T}(i)} - \varepsilon_{11}^{\text{T}(i)} - \varepsilon_{22}^{\text{T}(i)} - \varepsilon_{33}^{\text{T}(i)} \right) + 3\varepsilon_{11}^{\text{T}(i)} l_1^2 + 3\varepsilon_{22}^{\text{T}(i)} l_2^2 + 3\varepsilon_{33}^{\text{T}(i)} l_3^2 \right] \frac{l_k}{r^2}. \quad (6.1)$$

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Maxwell's methodology asserts that the far-field displacement field for the various sets of spheroidal inclusions in the cluster are then given by, for $k = 1, 2, 3$,

$$u_k^C = \frac{l_k}{6(1-\nu_m)r^2} \sum_{i=1}^N a_i b_i^2 \left[\begin{array}{l} (1-2\nu_m) \left(2\varepsilon_{kk}^{T(i)} - \varepsilon_{11}^{T(i)} - \varepsilon_{22}^{T(i)} - \varepsilon_{33}^{T(i)} \right) \\ + 3\varepsilon_{11}^{T(i)} l_1^2 + 3\varepsilon_{22}^{T(i)} l_2^2 + 3\varepsilon_{33}^{T(i)} l_3^2 \end{array} \right]. \quad (6.2)$$

When the equivalent single spheroidal transversely isotropic inclusion representing the cluster of aligned spheroids is subject to the same applied strain field, the far-field displacement distribution will have the form, for $k = 1, 2, 3$,

$$u_k^C = \frac{l_k \bar{a} \bar{b}^2}{6(1-\nu_m)r^2} \left[\begin{array}{l} (1-2\nu_m) \left(2\bar{\varepsilon}_{kk}^T - \bar{\varepsilon}_{11}^T - \bar{\varepsilon}_{22}^T - \bar{\varepsilon}_{33}^T \right) \\ + 3\bar{\varepsilon}_{11}^T l_1^2 + 3\bar{\varepsilon}_{22}^T l_2^2 + 3\bar{\varepsilon}_{33}^T l_3^2 \end{array} \right]. \quad (6.3)$$

The application of Maxwell's methodology demands that the far-field displacements defined by (6.2) and (6.3) are identical, and this leads to the following relationships that will be used to estimate the effective non-shear effective elastic constants of the composite

$$\bar{a} \bar{b}^2 \bar{\varepsilon}_{11}^T = \sum_{i=1}^N a_i b_i^2 \varepsilon_{11}^{T(i)}, \quad \bar{a} \bar{b}^2 \bar{\varepsilon}_{22}^T = \sum_{i=1}^N a_i b_i^2 \varepsilon_{22}^{T(i)}, \quad \bar{a} \bar{b}^2 \bar{\varepsilon}_{33}^T = \sum_{i=1}^N a_i b_i^2 \varepsilon_{33}^{T(i)}. \quad (6.4)$$

On using (2.1) these relationships may be written

$$\bar{\varepsilon}_{11}^T = \sum_{i=1}^N V_p^i \varepsilon_{11}^{T(i)}, \quad \bar{\varepsilon}_{22}^T = \sum_{i=1}^N V_p^i \varepsilon_{22}^{T(i)}, \quad \bar{\varepsilon}_{33}^T = \sum_{i=1}^N V_p^i \varepsilon_{33}^{T(i)}. \quad (6.5)$$

It is thus first necessary to determine the transformation strains $\varepsilon_{11}^{T(i)}$, $\varepsilon_{22}^{T(i)}$ and $\varepsilon_{33}^{T(i)}$ associated with the applied strain field ε_{11}^A , ε_{22}^A , ε_{33}^A for each type of inclusion in the cluster, and the strains $\bar{\varepsilon}_{11}^T$, $\bar{\varepsilon}_{22}^T$ and $\bar{\varepsilon}_{33}^T$ associated with the equivalent single inclusion.

Consider the linear equations (3.11)-(3.13) for a single inclusion that must be solved for the transformation strains ε_{11}^T , ε_{22}^T and ε_{33}^T in each inclusion of the cluster and in the effective medium representing the cluster. It can be shown that

when $\varepsilon_{33}^A = \varepsilon_{22}^A$:

$$\begin{aligned} \varepsilon_{11}^T &= \left[(QR - PS)(S_{2222} + S_{2233}) - (3k_m + \mu_m)P - \mu_m Q \right] \frac{\varepsilon_{11}^A}{\Delta} \\ &\quad + 2 \left[(PS - QR)S_{1122} - (3k_m + \mu_m)R - \mu_m S \right] \frac{\varepsilon_{22}^A}{\Delta}, \\ \varepsilon_{22}^T &= \left[(PS - QR)S_{2211} + \left(\frac{3}{2}k_m - \mu_m \right)P - \mu_m Q \right] \frac{\varepsilon_{11}^A}{\Delta} \\ &\quad + \left[(QR - PS)S_{1111} + (3k_m - 2\mu_m)R - 2\mu_m S \right] \frac{\varepsilon_{22}^A}{\Delta} = \varepsilon_{33}^T, \end{aligned} \quad (6.6)$$

when $\varepsilon_{33}^A = -\varepsilon_{22}^A$:

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$$\begin{aligned}\varepsilon_{11}^T &= \left[(QR - PS)(S_{2222} + S_{2233}) - (3k_m + \mu_m)P - \mu_m Q \right] \frac{\varepsilon_{11}^A}{\Delta}, \\ \varepsilon_{22}^T &= \left[(PS - QR)S_{2211} + \frac{1}{2}(3k_m - 2\mu_m)P - \mu_m Q \right] \frac{\varepsilon_{11}^A}{\Delta} - \frac{\varepsilon_{22}^A}{\Lambda}, \\ \varepsilon_{33}^T &= \left[(PS - QR)S_{2211} + \frac{1}{2}(3k_m - 2\mu_m)P - \mu_m Q \right] \frac{\varepsilon_{11}^A}{\Delta} + \frac{\varepsilon_{22}^A}{\Lambda},\end{aligned}\quad (6.7)$$

where

$$\begin{aligned}P &= E_A + 2\nu_A(2\nu_A - 1)k_T - 2\mu, \\ Q &= 6\nu_A k_T - 3k + 2\mu, \\ R &= (2\nu_A - 1)k_T + \mu, \\ S &= 3k_T - 3k - \mu = Q - 3R,\end{aligned}\quad (6.8)$$

$$\Lambda = S_{2222} - S_{2233} + \frac{\mu}{\mu_T - \mu}, \quad (6.9)$$

$$\Delta = L_1 P + L_2 Q + L_3 R + \Psi(QR - PS) + 9k_m \mu_m, \quad (6.10)$$

and where

$$\begin{aligned}L_1 &= (3k_m + \mu_m)S_{1111} - (3k_m - 2\mu_m)S_{1122}, \\ L_2 &= \left[S_{1111} + 2(S_{1122} + S_{2211} + S_{2222} + S_{2233}) \right] \mu_m, \\ L_3 &= 2(3k_m - 2\mu_m)S_{2211} - (3k_m + 4\mu_m)(S_{2222} + S_{2233}), \\ \Psi &= 2S_{1122}S_{2211} - S_{1111}(S_{2222} + S_{2233}).\end{aligned}\quad (6.11)$$

7. Solving for parameters defining properties of the effective medium

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It is useful to consider two special loading cases, which reduce the complexity of the approach. The first case is for uniaxial axial applied strains while the second considers plane strain equi-biaxial transverse loading.

7.1 Uniaxial axial loading

For the special case when ε_{11}^A is the only non-vanishing applied strain it follows from (6.6) or (6.7) that the strains $\varepsilon_{11}^{T(i)}$, $\varepsilon_{22}^{T(i)}$ and $\varepsilon_{33}^{T(i)}$ appearing in (6.5) are given by

$$\begin{aligned}\varepsilon_{11}^{T(i)} &= \left[(Q_i R_i - P_i S_i)(S_{2222} + S_{2233}) - (3k_m + \mu_m)P_i - \mu_m Q_i \right] \frac{\varepsilon_{11}^A}{\Delta_i}, \\ \varepsilon_{22}^{T(i)} &= \left[(P_i S_i - Q_i R_i)S_{2211} + \frac{1}{2}(3k_m - 2\mu_m)P_i - \mu_m Q_i \right] \frac{\varepsilon_{11}^A}{\Delta_i} = \varepsilon_{33}^T,\end{aligned}\quad (7.1)$$

where

$$\begin{aligned}
P_i &= E_A^i + 2v_A^i (2v_A^i - 1)k_T^i - 2\mu_m, \\
Q_i &= 6v_A^i k_T^i - 3k_m + 2\mu_m, \\
R_i &= (2v_A^i - 1)k_T^i + \mu_m, \\
S_i &= 3k_T^i - 3k_m - \mu_m = Q_i - 3R_i,
\end{aligned} \tag{7.2}$$

$$\Delta_i = L_1 P_i + L_2 Q_i + L_3 R_i + \Psi(Q_i R_i - P_i S_i) + 9k_m \mu_m. \tag{7.3}$$

Similarly for the effective composite

$$\begin{aligned}
\bar{\varepsilon}_{11}^T &= \left[(\bar{Q}\bar{R} - \bar{P}\bar{S})(S_{2222} + S_{2233}) - (3k_m + \mu_m)\bar{P} - \mu_m\bar{Q} \right] \frac{\varepsilon_{11}^A}{\bar{\Delta}}, \\
\bar{\varepsilon}_{22}^T &= \left[(\bar{P}\bar{S} - \bar{Q}\bar{R})S_{2211} + \frac{1}{2}(3k_m - 2\mu_m)\bar{P} - \mu_m\bar{Q} \right] \frac{\varepsilon_{11}^A}{\bar{\Delta}} = \bar{\varepsilon}_{33}^T,
\end{aligned} \tag{7.4}$$

$$\begin{aligned}
\bar{P} &= E_A^{\text{eff}} + 2v_A^{\text{eff}} (2v_A^{\text{eff}} - 1)k_T^{\text{eff}} - 2\mu_m, \\
\bar{Q} &= 6v_A^{\text{eff}} k_T^{\text{eff}} - 3k_m + 2\mu_m, \\
\bar{R} &= (2v_A^{\text{eff}} - 1)k_T^{\text{eff}} + \mu_m, \\
\bar{S} &= 3k_T^{\text{eff}} - 3k_m - \mu_m = \bar{Q} - 3\bar{R},
\end{aligned} \tag{7.5}$$

where

$$\bar{\Delta} = L_1 \bar{P} + L_2 \bar{Q} + L_3 \bar{R} + \Psi(\bar{Q}\bar{R} - \bar{P}\bar{S}) + 9k_m \mu_m. \tag{7.6}$$

It then follows from (6.5) that

$$\begin{aligned}
&\left[(\bar{Q}\bar{R} - \bar{P}\bar{S})(S_{2222} + S_{2233}) - (3k_m + \mu_m)\bar{P} - \mu_m\bar{Q} \right] \frac{1}{\bar{\Delta}} = A \\
&= \sum_{i=1}^N \left[(Q_i R_i - P_i S_i)(S_{2222} + S_{2233}) - (3k_m + \mu_m)P_i - \mu_m Q_i \right] \frac{V_p^i}{\Delta_i}, \\
&\left[(\bar{Q}\bar{R} - \bar{P}\bar{S})S_{2211} - \frac{1}{2}(3k_m - 2\mu_m)\bar{P} + \mu_m\bar{Q} \right] \frac{1}{\bar{\Delta}} = B \\
&= \sum_{i=1}^N \left[(Q_i R_i - P_i S_i)S_{2211} - \frac{1}{2}(3k_m - 2\mu_m)P_i + \mu_m Q_i \right] \frac{V_p^i}{\Delta_i}.
\end{aligned} \tag{7.7}$$

The values of the dimensionless parameters A and B are known as they can be calculated from inclusion and matrix parameters, and the volume fractions.

7.2 Plane-strain equi-biaxial transverse loading

For the special case when $\varepsilon_{11}^A = 0$ and $\varepsilon_{22}^A = \varepsilon_{33}^A$, it follows from (6.6) that the strains $\varepsilon_{11}^{T(i)}$, $\varepsilon_{22}^{T(i)}$ and $\varepsilon_{33}^{T(i)}$ appearing in (6.5) are given by

$$\begin{aligned}\varepsilon_{11}^{T(i)} &= 2[(P_i S_i - Q_i R_i) S_{1122} - (3k_m + \mu_m) R_i - \mu_m S_i] \frac{\varepsilon_{22}^A}{\Delta_i}, \\ \varepsilon_{22}^{T(i)} &= [(Q_i R_i - P_i S_i) S_{1111} + (3k_m - 2\mu_m) R_i - 2\mu_m S_i] \frac{\varepsilon_{22}^A}{\Delta_i} = \varepsilon_{33}^T.\end{aligned}\tag{7.8}$$

Similarly for the effective composite

$$\begin{aligned}\bar{\varepsilon}_{11}^T &= 2[(\bar{P}\bar{S} - \bar{Q}\bar{R}) S_{1122} - (3k_m + \mu_m) \bar{R} - \mu_m \bar{S}] \frac{\varepsilon_{22}^A}{\Delta}, \\ \bar{\varepsilon}_{22}^T &= [(\bar{Q}\bar{R} - \bar{P}\bar{S}) S_{1111} + (3k_m - 2\mu_m) \bar{R} - 2\mu_m \bar{S}] \frac{\varepsilon_{22}^A}{\Delta} = \bar{\varepsilon}_{33}^T.\end{aligned}\tag{7.9}$$

Since $S_i = Q_i - 3R_i$ and $\bar{S} = \bar{Q} - 3\bar{R}$, it then follows on using (6.5) that

$$\begin{aligned}[(\bar{Q}\bar{R} - \bar{P}\bar{S}) S_{1122} + \mu_m \bar{Q} + (3k_m - 2\mu_m) \bar{R}] \frac{1}{\Delta} &= C \\ &= \sum_{i=1}^N [(Q_i R_i - P_i S_i) S_{1122} + \mu_m Q_i + (3k_m - 2\mu_m) R_i] \frac{V_p^i}{\Delta_i}, \\ [(\bar{Q}\bar{R} - \bar{P}\bar{S}) S_{1111} - 2\mu_m \bar{Q} + (3k_m + 4\mu_m) \bar{R}] \frac{1}{\Delta} &= D \\ &= \sum_{i=1}^N [(Q_i R_i - P_i S_i) S_{1111} - 2\mu_m Q_i + (3k_m + 4\mu_m) R_i] \frac{V_p^i}{\Delta_i}.\end{aligned}\tag{7.10}$$

The values of the dimensionless parameters C and D are again known as they can be calculated from inclusion and matrix parameters, and the volume fractions. It is worth noting that if the relation (6.7) is used for shear states then no new information is provided as results obtained correspond exactly to results already given in Section 4.

7.3 Defining a soluble set of non-linear algebraic equations

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The problem that now remains is to find values of \bar{P} , \bar{Q} , \bar{R} and $\bar{S} = \bar{Q} - 3\bar{R}$ satisfying the non-linear relations (7.7) and (7.10), which are written

$$(S_{2222} + S_{2233})\Omega - (3k_m + \mu_m) \bar{P} - \mu_m \bar{Q} = A \bar{\Delta},\tag{7.11}$$

$$S_{2211}\Omega - \frac{1}{2}(3k_m - 2\mu_m) \bar{P} + \mu_m \bar{Q} = B \bar{\Delta},\tag{7.12}$$

$$S_{1122}\Omega + \mu_m \bar{Q} + (3k_m - 2\mu_m) \bar{R} = C \bar{\Delta},\tag{7.13}$$

$$S_{1111}\Omega - 2\mu_m \bar{Q} + (3k_m + 4\mu_m) \bar{R} = D \bar{\Delta},\tag{7.14}$$

where

$$\Omega = \bar{Q}\bar{R} - \bar{P}\bar{S} = \bar{Q}\bar{R} - \bar{P}\bar{Q} + 3\bar{P}\bar{R},\tag{7.15}$$

and where from (7.6)

$$\bar{\Delta} = L_1 \bar{P} + L_2 \bar{Q} + L_3 \bar{R} + \Psi \Omega + 9k_m \mu_m.\tag{7.16}$$

It can be shown that

$$(3k_m - 2\mu_m)A - 2(3k_m + \mu_m)B + (3k_m + 4\mu_m)C - (3k_m - 2\mu_m)D \equiv 0, \quad (7.17)$$

$$(3k_m - 2\mu_m)(S_{2222} + S_{2233}) - 2(3k_m + \mu_m)S_{2211} + (3k_m + 4\mu_m)S_{1122} - (3k_m - 2\mu_m)S_{1111} \equiv 0, \quad (7.18)$$

$$(3k_m - 2\mu_m)\left[-(3k_m + \mu_m)\bar{P} - \mu_m\bar{Q}\right] - 2(3k_m + \mu_m)\left[-\frac{1}{2}(3k_m - 2\mu_m)\bar{P} + \mu_m\bar{Q}\right] + (3k_m + 4\mu_m)\left[\mu_m\bar{Q} + (3k_m - 2\mu_m)\bar{R}\right] - (3k_m - 2\mu_m)\left[-2\mu_m\bar{Q} + (3k_m + 4\mu_m)\bar{R}\right] \equiv 0. \quad (7.19)$$

The relations (7.17)-(7.19) show that the four equations (7.11)-(7.14) are linearly dependent. It should be noted that (7.18) is an alternative to the last of the relations (3.10). The following three independent equations are derived from (7.12)-(7.14) and (7.16), which are then to be solved for the unknowns \bar{P} , \bar{Q} and \bar{R} in terms of Ω defined by (7.15), which is a non-linear functions of \bar{P} , \bar{Q} and \bar{R} .

$$\left(L_1 + \frac{3k_m - 2\mu_m}{2B}\right)\bar{P} + \left(L_2 - \frac{\mu_m}{B}\right)\bar{Q} + L_3\bar{R} = -\left(\Psi - \frac{S_{2211}}{B}\right)\Omega - 9k_m\mu_m, \quad (7.20)$$

$$L_1\bar{P} + \left(L_2 - \frac{\mu_m}{C}\right)\bar{Q} + \left(L_3 - \frac{3k_m - 2\mu_m}{C}\right)\bar{R} = -\left(\Psi - \frac{S_{1122}}{C}\right)\Omega - 9k_m\mu_m, \quad (7.21)$$

$$L_1\bar{P} + \left(L_2 + \frac{2\mu_m}{D}\right)\bar{Q} + \left(L_3 - \frac{3k_m + 4\mu_m}{D}\right)\bar{R} = -\left(\Psi - \frac{S_{1111}}{D}\right)\Omega - 9k_m\mu_m. \quad (7.22)$$

It follows from (7.7) and (7.10) that

$$\begin{aligned} A &= (S_{2222} + S_{2233})W - (3k_m + \mu_m)X - \mu_m Y, \\ B &= S_{2211}W - \frac{1}{2}(3k_m - 2\mu_m)X + \mu_m Y, \\ C &= S_{1122}W + \mu_m Y + (3k_m - 2\mu_m)Z, \\ D &= S_{1111}W - 2\mu_m Y + (3k_m + 4\mu_m)Z, \end{aligned} \quad (7.23)$$

where W , X , Y and Z are known constants defined by

$$W = \sum_{i=1}^N V_p^i \frac{Q_i R_i - P_i S_i}{\Delta_i}, \quad X = \sum_{i=1}^N V_p^i \frac{P_i}{\Delta_i}, \quad Y = \sum_{i=1}^N V_p^i \frac{Q_i}{\Delta_i}, \quad Z = \sum_{i=1}^N V_p^i \frac{R_i}{\Delta_i}. \quad (7.24)$$

Following a great deal of complex and laborious algebra (see supplementary information A for details), the equations (7.20)-(7.22) may be expressed

$$\Phi \bar{P} = (\beta_1 \Psi + \alpha_1) \hat{\Omega} - \beta_1, \quad (7.25)$$

$$\Phi \bar{Q} = (\beta_2 \Psi + \alpha_2) \hat{\Omega} - \beta_2, \quad (7.26)$$

$$\Phi \bar{R} = (\beta_3 \Psi + \alpha_3) \hat{\Omega} - \beta_3, \quad (7.27)$$

$$\text{where} \quad \Phi = L_1 X + L_2 Y + L_3 Z + 2\Psi W - 1, \quad (7.28)$$

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and where the dimensionless constant Ψ is given by (6.11). Using (7.15), $\hat{\Omega}$ is defined by

$$\hat{\Omega} = \frac{\Omega}{9k_m\mu_m} = \frac{\bar{Q}\bar{R} - \bar{P}\bar{Q} + 3\bar{P}\bar{R}}{9k_m\mu_m}, \quad (7.29)$$

and the remaining coefficients in (7.25)-(7.27) are defined by

$$\begin{aligned} \alpha_1 &= 2\mu_m(S_{2211} + S_{2222} + S_{2233})\Phi, \\ \beta_1 &= 9k_m\mu_m X - 2\mu_m(S_{2211} + S_{2222} + S_{2233})W, \\ \alpha_2 &= -\lambda\Phi, \\ \beta_2 &= 9k_m\mu_m Y + \lambda W, \\ \alpha_3 &= -\mu_m(S_{1111} + 2S_{1122})\Phi, \\ \beta_3 &= 9k_m\mu_m Z + \mu_m(S_{1111} + 2S_{1122})W, \end{aligned} \quad (7.30)$$

$$\begin{aligned} \text{with } \lambda &= (3k_m + 4\mu_m)S_{1122} - (3k_m - 2\mu_m)S_{1111} \\ &= 2(3k_m + \mu_m)S_{2211} - (3k_m - 2\mu_m)(S_{2222} + S_{2233}). \end{aligned} \quad (7.31)$$

8. Determination of effective composite properties

The elimination of the parameters \bar{P} , \bar{Q} and \bar{R} using (7.25)-(7.27) and (7.29) leads to the following quadratic equation for $\hat{\Omega}$

$$\alpha\hat{\Omega}^2 - \beta\hat{\Omega} - \gamma = 0, \quad (8.1)$$

$$\text{where } \begin{cases} \alpha = \Psi\beta + \Psi^2\gamma, \\ \beta = \{-(\alpha_1\beta_2 + \alpha_2\beta_1) + (\alpha_2\beta_3 + \alpha_3\beta_2) + 3(\alpha_1\beta_3 + \alpha_3\beta_1)\} \\ \quad + (-\alpha_1\alpha_2 + \alpha_2\alpha_3 + 3\alpha_1\alpha_3)/\Psi - 2\Psi\gamma, \\ \gamma = \beta_1\beta_2 - \beta_2\beta_3 - 3\beta_1\beta_3. \end{cases} \quad (8.2)$$

The solutions of (8.1) are given by

$$\hat{\Omega} = \frac{\Omega}{9k_m\mu_m} = \frac{1}{2\alpha} \left[\beta \pm \sqrt{\beta^2 + 4\alpha\gamma} \right]. \quad (8.3)$$

It has been noted from numerical calculations that the solutions for $\hat{\Omega}$ are either equal or they differ in value such that one of the roots leads to a zero value for $\bar{\Delta}$, indicating that the solutions (7.4) and (7.9) for special cases break down when applied to the single effective spheroid that represents the composite. When one of the roots leads to the result $\bar{\Delta} = 0$, the required value of $\hat{\Omega}$ is that value given by the other root of (8.3). It has also been observed numerically that the root leading to the result $\bar{\Delta} = 0$ does not depend on any of the elastic properties of the inclusions. For the case of cylinders it has been observed that the roots of (8.3) are always equal. These characteristics of the solution strongly suggest that simpler analytical solutions should exist.

It can be shown that

$$\Phi \bar{\Delta} = \left[(L_1 \beta_1 + L_2 \beta_2 + L_3 \beta_3) - 9k_m \mu_m \Phi \right] (\Psi \hat{\Omega} - 1), \quad (8.4)$$

indicating that the solution $\hat{\Omega}$ that leads to the situation $\bar{\Delta} = 0$ is simply

$$\hat{\Omega} = \frac{\Omega}{9k_m \mu_m} = \frac{1}{\Psi}. \quad (8.5)$$

From (6.11) it is clear that Ψ is independent of inclusion elastic properties, as observed numerically. It can then be shown that the required solution of (8.1) such that $\bar{\Delta} \neq 0$ is given by

$$\hat{\Omega} = \frac{\Omega}{9k_m \mu_m} = -\frac{\Psi \gamma}{\alpha} = \frac{\beta}{\alpha} - \frac{1}{\Psi}. \quad (8.6)$$

It should be noted that the quantity Φ defined by (7.28) may also be expressed in the form

$$\Phi = \sqrt{\frac{3\alpha_1 \alpha_3 - \alpha_1 \alpha_2 + \alpha_2 \alpha_3}{9k_m \mu_m \Psi}}. \quad (8.7)$$

It follows from (7.26) and (7.27) and the relation $\bar{S} = \bar{Q} - 3\bar{R}$ that

$$\Phi \bar{S} = \Phi (\bar{Q} - 3\bar{R}) = \left[(\beta_2 - 3\beta_3) \Psi + \alpha_2 - 3\alpha_3 \right] \hat{\Omega} - (\beta_2 - 3\beta_3). \quad (8.8)$$

It is now possible to calculate the required solution \bar{P} , \bar{Q} , \bar{R} and \bar{S} of the non-linear algebraic equations (7.20)-(7.22) and (7.15) using (7.25)-(7.27), (8.6) and (8.8). The relations (7.5) are then used to derive the following relationships for the effective non-shear properties of the composite

$$\begin{aligned} k_T^{\text{eff}} &= \frac{1}{3} (\bar{S} + 3k_m + \mu_m), \\ v_A^{\text{eff}} &= \frac{1}{2} \left(\frac{\bar{R} - \mu_m}{k_T^{\text{eff}}} + 1 \right) = \frac{\bar{Q} + 3k_m - 2\mu_m}{6k_T^{\text{eff}}}, \\ E_A^{\text{eff}} &= \bar{P} + 2v_A^{\text{eff}} (1 - 2v_A^{\text{eff}}) k_T^{\text{eff}} + 2\mu_m. \end{aligned} \quad (8.9)$$

The corresponding shear properties μ_A^{eff} and μ_T^{eff} have already been obtained and are given by (5.12) and (5.13). The corresponding values of the transverse Young's modulus E_T^{eff} and transverse Poisson's ratio v_T^{eff} are obtained using (3.8).

It has thus been shown that it is possible to apply the analysis of Eshelby [18], [19] together with Maxwell's methodology [13] to develop a method of estimating the effective elastic properties of a multi-phase distribution of aligned spheroidal transversely isotropic inclusions having the same aspect ratio that reinforce an isotropic matrix. The effective elastic properties may be estimated using the results (5.12), (5.13) and (8.9), and the various associated relationships that have been derived.

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9. Composites reinforced with isotropic spherical inclusions

It is useful to consider the special case when the reinforcing inclusions are spherical and isotropic, and distributed so that the composite is also isotropic. Rather than developing the required solution using (7.25)-(7.27), it is simpler to derive the results using the following method. When the inclusions are isotropic

$$\begin{aligned} E_A^i = E_T^i = E_p^i = 2\mu_p^i(1 + \nu_p^i), \quad \mu_A^i = \mu_T^i = \mu_p^i, \quad \nu_A^i = \nu_T^i = \nu_p^i, \\ E_A^{\text{eff}} = E_T^{\text{eff}} = E_p^{\text{eff}} = 2\mu_p^{\text{eff}}(1 + \nu_p^{\text{eff}}), \quad \mu_A^{\text{eff}} = \mu_T^{\text{eff}} = \mu_p^{\text{eff}}, \quad \nu_A^{\text{eff}} = \nu_T^{\text{eff}} = \nu_p^{\text{eff}}. \end{aligned} \tag{9.1}$$

The bulk modulus k_p and plane strain bulk modulus k_T are now introduced defined by

$$k_p = \frac{E_p}{3(1-2\nu_p)} = \frac{2\mu_p(1+\nu_p)}{3(1-2\nu_p)}, \quad k_T = \frac{\mu_p}{1-2\nu_p} = \frac{3k_p}{2(1+\nu_p)} = k_p + \frac{1}{3}\mu_p. \tag{9.2}$$

It then follows from (7.2) and (7.5) that

$$\begin{aligned} P_i &= 2(\mu_p^i - \mu_m), & \bar{P} &= 2(\mu_p^{\text{eff}} - \mu_m), \\ Q_i &= 3(k_p^i - k_m) - 2(\mu_p^i - \mu_m), & \bar{Q} &= 3(k_p^{\text{eff}} - k_m) - 2(\mu_p^{\text{eff}} - \mu_m), \\ R_i &= -(\mu_p^i - \mu_m), & \bar{R} &= -(\mu_p^{\text{eff}} - \mu_m), \\ S_i &= 3(k_p^i - k_m) + (\mu_p^i - \mu_m) = Q_i - 3R_i, & \bar{S} &= 3(k_p^{\text{eff}} - k_m) + (\mu_p^{\text{eff}} - \mu_m) = \bar{Q} - 3\bar{R}. \end{aligned} \tag{9.3}$$

It can be shown from (6.11), that

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$$\Psi = 2S_{1122}S_{2211} - S_{1111}(S_{2222} + S_{2233}) = -\frac{8-10\nu_m}{15(1-\nu_m)} \frac{1+\nu_m}{3(1-\nu_m)}, \tag{9.4}$$

and it follows from (7.3) and (7.6) that

$$\begin{aligned} \Delta_i &= \frac{1}{5(1-\nu_m)^2} [(1+\nu_m)k_p^i + 2(1-2\nu_m)k_m] [(8-10\nu_m)\mu_p^i + (7-5\nu_m)\mu_m], \\ \bar{\Delta} &= \frac{1}{5(1-\nu_m)^2} [(1+\nu_m)k_p^{\text{eff}} + 2(1-2\nu_m)k_m] [(8-10\nu_m)\mu_p^{\text{eff}} + (7-5\nu_m)\mu_m], \end{aligned} \tag{9.5}$$

and from (6.6) that

$$\begin{aligned} \varepsilon_{11}^{T(i)} = \varepsilon_{22}^{T(i)} = \varepsilon_{33}^{T(i)} &= -\frac{3(k_p^i - k_m)}{5(1-\nu_m)} [(8-10\nu_m)\mu_p^i + (7-5\nu_m)\mu_m] \frac{\varepsilon}{\Delta_i}, \\ \bar{\varepsilon}_{11}^T = \bar{\varepsilon}_{22}^T = \bar{\varepsilon}_{33}^T &= -\frac{3(k_p^{\text{eff}} - k_m)}{5(1-\nu_m)} [(8-10\nu_m)\mu_p^{\text{eff}} + (7-5\nu_m)\mu_m] \frac{\varepsilon}{\bar{\Delta}}, \end{aligned} \tag{9.6}$$

where ε is the equi-axial applied strain. The substitution of (9.5) in (9.6), using the relation $(1-2\nu_m)k_m = \frac{2}{3}(1+\nu_m)\mu_m$, leads to

$$\varepsilon_{11}^{T(i)} = \varepsilon_{22}^{T(i)} = \varepsilon_{33}^{T(i)} = -3 \frac{1-v_m}{1+v_m} \frac{k_p^i - k_m}{k_p^i + \frac{4}{3}\mu_m} \varepsilon, \quad (9.7)$$

$$\bar{\varepsilon}_{11}^T = \bar{\varepsilon}_{22}^T = \bar{\varepsilon}_{33}^T = -3 \frac{1-v_m}{1+v_m} \frac{k_p^{\text{eff}} - k_m}{k_p^{\text{eff}} + \frac{4}{3}\mu_m} \varepsilon.$$

On substituting (9.7) in (6.5), the bulk modulus of an isotropic particulate composite is obtained (see for example [17]), namely,

$$\frac{1}{k_p^{\text{eff}} + \frac{4}{3}\mu_m} = \sum_{i=1}^N \frac{V_p^i}{k_p^i + \frac{4}{3}\mu_m} + \frac{V_m}{k_m + \frac{4}{3}\mu_m}. \quad (9.8)$$

The corresponding expression for the shear modulus is obtained from (5.12) or (5.13), leading to the following result

$$\frac{1}{\mu_p^{\text{eff}} + \mu_m^*} = \sum_{i=1}^N \frac{V_p^i}{\mu_p^i + \mu_m^*} + \frac{V_m}{\mu_m + \mu_m^*}, \text{ where } \mu_m^* = \frac{7-5v_m}{8-10v_m} \mu_m = \frac{9k_m + 8\mu_m}{6(k_m + 2\mu_m)} \mu_m. \quad (9.9)$$

The results (9.8) and (9.9) have been shown [17] to predict effective properties that are in close agreement with accurate results that have been given in the literature for volume fractions of practical interest. It was shown that the f.c.c. and b.c.c. packing configurations considered by Arridge [23] lead to bulk moduli that are very close together for particulate volume fractions in the range $0 < V_p < 0.6$ (see [17], Fig.3). Furthermore, the results for spherical isotropic particles obtained using Maxwell's methodology lie between the f.c.c. and b.c.c. estimates for volume fractions in the range $0 < V_p < 0.4$. For the case of a simple cubic array of spherical particles with volume fractions in the range $0 < V_p < 0.4$, it was shown that bounds for shear modulus, obtained by Cohen and Bergman ([15], see Fig. 4) using a Fourier representation of an integro-differential equation for the displacement field, are very close to results obtained using Maxwell's methodology.

Further evidence that Maxwell's methodology can provide useful predictions for the bulk and shear moduli of isotropic composites reinforced with isotropic spherical particles is provided by the results of Sangani and Mo [24] for the extreme cases where the particles are either rigid or behave as cavities. Tables 1 and 2 compare the results of Sangani and Mo (labelled Accurate) with those based on the formulae (9.8) and (9.9) (labelled Maxwell). It is seen from Table 1 that predictions for the effective bulk modulus based on Maxwell's methodology applied to the extreme case of rigid particles become more accurate as the matrix Poisson's ratio increases and becomes exact when Poisson's ratio is 0.5 (i.e. an incompressible matrix) for all particle volume fractions considered, whereas predictions for the effective shear modulus become much less accurate, especially as Poisson's ratio for the matrix tends to the incompressible value 0.5. Significant differences in shear modulus are seen for this extreme case when the volume fraction has the value 0.6. As to be expected at low volume fractions there is good agreement for all values of the matrix Poisson's ratio ≤ 0.4 .

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Table 1: Effective properties for random distributions of rigid isotropic spherical particles.

V_p	v_m	k_{eff} / k_m			μ_{eff} / μ_m		
		Accurate	Maxwell	Dipole	Accurate	Maxwell	Dipole
0.1	0.2	1.23	1.222	1.226	1.236	1.222	1.228
0.1	0.3	1.183	1.179	1.18	1.242	1.233	1.234
0.1	0.4	1.145	1.143	1.143	1.257	1.25	1.249
0.1	0.5	1.111	1.111	1.111	1.311	1.278	1.291
0.45	0.2	2.93	2.636	-	3.21	2.636	-
0.45	0.3	2.55	2.322	-	3.43	2.718	-
0.45	0.4	2.18	2.052	-	3.9	2.841	-
0.45	0.5	1.818	1.818	-	5.7	3.045	-
0.6	0.2	5.28	4	-	6	4	-
0.6	0.3	4.4	3.423	-	6.7	4.15	-
0.6	0.4	3.59	2.929	-	8.2	4.375	-
0.6	0.5	2.502	2.5	-	17.6	4.75	-

Table 2: Effective properties for random distributions of spherical cavities.

V_p	v_m	k_{eff} / k_m			μ_{eff} / μ_m		
		Accurate	Maxwell	Dipole	Accurate	Maxwell	Dipole
0.1	0.2	0.8156	0.818	0.8169	0.813	0.818	0.816
0.1	0.3	0.771	0.774	0.772	0.82	0.825	0.823
0.1	0.4	0.662	0.667	0.664	0.829	0.833	0.832
0.1	0.5	0	0	0	0.856	0.844	0.859
0.45	0.2	0.362	0.379	0.377	0.346	0.379	0.379
0.45	0.3	0.299	0.3177	0.3143	0.355	0.39	0.391
0.45	0.4	0.1971	0.2136	0.2106	0.367	0.4044	0.405
0.45	0.5	0	0	0	0.399	0.423	0.427
0.6	0.2	0.223	0.25	-	0.198	0.25	-
0.6	0.3	0.177	0.2025	-	0.205	0.259	-
0.6	0.4	0.11	0.129	-	0.212	0.27	-
0.6	0.5	0	0	-	0.337	0.286	-

From Table 2 it is seen that predictions for the effective bulk modulus based on Maxwell's methodology applied to the other extreme case of cavities become more accurate as the matrix Poisson's ratio increases and becomes the exact value 0 when Poisson's ratio is 0.5 for all particle volume fractions considered, whereas predictions for the effective shear modulus become much less accurate. At low volume fractions there is again good agreement for all values of the matrix Poisson's ratio ≤ 0.4 . From Tables 1 and 2 it is clear that estimates of properties based on Maxwell's methodology are more accurate for cavities than they are for the case of rigid particles. Also shown in Tables 1 and 2 are predictions based on a dipole approximation that was considered by Sangani and Mo [24]. It is observed that corresponding predictions of effective elastic properties agree very closely with those obtained using Maxwell's methodology. This agreement suggests that, although particle

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Table 4: Predictions for C_{3333} for distributions of spheroidal particles.

μ_p / μ_m	Random (Maxwell)	Simple cubic	Random (Maxwell)	Simple cubic	Random (Maxwell)	Simple cubic
0	0	0	10	10	1000	1000
0.25	2.611	1.843	3.991	3.889	4.081	3.944
0.5	2.678	2.474	4.015	3.956	4.116	4.032
0.75	2.735	2.693	4.043	4.026	4.156	4.132
1.0	2.777	2.799	4.071	4.102	4.199	4.248
1.25	2.809	2.859	4.100	4.185	4.243	4.388
1.5	2.8315	2.897	4.129	4.281	4.289	4.565
1.75	2.848	2.923	4.157	4.397	4.335	4.810
2.0	2.859	2.941	4.184	4.555	4.3814	5.225
2.25	2.867	2.954	4.211	4.868	4.428	6.765

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For the effective property C_{3333} differences are less than 42% when $\mu_p / \mu_m = 0$, less than 14% when $\mu_p / \mu_m = 10$ and less than 35% when $\mu_p / \mu_m = 1000$. However, if spheroids are considered such that $0.5 \leq b/a \leq 2$ the differences for C_{1111} are less than 7.4% and those for C_{3333} are less than 16.2%. A contribution to the largest differences at the extreme values of the aspect ratio b/a will certainly be due to differences in the properties of composites reinforced with random and simple cubic arrays of particles. It is concluded from the results given in Tables 3 and 4 that there is reasonable agreement between the results of Kushch [25] for simple cubic arrays of spheroidal particles and those based on Maxwell's methodology for random distributions of aligned particles for cases of practical relevance (e.g. $\mu_p / \mu_m \approx 10$ including cavities having low volume fractions).

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10. Composites reinforced with aligned transversely isotropic cylindrical fibres

For the case of transversely isotropic cylindrical fibres, rather than developing the required solution using (7.25)-(7.27), it is again simpler to derive the results using another method that involves less laborious algebra (see supplementary information B for details). The principal results are given by

$$\Delta_i = 9k_m \mu_m \frac{k_T^i + \mu_m}{k_T^m + \mu_m}, \tag{10.1}$$

$$\bar{\Delta} = 9k_m \mu_m \frac{k_T^{eff} + \mu_m}{k_T^m + \mu_m}, \tag{10.2}$$

$$\frac{1}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{V_m}{k_T^m + \mu_m}, \quad (10.3)$$

$$\frac{v_A^{\text{eff}} k_T^{\text{eff}}}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N V_p^i \frac{v_A^i k_T^i}{k_T^i + \mu_m} + V_m \frac{v_m k_T^m}{k_T^m + \mu_m}, \quad (10.4)$$

$$E_A^{\text{eff}} + \frac{4k_T^{\text{eff}} (v_A^{\text{eff}})^2 \mu_m}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N V_p^i \left(E_A^i + \frac{4k_T^i (v_A^i)^2 \mu_m}{k_T^i + \mu_m} \right) + V_m \left(E_m + \frac{4k_T^m v_m^2 \mu_m}{k_T^m + \mu_m} \right), \quad (10.5)$$

where $k_T^m = k_m + \frac{1}{3} \mu_m$. (10.6)

The results (10.3) and (10.4) can also be derived using similar methods to those described for spherical inclusions in [17]. The result (10.5) for the axial Young's modulus for multi-phase unidirectional composite is thought to be a new formula. It follows from (5.12) and (5.13) that the corresponding effective shear moduli are given by

$$\frac{1}{\mu_A^{\text{eff}} + \mu_m} = \sum_{i=1}^N \frac{V_p^i}{\mu_A^i + \mu_m} + \frac{V_m}{\mu_m + \mu_m}, \quad (10.7)$$

$$\frac{1}{\mu_T^{\text{eff}} + \mu_T^*} = \sum_{i=1}^N \frac{V_p^i}{\mu_T^i + \mu_T^*} + \frac{V_m}{\mu_m + \mu_T^*}, \quad \text{where } \mu_T^* = \frac{k_T^m \mu_m}{k_T^m + 2\mu_m}. \quad (10.8)$$

Five independent effective elastic properties can thus be estimated, namely, k_T^{eff} , μ_T^{eff} , μ_A^{eff} , v_A^{eff} and E_A^{eff} . The transverse Young's modulus E_T^{eff} and Poisson's ratio v_T^{eff} can be estimated using the relations (3.8). For fibre reinforced composites subject to plane strain deformations so that the axial displacement and strain are everywhere zero, two other elastic constants E_T^* and v_T^* are relevant defined by the relations

$$\frac{4}{E_T^*} = \frac{1}{\mu_T^{\text{eff}}} + \frac{1}{k_T^{\text{eff}}}, \quad v_T^* = \frac{k_T^{\text{eff}} - \mu_T^{\text{eff}}}{k_T^{\text{eff}} + \mu_T^{\text{eff}}}. \quad (10.9)$$

The constant E_T^* is the transverse Young's modulus for plane strain conditions in the axial direction, while v_T^* is the corresponding transverse Poisson's ratio. It is useful to note that

$$E_T^* = 2\mu_T^{\text{eff}} (1 + v_T^*). \quad (10.10)$$

Predictions of effective properties for fibre reinforced composites will now be compared with those in the literature for two phases obtained using numerical methods, and which are expected to be accurate. Eischen and Torquato [26] have considered, using a boundary element method, the estimation of elastic constants for hexagonal arrays of aligned fibres subject to plane strain conditions. Three different materials systems were included having isotropic fibres and matrix properties given by $\mu_f/\mu_m = 135, 22.5, 6.75$, $\mu_f/k_f = 0.75$, $\mu_m/k_m = 0.33$, $v_m = 0.35$, $v_f = 0.2$, and the value $\mu_m/k_T^m = 0.3$ is also given. The matrix values $\mu_m/k_m = 0.33$, $v_m = 0.35$ and $\mu_m/k_T^m = 0.3$ are not consistent with the relation (10.6) due to rounding errors. To avoid this problem the matrix values

assumed here are $\mu_m / k_m = 1/3$, $\nu_m = 0.35$ and $\mu_m / k_T^m = 0.3$ which are consistent with the relation (10.6).

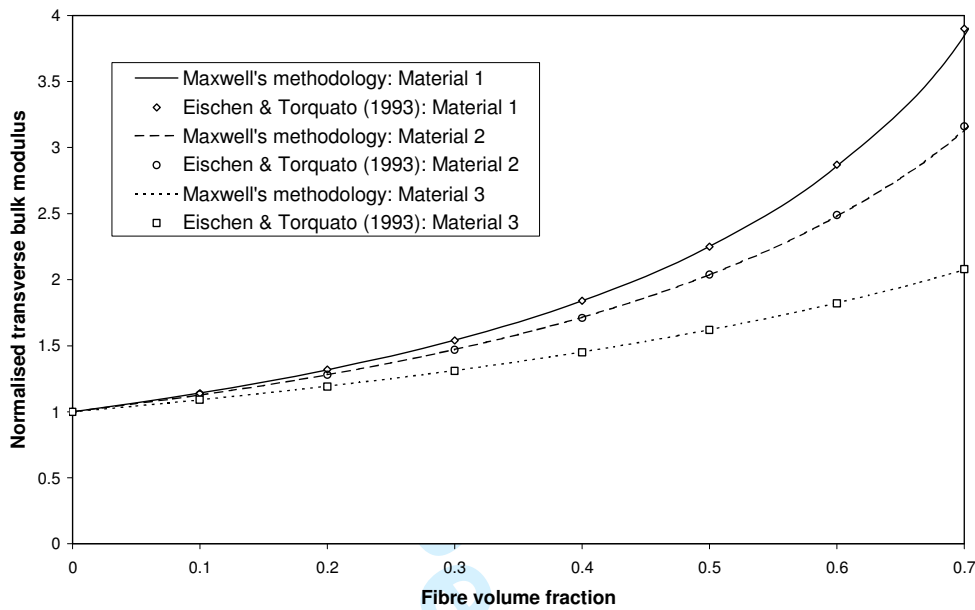


Figure 2 : Comparison of results for normalised effective plane strain bulk k_T^{eff} modulus obtained using Maxwell's methodology with those of Eischen and Torquato [26] for three different materials.

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Figure 2 shows a comparison of the effective plane strain bulk modulus k_T^{eff} , obtained using the relation (10.3) resulting from Maxwell's methodology, with the accurate boundary element results [26]. The normalised plane strain bulk modulus is defined by k_T^{eff} / k_T^m where k_T^m is defined by (10.6). For fibre volume fractions in the range $0 \leq V_f \leq 0.7$, the results predicted using Maxwell's methodology agree exceedingly well (errors less than 1.4%) with the results of Eischen and Torquato. For larger volume fractions significant differences arise especially when the fibres have a much larger shear modulus than the matrix.

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Figure 3 shows a comparison of transverse shear modulus μ_T^{eff} , obtained using the relation (10.8) resulting from Maxwell's methodology, with results of Eischen and Torquato [26]. The normalised effective transverse shear modulus is defined by $\mu_T^{\text{eff}} / \mu_m$. For fibre volume fractions in the range $0 \leq V_f \leq 0.4$, the results predicted using Maxwell's methodology agree well (errors less than 2.3%) with the results of Eischen and Torquato. For larger volume fractions significant differences arise especially when the fibres have a much larger shear modulus than the matrix.

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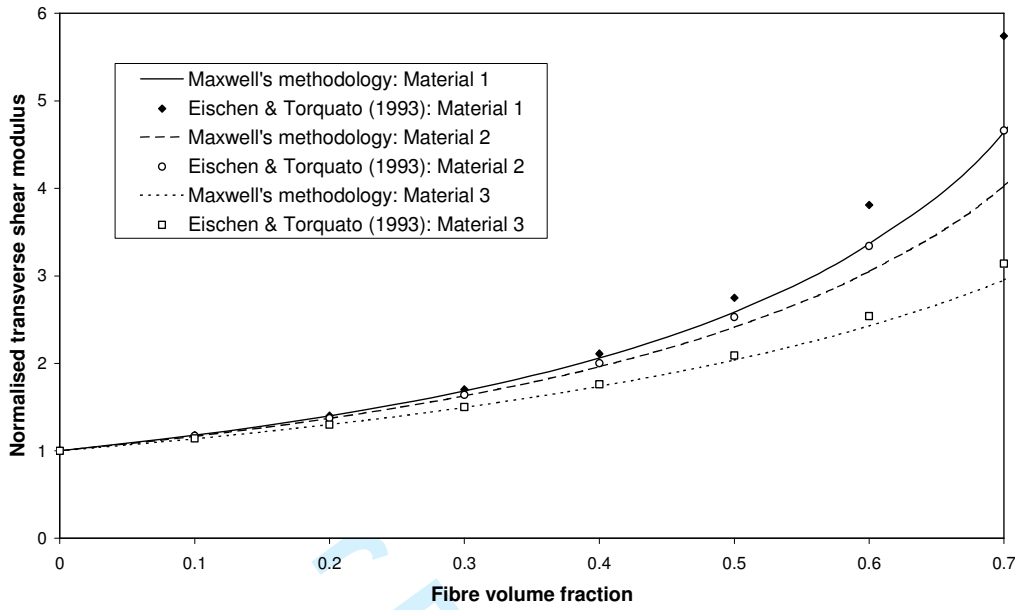


Figure 3 : Comparison of results for normalised effective transverse shear modulus μ_T^{eff} obtained using Maxwell's methodology with those of Eischen and Torquato [26] for three different materials.

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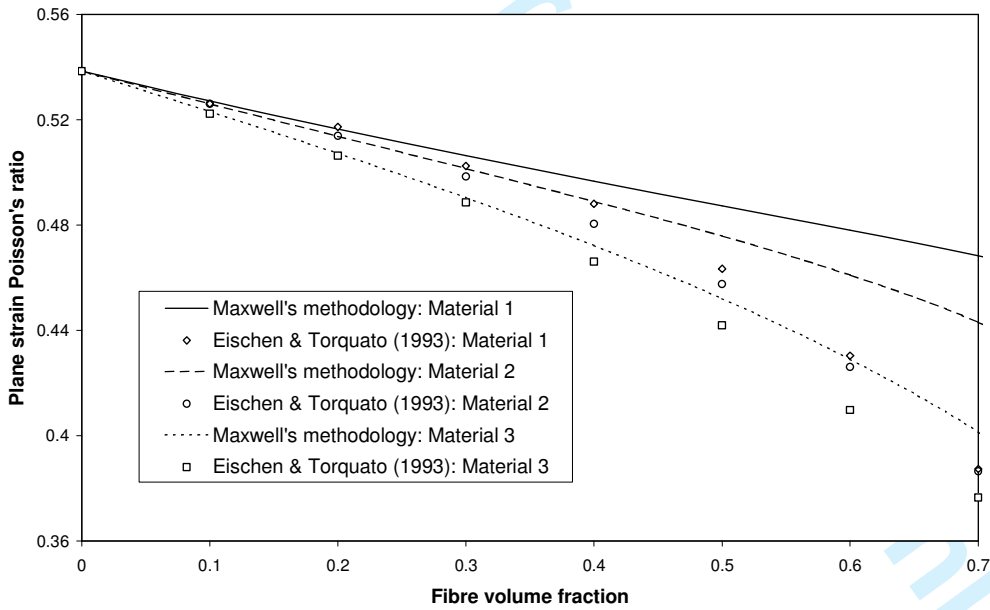


Figure 4 : Comparison of results for the effective plane strain transverse Poisson's ratio v_T^* obtained using Maxwell's methodology with those of Eischen and Torquato [26] for three different materials.

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Figure 4 shows a comparison of the plane strain transverse Poisson's ratio ν_T^* , defined by (10.9) and obtained from the relations (10.3) and (10.8) resulting from Maxwell's methodology, with the results of Eischen and Torquato [26]. For fibre volume fractions in the range $0 \leq V_f \leq 0.4$, the results predicted using Maxwell's methodology agree well (errors less than 1.8%) with the results of Eischen and Torquato. For larger volume fractions significant differences arise especially when the fibres have a much larger shear modulus than the matrix.

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Figure 5 shows a comparison of the plane strain transverse Young's modulus E_T^* , defined by (10.9) and obtained from the relations (10.3) and (10.8) resulting from Maxwell's methodology, with the results of Eischen and Torquato [26]. For fibre volume fractions in the range $0 \leq V_f \leq 0.4$, the results predicted using Maxwell's methodology agree well (errors less than 2.1%) with the results of Eischen and Torquato. For larger volume fractions significant differences arise especially when the fibres have a much larger shear modulus than the matrix.

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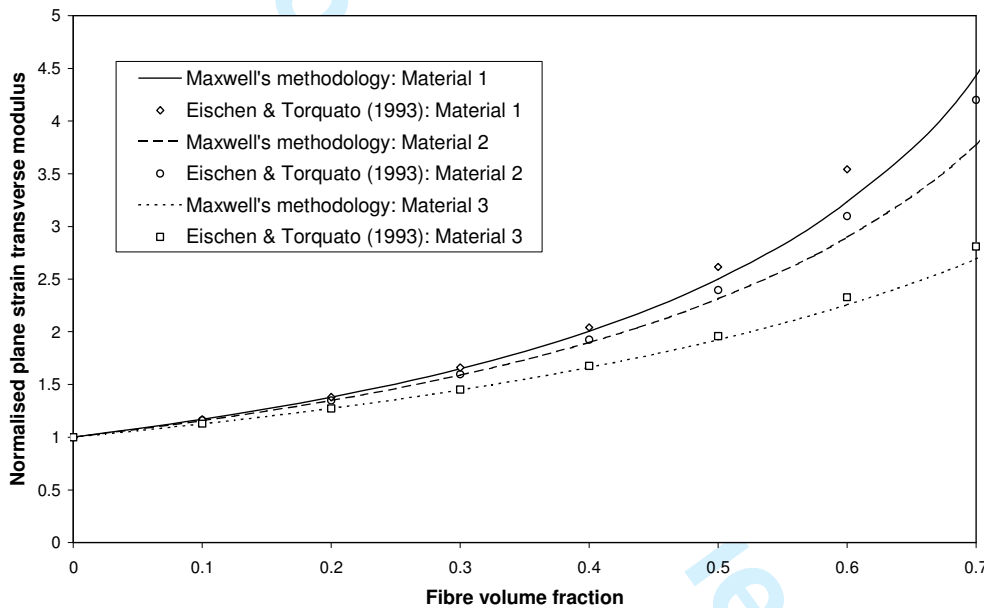


Figure 5 : Comparison of results for the normalised effective plane strain transverse Young's modulus E_T^* obtained using Maxwell's methodology with those of Eischen and Torquato [26] for three different materials.

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Figure 6 shows a comparison of axial shear modulus μ_A^{eff} , obtained using the relation (10.7) resulting from Maxwell's methodology, with the results of Symm [27]. The normalised effective axial shear modulus is defined by μ_A^{eff} / μ_m , and the four materials considered are for isotropic fibres and matrix such that $\mu_f / \mu_m = 6, 20, 120, \infty$. For fibre

volume fractions in the range $0 \leq V_f \leq 0.7$, the results predicted using Maxwell's methodology agree very well (errors less than 2.6%) with the results of Symm [27]. For larger volume fractions significant differences arise especially when the fibres have a much larger shear modulus than the matrix.

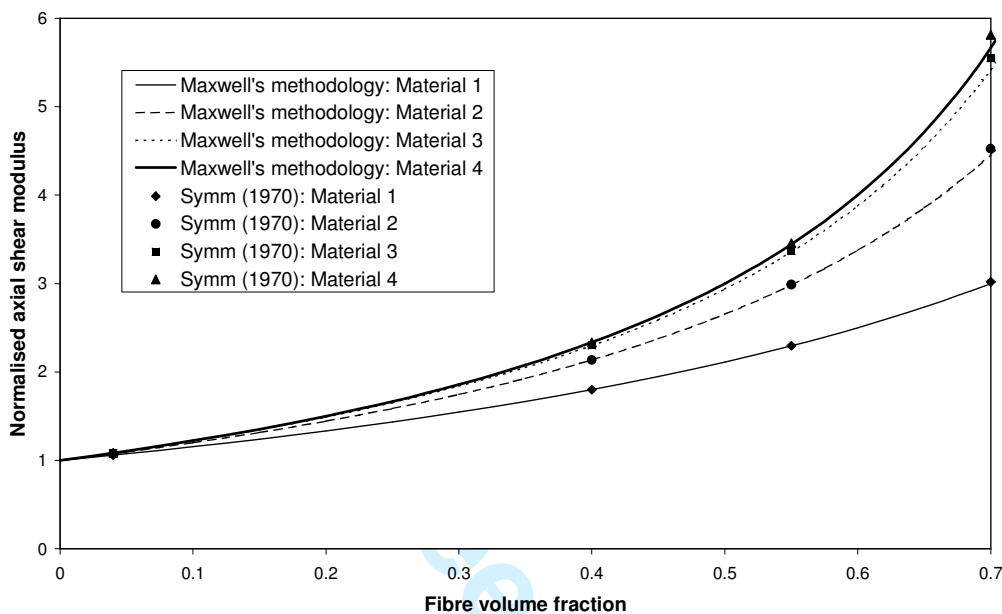


Figure 6 : Comparison of results for normalised effective axial shear modulus μ_A^{eff} obtained using Maxwell's methodology with those of Symm [27] for four different materials.

The above results have not tested the validity of the values of the elastic constants E_A^{eff} and ν_A^{eff} that can be estimated using the Equations (10.4) and (10.5). In reference [28], values were given for all the elastic constants calculated using a finite element analysis (Private communication, Li and Zou (2001)) for the two cases of hexagonal arrays of aligned carbon and glass fibres embedded in an isotropic epoxy matrix. The volume fraction for both types of fibre was taken to be 0.6. The fibre and matrix properties are given in Tables 1 and 3 of reference [28]. The carbon fibres were transversely isotropic while the glass fibres were assumed to be isotropic. Table 5 compares the finite element results to the various properties of two-phase composites that can be estimated using the results given in Section 10. It is seen that the effective properties E_A^{eff} , ν_A^{eff} , μ_A^{eff} and k_T^{eff} are estimated very accurately (errors less 0.5%) using Maxwell's methodology for both carbon and glass fibre composites. For the effective properties E_T^{eff} , ν_T^{eff} and μ_T^{eff} , the percentage errors for the carbon fibre composite are less than 3.7% while those for the glass fibre composite are less than 9.6%. These trends are consistent with those observed when comparing in Figures 2-6 predictions of results based on Maxwell's methodology with the boundary element results given by Eischen and Torquato [26] and by Symm [27].

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- Deleted: . Figure 6 shows a comparison of axial shear modulus μ_A^{eff} , obtained using the relation (10.7) resulting from Maxwell's methodology, with the results of Symm [18]. The normalised effective axial shear modulus is defined by μ_A^{eff} / μ_m , and the four materials considered are for isotropic fibres and matrix such that $\mu_f / \mu_m = 6, 20, 120, \infty$. For fibre volume fractions in the rang [... [90]
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Table 5: Comparison of effective properties estimated using Maxwell’s methodology with finite element results for carbon and glass fibre reinforced composites having a volume fraction of 0.6.

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Property	CFRP (FEA)	CFRP (Maxwell)	% Error	GRP (FEA)	GRP (Maxwell)	% Error
E_A^{eff}	136.70	136.7032	0.0023	45.76	45.7625	0.0055
E_T^{eff}	8.901	8.7564	1.62	11.80	11.0193	6.62
ν_A^{eff}	0.2526	0.2526	0	0.2515	0.2517	0.08
ν_T^{eff}	0.3082	0.3194	3.63	0.4020	0.4405	9.58
μ_A^{eff}	4.550	4.5365	0.30	4.339	4.3179	0.49
μ_T^{eff}	3.4020	3.3184	2.46	4.2083	3.8249	9.11
k_T^{eff}	6.5114	6.5110	0.0066	10.4355	10.4146	0.20

11. Discussion of results

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It is first noted that the Mori-Tanaka mean field result for the fourth order effective elastic property tensor L may be obtained (see for example [5]), in terms of those for the matrix L_m and the N phases of inclusion having properties L_r , $r = 1, \dots, N$, using a relation of the form

$$L^* = \left[L_m V_m + \sum_{r=1}^N V_p^r L_r \left[I + S L_m^{-1} (L_r - L_m) \right]^{-1} \right] \cdot \left(V_m I + \sum_{r=1}^N V_p^r \left[I + S L_m^{-1} (L_r - L_m) \right]^{-1} \right)^{-1}, \tag{11.1}$$

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where S is the fourth order tensor derived by Eshelby [18] and where I is the fourth order unit tensor. When a fourth order tensor L_m^* is defined by the relation

$$L_m^* = L_m (S^{-1} - I), \tag{11.2}$$

it can be shown that the relation (11.1) may be expressed in the following form that exhibits a mixtures structure for a fourth order tensor of the form $(L^* + L_m^*)^{-1}$

$$(L^* + L_m^*)^{-1} = V_m (L_m + L_m^*)^{-1} + \sum_{r=1}^N V_p^r (L_r + L_m^*)^{-1}. \tag{11.3}$$

The relation (11.3) follows directly from relations given in the literature (e.g. Norris [3], Weng [5], Walpole [6], [7], Qui and Weng [10], and Benveniste et al [11]). Its form is indicative of many of the results that have been derived in this paper (see Eqns. (5.12), (5.13), (9.8), (9.9), (10.7) and (10.8)).

When using Maxwell's methodology combined with Eshelby's far-field solution for the displacement in the matrix, the results (5.12) and (5.13) for the effective axial and shear moduli are easily derived for a composite having uniform distributions of aligned transversely isotropic spheroidal inclusions embedded in an isotropic matrix. These results are very simple in form showing that the effective shear moduli can be estimated using a mixtures relationship. It is noted that their form is a scalar equivalent of the result (11.3) defining the fourth order tensor L^* that is based on Mori-Tanaka theory. The situation regarding the non-shear elastic constants is far more complex. The analysis in Sections 7 and 8 leads to the results (8.9) for the effective elastic constants of the composite that depend on the parameters \bar{P} , \bar{Q} , \bar{R} and $\bar{S} = \bar{Q} - 3\bar{R}$, which are complicated functions of the properties and volume fractions of the inclusions, and of matrix properties.

For the case when the aspect ratios of all types of spheroidal inclusion are the same, Qiu and Weng [10] derived explicit and very complex expressions for the independent effective elastic constants p , m , k , l , and n describing the properties of a transversely isotropic composite, such that

$$p \equiv \mu_A, \quad m \equiv \mu_T, \quad k \equiv k_T, \quad l \equiv 2v_A k_T, \quad n \equiv E_A + 4k_T v_A^2. \quad (11.4)$$

The complexity of the expressions for the effective elastic constants arises because twelve different summations over the phases have to be performed. The approach derived in this paper involves only single summations when estimating the elastic constants μ_A and μ_T as seen from very simple results (5.12) and (5.13), and the four summations given by (7.24) when estimating the remaining elastic constants E_A , v_A and k_T using (8.9), i.e. six summations in total. In spite of their complexity, the results (8.9) and associated relations are, however, simpler than those given by Qiu and Weng [10].

As comparing algebraically the possible equivalence of the results of Qiu and Weng [10] for the elastic constants (11.4) with the corresponding results of this paper would be very laborious, values have been compared using exact numerical methods provided by the open source algebraic programming system REDUCE [29]. Exact agreement has been obtained for a wide range of parameter values. The numerical methods are based on the representation of the values of physical quantities and values of associated expressions by rational numbers having integer denominators and numerators. When performing the comparison, rational values are assumed for the tensor components S_{1111} , S_{2222} , S_{1122} , S_{2233} and S_{1212} . The remaining components of the S_{ijkl} tensor are then calculated using (3.10). It has been shown using these methods (based on the application of supplementary information C for a large range of values) that the results of this paper are exactly equivalent to the more complex results arising from the analysis of Qiu and Weng [10] based on Mori and Tanaka [1] theory. The principal result of this paper is thus obtained, namely, that the approximate Mori-Tanaka model, as implemented by Qiu and Weng [10] when estimating effective elastic properties, leads to results that are identical to the approximate relations obtained using Maxwell's methodology combined with Eshelby's far-field result for the displacement field. It is now useful to discuss other relevant results from the literature.

For the case of distributions of various types of isotropic spherical particles uniformly distributed in a matrix, having properties that are less than those of all reinforcements, it was shown [17] that the predictions of effective properties based on Maxwell's methodology were identical to the lower Hashin-Shtrikman bound [4], and that this bound was very close to

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accurate effective properties estimated by other methods. For the case of multi-phase fibre reinforced composites there is a similar situation. The lower bounds for the plane strain transverse bulk modulus and both the transverse and axial shear moduli of multi-phase composites derived by Hashin [30] can be manipulated so that they correspond exactly to the relations (10.3), (10.7) and (10.8) derived in this paper.

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For two-phase composites, predictions of many of the effective properties based on Maxwell's methodology are identical to those generated by the [composite sphere assemblage](#) and composite cylinder models, and they also correspond exactly to one of the variational bounds. These results strongly suggest that Maxwell's methodology is not restricted to dilute distributions of reinforcing inclusions. It can be shown for two-phase fibre reinforced composites that the relation (10.5) for the axial Young's modulus is identical to that which is obtained when using the concentric cylinders model of a unidirectional composite (Hashin and Rosen [31] for isotropic constituents, Hashin [32] for anisotropic constituents). While the comparisons of results shown in Figs. 2-6 indicate that estimates based on Maxwell's methodology become less accurate as the volume fraction of reinforcement increases, it is worth noting that as the volume fraction tends to unity, so that the system becomes almost homogenous without any matrix, the general relations (5.12), (5.13) for shear, and the specific relations (9.8), (9.9), (10.3)-(10.5), (10.7) and (10.8) all predict the expected result that the effective properties tend to the properties of the reinforcement. Distributions of very small inclusions would of course be needed to realise this limit in practice.

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For the special case where the matrix and spherical inclusions are isotropic and have the same shear modulus, the effective bulk modulus of an isotropic composite based on Mori-Tanaka theory [1] was shown by Weng [2] to correspond to the exact solution of Hill [33]. As Maxwell's methodology has been shown to predict elastic moduli that correspond exactly with Mori-Tanaka results, it follows that for this special case the results given in this paper for the effective bulk modulus must be exact for all volume fractions. For the special case where transversely isotropic spheroids have the form of aligned very thin circular discs, Weng [5] has shown that the Mori-Tanaka moduli are exact, implying that corresponding results derived for this limit using Maxwell's methodology will also be exact for all volume fractions.

It is indeed remarkable that the pioneering methodology developed by Maxwell [13], for predicting the effective properties of a composite, when combined with Eshelby's method [18] for predicting the far-field in the matrix for an isolated ellipsoid, are capable of predicting formulae for effective elastic properties that have been shown to correspond with many of those derived subsequently in the literature using alternative methods. A key characteristic of the approach is that a single method can be used to generate estimates for *all* the elastic constants of multi-phase composites having an isotropic matrix and reinforced with aligned transversely isotropic inclusions of the same aspect ratio, or with aligned fibres, or with spherical particles. The nature of the methodology is such that it has good potential for application to other situations of practical interest. For example, composites reinforced with inclusions having imperfect interfaces or one or more coatings of uniform thickness could easily be treated provided that the solution for an isolated inclusion is available. In the field of nano-composites, nano-inclusions (e.g. nano-particles and carbon nanotubes) are associated with interphase matrix regions adjacent to the inclusions having different properties to those of the bulk matrix. The interphase regions of these systems, which form due to nanoscale interactions between the embedded nano-inclusions and adjacent polymer chains, can occupy a substantial portion of the volume fraction of the composite due to the very large surface area

of nano-inclusion per unit volume available for interaction with local polymer chains. This effect results in significant changes to the effective properties of the polymer composite when compared to those of the bulk polymer (see for example Fisher et al [34], who consider nanotubes with adjacent interphase layers using the Mori-Tanaka method). Surface tension effects (see for example Duan et al [35], Mogilevskaya et al [36], [37]), which lead to discontinuities in the normal traction distributions at the nano-inclusion boundaries, can also modify the effective properties of the nano-composite.

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A common, and in fact incorrect, view of Maxwell's methodology is that it is expected to yield results valid only for very dilute concentrations of inclusions. However, it has been shown in this paper that, for spherical, fibrous and spheroidal inclusions, results based on Maxwell's methodology are in fact valid for a much wider range of volume fractions. An apparent contradiction has thus been identified. Maxwell's methodology implicitly neglects interactions between inclusions and yet it predicts accurate values (sometimes exact) for effective elastic constants for a wide range of volume fractions, up to 0.7 for some properties of fibre composites. In addition, for the case of spheroidal inclusions having the same aspect ratio, Maxwell's methodology has been shown to lead to expressions for all elastic constants that are apparently identical to the more complex results derived by Qiu and Weng [10], which are based on a mean field method of taking inclusion interactions into account. The only rational conclusion is that inclusion interaction effects for volume fractions of practical interest may not affect the far-field to any significant degree with the result that Maxwell's methodology, when combined with Eshelby's analysis, has much wider applicability than expected. Interesting questions beyond the scope of this paper are: why is this the case, and can a method be found of proving that the Maxwell procedure retains vital information concerning the effect of inclusion interactions?

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It is noted from the discussion in [17], that Bonnecaze and Brady [38] use a method of estimating the conductivity of a composite reinforced with cubic arrays of spherical particles that captures both far-field and near-field particle interactions. They compare their results with those of Sangani and Acrivos [39], providing numerical values of results in various tables. One type of estimate takes account only of dipole interactions, ignoring higher order terms, and it leads to results that appear coincident with results obtained using Maxwell's formula (agreement to 3 significant figures in most cases) for all volume fractions up to closest packing. This agreement was not noticed in the paper. They may have discovered a method that is showing why Maxwell's methodology works so well as the effect of a distribution of interacting dipoles at large distances may be identical to that for the case when they are all located at the same point, as assumed by Maxwell. As already mentioned above, further support of this idea is given in this paper for the case of rigid spherical particles and cavities (see Tables 1 and 2). It is noted that for the case of aligned cylindrical fibres, Mogilevskaya and Crouch [40]-[42], have developed a complex variable technique that can be used to investigate numerically the effect of fibre interactions at large distances from a cluster of fibres in an infinite matrix, including cases for homogeneously imperfect interfaces and uniform interphases.

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12. Conclusions

From the analysis presented in this paper the following conclusions may be drawn:

- (1) The methodology of Maxwell [13], which involves the use of the far-field deformation associated with a cluster of reinforcing inclusions, when combined with the far-field displacement result for an isolated inclusion of Eshelby [18], leads to a new method of estimating explicit formulae for all the elastic moduli of a multi-phase composite having an isotropic matrix that is reinforced by a uniform distribution of aligned transversely isotropic spheroidal inclusions having the same aspect ratio.
- (2) The use of the classical results of Maxwell and Eshelby has been shown capable of generating most of the explicit formulae that have been derived in the literature for the effective properties of composites having an isotropic matrix reinforced with aligned spheroidal inclusions, fibres or spherical particles. The results in this paper for the elastic moduli have been shown to correspond exactly to corresponding results based on a reformulation by Qiu and Weng [10] of Mori-Tanaka [1] mean field theory, and they have a simpler form. For the case of multi-phase composites reinforced with aligned transversely isotropic fibres and isotropic spherical particles, the Maxwell-Eshelby methodology leads to closed-form formulae for the relevant bulk modulus and shear moduli that correspond exactly with one of the Hashin bounds. A new relatively simple expression is derived for the effective axial modulus of a multi-phase fibre reinforced composite that reduces to the well-known composite cylinders assembly results when there are just two phases. The methodology leads to exact results for all volume fractions in two particular non-trivial cases.
- (3) Further research is needed to identify why Maxwell's methodology, which neglects inclusion interactions, is able to predict reasonably accurate effective properties for a range of volume fractions of reinforcement having practical relevance.

Acknowledgement

The author would like to acknowledge many fruitful technical interactions with Professor Anthony Kelly, CBE, FRS, over many years of collaboration in a wide variety of theoretical fields relating to the properties of, and damage formation in, composite materials. In particular, Professor Kelly is thanked for encouragement to investigate the literature for property predictions that enabled the author to become aware of the ingenious Maxwell methodology for predicting effective conductivities of particulate composites, which has led to results in a recent joint publication and then to the ideas and results presented in this paper.

The author would like to acknowledge suggestions by referees that have enabled improvements to this paper.

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For the case of multi-phase composites reinforced with aligned transverse isotropic fibres and isotropic spherical particles, the Maxwell-Eshelby methodology leads to closed-form formulae for the relevant bulk modulus and shear moduli that correspond exactly with one of the Hashin bounds. A new relatively simple expression is derived for the effective axial modulus of a multi-phase fibre reinforced composite that reduces to the well-known composite cylinders assembly results when there are just two phases.¶

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The results for effective properties can be expressed in a form having a mixtures structure that corresponds also to that for the fourth order elastic constants tensor that results from the mean field approach of Mori-Tanaka, which is the structure of many other results derived in the literature.¶

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<#>The use of the classical results of Maxwell and Eshelby has been shown capable of generating most of the explicit formulae that have been derived in the literature for the effective properties of composites having an isotropic matrix reinforced with aligned spheroidal inclusions, fibres or spherical particles.¶

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Figure captions

Fig.1 : Diagrams illustrating a) discrete model and b) effective medium model of a particulate composite having spherical reinforcements embedded in infinite isotropic matrix material.

Fig.2 : Comparison of results for normalised effective plane strain bulk modulus obtained using Maxwell's methodology with those of Eischen and Torquato [26], for three different materials.

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Fig.3 : Comparison of results for normalised effective transverse shear modulus obtained using Maxwell's methodology with those of Eischen and Torquato [26], for three different materials.

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Fig.4 : Comparison of results for the effective plane strain transverse Poisson's ratio obtained using Maxwell's methodology with those of Eischen and Torquato [26], for three different materials.

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Fig.5 : Comparison of results for the normalised effective plane strain transverse Young's modulus obtained using Maxwell's methodology with those of Eischen and Torquato [26], for three different materials.

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Fig.6 : Comparison of results for normalised effective axial shear modulus obtained using Maxwell's methodology with those of Symm [27], for four different materials.

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Table captions

Table 1: Effective properties for random distributions of rigid isotropic spherical particles.

Table 2: Effective properties for random distributions of spherical cavities.

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Table 3: Predictions for C_{1111} for distributions of spheroidal particles.

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Table 4: Predictions for C_{3333} for distributions of spheroidal particles.

Table 5: Comparison of effective properties estimated using Maxwell's methodology with finite element results for carbon and glass fibre reinforced composites having a volume fraction of 0.6.

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4 Figure 6 shows a comparison of axial shear modulus μ_A^{eff} , obtained using the
5 relation (10.7) resulting from Maxwell's methodology, with the results of Symm [18].
6 The normalised effective axial shear modulus is defined by $\mu_A^{\text{eff}} / \mu_m$, and the four
7 materials considered are for isotropic fibres and matrix such that $\mu_f / \mu_m = 6, 20, 120, \infty$.
8 For fibre volume fractions in the range $0 \leq V_f \leq 0.7$, the results predicted using
9 Maxwell's methodology agree very well (errors less than 2.6%) with the results of
10 Symm [18]. For larger volume fractions significant differences arise especially when the
11 fibres have a much larger shear modulus than the matrix.
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43 As establishing algebraically, using (11.4), the equivalence of the Qui and Weng
44 [10] results for the elastic constants k , l and n to the corresponding results of this paper is
45 very laborious, a comparison has again been undertaken using *exact* numerical methods
46 available in REDUCE [20, see supplementary information C]. Exact agreement has been
47 obtained for a wide range of parameter values. When performing the comparison in this
48 case, rational values are first assumed for the tensor components S_{1111} , S_{2222} , S_{1122} , S_{2233}
49 and S_{1212} . The remaining components of the tensor are then calculated using (3.10). This
50 approach avoids introducing irrational numbers, such as π , square roots and logarithmic
51 terms, which slow down computations and can require too much memory. It has been
52 shown using these methods that the results for k , l and n , and thus *all* effective elastic
53 constants derived in this paper, are exactly equivalent to the more complex results arising
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3 from the analysis of Qui and Weng [10] that is based on the approach of Mori and
4 Tanaka [1].
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7 It is noted that Torquato [21] has observed: i) this correspondence for the case of
8 isotropic ellipsoidal inclusions when using a formulation based on the fourth order elastic
9 constants described above, and ii) that the Mori and Tanaka results, and hence those in
10 this paper based on Maxwell's methodology, coincide with one of the bounds due to
11 Willis [22] depending on whether the matrix is stiffer or more compliant than all of the
12 inclusions, as shown by Weng [23].
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15 It has been shown that the general results for multi-phase spheroidal inclusions,
16 derived in Sections 5 and 8 based on Maxwell's methodology and Eshelby's far-field
17 results, reduce to the results, for homogeneous distributions of spherical particles leading
18 to isotropic effective elastic properties, which have been given in [14]. These equations
19 determine the isotropic bulk and shear moduli and they have exactly the same form as the
20 mixtures relation (11.3) that determines the fourth order elastic constants. Comparisons
21 in [14] with other methods of estimating effective properties indicate that the bulk
22 modulus predictions are expected to be more accurate than those for the shear modulus.
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26 For the special case when the spheroidal inclusions are aligned cylindrical fibres,
27 relatively simple results can be obtained, although the analysis remains complex. The
28 key non-shear results, which have a mixtures form, are given by the relations (10.3) for
29 the effective transverse bulk modulus k_T^{eff} (i.e. plane strain modulus), (10.4) determining
30 the effective axial Poisson's ratio ν_A^{eff} , and the relation (10.5) determining the effective
31 axial Young's modulus E_A^{eff} . The results (10.3) and (10.4) can also be derived using
32 similar methods to those described for spherical inclusions in [14], and their structure is
33 identical to the mixtures relation (11.3) for the fourth order effective elastic constants.
34 The result (10.5) for the axial Young's modulus for multi-phase unidirectional composite
35 is thought to be a new formula. Comparisons in Section 10 with other methods of
36 estimating effective properties indicate that predictions of the properties E_A^{eff} , ν_A^{eff} , μ_A^{eff}
37 and k_T^{eff} are expected to be more accurate than those for the properties E_T^{eff} , ν_T^{eff} and
38 μ_T^{eff} .
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SUPPLEMENTARY INFORMATION FOR ON-LINE VERSION

A: Derivation of solution of non-linear equations

In the paper it is required to solve the equations (7.20)-(7.22), which may be written in the form

$$L_1\bar{P} + L_2\bar{Q} + L_3\bar{R} + \Psi\Omega + 9k_m\mu_m + \frac{3k_m - 2\mu_m}{2B}\bar{P} - \frac{\mu_m}{B}\bar{Q} = \frac{S_{2211}}{B}\Omega, \quad (A1)$$

$$L_1\bar{P} + L_2\bar{Q} + L_3\bar{R} + \Psi\Omega + 9k_m\mu_m - \frac{\mu_m}{C}\bar{Q} - \frac{3k_m - 2\mu_m}{C}\bar{R} = \frac{S_{1122}}{C}\Omega, \quad (A2)$$

$$L_1\bar{P} + L_2\bar{Q} + L_3\bar{R} + \Psi\Omega + 9k_m\mu_m + \frac{2\mu_m}{D}\bar{Q} - \frac{3k_m + 4\mu_m}{D}\bar{R} = \frac{S_{1111}}{D}\Omega, \quad (A3)$$

where Ω is defined by (7.15). On subtracting (A1) and (A2)

$$\frac{3k_m - 2\mu_m}{2B}\bar{P} + \left(\frac{1}{C} - \frac{1}{B}\right)\mu_m\bar{Q} + \frac{3k_m - 2\mu_m}{C}\bar{R} = \left(\frac{S_{2211}}{B} - \frac{S_{1122}}{C}\right)\Omega. \quad (A4)$$

On subtracting (A1) and (A3)

$$\frac{3k_m - 2\mu_m}{2B}\bar{P} - \left(\frac{1}{B} + \frac{2}{D}\right)\mu_m\bar{Q} + \frac{3k_m + 4\mu_m}{D}\bar{R} = -\left(\frac{S_{1111}}{D} - \frac{S_{2211}}{B}\right)\Omega. \quad (A5)$$

On subtracting (A2) and (A3)

$$\left(\frac{1}{C} + \frac{2}{D}\right)\mu_m\bar{Q} + \left(\frac{3k_m - 2\mu_m}{C} - \frac{3k_m + 4\mu_m}{D}\right)\bar{R} = \left(\frac{S_{1111}}{D} - \frac{S_{1122}}{C}\right)\Omega. \quad (A6)$$

The relations (A4)-(A6) are now written

$$(3k_m - 2\mu_m)C\bar{P} + 2\mu_m(B - C)\bar{Q} + 2(3k_m - 2\mu_m)B\bar{R} = 2(CS_{2211} - BS_{1122})\Omega, \quad (A7)$$

$$(3k_m - 2\mu_m)D\bar{P} - 2\mu_m(2B + D)\bar{Q} + 2(3k_m + 4\mu_m)B\bar{R} = -2(BS_{1111} - DS_{2211})\Omega. \quad (A8)$$

$$(2C + D)\mu_m\bar{Q} + [(3k_m - 2\mu_m)D - (3k_m + 4\mu_m)C]\bar{R} = (CS_{1111} - DS_{1122})\Omega. \quad (A9)$$

The elimination of \bar{R} in (A7) and (A8) leads to the relation

$$\begin{aligned} & (3k_m - 2\mu_m)[(3k_m + 4\mu_m)C - (3k_m - 2\mu_m)D]\bar{P} \\ & + [9k_mB - (3k_m + 4\mu_m)C + (3k_m - 2\mu_m)D]2\mu_m\bar{Q} \\ & = \left[\begin{array}{l} \{(3k_m + 4\mu_m)C - (3k_m - 2\mu_m)D\}S_{2211} \\ - \{(3k_m + 4\mu_m)S_{1122} - (3k_m - 2\mu_m)S_{1111}\}B \end{array} \right] 2\Omega. \end{aligned} \quad (A10)$$

Using equations (7.17) and (7.18) of the paper, define parameters Γ and λ such that

$$\Gamma = (3k_m + 4\mu_m)C - (3k_m - 2\mu_m)D = 2(3k_m + \mu_m)B - (3k_m - 2\mu_m)A, \quad (\text{A11})$$

and

$$\begin{aligned} \lambda &= (3k_m + 4\mu_m)S_{1122} - (3k_m - 2\mu_m)S_{1111} \\ &= 2(3k_m + \mu_m)S_{2211} - (3k_m - 2\mu_m)(S_{2222} + S_{2233}). \end{aligned} \quad (\text{A12})$$

Substitution in (A10) then leads to

$$\Gamma \bar{P} + (A + B) 2\mu_m \bar{Q} = 2[(S_{2222} + S_{2233})B - S_{2211}A] \Omega. \quad (\text{A13})$$

The result (A9) can then be written

$$\Gamma \bar{R} - (2C + D)\mu_m \bar{Q} = (DS_{1122} - CS_{1111}) \Omega. \quad (\text{A14})$$

From relations (7.7) and (7.10) of the paper, the parameters A, B, C and D are given by

$$\begin{aligned} A &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)(S_{2222} + S_{2233}) - (3k_m + \mu_m)P_i - \mu_m Q_i] \frac{V_p^i}{\Delta_i}, \\ B &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)S_{2211} - \frac{1}{2}(3k_m - 2\mu_m)P_i + \mu_m Q_i] \frac{V_p^i}{\Delta_i}, \\ C &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)S_{1122} + \mu_m Q_i + (3k_m - 2\mu_m)R_i] \frac{V_p^i}{\Delta_i}, \\ D &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)S_{1111} - 2\mu_m Q_i + (3k_m + 4\mu_m)R_i] \frac{V_p^i}{\Delta_i}. \end{aligned} \quad (\text{A15})$$

Clearly

$$\begin{aligned} A + B &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)(S_{2211} + S_{2222} + S_{2233}) - \frac{9}{2}k_m P_i] \frac{V_p^i}{\Delta_i}, \\ 2C + D &= \sum_{i=1}^N [(Q_i R_i - P_i S_i)(S_{1111} + 2S_{1122}) + 9k_m R_i] \frac{V_p^i}{\Delta_i}, \end{aligned} \quad (\text{A16})$$

and

$$\begin{aligned} (S_{2222} + S_{2233})B - S_{2211}A &= \sum_{i=1}^N \left[\frac{1}{2}\lambda P_i + \mu_m (S_{2211} + S_{2222} + S_{2233})Q_i \right] \frac{V_p^i}{\Delta_i}, \\ DS_{1122} - CS_{1111} &= \sum_{i=1}^N \left[\lambda R_i - \mu_m (S_{1111} + 2S_{1122})Q_i \right] \frac{V_p^i}{\Delta_i}. \end{aligned} \quad (\text{A17})$$

The relations (A1)-(A3) are now written

$$L_1 B \bar{P} + L_2 B \bar{Q} + L_3 B \bar{R} + \Psi B \Omega + 9B k_m \mu_m + \frac{1}{2}(3k_m - 2\mu_m) \bar{P} - \mu_m \bar{Q} = S_{2211} \Omega, \quad (\text{A18})$$

$$L_1 C \bar{P} + L_2 C \bar{Q} + L_3 C \bar{R} + \Psi C \Omega + 9C k_m \mu_m - \mu_m \bar{Q} - (3k_m - 2\mu_m) \bar{R} = S_{1122} \Omega, \quad (\text{A19})$$

$$L_1 D \bar{P} + L_2 D \bar{Q} + L_3 D \bar{R} + \Psi D \Omega + 9 D k_m \mu_m + 2 \mu_m \bar{Q} - (3 k_m + 4 \mu_m) \bar{R} = S_{1111} \Omega. \quad (A20)$$

On multiplying (A19) by 2, adding to (A20) and then dividing by $(2C + D)$

$$L_1 \bar{P} + L_2 \bar{Q} + \left(L_3 - \frac{9 k_m}{2C + D} \right) \bar{R} = \left(\frac{S_{1111} + 2S_{1122}}{2C + D} - \Psi \right) \Omega - 9 k_m \mu_m. \quad (A21)$$

On using (A13) and (A14) to eliminate \bar{P} and \bar{R} in (A21), it follows that

$$\begin{aligned} & \left[(2C + D) \mu_m L_3 - 2(A + B) \mu_m L_1 + \Lambda L_2 - 9 k_m \mu_m \right] \bar{Q} \\ & = \left[\left(\frac{S_{1111} + 2S_{1122}}{2C + D} - \Psi \right) \Gamma - 2 \{ (S_{2222} + S_{2233}) B - S_{2211} A \} L_1 \right. \\ & \quad \left. - \frac{DS_{1122} - CS_{1111}}{2C + D} \{ (2C + D) L_3 - 9 k_m \} \right] \Omega - 9 k_m \mu_m \Gamma. \end{aligned} \quad (A22)$$

On using the relation (6.11) of the paper it follows that (A22) may be written

$$\begin{aligned} & (2C + D) \mu_m L_3 - 2(A + B) \mu_m L_1 + \Gamma L_2 - 9 k_m \mu_m \\ & = 9 k_m \mu_m \left[2S_{2211} C - S_{1111} A + 2S_{1122} B - (S_{2222} + S_{2233}) D - 1 \right], \end{aligned} \quad (A23)$$

and that

$$\begin{aligned} & \left(\frac{S_{1111} + 2S_{1122}}{2C + D} - \Psi \right) \Gamma - 2 \{ (S_{2222} + S_{2233}) B - S_{2211} A \} L_1 \\ & \quad - \frac{S_{1122} D - S_{1111} C}{2C + D} \{ (2C + D) L_3 - 9 k_m \} \\ & = \lambda \left[1 - 2S_{2211} C + (S_{2222} + S_{2233}) D + S_{1111} A - 2S_{1122} B \right] \\ & \quad + \left[(3 k_m - 2 \mu_m) A - 2(3 k_m + \mu_m) B \right] \left[S_{1111} (S_{2222} + S_{2233}) - 2S_{1122} S_{2211} \right]. \end{aligned} \quad (A24)$$

It then follows from (A22)-(A24) that

$$\Phi \bar{Q} = (\Gamma \Psi - \lambda \Phi) \frac{\Omega}{9 k_m \mu_m} - \Gamma, \quad (A25)$$

where

$$\Phi = 2S_{2211} C - (S_{2222} + S_{2233}) D - S_{1111} A + 2S_{1122} B - 1. \quad (A26)$$

On using (A15) and (A26)

$$\Phi = \sum_{i=1}^N \left[\begin{aligned} & 2(Q_i R_i - P_i S_i) \Psi + \{ S_{1111} (3 k_m + \mu_m) - S_{1122} (3 k_m - 2 \mu_m) \} P_i \\ & + \{ (S_{1111} + 2S_{1122}) + 2(S_{2211} + S_{2222} + S_{2233}) \} \mu_m Q_i \\ & + \{ 2S_{2211} (3 k_m - 2 \mu_m) - (S_{2222} + S_{2233}) (3 k_m + 4 \mu_m) \} R_i \end{aligned} \right] \frac{V_P^i}{\Delta_i} - 1. \quad (A27)$$

From (A13) and (A14)

$$\Gamma \bar{P} = 2[(S_{2222} + S_{2233})B - S_{2211}A] \Omega - 2(A + B)\mu_m \bar{Q}. \quad (A28)$$

$$\Gamma \bar{R} = (S_{1122}D - S_{1111}C)\Omega + (2C + D)\mu_m \bar{Q}. \quad (A29)$$

On substituting for \bar{Q} using (A25)

$$\Phi \bar{P} = \left[2\{(S_{2222} + S_{2233})B - S_{2211}A\} \Phi - \frac{2}{9k_m}(\Gamma\Psi - \lambda\Phi)(A + B) \right] \frac{\Omega}{\Gamma} + 2\mu_m(A + B), \quad (A30)$$

$$\Phi \bar{R} = \left[\{S_{1122}D - S_{1111}C\} \Phi + \frac{1}{9k_m}(\Gamma\Psi - \lambda\Phi)(2C + D) \right] \frac{\Omega}{\Gamma} - \mu_m(2C + D). \quad (A31)$$

It follows from (A15) that

$$\begin{aligned} A &= (S_{2222} + S_{2233})W - (3k_m + \mu_m)X - \mu_m Y, \\ B &= S_{2211}W - \frac{1}{2}(3k_m - 2\mu_m)X + \mu_m Y, \\ C &= S_{1122}W + \mu_m Y + (3k_m - 2\mu_m)Z, \\ D &= S_{1111}W - 2\mu_m Y + (3k_m + 4\mu_m)Z, \end{aligned} \quad (A32)$$

Where

$$W = \sum_{i=1}^N V_p^i \frac{Q_i R_i - P_i S_i}{\Delta_i}, \quad X = \sum_{i=1}^N V_p^i \frac{P_i}{\Delta_i}, \quad Y = \sum_{i=1}^N V_p^i \frac{Q_i}{\Delta_i}, \quad Z = \sum_{i=1}^N V_p^i \frac{R_i}{\Delta_i}. \quad (A33)$$

It follows from (A16) and (A17) that

$$\begin{aligned} A + B &= (S_{2211} + S_{2222} + S_{2233})W - \frac{9}{2}k_m X, \\ 2C + D &= (S_{1111} + 2S_{1122})W + 9k_m Z, \end{aligned} \quad (A34)$$

$$\begin{aligned} (S_{2222} + S_{2233})B - S_{2211}A &= \frac{1}{2}\lambda X + \mu_m(S_{2211} + S_{2222} + S_{2233})Y, \\ DS_{1122} - CS_{1111} &= \lambda Z - \mu_m(S_{1111} + 2S_{1122})Y, \end{aligned} \quad (A35)$$

and

$$\Gamma = \lambda W + 9k_m \mu_m Y, \quad (A36)$$

and from (A26)

$$\begin{aligned} \Phi &= 2\Psi W + [(3k_m + \mu_m)S_{1111} - (3k_m - 2\mu_m)S_{1122}]X \\ &\quad + [S_{1111} + 2(S_{1122} + S_{2211} + S_{2222} + S_{2233})] \mu_m Y \\ &\quad + [2(3k_m - 2\mu_m)S_{2211} - (3k_m + 4\mu_m)(S_{2222} + S_{2233})]Z - 1. \end{aligned} \quad (A37)$$

Consider now

$$\begin{aligned}
\lambda\Phi - \Gamma\Psi &= \lambda\Psi W + \lambda[(3k_m + \mu_m)S_{1111} - (3k_m - 2\mu_m)S_{1122}]X \\
&+ [\lambda S_{1111} + 2\lambda(S_{1122} + S_{2211} + S_{2222} + S_{2233}) - 9k_m\Psi] \mu_m Y \quad (A38) \\
&+ \lambda[2(3k_m - 2\mu_m)S_{2211} - (3k_m + 4\mu_m)(S_{2222} + S_{2233})]Z - \lambda.
\end{aligned}$$

Substitution in (A25), (A30) and (A31) leads to

$$\begin{aligned}
\Phi \bar{P} &= \left[\begin{aligned} &\{9k_m\mu_m X - 2\mu_m(S_{2211} + S_{2222} + S_{2233})W\} \Psi \\ &+ 2\mu_m(S_{2211} + S_{2222} + S_{2233})\Phi \end{aligned} \right] \frac{\Omega}{9k_m\mu_m} \\
&- \{9k_m\mu_m X - 2\mu_m(S_{2211} + S_{2222} + S_{2233})W\}, \quad (A39)
\end{aligned}$$

$$\Phi \bar{Q} = \left[\{9k_m\mu_m Y + \lambda W\} \Psi - \lambda\Phi \right] \frac{\Omega}{9k_m\mu_m} - \{9k_m\mu_m Y + \lambda W\}, \quad (A40)$$

$$\begin{aligned}
\Phi \bar{R} &= \left[\{9k_m\mu_m Z + \mu_m(S_{1111} + 2S_{1122})W\} \Psi - \mu_m(S_{1111} + 2S_{1122})\Phi \right] \frac{\Omega}{9k_m\mu_m} \\
&- \{9k_m\mu_m Z + \mu_m(S_{1111} + 2S_{1122})W\}. \quad (A41)
\end{aligned}$$

These results are of the form (7.25)-(7.31) given in Section 7.

B: Analysis for transversely isotropic cylindrical inclusions

Since $S_{1111} = S_{1122} = 0$ for the fibre case, it follows from (6.11), (7.3) and (7.6) that $\Psi = 0$ and

$$\Delta_i = \mu_m \frac{1+v_m}{1-v_m} (Q_i - 2R_i) - 3k_m \frac{1-2v_m}{2(1-v_m)} R_i + 9k_m \mu_m, \quad (B1)$$

$$\bar{\Delta} = \mu_m \frac{1+v_m}{1-v_m} (\bar{Q} - 2\bar{R}) - 3k_m \frac{1-2v_m}{2(1-v_m)} \bar{R} + 9k_m \mu_m.$$

The equations (7.11)-(7.14) reduce to

$$\frac{\Omega}{2(1-v_m)} - (3k_m + \mu_m) \bar{P} - \mu_m \bar{Q} = A \bar{\Delta}, \quad (B2)$$

$$\frac{v_m \Omega}{2(1-v_m)} - \frac{1}{2}(3k_m - 2\mu_m) \bar{P} + \mu_m \bar{Q} = B \bar{\Delta}. \quad (B3)$$

$$\mu_m \bar{Q} + (3k_m - 2\mu_m) \bar{R} = C \bar{\Delta}, \quad (B4)$$

$$-2\mu_m \bar{Q} + (3k_m + 4\mu_m) \bar{R} = D \bar{\Delta}. \quad (B5)$$

From (B4) and (B5)

$$9k_m \bar{R} = (2C + D) \bar{\Delta}, \quad (B6)$$

$$3\mu_m (\bar{Q} - 2\bar{R}) = (C - D) \bar{\Delta}. \quad (B7)$$

Substitution in (B1) leads to the following expression for $\bar{\Delta}$

$$\frac{1}{\bar{\Delta}} = \frac{1-v_m - v_m C + \frac{1}{2} D}{9k_m \mu_m (1-v_m)} = \frac{1}{9k_m \mu_m} \left(1 + \frac{D - 2v_m C}{2(1-v_m)} \right). \quad (B8)$$

Consider now the constants C and D defined by (7.10). As $S_{1111} = S_{1122} = 0$, it follows using (B1) that

$$\Delta_i = \frac{9k_m \mu_m}{3k_m + 4\mu_m} (Q_i - 3R_i + 3k_m + 4\mu_m). \quad (B9)$$

$$C = \sum_{i=1}^N [\mu_m Q_i + (3k_m - 2\mu_m) R_i] \frac{V_p^i}{\Delta_i}, \quad (B10)$$

$$D = \sum_{i=1}^N [-2\mu_m Q_i + (3k_m + 4\mu_m) R_i] \frac{V_p^i}{\Delta_i}. \quad (B11)$$

On using (7.2) it then follows that

$$\Delta_i = 27k_m \mu_m \frac{k_T^i + \mu_m}{3k_m + 4\mu_m}, \quad (B12)$$

$$C = \frac{3k_m + 4\mu_m}{27k_m \mu_m} \sum_{i=1}^N V_p^i \left[2(3k_m + \mu_m) v_A^i - (3k_m - 2\mu_m) \right] \frac{k_T^i}{k_T^i + \mu_m}, \quad (B13)$$

$$D = \frac{3k_m + 4\mu_m}{27k_m \mu_m} \sum_{i=1}^N V_p^i \left[2(3k_m - 2\mu_m) v_A^i - (3k_m + 4\mu_m) + \frac{9k_m \mu_m}{k_T^i} \right] \frac{k_T^i}{k_T^i + \mu_m}. \quad (B14)$$

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It can be shown that

$$2C + D = \frac{3k_m + 4\mu_m}{3\mu_m} \sum_{i=1}^N V_p^i \left(2v_A^i - 1 + \frac{\mu_m}{k_T^i} \right) \frac{k_T^i}{k_T^i + \mu_m}, \quad (B15)$$

$$C - D = \frac{3k_m + 4\mu_m}{9k_m} \sum_{i=1}^N V_p^i \left[2(1 + v_A^i) - \frac{3k_m}{k_T^i} \right] \frac{k_T^i}{k_T^i + \mu_m}. \quad (B16)$$

It then follows from (B6) and (B7) that

$$\bar{Q} = \bar{\Delta} \frac{3k_m + 4\mu_m}{27k_m\mu_m} \sum_{i=1}^N V_p^i \left[6v_A^i - \frac{(3k_m - 2\mu_m)}{k_T^i} \right] \frac{k_T^i}{k_T^i + \mu_m}, \quad (B17)$$

$$\bar{R} = \bar{\Delta} \frac{3k_m + 4\mu_m}{27k_m\mu_m} \sum_{i=1}^N V_p^i \left(2v_A^i - 1 + \frac{\mu_m}{k_T^i} \right) \frac{k_T^i}{k_T^i + \mu_m}. \quad (B18)$$

From (B8)

$$\frac{1}{\bar{\Delta}} = \left(\frac{3k_m + 4\mu_m}{9k_m\mu_m} + \frac{2C + D}{9k_m} - \frac{C - D}{3\mu_m} \right) \frac{1}{3k_m + 4\mu_m}, \quad (B19)$$

so that on using (B15) and (B16)

$$\frac{1}{\bar{\Delta}} = \frac{1}{9k_m\mu_m} \left(V_m + \left(k_m + \frac{4}{3}\mu_m \right) \sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} \right). \quad (B20)$$

It then follows from (7.5), (B17) and (B18) that

$$\bar{Q} \equiv 6v_A^{\text{eff}} k_T^{\text{eff}} - 3k_m + 2\mu_m = \frac{\sum_{i=1}^N \left[6v_A^i k_T^i - 3k_m + 2\mu_m \right] \frac{V_p^i}{k_T^i + \mu_m}}{\sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{3V_m}{3k_m + 4\mu_m}}, \quad (B21)$$

$$\bar{R} \equiv \left(2v_A^{\text{eff}} - 1 \right) k_T^{\text{eff}} + \mu_m = \frac{\sum_{i=1}^N \left[\left(2v_A^i - 1 \right) k_T^i + \mu_m \right] \frac{V_p^i}{k_T^i + \mu_m}}{\sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{3V_m}{3k_m + 4\mu_m}}. \quad (B22)$$

On eliminating v_A^{eff} in (B21) and (B22)

$$3(k_T^{\text{eff}} + \mu_m) - (3k_m + 4\mu_m) = \frac{\sum_{i=1}^N \left[3(k_T^i + \mu_m) - (3k_m + 4\mu_m) \right] \frac{V_p^i}{k_T^i + \mu_m}}{\sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{3V_m}{3k_m + 4\mu_m}}. \quad (B23)$$

It then follows, on using the relation $k_T^m = k_m + \frac{1}{3}\mu_m$, that the result (B23) may be written as the following mixtures relationship

$$\frac{1}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{V_m}{k_T^m + \mu_m}. \quad (B24)$$

From (B21)

$$v_A^{\text{eff}} k_T^{\text{eff}} = \frac{(3k_m - 2\mu_m) V_m + \sum_{i=1}^N V_p^i \frac{v_A^i k_T^i}{k_T^i + \mu_m}}{\sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{V_m}{k_T^m + \mu_m}}. \quad (\text{B25})$$

Since

$$\frac{v_m k_T^m}{k_T^m + \mu_m} = \frac{3k_m - 2\mu_m}{2(3k_m + 4\mu_m)}, \quad (\text{B26})$$

it follows that

$$v_A^{\text{eff}} k_T^{\text{eff}} = \frac{\sum_{i=1}^N V_p^i \frac{v_A^i k_T^i}{k_T^i + \mu_m} + \frac{V_m v_m k_T^m}{k_T^m + \mu_m}}{\sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + \frac{V_m}{k_T^m + \mu_m}}. \quad (\text{B27})$$

On using (B24) the following mixtures relation is obtained

$$\frac{v_A^{\text{eff}} k_T^{\text{eff}}}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N V_p^i \frac{v_A^i k_T^i}{k_T^i + \mu_m} + \frac{V_m v_m k_T^m}{k_T^m + \mu_m}. \quad (\text{B28})$$

On subtracting (B2) and (B3) to obtain a value for Ω and then substituting in (7.15)

$$[(3k_m + 4\mu_m) + \bar{Q} - 3\bar{R}] \bar{P} = \bar{Q}\bar{R} - 4\mu_m \bar{Q} - 2(A - B) \bar{\Delta}. \quad (\text{B29})$$

Thus, on using (7.5)

$$3(k_T^{\text{eff}} + \mu_m) \bar{P} = [(2v_A^{\text{eff}} - 1)k_T^{\text{eff}} - 3\mu_m] (6v_A^{\text{eff}} k_T^{\text{eff}} - 3k_m + 2\mu_m) - 2(A - B) \bar{\Delta}. \quad (\text{B30})$$

From (7.7) and on using the appropriate values of S_{ijk} relevant to cylinders

$$A = \sum_{i=1}^N \left[\frac{Q_i R_i - P_i S_i}{2(1 - v_m)} - (3k_m + \mu_m) P_i - \mu_m Q_i \right] \frac{V_p^i}{\Delta_i}, \quad (\text{B31})$$

$$B = \sum_{i=1}^N \left[\frac{(Q_i R_i - P_i S_i) v_m}{2(1 - v_m)} - \frac{1}{2}(3k_m - 2\mu_m) P_i + \mu_m Q_i \right] \frac{V_p^i}{\Delta_i}. \quad (\text{B32})$$

It then follows that

$$2(A - B) = \sum_{i=1}^N [(Q_i R_i - P_i S_i) - (3k_m + 4\mu_m) P_i - 4\mu_m Q_i] \frac{V_p^i}{\Delta_i}. \quad (\text{B33})$$

This relation is now written on using (7.2) and (B12)

$$2(A - B) \frac{9k_m \mu_m}{3k_m + 4\mu_m} = 2\mu_m \left(k_m - \frac{2}{3}\mu_m \right) \sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} - 2 \left(k_m + \frac{4}{3}\mu_m \right) \sum_{i=1}^N \frac{V_p^i k_T^i v_A^i}{k_T^i + \mu_m} - \sum_{i=1}^N V_p^i E_A^i - 4\mu_m \sum_{i=1}^N V_p^i (v_A^i)^2 + \left(k_m + \frac{4}{3}\mu_m \right) \sum_{i=1}^N V_p^i + 4\mu_m^2 \sum_{i=1}^N V_p^i \frac{(v_A^i)^2}{k_T^i + \mu_m}. \quad (\text{B34})$$

It should be noted from (B20) and (B24) that

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$$\bar{\Delta} = 27k_m\mu_m \frac{k_T^{\text{eff}} + \mu_m}{3k_m + 4\mu_m}. \quad (\text{B35})$$

Substitution in (B30) leads to

$$\bar{P} = \left[(2v_A^{\text{eff}} - 1)k_T^{\text{eff}} - 3\mu_m \right] \frac{6v_A^{\text{eff}}k_T^{\text{eff}} - (3k_m - 2\mu_m)}{3(k_T^{\text{eff}} + \mu_m)} - 2(A - B) \frac{9k_m\mu_m}{3k_m + 4\mu_m}. \quad (\text{B36})$$

| On using (7.5) it follows that

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$$E_A^{\text{eff}} = 2\mu_m(1 - 3v_A^{\text{eff}}) - \left[(2v_A^{\text{eff}} - 1)k_T^{\text{eff}} - 3\mu_m \right] \frac{6v_A^{\text{eff}}\mu_m + (3k_m - 2\mu_m)}{3(k_T^{\text{eff}} + \mu_m)} - 2(A - B) \frac{9k_m\mu_m}{3k_m + 4\mu_m}. \quad (\text{B37})$$

From (B34) and (B37)

$$E_A^{\text{eff}} = (1 - 2v_A^{\text{eff}})k_m + \frac{4}{3}\mu_m(1 + v_A^{\text{eff}})(1 - 3v_A^{\text{eff}}) + \frac{2\mu_m(1 + v_A^{\text{eff}})}{k_T^{\text{eff}} + \mu_m} \left[2v_A^{\text{eff}}\mu_m + (k_m - \frac{2}{3}\mu_m) \right] - 2\mu_m(k_m - \frac{2}{3}\mu_m) \sum_{i=1}^N \frac{V_p^i}{k_T^i + \mu_m} + 2(k_m + \frac{4}{3}\mu_m) \sum_{i=1}^N \frac{V_p^i k_T^i v_A^i}{k_T^i + \mu_m} + \sum_{i=1}^N V_p^i E_A^i + 4\mu_m \sum_{i=1}^N V_p^i (v_A^i)^2 - (k_m + \frac{4}{3}\mu_m) \sum_{i=1}^N V_p^i - 4\mu_m^2 \sum_{i=1}^N V_p^i \frac{(v_A^i)^2}{k_T^i + \mu_m}. \quad (\text{B38})$$

On using (B24) and (B28) it can be shown after some calculation that the following mixtures relationship is valid that determines the effective axial modulus E_A^{eff} of the composite

$$E_A^{\text{eff}} + \frac{4k_T^{\text{eff}}(v_A^{\text{eff}})^2\mu_m}{k_T^{\text{eff}} + \mu_m} = \sum_{i=1}^N V_p^i \left(E_A^i + \frac{4k_T^i(v_A^i)^2\mu_m}{k_T^i + \mu_m} \right) + V_m \left(E_m + \frac{4k_T^m v_m^2 \mu_m}{k_T^m + \mu_m} \right), \quad (\text{B39})$$

where

$$k_T^m = k_m + \frac{1}{3}\mu_m. \quad (\text{B40})$$

C: REDUCE program

The following code can be used with the algebraic programming system REDUCE to check that general results given in the paper agree *exactly* with the results of Qiu and Weng.

```

1 % symbol at start of a line, or towards end of a line, denotes a comment which is ignored by
2 % REDUCE
3
4 % A REDUCE program to check analysis in paper agrees exactly with results of Qiu & Weng (1992)
5
6 % Written by L N McCartney, October 2009
7
8 % The following code calculates and outputs various residuals denoted res(x) which should all
9 % be zero
10
11 off echo$ % Stops input instructions being shown in output file
12
13 % Only statements terminated by ; are printed in output file, i.e. echoed
14 % Statements terminated with $ are not echoed
15
16 sn:= 5$ % No. of reinforcing phases in composite
17
18 % array declarations
19
20 array v(sn), vp(sn), ea(sn), et(sn), na(sn), nt(sn), ga(sn), gt(sn), kt(sn)$
21 array k(sn), l(sn), ll(sn), m(sn), n(sn), p(sn), q(sn), r(sn), s(sn), del(sn)$
22 array c(sn), d(sn), e(sn), f(sn), g(sn), h(sn)$
23
24 % on rounded;
25 % When switched on, this outputs approximate results in decimal form rather than as exact
26 % fractions
27
28 % Set properties (in rational number form) for matrix
29
30 em:= 5$ % Young's modulus
31 nm:= 3/11$ % Poisson's ratio
32 km:= em/3/(1 - 2*nm)$ % Bulk modulus
33 gm:= em/2/(1 + nm)$ % Shear modulus
34 ktm:= km + gm/3$ % Plane strain transverse bulk modulus
35
36 vm:= 4/10$ % Volume fraction of matrix
37
38 % Matrix values for parameters k, l, m, n and p used by Qiu & Weng
39
40 k(0):= ktm$
41 l(0):= 2*ktm*nm$
42 m(0):= gm$
43 n(0):= em + l(0)*l(0)/k(0)$
44 p(0):= gm$
45
46 % Set properties (in rational number form) for each reinforcing species
47
48 for i:=1:sn do
49 << ea(i):= 200 * (1 + 4/10*i)/sn;
50 na(i):= 1/2 * (11/10 + 1/2*i)/sn;
51 kt(i):= 200 * (4/10 + 3/10*i)/sn;
52 ga(i):= 200 * (6/10 + 6/10*i)/sn;
53 gt(i):= 200 * (12/10 + 9/10*i)/sn;
54 vp(i):= 1;
55 k(i):= kt(i)$
56 l(i):= 2*k(i)*na(i)$
57 m(i):= gt(i)$
58 n(i):= ea(i) + l(i)*l(i)/k(i)$
59 p(i):= ga(i) >>$
60
61 % Normalise volume fractions so they sum to 1 exactly
62
63 s1:= 0$
64 for i:= 1:sn do
65 << s1:= s1 + vp(i) >>$
66 for i:=1:sn do
67 << vp(i):= vp(i) / s1 * (1 - vm) >>$
68
69 % Check sum of volume fractions is unity
70
71 s1:= 0$
72 for i:=1:sn do
73 << s1:= s1 + vp(i)$
74 v(i):= vp(i) >>$
75 s1:= s1 + vm$
76 res1:= 1 - s1;
77
78 % Set matrix volume fraction in the v array

```

```

1
2 v(0):= vm$
3
4 % Calculate Sijkl for spheroid using only rational values
5 % Method can use any Sijkl satisfying some essential conditions
6
7 S1111:= 3/10$
8 S2222:= 6/10$
9 S1122:= 75/10000$
10 S2233:= 46/1000$
11 S1212:= 22/100$
12
13 % Calculate dependent values of Sijkl
14
15 S2323:= (S2222 - S2233)/2$
16 S2211:= nm*(S2222 + S2233 - S1111 - S1122) + S1122$
17 S3311:= S2211$
18 S1133:= S1122$
19 S3333:= S2222$
20 S3322:= S2233$
21
22 % Qiu and Weng (1990) solution
23
24 for r:=0:sn do
25   << w1:= 2*(k(r) - k(0))*(1 - nm)*(S2222 + S2233) - 2*nm*S2211$
26     w2:= 2*(l(r) - l(0))*(1 - nm)*S1122 - nm*S1111$
27     c(r):= 1 + w1/em + w2/em$
28     w1:= (n(r) - n(0))*(S1111 - 2*nm*S1122)$
29     w2:= 2*(l(r) - l(0))*(1 - nm)*S1122 - nm*S1111$
30     d(r):= 1 + w1/em + w2/em$
31     e(r):= 1 + 2*(m(r) - m(0))/m(0)*S2323$
32     f(r):= 1 + 2*(p(r) - p(0))/p(0)*S1212$
33     w1:= 2*(k(r) - k(0))*(1 - nm)*S1122 - nm*S1111$
34     w2:= (l(r) - l(0))*(S1111 - 2*nm*S1122)$
35     g(r):= w1/em + w2/em$
36     w1:= (n(r) - n(0))*(S2211 - nm*(S2222 + S2233))$
37     w2:= (l(r) - l(0))*(1 - nm)*(S2222 + S2233) - 2*nm*S2211$
38     h(r):= w1/em + w2/em$
39     ll(r):= c(r)*d(r) - 2*g(r)*h(r) >>$
40
41 res2:= l(0) - km + 2/3*gm;
42 res3:= n(0) - km - 4/3*gm;
43
44 s1:= 0$
45 s2:= 0$
46 s3:= 0$
47 s4:= 0$
48 s5:= 0$
49 s6:= 0$
50
51 for r:=0:sn do
52   << s1:= s1 + 2*v(r)*(k(r)*d(r) - l(r)*g(r))/l1(r)$
53     s2:= s2 + v(r)*(n(r)*c(r) - 2*l(r)*h(r))/l1(r)$
54     s3:= s3 + 2*v(r)*m(r)/e(r)$
55     s4:= s4 + 2*v(r)*p(r)/f(r)$
56     s5:= s5 + v(r)*(l(r)*d(r) - n(r)*g(r))/l1(r)$
57     s6:= s6 + v(r)*(l(r)*c(r) - 2*k(r)*h(r))/l1(r) >>$
58
59 ccla:= s1$
60 dcla:= s2$
61 ecla:= s3$
62 fcla:= s4$
63 gcla:= s5$
64 hcla:= s6$
65
66 s1:= 0$
67 s2:= 0$
68 s3:= 0$
69 s4:= 0$
70 s5:= 0$
71 s6:= 0$
72
73 for r:=0:sn do
74   << s1:= s1 + v(r)*c(r)/l1(r)$
75     s2:= s2 + v(r)*d(r)/l1(r)$
76     s3:= s3 + v(r)/e(r)$
77     s4:= s4 + v(r)/f(r)$
78     s5:= s5 + v(r)*g(r)/l1(r)$
79     s6:= s6 + v(r)*h(r)/l1(r) >>$
80
81 lcai:= s1*s2 - 2*s5*s6$
82 ccgai:= s1/lcai$
83 dcgai:= s2/lcai$
84 ecgai:= 1/s3$
85 fcgai:= 1/s4$
86 gcgai:= s5/lcai$
87 hcgai:= s6/lcai$
88
89 kc:= (ccla*ccgai + 2*hcla*gcgai)/2$

```

```

1
2 lc:= gcla*ccai + dcla*gcasi
3 lcp:= hcla*dcai + ccla*hcai$
4 res4:= lc - lcp;
5 nc:= dcla*dcai + 2*gcla*hcai$
6 mc:= ecla*ecai/2$
7 pc:= fcla*fcasi/2$
8
9 EAC1:= nc - lc*lc/kc$
10 NAC1:= lc/2/kc$
11 KTC1:= kc$
12 GTC1:= mc$
13 GAC1:= pc$
14
15 % Solution based on analysis in this paper
16
17 s1:= 0$
18 gts:= (1/2/S2323 - 1)*gm$
19 for i:= 1:sn do
20   << w:= v(i)/(gt(i) + gts)$
21     s1:= s1 + w >>$
22 s1:= s1 + vm/(gm + gts)$
23 GTC2:= 1/s1 - gts$
24
25 s1:= 0$
26 for i:=1:sn do
27   << w:= v(i)/(gm/(gt(i) - gm) + 2*S2323)$
28     s1:= s1 + w >>$
29 s1:= 1/s1 - 2*S2323$
30 GTC3:= gm/s1 + gm$
31
32 s1:= 0$
33 gas:= (1/2/S1212 - 1)*gm$
34 for i:= 1:sn do
35   << w:= v(i)/(ga(i) + gas)$
36     s1:= s1 + w >>$
37 s1:= s1 + vm/(gm + gas)$
38 GAC2:= 1/s1 - gas$
39
40 s1:= 0$
41 for i:=1:sn do
42   << w:= v(i)/(gm/(ga(i) - gm) + 2*S1212)$
43     s1:= s1 + w >>$
44 s1:= 1/s1 - 2*S1212$
45 GAC3:= gm/s1 + gm$
46
47 % Calculate L coefficients and P, Q, R, S, DEL for each I using (6.8), (6.10) and (6.11)
48
49 l1:= (3*km + gm)*S1111 - (3*km - 2*gm)*S1122$
50 l2:= (S1111 + 2*(S1122 + S2211 + S2222 + S2233))*gm$
51 l3:= 2*(3*km - 2*gm)*S2211 - (3*km + 4*gm)*(S2222 + S2233)$
52 psi:= 2*S1122*S2211 - S1111*(S2222 + S2233)$
53
54 for i:=1:sn do
55   << p(i):= ea(i) + 2*na(i)*(2*na(i) - 1)*kt(i) - 2*gm$
56     q(i):= 6*na(i)*kt(i) - 3*km + 2*gm$
57     r(i):= (2*na(i) - 1)*kt(i) + gm$
58     s(i):= 3*kt(i) - 3*km - gm$
59     del(i):= l1*p(i) + l2*q(i) + l3*r(i) + psi*(q(i)*r(i) - p(i)*s(i)) + 9*km*gm >>$
60
61 % Calculate W, X, Y, Z using (7.24)
62
63 s1:= 0$
64 s2:= 0$
65 s3:= 0$
66 s4:= 0$
67 for i:=1:sn do
68   << s1:= s1 + v(i)*(q(i)*r(i) - p(i)*s(i))/del(i)$
69     s2:= s2 + v(i)*p(i)/del(i)$
70     s3:= s3 + v(i)*q(i)/del(i)$
71     s4:= s4 + v(i)*r(i)/del(i) >>$
72
73 W:= s1$
74 X:= s2$
75 Y:= s3$
76 Z:= s4$
77
78 % Calculate SLA, PHI using (7.31) and (7.28)
79
80 sla:= (3*km + 4gm)*S1122 - (3*km - 2*gm)*S1111$
81 phi:= l1*X + l2*Y + l3*Z + 2*psi*W - 1$
82
83 % Calculate alpha and beta coefficients using (7.30)
84
85 a1:= 2*gm*(S2211 + S2222 + S2233)*phi$
86 be1:= 9*km*gm*X - 2*gm*(S2211 + S2222 + S2233)*W$
87 a2:= - sla*phi$
88 be2:= 9*km*gm*Y + sla*W$

```

```

1
2 a13:= - gm*(S1111 + 2*S1122)*phi$
3 be3:= 9*km*gm*Z + gm*(S1111 + 2*S1122)*W$
4
5 % Calculate alpha, beta, gamma using (8.2)
6 w1:= -(a11*be2 + a12*be1) + (a12*be3 + a13*be2) + 3*(a11*be3 + a13*be1)$
7 w2:= -a11*a12 + a12*a13 + 3*a11*a13$
8 gam:= be1*be2 - be2*be3 - 3*be1*be3$
9 be:= w1 + w2/psi - 2*psi*gam$
10 al:= be*psi + gam*psi**2$
11
12 res5:= al - be*psi - gam*psi**2;
13 res6:= w2/psi/(9*km*gm) - phi**2;
14
15 % Calculate Omega using (8.6)
16 omb:= - psi*gam/al$
17
18 res7:= al*omb**2 - be*omb - gam;
19
20 om:= -9*km*gm*psi*gam/al$
21
22 w1:= 9*km*gm*(be/al - 1/psi)$
23 res8:= OM - w1;
24
25 % Calculate solution using (7.25-7.27) and relation S = Q - 3R
26 P:= ((be1*psi + a11)*omb - be1)/phi$
27 Q:= ((be2*psi + a12)*omb - be2)/phi$
28 R:= ((be3*psi + a13)*omb - be3)/phi$
29 S:= Q - 3*R$
30
31 res9:= Q*R - P*S - om;
32
33 % Calculate non-shear effective properties of composite using (8.9)
34 KTC2:= (S + 3*km + gm)/3$
35 NAC2:= ( (R - gm)/KTC2 + 1 )/2$
36 EAC2:= P + 2*NAC2*(1 - 2*NAC2)*KTC2 + 2*gm$
37
38 res10:= EAC1 - EAC2;
39 res11:= NAC1 - NAC2;
40 res12:= KTC1 - KTC2;
41 res13:= GAC1 - GAC2;
42 res14:= GTC1 - GTC2;
43
44 % Output values of elastic constants if needed
45 % Switch rounded on after array declarations if compact decimal form is needed
46
47 % write EAC:= EAC2;
48 % write NAC:= NAC2;
49 % write KTC:= KTC2;
50 % write MUA:= GAC2;
51 % write MUT:= GTC2;
52
53 end;
54
55
56
57
58
59
60

```