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Commutator length of annulus diffeomorphisms

E. Militon

May 23, 2011

Abstract

We study the group $\text{Diff}_0^r(\mathbb{A})$ of C^r -diffeomorphisms of the closed annulus that are isotopic to the identity. We show that, for $r \neq 3$, the linear space of homogeneous quasi-morphisms on the group $\text{Diff}_0^r(\mathbb{A})$ is one dimensional. Therefore, the commutator length on this group is (stably) unbounded. In particular, this provides an example of a manifold whose diffeomorphisms group is unbounded in the sense of Burago, Ivanov and Polterovich.

1 Introduction

Let M be a manifold. For r in $\mathbb{N} \cup \{\infty\}$, denote by $\text{Diff}_0^r(M)$ the identity component of the group of C^r -diffeomorphism of the M . We also write $\text{Homeo}_0(M) = \text{Diff}_0^0(M)$. We study algebraic properties of these groups.

A commutator in $\text{Diff}_0^r(M)$ is an element of the form $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$. It is known that, for $r \neq 3$ every element in $\text{Diff}_0^r(M)$ can be written as a product of commutators. A question naturally arises: how many commutators shall we need to write a given diffeomorphism as a product of commutators? For an element f in $\text{Diff}_0^r(M)$, the *commutator length* $cl_r(f)$ is the minimal number of commutators needed to write f as a product of commutators. Burago, Ivanov and Polterovich showed in [4] that, when the manifold M is a sphere or compact and three-dimensional, the commutator length is bounded (by 4 in the case of the sphere and 10 in the case of three-dimensional manifolds). They also exhibited a wide class of open manifolds (portable manifolds), including \mathbb{R}^n , for which the commutator length is bounded by 2. Tsuboi generalized these results for odd-dimensional compact manifolds and gave a better bound : the commutator length is bounded by 6 in those cases.

The *support* $\text{supp}(f)$ of a homeomorphism f of M is defined to be the closure of the set:

$$\{x \in M, f(x) \neq x\}.$$

A homeomorphism f is said to be *supported in a ball* if there exists a topological embedding $i : \mathbb{B} \rightarrow M$ of the closed unit ball \mathbb{B} of the same dimension as M such that $\text{supp}(f) \subset i(\mathbb{B})$ and $i(\mathbb{B}) \cap \partial M$ is homeomorphic to an open cell. Like for commutators, every element f in $\text{Diff}_0^r(M)$ can be written as a product of

diffeomorphisms in $\text{Diff}_0^r(M)$ supported in balls and we define $\text{Frag}_r(f)$ to be the minimal number of diffeomorphisms in such a decomposition. Burago, Ivanov, Polterovich and Tsuboi showed also that the fragmentation norm is bounded in the case of odd-dimensional manifolds and of open portable manifolds.

The commutator length and the fragmentation norm are two examples of the more general notion of conjugation-invariant norm on a group introduced by Burago, Ivanov and Polterovich in [4]. They proved that the fragmentation norm plays a crucial role: every conjugation-invariant norm on $\text{Diff}_0^r(M)$ is bounded if and only if the fragmentation norm is bounded.

In this paper, we consider the case of the closed annulus \mathbb{A} . We prove some estimates on the commutator length and the fragmentation norm, which imply in particular that they are unbounded, contrarily to the above examples¹. For a diffeomorphism f in $\text{Diff}_0^r(\mathbb{A})$, let $\rho(f)$ be the difference of the translation numbers of f on the two boundary components (for a precise definition of $\rho(f)$, see the next section). The following theorem shows that ρ is a quasi-isometry from $\text{Diff}_0^r(M)$, endowed with the fragmentation norm or the commutator length, to \mathbb{R} :

Theorem 1.1 *Let r be an integer different from 3. For any diffeomorphism f in $\text{Diff}_0^r(\mathbb{A})$,*

$$\left| \frac{|\rho(f)|}{4} - cl_r(f) \right| \leq 12$$

and

$$|\rho(f) - \text{Frag}_r(f)| \leq 40.$$

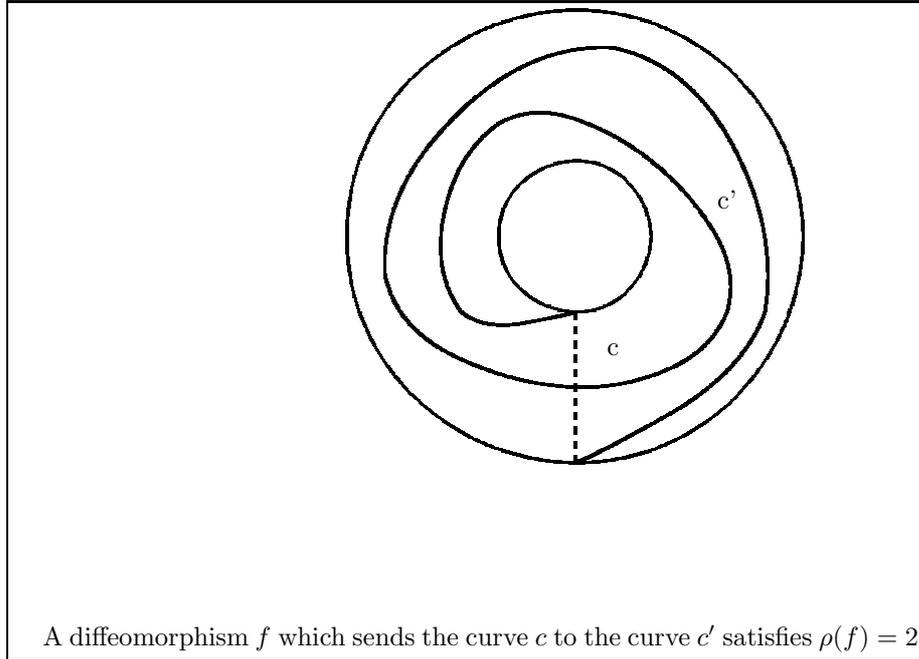
In fact, the map ρ is a quasi-morphism (this notion will be defined in the next section) on the group $\text{Diff}_0^r(\mathbb{A})$ and is essentially the only quasi-morphism on this group:

Theorem 1.2 *If $r \neq 3$, every homogeneous quasi-morphism on $\text{Diff}_0^r(\mathbb{A})$ is colinear to ρ .*

Remark: for $r = 3$, the result holds if we replace $\text{Diff}_0^r(\mathbb{A})$ by $[\text{Diff}_0^r(\mathbb{A}), \text{Diff}_0^r(\mathbb{A})]$, the subgroup generated by commutators. Whether the equality $[\text{Diff}_0^3(\mathbb{A}), \text{Diff}_0^3(\mathbb{A})] = \text{Diff}_0^3(\mathbb{A})$ holds or not is still an open question.

In the next section, we state a proposition which give fine estimates on the commutator length. We also show that this proposition imply the first part of Theorem 1.1 and Theorem 1.2. The third section is devoted to the proof of this proposition. In the fourth section, we prove an analogous proposition on the fragmentation norm which implies the second part of Theorem 1.1. Finally, in the last section, we discuss generalizations of these results to other surfaces.

¹Observe in particular that, for a manifold M with interior $\text{int}(M)$, the boundedness of $\text{Diff}_0^r(\partial M)$ and of $\text{Diff}_0^r(\text{int}(M))$ does not necessarily imply that $\text{Diff}_0^r(M)$ is bounded (compare to [1]).



2 Quasimorphisms on $\text{Diff}_0^r(\mathbb{A})$

Definition 2.1 Given a group G , a homogeneous quasi-morphism on G is a map $q : G \rightarrow \mathbb{R}$ which satisfies:

1. $\exists D > 0, \forall a, b \in G, |q(ab) - q(a) - q(b)| \leq D.$
2. $\forall a \in G, \forall n \in \mathbb{Z}, q(a^n) = nq(a).$

The least constant $D(q)$ which satisfies 1) is called the defect of the quasimorphism q . The trivial quasimorphism is the quasimorphism which maps every element of G to 0.

A first connection between commutator length and quasi-morphisms is given by the following classical lemma:

Lemma 2.1 Suppose G is perfect. If G admits a non-trivial homogeneous quasi-morphism q , then cl_G is unbounded. Moreover,

$$\forall g \in G, |q(g)| \leq 2cl_G(g)D(q).$$

Proof. First, we show that q is conjugation invariant. Indeed, for elements g and h in G :

$$|q(hgh^{-1}) - q(g)| = \frac{1}{n} |q(hg^n h^{-1}) - q(g^n)| \leq \frac{1}{n} D(q)$$

and $q(hgh^{-1}) = q(g)$. Let g be an element of G with commutator length $cl_G(g) = n$. Then g can be written the following way:

$$g = [g_1, h_1][g_2, h_2] \dots [g_n, h_n],$$

where the g_i and the h_i lie in G . Then,

$$\left| q(g) - \sum_{i=1}^n (q(g_i) + q(h_i g_i^{-1} h_i^{-1})) \right| \leq (2n - 1)D(q)$$

and, as $q(h_i g_i^{-1} h_i^{-1}) = -q(g_i)$,

$$|q(g)| \leq 2cl_G(g)D(q)$$

for all g in G . Therefore, for every natural integer p ,

$$|q(g^p)| = p |q(g)| \leq 2cl_G(g^p)D(q).$$

Hence, $cl_G(g^p) \rightarrow +\infty$ when $p \rightarrow +\infty$ as soon as $q(g) \neq 0$. ■

Quasi-morphisms on G are in fact more closely related to commutator length via the Bavard duality (see [2] and [5]). A typical example is the *translation number* on the group $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of \mathbb{R} which commute with integral translations (which is also the group of lifts of orientation-preserving homeomorphisms of the circle): for every homeomorphism f in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$, the sequence $(\frac{f^n(x) - x}{n})_n$ converges for any x and the limit is independant of the chosen point x . This limit is called the translation number of f . It actually defines a quasi-morphism on the group $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ (see [6] for more information on the translation number).

We now consider the closed annulus $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$. We build a quasi-morphism on the identity component $\text{Homeo}_0(\mathbb{A})$ of the group of homeomorphisms of \mathbb{A} . The following construction is due to Frederic Le Roux. We denote by $\pi : \mathbb{R} \times [0, 1] \rightarrow \mathbb{A}$ the universal covering of \mathbb{A} . Given a homeomorphism f in $\text{Homeo}_0(\mathbb{A})$, consider a lift $F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ of f , *i.e.* a homeomorphism of $\mathbb{R} \times [0, 1]$ which satisfies $\pi \circ F = f \circ \pi$. The maps $F_0 = F_{\mathbb{R} \times \{0\}}$ and $F_1 = F_{\mathbb{R} \times \{1\}}$ belong to the group $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$. If we replace F by some other lift $F + k$, then both translation numbers of F_0 and F_1 will increase by k . Thus the difference of the translation numbers of these homeomorphisms is independant of the lift F chosen. We denote this number by $\rho(f)$ and call it the *torsion number*. As the translation number is a homogeneous quasi-morphism on the group $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$, ρ is a (nontrivial) homogeneous quasi-morphism on $\text{Homeo}_0(\mathbb{A})$ which restricts to a non-trivial quasi-morphism on $\text{Diff}_0^r(\mathbb{A})$ for every r in $\mathbb{N} \cup \{\infty\}$.

We first reduce Theorem 1.2 to the following proposition, which gives estimates on the commutator length. We note E the lower integer part and, for an element f in $\text{Diff}_0^r(\mathbb{A})$:

$$\alpha(f) = \min_{r \in \mathbb{R}} |F_1(r) - F_0(r)|,$$

where F is a lift of f (note that this quantity is independent of the choice of the lift).

Proposition 2.2 *Let $f \neq Id_{\mathbb{A}}$ be a homeomorphism in $\text{Diff}_0^r(\mathbb{A})$ and F be a lift of f . Then, for any $r \neq 3$:*

$$E\left(\frac{\alpha(f) + 3}{4}\right) + 9 \geq cl_r(f) \geq E\left(\frac{\alpha(f)}{4}\right) + 1.$$

If $r = 0$, then the 9 appearing in the upper bound may be improved to 5.

This proposition will be shown in the next section. Now, we deduce the first part of Theorem 1.1 and Theorem 1.2 from Proposition 2.2.

Proof of the first part of Theorem 1.1. It suffices to show that, for every element f in $\text{Diff}_0^r(\mathbb{A})$:

$$|\rho(f) - \alpha(f)| \leq 2.$$

Take a real number r_0 in \mathbb{R} such that $\alpha(f) = |F_1(r_0) - F_0(r_0)|$, where F is a lift of f . We will prove that :

$$|\rho(f) - F_1(r_0) - F_0(r_0)| \leq 2.$$

For every homeomorphisms G and H in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we have the following classical inequality:

$$\forall r \in \mathbb{R}, |(G(H(r)) - r) - (G(r) - r) - (H(r) - r)| \leq 1.$$

Using this formula, we obtain by induction, for $i \in \{0, 1\}$:

$$\forall n \in \mathbb{N}, |F_i^n(r_0) - nF_i(r_0) + (n-1)r_0| \leq n-1$$

and $|F_1^n(r_0) - F_0^n(r_0) - n(F_1(r_0) - F_0(r_0))| \leq 2n-2$. Dividing by n and taking the limit $n \rightarrow +\infty$ allows us to conclude the proof. ■

Proof of Theorem 1.2. The strategy of the proof is the same as in the case of the translation number on $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ (see [6]). Let q be a homogeneous quasi-morphism on $\text{Diff}_0^r(\mathbb{A})$. Let t be a C^∞ -diffeomorphism in $\text{Diff}_0^\infty(\mathbb{A})$ which is a "twist": there exists a lift T of t which satisfies:

$$\forall x \in \mathbb{R}, \begin{cases} T(x, 0) = (x, 0). \\ T(x, 1) = (x + 1, 1). \end{cases}$$

(For instance, one can take $t(x, r) = (x+r, r)$.) Then $q - q(t)\rho$ is a homogeneous quasi-morphism which vanishes on t and the next lemma allows us to complete the proof of Theorem 1.2. ■

Remark: observe that t fixes the boundary but there is no continuous path of diffeomorphisms fixing the boundary between the identity and t .

Lemma 2.3 *Every homogeneous quasi-morphism on $\text{Diff}_0^r(\mathbb{A})$ which vanishes on t is trivial, if $r \neq 3$.*

Proof of lemma 2.3. Let q be such a quasi-morphism. Then, for every integer $n \in \mathbb{Z}$ and every homeomorphism f in $\text{Diff}_0^r(\mathbb{A})$, we have

$$|q(t^n f) - q(f)| \leq D(q).$$

Let us fix f in $\text{Diff}_0^r(\mathbb{A})$ and choose an integer n_0 such that $\alpha(t^{n_0} f) < 1$ (if $\alpha(f) = \min_{r \in \mathbb{R}} (F_1(r) - F_0(r))$, then $\alpha(t^p f) = \alpha(f) + p$ as long as $p > -\alpha(f)$ and the other cases are similar or easier). Then, by proposition 2.2, the homeomorphism $t^{n_0} f$ may be written as a product of at most 9 commutators. Therefore, using lemma 1.2:

$$|q(t^{n_0} f)| \leq 18D(q)$$

and $|q(f)| \leq 19D(q)$. Hence, q is a bounded homogeneous quasimorphism: it is trivial. ■

3 Estimation of the commutator length

This section is devoted to the proof of Proposition 2.2. We will first make the proof in the case $r = 0$ and then an approximation argument will give the $r > 0$ case. We will need a theorem by Eisenbud, Hirsch and Neumann on the commutator length for lifts of circle homeomorphisms.

Theorem 3.1 *(Eisenbud-Hirsch-Neumann (see [7] Theorem 2.3))*

A homeomorphism F in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ can be written as a product of n commutators in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ if and only if:

$$\min_{x \in \mathbb{R}} |F(x) - x| < 2n - 1.$$

3.1 Lower bound of the commutator length for $r = 0$

Let f be a homeomorphism in $\text{Homeo}_0(\mathbb{A})$ with $cl_0(f) = n$. Let F denote a lift of f which can be written as a product of n commutators in the group of lifts of elements in $\text{Homeo}_0(\mathbb{A})$ namely

$$\{G \in \text{Homeo}_0(\mathbb{R} \times [0, 1]), \forall (x, t) \in \mathbb{R} \times [0, 1], G(x+1, t) = G(x, t) + (1, 0)\}.$$

Then F_0 and F_1 , which belong to $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$, can be written as a product of n commutators. Using the Eisenbud-Hirsch-Neumann theorem, we get:

$$\min_{x \in \mathbb{R}} |F_0(x) - x| = |F_0(x_0) - x_0| < 2n - 1$$

and:

$$\min_{x \in \mathbb{R}} |F_1(x) - x| = |F_1(x_1) - x_1| < 2n - 1.$$

We may assume that: $x_0 < x_1 < x_0 + 1$. These inequalities imply that:

$$\begin{aligned}\alpha(f) &\leq |F_0(x_0) - F_1(x_0)| \\ &\leq |F_0(x_0) - x_0| + |(F_1(x_0) - x_0) - (F_1(x_1) - x_1)| + |F_1(x_1) - x_1| \\ &< (2n - 1) + 2 + (2n - 1) \\ &< 4n\end{aligned}$$

The lower bound in Proposition 2.2 is therefore proved.

3.2 Upper bound of the commutator length for $r = 0$

To get an upper bound, we first compose a given homeomorphism by some number of commutators to get a homeomorphism which admits a lift which pointwise fixes the boundary. Then we compose by 2 commutators to get a homeomorphism with a lift which is the identity in a neighbourhood of the boundary. Finally, such a homeomorphism can be written as a product of 2 commutators, by a result by Burago, Ivanov and Polterovich.

Let f be a homeomorphism in $\text{Homeo}_0(\mathbb{A})$. Let $k = E(\frac{\alpha(f)+3}{4}) + 1$.

3.2.1 First step: getting a homeomorphism with a lift which fixes pointwise the boundary by composing with k commutators

Up to conjugating by $(x, r) \rightarrow (x, 1 - r)$, we may assume that f is a "positive twist": there exists a lift F of f and a real number x_0 which satisfy $\alpha(f) = F_1(x_0) - F_0(x_0)$.

By composing the lift F by an integral translation if necessary, we may suppose that:

$$2k - 2 \leq F_1(x_0) - x_0 < 2k - 1.$$

Then, as $4k > \alpha(f) + 3$:

$$F_0(x_0) - x_0 = F_1(x_0) - x_0 - \alpha(f) \geq 2k - 2 - \alpha(f) > 1 - 2k.$$

By using the Eisenbud-Hirsch-Neumann theorem, we get that F_1 and F_0 can be written as products of k commutators in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$:

$$\begin{cases} F_1 = \prod_{i=1}^k [g_{i,1}, h_{i,1}] \\ F_0 = \prod_{i=1}^k [g_{i,0}, h_{i,0}] \end{cases},$$

where the $g_{i,j}$'s and the $h_{i,j}$'s belong to $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

For every index i , let us take $G_i : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ (respectively H_i) a lift of an element g_i (respectively h_i) of $\text{Homeo}_0(\mathbb{A})$ which satisfies:

$$\begin{cases} p_1 \circ G_i(\cdot, 1) = g_{i,1} \\ p_1 \circ G_i(\cdot, 0) = g_{i,0} \end{cases}$$

(respectively:

$$\begin{cases} p_1 \circ H_i(\cdot, 1) = h_{i,1} \\ p_1 \circ H_i(\cdot, 0) = h_{i,0} \end{cases}.$$

Note that the homeomorphisms H_i and G_i exist because the $g_{i,j}$'s and the $h_{i,j}$'s are isotopic to the identity. Then the homeomorphism $[h_k, g_k] \circ [h_{k-1}, g_{k-1}] \circ \dots \circ [h_1, g_1] \circ f$ admits a lift which pointwise fixes the boundary.

3.2.2 Second step: a homeomorphism with a lift which pointwise fixes the boundary can be written as a product of 4 commutators

Fix a homeomorphism f in $\text{Homeo}_0(\mathbb{A})$ with a lift F which is the identity on the boundary.

Consider a path $c : [0, 1] \rightarrow \mathbb{A}$ which satisfies the following conditions, where p_1 and p_2 are respectively the first and the second projection of $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$:

- $c(0) = (-\frac{1}{4}, 0)$ and $c(1) = (\frac{1}{4}, 0)$.
- $\forall t \in (0, 1), 0 < p_2 \circ c(t) < \frac{1}{4}$.
- $\forall t \in [0, 1], p_2 \circ F \circ c(t) < \frac{1}{4}$.
- $\forall t \in [0, 1], p_1 \circ F \circ c(t) \in (-\frac{3}{8}, \frac{3}{8})$.
- $\forall t \in [0, 1], p_1 \circ c(t) \in (-\frac{3}{8}, \frac{3}{8})$.

Let D be the connected component of $\mathbb{A} - ([-\frac{1}{4}, \frac{1}{4}] \times \{0\} \cup c([0, 1]))$ whose boundary contains the point $(0, 0)$. According to the Schönflies theorem, there exists a homeomorphism $f_{1,0}$ in $\text{Homeo}_0(\mathbb{A})$ which coincides with f on the closure of D (which is a closed topological disc) and with support included in $(-\frac{3}{8}, \frac{3}{8}) \times [0, \frac{1}{2}] = D_{1,0}$. By taking a lace homotopic to the lower boundary component and close to it and applying the Schönflies theorem, we can find a homeomorphism $f_{2,0}$ which coincides with $f_{1,0}^{-1} \circ f$ (which is the identity on D) in a neighbourhood of the lower boundary $\mathbb{R} \times \{0\}$ and which satisfies:

$$\text{supp}(f_{2,0}) \subset (\frac{1}{8}, \frac{7}{8}) \times [0, \frac{1}{2}] = D_{2,0}.$$

With the same technique, we can build homeomorphisms $f_{1,1}$ and $f_{2,1}$ in $\text{Homeo}_0(\mathbb{A})$ which satisfy:

- the homeomorphism $f_{1,1} \circ f_{2,1}$ coincides with f in a neighbourhood of the upper boundary $\mathbb{R} \times \{1\}$.
- the support of $f_{1,1}$ is included in $(-\frac{3}{8}, \frac{3}{8}) \times (\frac{1}{2}, 1] = D_{1,1}$.
- the support of $f_{2,1}$ is included in $(\frac{1}{8}, \frac{7}{8}) \times (\frac{1}{2}, 1] = D_{2,1}$.

Thus, we have $f = f_{1,1} \circ f_{2,1} \circ f_{1,0} \circ f_{2,0}$ near the boundary.

Lemma 3.2 *Let \mathbb{H} denote the closed upper half plane. Then every element in $\text{Homeo}_0(\mathbb{H})$ can be written as one commutator.*

Proof. Let h be an element in $\text{Homeo}_0(\mathbb{H})$. Let U be a neighbourhood of $\text{supp}(h)$ and φ be a homeomorphism in $\text{Homeo}_0(\mathbb{H})$ which satisfies the following conditions:

- the open sets $\varphi^k(U)$, for k in \mathbb{N} , are pairwise disjoint.
- the sequence $(\varphi^k(U))_{k \in \mathbb{N}}$ converges to a singleton $\{p\}$ which lies on the boundary as k tends to $+\infty$.

Now, consider the homeomorphism g in $\text{Homeo}_0(\mathbb{H})$ which satisfies:

- $g = \text{Id}$ outside $\bigcup_{k \in \mathbb{N}} \varphi^k(U)$.
- for every non-negative integer k , $g|_{\varphi^k(U)} = \varphi^k \circ h \circ \varphi^{-k}$.

Then $h = [g, \varphi]$. ■

According to the lemma applied in the discs $D_{i,j}$, each $f_{i,j}$ is a commutator, thus f coincides with a product of 4 commutators in a neighbourhood of the boundary. We will see that we can improve this to 2. Using this last lemma, we may consider homeomorphisms $g_{i,j}$ and $h_{i,j}$ supported in $D_{i,j}$ which satisfy $f_{i,j} = [g_{i,j}, h_{i,j}]$. Note that:

$$\begin{aligned} \text{supp}(g_{i,0}) \cap \text{supp}(g_{i,1}) &= \emptyset, \\ \text{supp}(h_{i,0}) \cap \text{supp}(h_{i,1}) &= \emptyset, \\ \text{supp}(g_{i,0}) \cap \text{supp}(h_{i,1}) &= \emptyset, \\ \text{supp}(h_{i,0}) \cap \text{supp}(g_{i,1}) &= \emptyset, \end{aligned}$$

and

$$\text{supp}(f_{i,0}) \cap \text{supp}(f_{i,1}) = \emptyset.$$

Thus, those pairs of homeomorphisms commute. Therefore:

$$\begin{aligned} g = f_{1,1} \circ f_{2,1} \circ f_{1,0} \circ f_{2,0} &= f_{1,1} \circ f_{1,0} \circ f_{2,1} \circ f_{2,0} \\ &= [g_{1,1}, h_{1,1}] \circ [g_{1,0}, h_{1,0}] \circ [g_{2,1}, h_{2,1}] \circ [g_{2,0}, h_{2,0}] \\ &= [g_{1,1} \circ g_{1,0}, h_{1,1} \circ h_{1,0}] \circ [g_{2,1} \circ g_{2,0}, h_{2,1} \circ h_{2,0}] \end{aligned}$$

Moreover, this homeomorphism coincides with f on a neighbourhood of the boundary. Thus $g^{-1} \circ f$ admits a lift which is the identity near the boundary. As the open annulus is a portable manifold (see [4] Theorem 1.18), the homeomorphism $g^{-1} \circ f$ can be written as a product of two commutators. This ends the proof of Proposition 2.2 in the case $r = 0$.

3.3 Case $r > 0$

It follows directly from the result on homeomorphisms that:

$$\forall f \in \text{Diff}_0^r(\mathbb{A}), \quad cl_r(f) \geq cl_0(f) \geq E\left(\frac{\alpha(f)}{4}\right) + 1.$$

To get an upper bound (for $r \neq 3$), the process is the following. We first write a diffeomorphism f as a product of $E\left(\frac{\alpha(f)+3}{4}\right) + 5$ commutators of homeomorphisms and we approximate every homeomorphism appearing in this product by a diffeomorphism. Hence, arbitrarily close to f in the C^0 topology, there is a

product of $E(\frac{\alpha(f)}{4}) + 5$ commutators of diffeomorphisms. To conclude, it suffices to notice that, for $r \neq 3$, a diffeomorphism sufficiently close to the identity can be written as a product of 4 commutators. Let us show this last fact. We need the following lemma, which can be proved by the same methods as in the proof of Theorem 3.1 in [8]:

Lemma 3.3 *There exists a C^0 neighbourhood Υ of the identity in $\text{Diff}_0^r(\mathbb{A})$ such that:*

$$\forall f \in \Upsilon, \exists f_1, f_2 \in \text{Diff}_0^r(\mathbb{A}), \begin{cases} f = f_1 \circ f_2 \\ \text{supp}(f_1) \subset (-\frac{3}{8}, \frac{3}{8}) \times [0, 1] \\ \text{supp}(f_2) \subset (\frac{1}{8}, \frac{7}{8}) \times [0, 1] \end{cases} .$$

Now, the manifolds $(-\frac{3}{8}, \frac{3}{8}) \times [0, 1]$ and $(\frac{1}{8}, \frac{7}{8}) \times [0, 1]$ are portable, in the sense of [4]. It follows from [4] Theorem 1.18 that, if $r \neq 3$, the diffeomorphisms f_1 and f_2 may be each written as a product of two commutators.

This concludes the proof in the case $r > 0$.

4 Estimation of the fragmentation norm

The analog of Proposition 2.2 for the fragmentation norm is the following proposition, which implies the second part of Theorem 1.1:

Proposition 4.1 *Let f be a homeomorphism in $\text{Diff}_0^r(\mathbb{A})$ and $F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ be a lift of f . Suppose $f \neq \text{Id}$. Then, for any r in $\mathbb{N} \cup \{\infty\}$:*

$$E(\alpha(f)) + 40 \geq \text{Frag}_r(f) \geq E(\alpha(f)) + 1.$$

If $r = 0$, then the 40 appearing in the upper bound may be improved to 36.

Actually, we can have far better upper bound in the preceding proposition by using the following result, which will be showed in a next paper:

Proposition 4.2 *For any $r \in \mathbb{N} \cup \{\infty\}$, the fragmentation norm on $\text{Diff}_0^r(\text{int}(\mathbb{A}))$ is bounded by 4.*

If we admit this last proposition, we get more precise estimates on the fragmentation norm:

Proposition 4.3 *Let f be a homeomorphism in $\text{Diff}_0^r(\mathbb{A})$ and $F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ be a lift of f . Suppose $f \neq \text{Id}$. Then, for any r in $\mathbb{N} \cup \{\infty\}$:*

$$E(\alpha(f)) + 16 \geq \text{Frag}_r(f) \geq E(\alpha(f)) + 1.$$

If $r = 0$, then the 16 appearing in the upper bound may be improved to 12.

To make estimates on the fragmentation norm for the closed annulus, we need an analog of the Eisenbud-Hirsch-Neumann theorem.

4.1 Fragmentation norm on $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$

Let A be the subset of $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ given by elements which fix pointwise a nonempty open interval. This subset generates $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ as a group. If F is a homeomorphism in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we denote $\text{Frag}(F)$ the minimal number of elements of A necessary to write F as a product of elements of A . The analog of the Eisenbud-Hirsch-Neumann theorem for the fragmentation norm is the following:

Proposition 4.4 *Let F be a homeomorphism in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$. Then, for any $k \geq 0$:*

$$\text{Frag}(F) \leq k + 2 \iff \min_{x \in \mathbb{R}} |F(x) - x| < k + 1.$$

Proof. Let us start with the direct implication. Write:

$$F = F_1 F_2 \dots F_{k+1} F_{k+2},$$

where each F_i belongs to A . Take a point x_0 such that $F_{k+2}(x_0) = x_0$. We have, for every integer $i \in [1, k + 1]$:

$$|F_i F_{i+1} \dots F_{k+1} F_{k+2}(x_0) - F_{i+1} \dots F_{k+1} F_{k+2}(x_0)| < 1$$

and by summing these inequalities:

$$|F(x_0) - x_0| < k + 1.$$

This proves the direct implication. Let us show the converse by induction on k . For $k = 0$, up to conjugating by $(x, r) \rightarrow (x, 1 - r)$, we may assume:

$$0 \leq \min_{x \in \mathbb{R}} |F(x) - x| = F(x_0) - x_0 < 1.$$

Choose a point x_1 such that $F(x_0) < x_1 < x_0 + 1$. Then $F(x_0) < F(x_1) < F(x_0) + 1$ and we can find a homeomorphism h in $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ which fixes a neighbourhood of $F(x_0)$ and satisfies $h \circ F = Id$ in a neighbourhood of x_1 . The decomposition $F = h^{-1} \circ (h \circ F)$ shows that $\text{Frag}(F) \leq 2$.

Suppose the converse holds for an integer k . Let us prove it for the integer $k + 1$. Suppose $\min_{x \in \mathbb{R}} |F(x) - x| < k + 2$. We may also suppose that $\min_{x \in \mathbb{R}} |F(x) - x| \geq k + 1$ (otherwise, we can conclude directly from the induction hypothesis). As usual, we may assume:

$$k + 1 \leq \min_{x \in \mathbb{R}} (F(x) - x) < k + 2.$$

Let x_0 be a point which satisfies:

$$\min_{x \in \mathbb{R}} (F(x) - x) = F(x_0) - x_0.$$

The same way as for the initialization, we can find a point $x_1 \in (x_0, x_0 + 1)$ such that $F(x_0) < x_1 + k + 1 < F(x_0) + 1$. Then $F(x_0) < F(x_1) < F(x_0) + 1$.

Therefore, there exists a homeomorphism h which fixes a neighbourhood of $F(x_0)$ and which satisfies:

$$h(F(x_1)) < x_1 + k + 1.$$

The induction hypothesis allows then us to finish the induction and the proof of Proposition 4.4. ■

4.2 Proof of Proposition 4.1

The technique is exactly the same as in the proof of Proposition 2.2, using Proposition 4.4 instead of the Eisenbud-Hirsch-Neumann theorem. Thus the explanations will be briefer.

4.2.1 Lower bound for the fragmentation norm

Let f be an element in $\text{Homeo}_0(\mathbb{A})$ with $\text{Frag}_0(f) \geq 1$ and F be a lift of f . Fix a decomposition of f as a product of $\text{Frag}_0(f)$ homeomorphisms supported in discs. Let k_0 (respectively k_1) be the number of homeomorphisms appearing in this decomposition whose support meets the lower boundary $\mathbb{R}/\mathbb{Z} \times \{0\}$ (resp. the upper boundary $\mathbb{R}/\mathbb{Z} \times \{1\}$) of \mathbb{A} . Note that $k_0 + k_1 \leq \text{Frag}_0(f)$ as a disc doesn't touch both components of the boundary, by definition. First, suppose $k_0 \geq 2$ and $k_1 \geq 2$. By Proposition 4.4, there exist points x_0 and x_1 in \mathbb{R} such that:

$$|F_0(x_0) - x_0| < k_0 - 1$$

and:

$$|F_1(x_1) - x_1| < k_1 - 1.$$

From these inequalities, we get:

$$\alpha(f) < k_0 - 1 + 2 + k_1 - 1 \leq \text{Frag}_0(f),$$

which gives the lower bound in Proposition 4.1. In the cases $k_0 = 1$ and $k_1 \geq 2$ or $k_1 = 1$ and $k_0 \geq 2$, one of the inequalities

$$|F_0(x_0) - x_0| \leq k_0 - 1$$

and

$$|F_1(x_1) - x_1| \leq k_1 - 1$$

is indeed strict, which allows us to conclude.

In the case $k_0 = 0$ and $k_1 \geq 2$ (the case $k_1 = 0$ and $k_0 \geq 2$ is symmetric), we have, as $F_0 = Id$:

$$\alpha(f) < |F_1(x_1) - x_1| + 1 \leq k_1 \leq \text{Frag}_0(f).$$

In the case $(k_0, k_1) = (1, 1)$, we have $\alpha(f) < 1$ and $\text{Frag}_0(f) \geq 2$ so the inequality $\alpha(f) < \text{Frag}_0(f)$ holds.

In the cases $(k_0, k_1) \in \{(0, 0), (0, 1), (1, 0)\}$, we have $\alpha(f) = 0$ so the inequality $\alpha(f) < \text{Frag}_0(f)$ holds.

4.2.2 Upper bound for the fragmentation norm

Let f be an element in $\text{Homeo}_0(\mathbb{A})$ with lift F . As usual, we may assume: $\alpha(f) = p_1 \circ F(x_0, 1) - p_1 \circ F(x_0, 0)$. Consider a lift of f which satisfies:

$$-1 < F_0(x_0) - x_0 \leq 0.$$

Then $F_1(x_0) = F_0(x_0) + \alpha(f) < E(\alpha(f)) + 1$. Using Proposition 4.4, we can see that, after composing by at most $E(\alpha(f)) + 2$ homeomorphisms supported in discs which touch the upper boundary, we get a homeomorphism with a lift which fixes the upper boundary and, by composing by two more homeomorphisms supported in discs we get a homeomorphism with a lift which fixes both boundary components. Then, by composing by four homeomorphisms supported in discs, we get a homeomorphism with a lift which fixes a neighbourhood of the boundary. Finally, by the result by Burago, Ivanov and Polterovich (see the proof of Theorem 1.17 in [4]), such a homeomorphism may be written as a product of 28 homeomorphisms supported in discs. Using proposition 4.2, we see that this might be improved to 4.

Then, an approximation argument combined with the fragmentation lemma yields the case $r > 0$.

5 Generalization to other surfaces

A similar construction as the one made in section 2 can be carried out on every compact surface M with boundary to obtain quasi-morphisms. However, in those cases, we do not know the dimension of the space of quasi-morphism: we just have a minoration of it.

5.1 Case of open surfaces

Suppose M is a non-compact surface with p boundary components which are circles. Let us fix such a boundary component of M : C_1 . Take a path homeomorphic to the half-line which begins on C_1 , goes to infinity, and touches the boundary component C_1 only at endpoint. Consider the cyclic covering $p : \tilde{M} \rightarrow M$ associated to this path. Now, a homeomorphism f in $\text{Homeo}_0(M)$ admits a unique lift $F : \tilde{M} \rightarrow \tilde{M}$ which is compactly supported. The translation number of the restriction of F to $p^{-1}(C_1)$ gives rise to a quasi-morphism. With this method, p independant quasi-morphisms can be built. Moreover, if the commutator length is bounded on the group $\text{Diff}_0^r(\text{int}(M))$, where $\text{int}(M)$ denotes $M - \partial M$, then it can be shown with the same techniques as in the case of the annulus that the vector space of homogeneous quasi-morphisms is p -dimensional. In particular, the vector space of homogeneous quasi-morphism is one-dimensional in the case of the half-open annulus $\mathbb{R}/\mathbb{Z} \times [0, 1)$. However, it is not known whether the commutator length is bounded or not on the group $\text{Diff}_0^r(\text{int}(M))$ when $\text{int}(M)$ is different from the open annulus or from \mathbb{R}^2 .

5.2 Case of other compact oriented surfaces with boundary

When M is a compact oriented surface with boundary which is different from the closed disc or the closed annulus, its universal covering \tilde{M} may be seen as a subspace of the Poincaré disc \mathbb{D} . We endow M and \mathbb{D} with riemannian metrics of constant curvature -1 such that M has a geodesic boundary and the projection $\tilde{M} \rightarrow M$ is a riemannian covering. Denote by d the associated distance. Given two points x and y on \mathbb{D} and an oriented geodesic g which contains both x and y , we define $a(x, y, g) = d(x, y)$ if the geodesic from x to y has the same orientation as g and $a(x, y, g) = -d(x, y)$ otherwise.

Let us take a homeomorphism f in $\text{Homeo}_0(M)$. The homeomorphism f admits a canonical lift $F : \tilde{M} \rightarrow \tilde{M}$ which is the identity on the limit set of \tilde{M} at infinity. Given a point x on the boundary component C of \tilde{M} , which is a geodesic, define:

$$q(x) = \lim_{n \rightarrow +\infty} \left(\frac{a(x, F^n(x), C)}{n} \right).$$

This number is the translation number of F restricted to C and does not depend on the point x chosen on C . With this construction, we can build p independant quasi-morphisms, where p is equal to the number of boundary component of M . Here also, the vector space of quasi-morphisms on $\text{Diff}_0^r(M)$ is p -dimensional if the commutator length on $\text{Diff}_0^r(\text{int}(M))$ is bounded.

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