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EXPONENTIALITY OF FIRST PASSAGE TIMES OF CONTINUOUS TIME MARKOV CHAINS

ROMAIN BOURGET, LOÏC CHAUMONT, AND NATALIA SAPOUKHINA

ABSTRACT. Let (X, \mathbb{P}_x) be a continuous time Markov chain with finite or countable state space S and let T be its first passage time in a subset D of S . It is well known that if μ is a quasi-stationary distribution relatively to T , then this time is exponentially distributed under \mathbb{P}_μ . However, quasi-stationarity is not a necessary condition. In this paper, we determine more general conditions on an initial distribution μ for T to be exponentially distributed under \mathbb{P}_μ . We show in addition how quasi-stationary distributions can be expressed in terms of any initial law which makes the distribution of T exponential. We also study two examples in branching processes where exponentiality does imply quasi-stationarity.

1. INTRODUCTION

Let us denote by $P(t) = \{p_{ij}(t) : i, j \in S\}$, $t \geq 0$ the transition probability of a continuous time irreducible Markov chain $X = \{(X_t)_{t \geq 0}, (\mathbb{P}_i)_{i \in S}\}$, with finite or countable state space S and let $Q = \{q_{ij} : i, j \in S\}$ be the associated q -matrix, that is $q_{ij} = p'_{ij}(0)$. We assume that Q is conservative, that is $\sum_{j \in S} q_{ij} = 0$, for all $i \in S$, and that X is not explosive. The transition probability (that will also be called the transition semigroup) of X satisfies the backward Kolmogorov's equation:

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t). \quad (1.1)$$

Let $D \subset S$ be some domain and define the first passage time by X in D by,

$$T = \inf\{t \geq 0 : X_t \in D\}. \quad (1.2)$$

This work aims at characterizing probability measures μ on $E = S \setminus D$ such that under \mathbb{P}_μ , the time T is exponentially distributed, that is, there exists $\alpha > 0$, such that:

$$\mathbb{P}_\mu(T > t) = e^{-\alpha t}. \quad (1.3)$$

It is well known that when μ is a quasi-stationary distribution with respect to T , that is if

$$\mathbb{P}_\mu(X_t = i | T > t) = \mu_i, \quad \text{for all } i \in E \text{ and } t \geq 0, \quad (1.4)$$

then (1.3), for some value $\alpha > 0$, follows from a simple application of the Markov property, see [15] or [6] for example. Quasi-stationarity of μ holds if and only if μ is a left eigenvector of the q -matrix of the process X killed at time T , associated to the eigenvalue $-\alpha$, see [17]. However, quasi-stationarity is not necessary to obtain (1.3). Some examples of non

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quasi-stationary distribution μ such that (1.3) holds are given later on in this paper.

Our work was first motivated by population dynamics, where it is often crucial to determine the extinction time of a population or the emergence time of a new mutant, see [2, 3, 11, 18] for example. In many situations, those times can be represented as first passage times of Markov processes in some particular domain. Then it is often much easier to find an initial distribution, under which this first passage time is exponentially distributed than to compute its distribution under any initial conditions.

Let us be more specific about applications to emergence times in biology which is the central preoccupation of the authors in [2]. Adaptation to a new environment passes by the emergence of new mutants. In adaptation theory, emergence can be described by the estimation of the fixation time of an allele in the population. We may also imagine a parasite infecting a resistant or new host, a pathogen evading chemical treatment, a cancer cell escaping from chemotherapy, etc. [10, 11, 13, 22]. An interesting and important point is to estimate the law of the time at which these new mutant individuals emerge in the population, for example to estimate the durability or the success probability of a new treatment or a new resistance. Emergence problem has already been considered in the setting of branching processes [21, 22], for multitype Moran models in [7], and for competition processes, in [2]. In order to explain the latter case in more details, let us recall that a competition process is a continuous time Markov chain $X = (X^{(1)}, \dots, X^{(d)})$ with state space $S = \mathbb{N}^d$, for $d \geq 2$, whose transition probabilities only allow jumps to certain nearest neighbors. Competition processes were introduced by Reuter [20] as the natural extensions of birth and death processes and are often involved in epidemic models [4, 10, 13]. In [2], the authors were interested in some estimation of the law of the first passage time T , when an individual of type $r = 1, \dots, d$ first emerges from the population, that is

$$T = \inf\{t \geq 0 : X_t^{(r)} = 1\}.$$

Then varying the birth, mutation, migration and death rates, some simulations of the law of the time T allow us to conclude that the consideration of interactions among two stochastic evolutionary forces, mutation and migration, can expand our understanding of the adaptation process at the population level. In particular, it shows under which conditions on mutation and migration rates, pathogen can adapt swiftly to a given multicomponent treatment.

This paper is organized as follows. In section 2, we establish a general criterion for a measure μ to satisfy (1.3) and we study the connections between such measures and quasi-stationary or quasi-limiting distributions. Then, in the third section, we give some sufficient conditions for (1.3) involving the special structure of the chain on a partition of the state space E . In particular, Theorem 3 and its consequences allow us to provide some examples where exponentiality may hold without quasi-stationarity. An example of application in adaptation theory is provided in Subsection 3.2. The fourth section is devoted to the presentation of some examples in the setting of branching processes where exponentiality implies quasi-stationarity.

2. FROM EXPONENTIALITY TO QUASI STATIONARITY

We first introduce the killed process at time T , as follows:

$$X_t^T = \begin{cases} X_t, & \text{if } t < T, \\ \Delta, & \text{if } t \geq T, \end{cases} \quad (2.1)$$

where Δ is a cemetery point. Then X^T is a continuous time Markov chain which is valued in $E_\Delta := E \cup \{\Delta\}$. Moreover if we define the killing rate by

$$\eta_i = \sum_{j \in D} q_{ij}, \quad (2.2)$$

then the q -matrix $Q^T = (q_{ij}^T)$ of X^T is given by

$$q_{ij}^T = \begin{cases} q_{ij}, & i, j \in E \\ q_{i\Delta} = \eta_i, & i \in E \\ q_{\Delta j} = 0, & j \in E_\Delta. \end{cases} \quad (2.3)$$

From our assumptions, Q^T is obviously conservative and X^T is non explosive. In particular, Q^T is the q -matrix of a unique transition probability that we will denote by $P^T(t) = (p_{ij}^T(t))_{i,j \in E_\Delta}$, $t \geq 0$, and which is expressed as

$$p_{ij}^T(t) = \begin{cases} \mathbb{P}_i(X_t = j, t < T), & \text{if } i, j \in E, \\ \mathbb{P}_i(t \geq T), & \text{if } i \in E \text{ and } j = \Delta, \\ 1_{j=\Delta}, & \text{if } i = \Delta \text{ and } j \in E_\Delta. \end{cases} \quad (2.4)$$

Then this semigroup inherits the Kolmogorov backward equation from (1.1):

$$\frac{d}{dt} p_{ij}^T(t) = \sum_{k \in E_\Delta} q_{ik}^T p_{kj}^T(t). \quad (2.5)$$

Henceforth, all distributions ν on E_Δ that will be considered will not charge the state Δ , i.e. $\nu_\Delta = 0$. In this section, we shall often consider initial distributions $\mu = (\mu_i)_{i \in E_\Delta}$ for (X_t^T) , on E_Δ satisfying the following differentiability condition:

$$\mu P^T(t) \text{ is differentiable and } \frac{d}{dt} \mu P^T(t) = \mu \frac{d}{dt} P^T(t), \quad t > 0. \quad (2.6)$$

We extend the family of probability $(\mathbb{P}_i)_{i \in E}$ to $i = \Delta$, in accordance with the definition of $(P^T(t))$ and for each $t \geq 0$, we define the probability distribution $\mu(t)$ on E_Δ as follows:

$$\mu_i(t) = \mathbb{P}_\mu(X_t^T = i | T > t), \quad i \in E_\Delta. \quad (2.7)$$

We define the vector δ by $\delta_i = 0$, if $i \in E$ and $\delta_\Delta = 1$.

Theorem 1. *Let μ be a distribution on E_Δ .*

(i) *Assume that μ satisfies condition (2.6), then there is $\alpha > 0$ such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, for all $t \geq 0$ if and only if*

$$\mu(t) \text{ is differentiable and } \mu'(t) = e^{\alpha t} (\mu Q^T + \alpha(\mu - \delta)) P^T(t), \quad t > 0. \quad (2.8)$$

(ii) *Assume that there is $\alpha > 0$ such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, for all $t \geq 0$, then conditions (2.6) and (2.8) are equivalent.*

(iii) *When (2.8) is satisfied, the rate α may be expressed as*

$$\alpha = \sum_{i \in E} \eta_i \mu_i. \quad (2.9)$$

Proof. Note that the condition $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$ is equivalent to $\mathbb{P}_\mu(X_t^T = i, t < T) = e^{-\alpha t} \mu_i(t)$. So since

$$\mathbb{P}_\mu(X_t^T = i) = \mathbb{P}_\mu(X_t^T = i, t < T) + \mathbb{1}_{i=\Delta} \mathbb{P}_\mu(t \geq T),$$

the transition function $P^T(t)$ of X^T satisfies

$$\mu P^T(t) = e^{-\alpha t} \mu(t) + (1 - e^{-\alpha t}) \delta. \quad (2.10)$$

Then from the differentiability condition (2.6), we see that $\mu(t)$ is differentiable and from the Kolmogorov backward equation (2.5), we obtain

$$\begin{aligned} \mu \frac{d}{dt} P^T(t) &= -\alpha e^{-\alpha t} \mu(t) + e^{-\alpha t} \mu'(t) + \alpha e^{-\alpha t} \delta \\ &= \mu Q^T P^T(t). \end{aligned} \quad (2.11)$$

Then from (2.10), we have $e^{-\alpha t} \mu(t) = \mu P^T(t) - (1 - e^{-\alpha t}) \delta$ and since $\delta P^T(t) = \delta$, for all $t \geq 0$, we see that equation (2.11) may be expressed as

$$\mu'(t) = e^{\alpha t} (\mu Q^T + \alpha(\mu - \delta)) P^T(t), \quad t \geq 0.$$

Conversely, if condition (2.8) is satisfied, then from (2.6), we can write equation (2.11). Integrating this expression, we get (2.10) which implies that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, for all $t \geq 0$. The first assertion of the theorem is proved.

Now if $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, for all $t \geq 0$, then we have (2.10), so that if condition (2.6) is satisfied, then $\mu(t)$ is differentiable and

$$\frac{d}{dt} \mu P^T(t) = -\alpha e^{-\alpha t} \mu(t) + e^{-\alpha t} \mu'(t) + \alpha e^{-\alpha t} \delta. \quad (2.12)$$

Moreover from the Kolmogorov backward equation and (2.12), we have $\mu Q^T P^T(t) = -\alpha e^{-\alpha t} \mu(t) + e^{-\alpha t} \mu'(t) + \alpha e^{-\alpha t} \delta$, which is (2.8). The converse is easily derived from similar arguments, so the second assertion is proved.

Then from equation (2.8), we obtain

$$\lim_{t \rightarrow 0} \mu'(t) = (\mu Q^T + \alpha(\mu - \delta)) P^T(0). \quad (2.13)$$

On the other hand, note that $\mu_\Delta(t) = 0$, for all $t \geq 0$, so that in particular $\mu_\Delta = \mu_\Delta(0) = 0$ and $\lim_{t \rightarrow 0} \mu'_\Delta(t) := \mu'_\Delta(0) = 0$. Finally, taking equality (2.13) at Δ yields

$$\mu Q_\Delta^T = \sum_{i \in E_\Delta} \mu_i q_{i\Delta}^T = \sum_{i \in E} \mu_i \eta_i = \mu'_\Delta(0) - \alpha(\mu_\Delta - \delta_\Delta) = \alpha,$$

which proves the third assertion of the theorem. □

Remarks 1. *It is important to note that a distribution μ on E_Δ may satisfy $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, $t \geq 0$, whereas (2.6) does not hold. Examples can be constructed from Corollary 4 in the next section.*

2. *When E is finite, condition (2.6) is clearly satisfied. In the infinite case, this condition may appear theoretical to some extent and sometimes difficult to check when not much is known on the transition probability. However it is possible to obtain quite simple conditions*

implying (2.6). For instance, observe that from (2.5), for all $i, j \in E_\Delta$ and $t > 0$,

$$\begin{aligned} \left| \frac{d}{dt} p_{ij}^T(t) \right| &\leq \sum_{k \in E_\Delta} |q_{ik}^T p_{kj}^T(t)| \\ &\leq \sum_{k \in E_\Delta} |q_{ik}^T| \\ &= -2q_{ii}^T. \end{aligned} \quad (2.14)$$

A sufficient condition for (2.6) to hold is then

$$\sum_{i \in E} q_i \mu_i < \infty, \quad (2.15)$$

where $q_i = -q_{ii}^T$. The later condition is satisfied in particular when the q_i 's are bounded.

Recall definition (1.4) of quasi-stationarity. In our setting, it is equivalent to the following statement: a distribution μ on E_Δ , is quasi-stationary if

$$\mu_i = \mu_i(t), \quad \text{for all } t \geq 0 \text{ and } i \in E_\Delta. \quad (2.16)$$

We will simply say that μ is a quasi-stationary distribution. Then, let us state the following classical result, already mentioned in the introduction.

Theorem 2 ([19]). *A distribution μ on E_Δ is quasi-stationary if and only if the equation*

$$\mu Q^T = -\alpha \mu + \alpha \delta, \quad (2.17)$$

holds for some $\alpha > 0$. (Note that (2.17) is equivalent to $\mu Q_i^T = -\alpha \mu_i$, for all $i \in E$.)

In [19] it is proved that (2.17) is equivalent to the fact that P^T satisfies the Kolmogorov forward equation, which is the case under our assumptions, that is

$$\frac{d}{dt} p_{ij}^T(t) = \sum_{k \in E} p_{ik}^T(t) q_{kj}^T. \quad (2.18)$$

Knowing condition (2.6), it is actually quite easy to prove Theorem 2. Note also that under this assumption, Theorem 2 is a consequence of Theorem 1. As a consequence of both these theorems we also obtain that (2.6) holds whenever μ is quasi-stationary.

Corollary 1. *If μ is a quasi-stationary distribution then condition (2.6) holds.*

Proof. If μ is quasi-stationary, then it follows from (2.16) and the Markov property that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, for some $\alpha > 0$ (this fact is well known, see [6], for instance). Moreover the function $\mu(t)$ is differentiable and $\mu'(t) = 0$, for all $t \geq 0$. On the other hand, from Theorem 2, equation (2.17) holds. Therefore, condition (2.8) holds, so that (2.6) is satisfied from part (ii) of Theorem 1. \square

A distribution π on E_Δ is called the quasi-limiting distribution (or the Yaglom limit) of a distribution μ on E_Δ , if it satisfies

$$\lim_{t \rightarrow \infty} \mathbb{P}_\mu(X_t^T = i | T > t) = \pi(i), \quad \text{for all } i \in E_\Delta. \quad (2.19)$$

Then a well known result asserts that any quasi-limiting distribution is also a quasi-stationary distribution, see for example [15, 6, 16]. Recall also that if π is the quasi-limiting

distribution of some distribution μ , then the rate α satisfying (1.3) is given by the expression

$$\alpha = \inf \left\{ a \geq 0 : \int_0^\infty e^{at} P_i(T > t) dt = \infty \right\} > 0, \quad (2.20)$$

which does not depend on the state $i \in E$. As an application of Theorem 1 and the above remarks, we show in the next corollary how to construct quasi-stationary distributions from distributions satisfying (1.3).

Corollary 2. *Let μ be a distribution on E_Δ such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, $t \geq 0$, for some $\alpha > 0$ and satisfying (2.6). If μ admits a quasi-limiting distribution, π , then the later is given by:*

$$\pi = \mu + \int_0^\infty (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha t} P^T(t) dt,$$

where $\int_0^\infty (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha t} P^T(t) dt$ should be understood as a possibly improper integral. In particular, π is a quasi-stationary distribution on E_Δ .

Proof. Under these assumptions, it follows from Theorem 1 that for all $t \geq 0$, $\mu'(t) = e^{\alpha t} (\mu Q^T + \alpha(\mu - \delta)) P^T(t)$. Moreover, since $\mathbb{P}_\mu(T > 0) = 1$, $\mu(t)$ is continuous at 0 and $\mu(0) = \mu$, so that

$$\mu(t) - \mu = \int_0^t (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha u} P^T(u) du.$$

Since $\mu(t)$ converges to a proper distribution μ , as t tends to ∞ , it follows that the improper integral $\int_0^\infty (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha u} P^T(u) du = \lim_{t \rightarrow +\infty} \int_0^t (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha u} P^T(u) du$ exists and is finite. The fact that μ is a quasi-stationary distribution follows from the results which are recalled before the statement of the corollary. \square

Corollary 2 may be interpreted as follows: if μ is such that T is exponentially distributed under \mathbb{P}_μ and admits a Yaglom limit, then the correction term which allows us to obtain a quasi-stationary distribution from μ is $\int_0^\infty (\mu Q^T + \alpha(\mu - \delta)) e^{\alpha t} P^T(t) dt$.

The next result shows that whenever there exists a non quasi-stationary distribution which makes the time T exponentially distributed, then under some conditions, we may construct a whole family of distributions having the same property.

Proposition 1. *Let μ be a distribution on E_Δ satisfying (2.6) and such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, $t \geq 0$, for some $\alpha > 0$. For $n \geq 1$, let us denote by $q_{ij}^{n,T}$ the entries of $(Q^T)^n$ and let us define the vector $(\mu_i^{(n)})_{i \in E_\Delta}$, by*

$$\mu_j^{(n)} = \frac{(-1)^n}{\alpha^n} \sum_{i \in E_\Delta} \mu_i q_{ij}^{n,T}, \quad j \in E, \quad \mu_\Delta^{(n)} = 0. \quad (2.21)$$

If for all $j \in E$,

$$0 \leq (-1)^n \sum_{i \in E_\Delta} \mu_i q_{ij}^{n,T} \leq \alpha^n, \quad (2.22)$$

then $(\mu_i^{(n)})_{i \in E_\Delta}$ is a distribution on E_Δ which satisfies $\mathbb{P}_{\mu^{(n)}}(T > t) = e^{-\alpha t}$, for all $t \geq 0$.

Proof. The assumption $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$ is equivalent to

$$\sum_{i \in E} \mu_i p_{i\Delta}^T(t) = 1 - e^{-\alpha t}. \quad (2.23)$$

Using condition (2.6) and the Kolmogorov backward equation (2.5), we obtain by differentiating the latter equality

$$\sum_{i \in E} \left(\sum_{j \in E_\Delta} q_{ij}^T p_{j\Delta}^T(t) \right) \mu_i = \alpha e^{-\alpha t}.$$

Decomposing the left hand side and using (2.3) and (2.9), we obtain

$$\begin{aligned} \sum_{i \in E} \left(\sum_{j \in E_\Delta} q_{ij}^T p_{j\Delta}^T(t) \right) \mu_i &= \sum_{i \in E} \left(q_{i\Delta}^T + \sum_{j \in E} q_{ij}^T p_{j\Delta}^T(t) \right) \mu_i \\ &= \alpha + \sum_{i,j \in E} q_{ij}^T p_{j\Delta}^T(t) \mu_i \\ &= \alpha e^{-\alpha t}, \end{aligned}$$

which gives

$$\sum_{j \in E} p_{j\Delta}^T(t) \left(\frac{-1}{\alpha} \sum_{i \in E} \mu_i q_{ij}^T \right) = 1 - e^{-\alpha t}. \quad (2.24)$$

Then from condition (2.22), we may let t tend to ∞ in (2.24), in order to obtain by monotone convergence that $\sum_{j \in E} \mu_j^{(1)} = 1$, so that

$$\mu_j^{(1)} = \frac{-1}{\alpha} \sum_{i \in E} \mu_i q_{ij}^T, \quad j \in E, \quad \mu_\Delta^{(1)} = 0$$

is a distribution on E_Δ . Moreover (2.24) is equation (2.23) where we have replaced μ by $\mu^{(1)}$, so that $\mu^{(1)}$ satisfies $\mathbb{P}_{\mu^{(1)}}(T > t) = e^{-\alpha t}$.

The result is proved for $n = 1$. Then the proof is completed by iterating these arguments. \square

Let μ be a distribution on E_Δ such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, $t \geq 0$, for some $\alpha > 0$. As we have already observed, if $\sup_{i \in E} q_i \leq \alpha$, where $q_i := -q_{ii}^T$, then condition (2.6) is satisfied, but moreover, it is easy to check that for all $n \geq 1$ and for all $j \in E$,

$$(-1)^n \sum_{i \in E_\Delta} \mu_i q_{ij}^{n,T} \leq \alpha^n, \quad (2.25)$$

which provides the second inequality in (2.21). An interesting problem is then to determine simple conditions insuring the first inequality in (2.21), that is nonnegativity of the term $(-1)^n \sum_{i \in E_\Delta} \mu_i q_{ij}^{n,T}$.

Corollary 3. *Let μ be a distribution on E_Δ satisfying (2.6) and such that $\mathbb{P}_\mu(T > t) = e^{-\alpha t}$, $t \geq 0$, for some $\alpha > 0$. Define $\mu^{(n)}$ as in (2.21). Then,*

1. *if for some $n \geq 1$, $\mu^{(n)}$ is a quasi-stationary distribution, then $\mu^{(k)} = \mu^{(k+1)}$, for all $k \geq n$.*
2. *If the sequence of distributions $\mu^{(n)}$ converges, as $n \rightarrow \infty$, toward a proper distribution $\mu^{(\infty)}$, then $\mu^{(\infty)}$ is a quasi-stationary distribution.*

Proof. The proof of the first part simply follows from the identity:

$$\mu_i^{(n+1)} = \frac{-1}{\alpha} \mu^{(n)} Q_i^T, \quad i \in E,$$

and Theorem 2. The second assertion is a consequence of the same observation, which leads, under the assumption to, $\mu^{(\infty)}Q_i^T = -\alpha\mu_i^{(\infty)}$, $i \in E$. Then we conclude by applying Theorem 2. \square

In particular if the state space E_Δ is finite and $-Q^T$ is an honest matrix then from Perron-Frobenius theorem, assertion 2. of Corollary 3 is satisfied for the maximal eigenvalue α of $-Q$.

3. SUFFICIENT CONDITIONS FOR EXPONENTIALITY.

3.1. General results. Let us keep the notation of the previous sections. The next theorem provides sufficient conditions for a distribution μ to insure that T is exponentially distributed under \mathbb{P}_μ . This result together with its Corollary 4 allow us to construct examples for which such distributions exist.

Theorem 3. *Let $\{E_1, E_2, \dots\}$ be a finite or infinite partition of S containing at least two elements and with $E_1 = D$ (in particular $\{E_2, E_3, \dots\}$ is a partition of E). Assume that μ is a distribution with support in E that satisfies the following condition:*

- (i) *For all $k \geq 2$ and $l \geq 1$, with $k \neq l$ and for all $i \in E_k$, the quantity $\sum_{j \in E_l} q_{ij}$ does not depend on i . For $i \in E_k$, we set*

$$\bar{q}_{kl} := \sum_{j \in E_l} q_{ij}. \quad (3.1)$$

Then the following two conditions are equivalent.

- (ii) *For all $k \geq 1$, the quantity $\mathbb{P}_\mu(X_t \in E_k \mid T > t)$ does not depend on $t \geq 0$. More specifically, we have,*

$$\mathbb{P}_\mu(X_t \in E_k \mid T > t) = \bar{\mu}_k, \quad t \geq 0, \quad (3.2)$$

where $\bar{\mu}_k = \sum_{i \in E_k} \mu_i$.

- (iii) *There exists $\alpha > 0$, such that*

$$\bar{\mu}\bar{Q} = -\alpha\bar{\mu} + \alpha\mathbf{d},$$

where $\bar{Q} = (\bar{q}_{kl})_{k,l \geq 1}$, $\bar{q}_{1k} = 0$, for $k \geq 1$, $\bar{q}_{kk} = -\sum_{l \geq 1} \bar{q}_{kl}$, for $k \geq 1$, $\bar{\mu} = (\bar{\mu}_k)_{k \geq 1}$ and $\mathbf{d} = (1, 0, 0, \dots)$.

Moreover, if conditions (i) and (ii) (or equivalently conditions (i) and (iii)) are satisfied, then T is exponentially distributed under \mathbb{P}_μ , with parameter α given by

$$\alpha = \sum_{k \geq 1} \bar{q}_{k1} \bar{\mu}_k. \quad (3.3)$$

Proof. Let $(Y_t)_{t \geq 0}$ be the continuous time process with values in $\{1, 2, \dots\}$ which is defined by $Y_t = k$, if $X_t \in E_k$, that is

$$Y_t = \sum_{k \geq 1} k \mathbb{I}_{\{X_t \in E_k\}}, \quad t \geq 0.$$

Observe that $T = \inf\{t : Y_t = 1\}$. Then we will first show that under assumption (i), the absorbed process

$$Y_t^T = \begin{cases} Y_t, & \text{if } t < T, \\ 1, & \text{if } t \geq T, \end{cases} \quad (3.4)$$

is a continuous time Markov chain with q -matrix $\bar{Q} = (\bar{q}_{kl})_{k,l \geq 1}$, as defined in (iii).

Let us first assume that the number of sets in the partition (E_k) is finite and is $n \geq 2$. Recall from Section 2, the definition of $\{p_{ij}^T(t) : i, j \in S\}$ and $\{q_{ij}^T : i, j \in S\}$, the transition probability and the q -matrix of the Markov chain X , killed at T . For all $t \geq 0$, define

$$k_{il}(t) := \begin{cases} \mathbb{P}_i(X_t \in E_l, t < T) = \sum_{j \in E_l} p_{ij}^T(t), & \text{if } i \in E_\Delta \text{ and } 2 \leq l \leq n, \\ \mathbb{P}_{i\Delta}^T(t), & \text{if } i \in E_\Delta \text{ and } l = 1. \end{cases}$$

From the corollary, on page 132 in [5], the function $t \mapsto k_{il}(t)$ is differentiable and for $2 \leq l \leq n$,

$$\frac{d}{dt}k_{il}(t) = \sum_{j \in E_l} \frac{d}{dt}p_{ij}^T(t), \quad t \geq 0. \quad (3.5)$$

Applying the Kolmogorov forward equation (2.18), we obtain, for $1 \leq l \leq n$,

$$\begin{aligned} \frac{d}{dt}k_{il}(t) &= \sum_{j \in E_l} \sum_{m \in E} p_{im}^T(t) q_{mj} \\ &= \sum_{x=2}^n \sum_{m \in E_x} p_{im}^T(t) \sum_{j \in E_l} q_{mj} \\ &= \sum_{x=2}^n \sum_{m \in E_x} p_{im}^T(t) \bar{q}_{xl} \\ &= \sum_{x=2}^n k_{ix}(t) \bar{q}_{xl} = \sum_{x=1}^n k_{ix}(t) \bar{q}_{xl}. \end{aligned}$$

Let $i_1 \in E_1, \dots, i_n \in E_n$. Then from the above equality, the function $K(t) = \{k_{i_j l}(t) : 1 \leq j, l \leq n\}$ satisfies

$$\frac{d}{dt}K(t) = K(t)\bar{Q}, \quad t \geq 0, \quad \text{with } K(0) = Id. \quad (3.6)$$

Since $n < \infty$, the solution of (3.6) is unique, this shows that for all $l \geq 1$, the functions $t \mapsto k_{il}(t)$ does not depend on $i \in E_k$. Then we set

$$\bar{p}_{kl}(t) = k_{il}(t), \quad \text{for } 1 \leq k, l \leq n \text{ and } i \in E_k,$$

and $\bar{P}(t) = \{\bar{p}_{kl}(t) : 1 \leq k, l \leq n\}$, $t \geq 0$. Form (3.6), we can write

$$\frac{d}{dt}\bar{P}(t) = \bar{P}(t)\bar{Q}, \quad t \geq 0, \quad \text{with } \bar{P}(0) = Id. \quad (3.7)$$

Then let us check that Y^T is a Markov chain with transition semigroup $\bar{P}(t)$, $t \geq 0$ and q -matrix \bar{Q} . For all $k_1, \dots, k_n, k_{n+1} \geq 2$, $0 \leq s_1 < s_2 < \dots < s_{n+1}$ and for any measure ν on E ,

$$\begin{aligned} &\mathbb{P}_\nu(Y_{s_{n+1}}^T = k_{n+1} | Y_{s_1}^T = k_1, \dots, Y_{s_n}^T = k_n) \\ &= \mathbb{P}_\nu(X_{s_{n+1}} \in E_{k_{n+1}}, s_{n+1} < T | X_{s_1} \in E_{k_1}, \dots, X_{s_n} \in E_{k_n}, s_n < T) \\ &= \frac{\mathbb{E}_\nu(\mathbb{1}_{\{X_{s_1} \in E_{k_1}, \dots, X_{s_n} \in E_{k_n}, s_n < T\}} \mathbb{P}_{X_{s_n}}(X_{s_{n+1}-s_n} \in E_{k_{n+1}}, s_{n+1} - s_n < T))}{\mathbb{P}_\nu(X_{s_1} \in E_{k_1}, \dots, X_{s_n} \in E_{k_n}, s_n < T)} \\ &= \bar{p}_{k_n k_{n+1}}(s_{n+1} - s_n). \end{aligned}$$

The last identity follows from the fact that for any $i \in E_{k_n}$,

$$k_{ik_{n+1}}(s_{n+1} - s_n) = \mathbb{P}_i(X_{s_{n+1}-s_n} \in E_{k_{n+1}}, s_{n+1} - s_n < T) = \bar{p}_{k_n k_{n+1}}(s_{n+1} - s_n),$$

which has been proved above. The case where k_1, \dots, k_n, k_{n+1} may possibly be equal to 1, is proved similarly.

Then recall from (iii), the definition of the measure $\bar{\mu}$ on $\{1, \dots, n\}$: $\bar{\mu}_k = \sum_{i \in E_k} \mu_i$, $k \geq 2$ and $\bar{\mu}_1 = 0$. Let $(\bar{\mathbb{P}}_k)_{k \geq 1}$ be the family of probability laws associated to the Markov process $(Y_t)_{t \geq 0}$. For all $k = 2, \dots, n$,

$$\begin{aligned} \mathbb{P}_\mu(X_t \in E_k, t < T) &= \sum_{i \in E} \mu_i \mathbb{P}_i(X_t \in E_k, t < T) \\ &= \sum_{l=2}^n \sum_{i \in E_l} \mu_i \mathbb{P}_i(X_t \in E_k, t < T) \\ &= \sum_{l=2}^n \bar{\mu}_l \bar{\mathbb{P}}_l(Y_t = k, t < T) \\ &= \bar{\mathbb{P}}_{\bar{\mu}}(Y_t = k, t < T), \end{aligned} \tag{3.8}$$

where the third equality follows from the fact that $\mathbb{P}_i(X_t \in E_k, t < T) = \bar{\mathbb{P}}_l(Y_t = k, t < T)$, for all $i \in E_l$. Assume that condition (ii) holds, then we derive from (3.2) and (3.8) that for all $k = 2, \dots, n$,

$$\bar{\mathbb{P}}_{\bar{\mu}}(Y_t = k | t < T) = \bar{\mu}_k,$$

which means that $\bar{\mu}$ is a quasi stationary distribution with respect to the lifetime of the Markov process Y^T . In particular, thanks to Theorem 2, (ii) and (iii) are equivalent. Moreover, since (2.8) in Theorem 1 is satisfied, then from (iii) in this theorem, T is exponentially distributed under $\mathbb{P}_{\bar{\mu}}$, with parameter $\alpha = \sum_{k=2}^n \bar{q}_{k1} \bar{\mu}_k$. We conclude from equality (3.8) which shows that $\mathbb{P}_\mu(t < T) = \bar{\mathbb{P}}_{\bar{\mu}}(t < T)$.

Now assume that the number of elements in the partition (E_k) is countable. For all $n \geq 1$, define the set $E_1^{(n)} = E_1 \cup E_{n+1} \cup E_{n+2} \cup \dots$ and set $T_n = \inf\{t : X_t \in E_1^{(n)}\}$. Let us consider the process $Y^{(n)}$ defined by

$$Y_t^{(n)} = \mathbb{I}_{\{X_t \in E_1^{(n)}\}} + \sum_{k=2}^n k \mathbb{I}_{\{X_t \in E_k\}}, \quad t \geq 0.$$

Then we have $T_n = \inf\{t : Y_t^{(n)} = 1\}$, and the absorbed process

$$Y_t^{(n), T_n} = \begin{cases} Y_t^{(n)}, & \text{if } t < T_n, \\ 1, & \text{if } t \geq T_n, \end{cases} \tag{3.9}$$

is a continuous time Markov chain. Indeed, the Markov chain X satisfies assumption (ii) of the theorem with respect to the partition $\{E_1^{(n)}, E_2, \dots, E_n\}$ of S . Therefore, from what has been proved before, $Y^{(n), T_n}$ is a continuous time Markov chain. Moreover, we easily check that $Y^{(n), T_n}$ converges in law in the sense of finite dimensional distributions to Y^T and that the later has \bar{Q} for q -matrix. Then the equivalence between (ii) and (iii) follows in the same way as in the finite case. Moreover, when (i) and (ii) are satisfied then we conclude that T is exponentially distributed with parameter $\alpha = \sum_{k \geq 1} \bar{q}_{k1} \bar{\mu}_k$ in the same way as in the finite case. \square

In the particular case where the partition $\{E_2, E_3, \dots\}$ of E is reduced to the singletons of E , then condition (ii) is obviously satisfied and condition (i) simply means that μ is quasi-stationary with respect to T , hence the conclusion follows from Theorem 2. Conversely, by

considering $\{E\}$ as a partition of E , we obtain the following consequence of Theorem 3. Recall the definition of η_i in (2.2).

Corollary 4. *If there exists $\alpha > 0$ such that $\eta_i = \alpha$, for all $i \in E$, then the first passage time T has an exponential distribution with parameter α under \mathbb{P}_μ , for all initial distribution μ with support in E .*

Proof. We apply Theorem 3 with the finite partition $\{E_1, E_2\}$ of S , given by $E_1 = D$ and $E_2 = E$. Then we see that part (i) of Theorem 3 is precisely the assumption of the present corollary. Moreover, part (ii) simply follows from the fact for all $i \in E_2$, $\mathbb{P}_i(X_t \in E_2, t < T) = \mathbb{P}_i(t < T)$, in this particular case. Then we conclude from Theorem 3. \square

Remarks 1. *From Corollary 4, it becomes clear that there exist many instances of continuous time Markov chains for which we can find initial distributions μ such that T is exponentially distributed under \mathbb{P}_μ , whereas (2.6) is not satisfied.*

2. *Under the assumption of this corollary, if moreover there is a Yaglom limit, μ , as recalled in the previous section, then μ is explicitly given by*

$$\mu = 1_{\{i\}} + \int_0^\infty (1_{\{i\}} Q^T + \alpha(1_{\{i\}} - \delta)) e^{\alpha t} P^T(t) dt.$$

In particular, this expression does not depend on i .

3. *In the case where S is finite, it is stated in Proposition 2.1, (ii) of [6] that if the sum of the rows of the matrix $(q_{ij})_{i,j \in E}$ are constant, then T is exponentially distributed under \mathbb{P}_μ , for all probability measure μ . This result is a consequence of Corollary 4 since when the sum of the rows of the matrix $(q_{ij})_{i,j \in E}$ are constant, the rates η_i are constant.*

Actually it is always possible to compare the distribution of T with the exponential law, as Proposition 1 shows. Let us first define the continuous time Markov chain \hat{X} by removing D from the state space of X . More specifically, the state space of \hat{X} is E and its q -matrix is given by $\hat{q}(i, j) = q(i, j)$, if $i \neq j$ and $i, j \in E$ and $\hat{q}(i, i) = -\sum_{j \in E, i \neq j} \hat{q}(i, j)$, for all $i \in E$. We denote by $\{\hat{\mathbb{P}}_i : i \in E\}$ the family of probability distributions associated to \hat{X} . Then we may check that $\{(\hat{X}_t)_{t \geq 0}, (\hat{\mathbb{P}}_i)_{i \in E}\}$ is a continuous time irreducible and non explosive Markov chain on E . We first establish the following lemma which presents a decomposition of the first passage time T .

Lemma 1. *Let $\{\varepsilon(i) : i \in E\}$ be a family of random variables which is independent of the process $\{(\hat{X}_t)_{t \geq 0}, (\hat{\mathbb{P}}_i)_{i \in E}\}$. We suppose that each random variable $\varepsilon(i)$ is exponentially distributed with parameter η_i . Let $(T_n)_{n \geq 1}$ be the sequence of jump times of \hat{X} , set $T_0 = 0$, define $I_n = T_n - T_{n-1}$, $n \geq 1$ and*

$$\hat{T} = \sum_{n=0}^{\infty} (T_n + \varepsilon(\hat{X}_{T_n})) \mathbb{1}_{\Omega_n}, \quad (3.10)$$

where $\Omega_n = \{\varepsilon(\hat{X}_{T_0}) > I_1, \dots, \varepsilon(\hat{X}_{T_{n-1}}) > I_n, \varepsilon(\hat{X}_{T_n}) \leq I_{n+1}\}$, for $n \geq 1$ and $\Omega_0 = \{\varepsilon(\hat{X}_0) \leq I_1\}$. Then for all $i \in E$, we have the identity in law

$$[(X_u, u < T), \mathbb{P}_i] = [(\hat{X}_u, u < \hat{T}), \hat{\mathbb{P}}_i].$$

Proof. This result is a direct consequence of the general structure of continuous time Markov chains. Indeed, it suffices to observe that when the process X is in a state $i \in E$, the waiting time for hitting D is independent from the past and exponentially distributed, with

parameter η_i . Hence it is clear that before time T , the process X behaves like a process in E , i.e. \hat{X} killed at a time \hat{T} , whose decomposition is given by equation (3.10). \square

The following result provides exponential bounds for the distribution function of the first passage time.

Proposition 2. *Define the rates $\alpha_0 = \inf_{i \in E} \eta_i$ and $\alpha_1 = \sup_{i \in E} \eta_i$, where η_i is defined in (2.2). Then the tail distribution of the first passage time T satisfies the inequalities:*

$$e^{-\alpha_1 t} \leq \mathbb{P}_i(t < T) \leq e^{-\alpha_0 t}, \quad (3.11)$$

for all $t \geq 0$ and for all $i \in E$.

Proof. It follows from Lemma 1 that for all $i \in E$ and $t \geq 0$,

$$\begin{aligned} & \mathbb{P}_i(t < T) & (3.12) \\ = & \hat{\mathbb{P}}_i(t < \hat{T}) \\ = & \sum_{n=0}^{\infty} \sum_{i_0, \dots, i_n \in E} \hat{\mathbb{P}}_i(t < T_n + \varepsilon(\hat{X}_{T_n}), \Omega_n \mid \hat{X}_{T_0} = i_0, \dots, \hat{X}_{T_n} = i_n) \times \\ & \hat{\mathbb{P}}_i(\hat{X}_{T_0} = i_0, \dots, \hat{X}_{T_n} = i_n), & (3.13) \end{aligned}$$

where $i_0 = i$. Then, from the Markov property and the assumption on the random variables $\{\varepsilon(i) : i \in E\}$ in Lemma 1, under $\hat{\mathbb{P}}_i$, conditionally on $\{\hat{X}_{T_0} = i_0, \dots, \hat{X}_{T_n} = i_n\}$, the random variables $I_1, \dots, I_{n+1}, \varepsilon(i_0), \dots, \varepsilon(i_n)$ are independent. So with $I = (I_1, \dots, I_{n+1})$ and $x = (x_1, \dots, x_{n+1})$, one has

$$\begin{aligned} & \hat{\mathbb{P}}_i(t < T_n + \varepsilon(\hat{X}_{T_n}), \Omega_n \mid \hat{X}_{T_0} = i_0, \dots, \hat{X}_{T_n} = i_n) & (3.14) \\ = & \int_{\mathbb{R}_+^{n+1}} \hat{\mathbb{P}}_i(I \in dx) \hat{\mathbb{P}}_i(\varepsilon(i_0) > x_1, \dots, \varepsilon(i_{n-1}) > x_n, t - t_n < \varepsilon(i_n) < x_{n+1}), \end{aligned}$$

where $t_n = x_1 + \dots + x_n$. The integrand in the above integral may be written as

$$\begin{aligned} & \hat{\mathbb{P}}_i(\varepsilon(i_0) > x_1, \varepsilon(i_1) > x_2, \dots, \varepsilon(i_{n-1}) > x_n, t - t_n < \varepsilon(i_n) < x_{n+1}) & (3.15) \\ = & e^{-\eta(i_0)x_1 - \eta(i_1)x_2 - \dots - \eta(i_{n-1})x_n} (e^{-\eta(i_n)[(t-t_n) \vee 0]} - e^{-\eta(i_n)x_{n+1}}) \mathbb{1}_{\{t < t_{n+1}\}}, \end{aligned}$$

where for $n = 0$, the term $e^{-\eta(i_0)x_1 - \eta(i_1)x_2 - \dots - \eta(i_{n-1})x_n}$ is understood to be 1. Hence by applying successively (3.15), (3.14) and (3.13), it follows

$$\begin{aligned} & \mathbb{P}_i(t < T) \\ = & \hat{\mathbb{E}}_i \left(\sum_{n=0}^{\infty} e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{n-1})I_n} \left[e^{-\eta(X'_n)[(t-T_n) \vee 0]} - e^{-\eta(X'_n)I_{n+1}} \right] \mathbb{1}_{\{t < T_{n+1}\}} \right), \end{aligned}$$

where for each n , $\{X'_n : n \geq 0\}$ is a sequence of random variables which has the same law as $\{\hat{X}_n : n \geq 0\}$ under $\hat{\mathbb{P}}_x$ and which is independent of the sequence $\{T_n : n \geq 0\}$. Let k be the (random) index such that $T_k \leq t < T_{k+1}$, then we easily check that the term which is

in the expectation of the right hand side of the above equality is

$$\begin{aligned}
 & \sum_{n=0}^{\infty} e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{n-1})I_n} \left[e^{-\eta(X'_n)[(t-T_n) \vee 0]} - e^{-\eta(X'_n)I_{n+1}} \right] \mathbb{1}_{\{t < T_{n+1}\}} \\
 = & e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{k-1})I_k} \left[e^{-\eta(X'_k)(t-T_k)} - e^{-\eta(X'_k)I_{k+1}} \right] \\
 & + \sum_{n=k+1}^{\infty} e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{n-1})I_n} \left[1 - e^{-\eta(X'_n)I_{n+1}} \right] \\
 = & e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{k-1})I_k} e^{-\eta(X'_k)(t-T_k)}.
 \end{aligned}$$

But from the assumption, we have $\eta(X'_n) \geq \alpha_0$, a.s., for each $n \geq 0$, so that it follows $e^{-\eta(X'_0)I_1 - \dots - \eta(X'_{k-1})I_k} e^{-\eta(X'_k)(t-T_k)} \leq e^{-\alpha_0(I_1 + \dots + I_k)} e^{-\alpha_0(t-T_k)} = e^{-\alpha_0 t}$, a.s. which gives the second inequality in (3.11).

We obtain the first inequality in (3.11) by bounding from above the term $\mathbb{P}_i(T \leq t)$. More precisely, we show that $\mathbb{P}_i(T \leq t) = 1 - \mathbb{P}_i(t < T) \leq e^{-\alpha_1 t}$ in the same way as we obtained the other bound. \square

Note that Corollary 4 may also be obtained as a direct consequence on Proposition 1. It suffices to assume that $\alpha_0 = \alpha_1$ in this proposition.

3.2. Application to the emergence time of a mutant escaping treatment. Let us consider the case of a pathogen population living on a host population. At each time t , the whole host population is either treated or not. The mutation rate depends on the pathogen population size. Since treatment reduces the pathogen population size, we assume that the mutation rate takes two different values, according to the presence or absence of the treatment. Then, the dynamics of the pathogen population size is described as a Markov chain X whose state space S is split up in three parts, that is $S = E_1 \cup E_2 \cup E_3$, with :

- . E_1 , the set of values of the pathogen population size when the population contains at least one mutant,
- . E_2 , the set of values of the pathogen population size when the population contains no mutants and its size is less than a given value K ,
- . E_3 , the set of values of the pathogen population size when the population contains no mutants and its size is greater than K .

The set E_2 corresponds to the presence of treatment and E_3 corresponds to its absence. Recall the notation of Theorem 3. The transition rates between E_2 and E_3 depend only on the treatment strategy and are considered to be constant equal to \bar{q}_{23} and \bar{q}_{32} . We also note the two transition rates to E_1 , that is the mutations rates, by \bar{q}_{21} and \bar{q}_{31} . The emergence time is then defined as $T = \inf\{t \geq 0 : X_t \in E_1\}$.

Let us consider a treatment strategy ensuring that the probability for the pathogen population size to be less than K , before a mutation occurs, is $\mu(E_2) = \bar{\mu}_2$. Similarly, the probability for the size to be greater than K , before mutation, is $\mu(E_3) = \bar{\mu}_3 = 1 - \bar{\mu}_2$. From Theorem 3, T is exponentially distributed with parameter $\alpha > 0$, if $\bar{\mu}$ solves the equation :

$$\bar{\mu} \bar{Q}^T = -\alpha \bar{\mu}, \tag{3.16}$$

with

$$\bar{Q}^T = \begin{pmatrix} -\bar{q}_{23} - \bar{q}_{21} & \bar{q}_{23} \\ \bar{q}_{32} & -\bar{q}_{32} - \bar{q}_{31} \end{pmatrix}.$$

Set $\alpha = \bar{\mu}_2 \bar{q}_{21} + \bar{\mu}_3 \bar{q}_{31}$ and

$$\bar{\mu}_2 = \frac{\bar{q}_{21} - \bar{q}_{31} + \bar{q}_{23} + \bar{q}_{32} - \sqrt{(\bar{q}_{21} - \bar{q}_{31} + \bar{q}_{23} - \bar{q}_{32})^2 + 4\bar{q}_{23}\bar{q}_{32}}}{2(\bar{q}_{21} - \bar{q}_{31})},$$

then we can check that $\bar{\mu} = (\bar{\mu}_2, \bar{\mu}_3)$ is a solution of (3.16), so that with this choice for α and $\bar{\mu}$, the time T is exponentially distributed with parameter $\alpha > 0$.

From a biological point of view, these results may be interpreted and used as follows. The rate $\bar{\mu}_2$ represents the proportion of time during which the host population has been treated after some time. From this proportion of time, we can determine the emergence time of a mutant pathogen. Then a treatment strategy can be designed.

4. WHEN EXPONENTIALITY IMPLIES QUASI-STATIONARITY

In this section, we present two examples where exponentiality implies quasi-stationarity. The first one concerns extinction times in branching processes and in the second one we set out the case of the emergence time of a new type in a multitype branching process.

4.1. Extinction time in branching processes. Let $(Z_t, t \geq 0)$ be an irreducible critical or subcritical continuous time branching process on the set $S = \{0, 1, \dots\}$ of nonnegative integers, as it is defined in Chapter III of [1]. Set $D = \{0\}$ and recall the definition of the first passage time:

$$T = \inf\{t : Z_t = 0\},$$

which is an *absorption time* in the present case. Then it is well known that under these assumptions, $\mathbb{P}_k(T < \infty) = 1$, for all $k \geq 1$ and from the branching property, we have for all $k \in E = \{1, 2, \dots\}$ and all $t > 0$,

$$\begin{aligned} \mathbb{P}_k(T \leq t) &= \mathbb{P}_k(Z_t = 0) \\ &= [\mathbb{P}_1(Z_t = 0)]^k. \end{aligned} \tag{4.1}$$

Let us set $q_t = \mathbb{P}_1(Z_t = 0)$, then from (4.1), for any probability measure μ on E , the quantity $\mathbb{P}_\mu(Z_t = 0)$ corresponds to the generating function G_μ of μ evaluated at the point q_t , that is

$$\begin{aligned} \mathbb{P}_\mu(Z_t = 0) &= \sum_{k=0}^{\infty} \mathbb{P}_k(Z_t = 0) \mu(k) \\ &= \sum_{k=0}^{\infty} q_t^k \mu(k) \\ &= G_\mu(q_t). \end{aligned} \tag{4.2}$$

Note that q_t is a non decreasing function. Let q_t^{-1} be its right continuous inverse, i.e. $q_t^{-1} = \inf\{s : q_s > t\}$. Then from (4.1) and (4.2), for T to be exponentially distributed under \mathbb{P}_μ with parameter $\alpha > 0$, we should have for all $t \in [0, 1)$:

$$G_\mu(t) = 1 - e^{-\alpha q_t^{-1}}. \tag{4.3}$$

Equation (4.3) shows that for each $\alpha > 0$, there is exactly one measure $\mu = \mu_\alpha$, with support on E such that T is exponentially distributed with parameter α , under \mathbb{P}_μ . According to Theorem 3.1 in [14], that is easily extended to discrete state space, continuous time branching processes, if the Malthusian parameter ρ of $(Z_t, t \geq 0)$, i.e. $\mathbb{E}_1(Z_t) = e^{-\rho t}$, is strictly positive, then for $\alpha \leq \rho$, μ_α is a quasi-stationary distribution, whereas it is not in

the case where $\alpha > \rho$. When $\rho = 0$ there is no quasi-stationary distribution associated to T .

Observe that sometimes the generating function G_{μ_α} may be derived explicitly. For instance in [1], p.109, it is proved that if Z is a birth and death process with rates $\lambda_n = n\lambda$ and $\nu_n = n\mu$, when the process is in state n , then

$$q_t = \frac{\nu e^{(\nu-\lambda)t} - \nu}{\nu e^{(\nu-\lambda)t} - \lambda}.$$

With $q_t^{-1} = \ln \left(\frac{\nu-\lambda t}{\nu(1-t)} \right)^{\frac{1}{\nu-\lambda}}$, we obtain,

$$G_{\mu_\alpha}(t) = 1 - \left(\frac{\nu - \lambda t}{\nu(1-t)} \right)^{\frac{-\alpha}{\nu-\lambda}}, \quad t \in [0, 1),$$

which allows us to recover the generating function of the unique quasi stationary distribution associated to T and α . In the case of continuous state branching processes, a similar expression for the Laplace transform of μ_α has been obtained in [14], see p. 438 therein.

4.2. Emergence time in multitype branching processes. Let $\mathbf{Z} = (Z^{(1)}, \dots, Z^{(d)})$ be a d -type, irreducible, supercritical branching process, see Section V.7 in [1]. The state space of \mathbf{Z} is then $S = \{0, 1, \dots\}^d$. Let $\nu = (\nu^{(1)}, \dots, \nu^{(d)})$ be the offspring distribution of \mathbf{Z} . We assume that for all $i = 1, \dots, d$,

$$\nu^{(i)}(0) = 0,$$

so that, each individual has at least one child, with probability 1. In particular, the extinction time is almost surely infinite. Then we are concerned with the *emergence time* of the subpopulation of type d , that is:

$$T = \inf\{t : Z_t^{(d)} \geq 1\}.$$

For $\mathbf{k} = (k_1, \dots, k_d) \in S$, let $\mathbb{P}_{\mathbf{k}}$ be the probability under which \mathbf{Z} starts from \mathbf{k} , i.e. $\mathbf{Z}_0 = \mathbf{k}$, $\mathbb{P}_{\mathbf{k}}$ -a.s. It is not difficult to check that under our assumptions, $\mathbb{P}_{\mathbf{k}}(T < \infty) = 1$, for all $\mathbf{k} \in S \setminus \{0\}$. In the present case, the domain in which T is the first passage time is $D = S \setminus (\{0, 1, \dots\}^{d-1} \times \{0\})$, that is $E = \{0, 1, \dots\}^{d-1} \times \{0\}$. Let $\mathbf{e}_i \in S$ be defined as $\mathbf{e}_1 = (1, 0, \dots, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, \dots, 0)$,... Then we derive from the branching property of \mathbf{Z} that for all $\mathbf{k} \in E$,

$$\mathbb{P}_{\mathbf{k}}(T > t) = \prod_{i=1}^{d-1} \mathbb{P}_{\mathbf{e}_i}(T > t)^{k_i}. \quad (4.4)$$

(Recall that $k_d = 0$, when $\mathbf{k} \in E$.) Let μ be a distribution whose support is included in E . Recall that the (multidimensional) generating function of μ is defined by

$$\mathbf{G}_\mu(\mathbf{t}) = \sum_{\mathbf{k} \in E} t_1^{k_1} \dots t_d^{k_d} \mu(\mathbf{k}), \quad \text{for } \mathbf{t} = (t_1, \dots, t_d) \in [0, 1)^d. \quad (4.5)$$

Let us set $q_{i,t} = \mathbb{P}_{\mathbf{e}_i}(T > t)$ and for $\mathbf{k} \in E$, let $\mathbf{q}_t = (q_{1,t}, \dots, q_{d,t})$ and $\mathbf{q}_t^{\mathbf{k}} = q_{1,t}^{k_1} \times \dots \times q_{d,t}^{k_d} = q_{1,t}^{k_1} \times \dots \times q_{d-1,t}^{k_{d-1}}$ (recall that $q_{d,t} \equiv 1$). Then we derive from (4.4) and (4.5) that

$$\begin{aligned} \mathbb{P}_\mu(T > t) &= \sum_{\mathbf{k} \in E} \mathbf{q}_t^{\mathbf{k}} \mu(\mathbf{k}), \quad t \in [0, 1), \\ &= \mathbf{G}_\mu(\mathbf{q}_t). \end{aligned} \quad (4.6)$$

Let us assume that $d = 2$. Then in this case, the state space of the process \mathbf{Z} killed at time T can be identified to $\{0, 1, \dots\}$, so we write \mathbb{P}_k , $k \in \{0, 1, \dots\}$ for the probability distributions associated to the killed process. Similarly to the situation presented in 4.1, for each $\alpha > 0$, the distribution μ_α such that $\mathbb{P}_{\mu_\alpha}(T > t) = e^{-\alpha t}$, is uniquely determined by equation (4.6). Indeed, if $q_{1,t}^{-1}$ is the right continuous inverse of $q_{1,t} = \mathbb{P}_1(T > t)$, then

$$\mathbf{G}_{\mu_\alpha}(t) = e^{-\alpha q_{1,t}^{-1}}, \quad t \geq 0.$$

An open question is then to determine whether if μ_α is quasi-stationary or not. According to Theorem 2, it amounts to check if μ_α satisfies the equation $\mu_\alpha Q^T = -\alpha \mu_\alpha + \alpha \delta$. Besides, from Theorem 1.1 in [8] this can be determined at least for the rate α which is defined in (2.20), that is

$$\alpha = \inf \left\{ a \geq 0 : \int_0^\infty e^{at} P_i(T > t) dt = \infty \right\}.$$

Then provided α is strictly positive and $\lim_{k \rightarrow \infty} \mathbb{P}_k(T < t) = 0$, for all $t \geq 0$ (the latter condition is clearly satisfied in our case) the distribution μ_α is a quasi-stationary distribution associated to the rate α .

Remarks 1. *In the case where S is a finite set, another example where exponentiality implies quasi-stationarity is given in part (iii) of Proposition 2.1 of [6]. The Markov chain that is considered in this work is a random walk in the finite set $\{0, 1, \dots, N\}$ killed at 0.*
2. In the case of continuous state space Markov processes, other examples where exponentiality implies quasi-stationarity have been emphasized in [9]. In this work it is proved that if the absorption time of a positive selfsimilar Markov process is exponentially distributed under some initial distribution, then the latter is necessarily quasi-stationary.

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