Low Field Regime for the Relativistic Vlasov-Maxwell-Fokker-Planck System; the One and One Half Dimensional Case

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Abstract

We study the asymptotic regime for the relativistic Vlasov-Maxwell-Fokker-Planck system which corresponds to a mean free path small compared to the Debye length, chosen as an observation length scale, combined to a large thermal velocity assumption. We are led to a convection-diffusion equation, where the convection velocity is obtained by solving a Poisson equation. The analysis is performed in the one and one half dimensional case and the proof combines dissipation mechanisms and finite speed of propagation properties.

Keywords: Vlasov-Maxwell-Fokker-Planck system, Asymptotic behavior, Diffusion approximation.

AMS classification: 35Q99, 35B40.

1 Introduction

This paper is devoted to the asymptotic analysis of a system of PDEs describing the evolution of charged particles. The unknown is the distribution function of particles, which depends on time t, space x and momentum p. The particles are subject to collisional mechanisms and to the action of electro-magnetic forces. The latter are defined in a self-consistent way by the Maxwell equations. We are interested in

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hydrodynamic limits where the relaxation effects induced by the collisional processes are strong enough and force the distribution function to tend towards an equilibrium state. Hence, in such a regime the behavior of the particles can be described by means of a finite set of macroscopic quantities, that is certain averages with respect to p of the distribution function, see e.g. [23, 24, 37]. We distinguish two asymptotic regimes:

- the high-field regime corresponds to a situation where the force field has the same order as the collision term,

- the low-field regime corresponds to a situation where the convection and the force field are also singular terms within the equations, but at lower order than the leading contribution of the collisions.

Roughly speaking, the latter regime leads to convection-diffusion limit equations, while the former yields a purely hyperbolic model. The question has been pointed out by Poupaud [33], see also [16], motivated by the modeling of semi-conductors devices; we also refer to the modeling discussions and numerical studies in [1].

In this paper, we assume that the evolution of the particles is governed by the relativistic Vlasov-Maxwell-Fokker-Planck (VMFP) equations. We write the equations in dimensionless form, detailing in the Appendix the discussion on the scaling. The system depends on three dimensionless parameters: $\theta > 0$, $0 < \delta < 1$ and $\varepsilon > 0$ which is intended to tend to 0. For a momentum $p \in \mathbb{R}^3$, we define the energy

$$\mathcal{E}(p) = \frac{1/\delta^2 - 1}{2}\left(\sqrt{1 + \frac{4\ |p|^2}{\delta^2(1/\delta^2 - 1)^2}} - 1\right) \tag{1}$$

and the velocity is given by

$$v(p) = \nabla_p \mathcal{E}(p) = \frac{2}{1-\delta^2}\ \frac{p}{\sqrt{1 + \dfrac{4\ |p|^2}{\delta^2(1/\delta^2 - 1)^2}}}. \tag{2}$$

As a matter of fact, notice that the velocity remains dominated by

$$|v(p)| < \frac{\delta(1/\delta^2 - 1)}{1-\delta^2} = \frac{1}{\delta}. \tag{3}$$

We are concerned with the following equations

$$\begin{aligned}\partial_t f^\varepsilon + \frac{1}{\varepsilon} v(p) \cdot \nabla_x f^\varepsilon + \left(\frac{1}{\varepsilon}\mathbf{E}^\varepsilon(t,x) + \delta^2 v(p) \wedge \mathbf{B}^\varepsilon(t,x)\right) \cdot \nabla_p f^\varepsilon \\ = \frac{\theta}{\varepsilon^2} \mathrm{div}_p(\nabla_p f^\varepsilon + v(p) f^\varepsilon), \qquad (t,x,p) \in]0,T[\times \mathbb{R}^3 \times \mathbb{R}^3,\end{aligned} \tag{4}$$

$$\partial_t \mathbf{E}^\varepsilon - \mathrm{curl}_x \mathbf{B}^\varepsilon = -\frac{j^\varepsilon}{\varepsilon} + J, \qquad \varepsilon^2\delta^2 \partial_t \mathbf{B}^\varepsilon + \mathrm{curl}_x \mathbf{E}^\varepsilon = 0, \qquad (t,x) \in]0,T[\times \mathbb{R}^3, \tag{5}$$

$$\mathrm{div}_x \mathbf{E}^\varepsilon = \rho^\varepsilon - D, \qquad \mathrm{div}_x \mathbf{B}^\varepsilon = 0, \qquad (t,x) \in]0,T[\times \mathbb{R}^3, \tag{6}$$

where $\rho^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon \, dp$ and $j^\varepsilon = \int_{\mathbb{R}^3} v(p) f^\varepsilon \, dp$ are the charge and current densities associated to the distribution f^ε, respectively while $D \geq 0$ and $J \in \mathbb{R}^3$ are the (given) charge and current densities of a background particle distribution of opposite charge. Throughout the paper we suppose that the global neutrality condition

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^\varepsilon(t,x,p) \, dp \, dx = \int_{\mathbb{R}^3} D(t,x) \, dx, \qquad t \in [0,T], \ \varepsilon > 0$$

holds. We are interested in the asymptotic regime $0 < \varepsilon \ll 1$, with $0 < \delta < 1$ and $\theta = \mathcal{O}(1)$ kept fixed.

In plasma physics or semiconductors theory, one often uses a simplified model where on the one hand relativistic corrections are neglected (which means replacing $v(p)$ by p), and on the other hand, the full set of Maxwell equations is replaced by assuming that the force derives from a potential Φ, which itself obeys the Poisson equation with a right hand side depending on the density of particles. Namely, one considers the Vlasov-Poisson-Fokker-Planck (VPFP) system

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} p \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon} \nabla_x \Phi^\varepsilon \cdot \nabla_p f^\varepsilon = \frac{\theta}{\varepsilon^2} \mathrm{div}_p(\nabla_p f^\varepsilon + p f^\varepsilon), \tag{7}$$

coupled to

$$-\Delta_x \Phi^\varepsilon = \rho^\varepsilon - D. \tag{8}$$

The asymptotic behavior of the system (7), (8) when ε goes to 0 has been studied in [34], where mathematical difficulties depending on the space dimension are clearly pointed out. It was shown that the limit $(\rho, \Phi) := \lim_{\varepsilon \searrow 0} (\rho^\varepsilon, \Phi^\varepsilon)$ solves the following drift-diffusion system

$$\partial_t \rho - \frac{1}{\theta} \mathrm{div}_x(\nabla_x \rho + \rho \nabla_x \Phi) = 0, \qquad -\Delta_x \Phi = \rho(t,x) - D(t,x). \tag{9}$$

The convergence statement is proven in full generality in dimension one, and two (we refer to [26] for this case) but with restriction on initial data and on a small enough time interval in higher dimension. We also mention the tricky analysis recently performed in [29] for the Boltzmann-Poisson system in a bounded domain which leads to quite general results. The high field regime relies on the analysis of the behavior for $\varepsilon \to 0$ of

$$\partial_t f^\varepsilon + p \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon} \nabla_x \Phi^\varepsilon \cdot \nabla_p f^\varepsilon = \frac{\theta}{\varepsilon} \mathrm{div}_p(\nabla_p f^\varepsilon + p f^\varepsilon), \tag{10}$$

where now the non linear force term $\nabla_x \Phi^\varepsilon \cdot \nabla_p f^\varepsilon$ is of the same order of magnitude that the diffusion Fokker-Planck term. The high-field limit of the VPFP system has been studied in [31], [25] and leads to

$$\partial_t \rho - \frac{1}{\theta} \mathrm{div}_x(\rho \nabla_x \Phi) = 0, \qquad -\Delta_x \Phi = \rho(t,x) - D(t,x). \tag{11}$$

This is a pure transport equation, where the velocity field depends on the density ρ through the Poisson equation. We refer to [31] for comments on this problem which shares some features with the pressureless gases model. Therefore, high field combines with hydrodynamic limits and yields interesting phenomena. We also mention in this direction the recent developments in [7] and [5]. Of course, another natural question consists in investigating a vanishing viscosity limit of (9) so that we recover (11); this has been analyzed in [30]. Clearly, in these asymptotic problems the mathematical difficulty relies on the treatment of the non linear term $\nabla_x\Phi^\varepsilon f^\varepsilon$. When analyzing the behavior for small ε's in (7) (or (10)) with (8) we appeal to the very specific form of the coupling with the Poisson equation: it allows us to make use of nice convolution formulae to write the force field by means of the density (that also makes the role of the space dimension clear). Hence the motivation of the questions we address is two-fold. First, on a modeling viewpoint, the coupling with the Maxwell equations takes into account more details of the physics. Second, on a mathematical viewpoint, we investigate how robust the derivation of low and high field limits is or if it cucially depends on the original coupling. In [10], we deal with the high field asymptotics for the (non relativistic) VMFP equations

$$\partial_t f^\varepsilon + p\cdot\nabla_x f^\varepsilon + \left(\frac{1}{\varepsilon}\mathbf{E}^\varepsilon(t,x) + p\wedge\mathbf{B}^\varepsilon(t,x)\right)\cdot\nabla_p f^\varepsilon = \frac{1}{\varepsilon}\mathrm{div}_p(pf^\varepsilon + \nabla_p f^\varepsilon), \quad (12)$$

$$\partial_t\mathbf{E}^\varepsilon - \mathrm{curl}_x\mathbf{B}^\varepsilon = J(t,x) - j^\varepsilon(t,x),\ \varepsilon\partial_t\mathbf{B}^\varepsilon + \mathrm{curl}_x\mathbf{E}^\varepsilon = 0,\ (t,x)\in]0,T[\times\mathbb{R}^3, \quad (13)$$

$$\mathrm{div}_x\mathbf{E}^\varepsilon = \rho^\varepsilon(t,x) - D(t,x), \qquad \mathrm{div}_x\mathbf{B}^\varepsilon = 0, \qquad (t,x)\in]0,T[\times\mathbb{R}^3. \quad (14)$$

It yields the following limit system

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(\rho\mathbf{E}) = 0, \qquad (t,x)\in]0,T[\times\mathbb{R}^3, \\ \mathrm{div}_x\mathbf{E} = \rho(t,x) - D(t,x), \qquad \mathrm{curl}_x\mathbf{E} = 0, \qquad (t,x)\in]0,T[\times\mathbb{R}^3, \\ \partial_t\mathbf{E} - \mathrm{curl}_x\mathbf{B} = J(t,x) - \rho(t,x)\mathbf{E}(t,x), \qquad \mathrm{div}_x\mathbf{B} = 0, \qquad (t,x)\in]0,T[\times\mathbb{R}^3. \end{cases} \quad (15)$$

We analyze here the low field regime taking into account relativistic corrections. Furthermore, we restrict ourselves to the one and one half dimensional framework which means that $f = f(t,x,p_1,p_2), \mathbf{E} = (E_1(t,x), E_2(t,x), 0), \mathbf{B} = (0,0,B(t,x))$ for any $(t,x,p_1,p_2)\in[0,T]\times\mathbb{R}^3$. Precisely, we deal with the equations

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}v_1(p)\partial_x f^\varepsilon + \left(\frac{1}{\varepsilon}E_1^\varepsilon + \delta^2 v_2(p)B^\varepsilon\right)\partial_{p_1}f^\varepsilon + \left(\frac{1}{\varepsilon}E_2^\varepsilon - \delta^2 v_1(p)B^\varepsilon\right)\partial_{p_2}f^\varepsilon$$
$$= \frac{\theta}{\varepsilon^2}\mathrm{div}_p(\nabla_p f^\varepsilon + v(p)f^\varepsilon), \qquad (t,x,p)\in]0,T[\times\mathbb{R}\times\mathbb{R}^2, \quad (16)$$

$$\partial_t E_1^\varepsilon = -\frac{1}{\varepsilon}j_1^\varepsilon(t,x) + J(t,x), \qquad (t,x)\in]0,T[\times\mathbb{R}, \quad (17)$$

$$\partial_t E_2^\varepsilon + \partial_x B^\varepsilon = -\frac{1}{\varepsilon}j_2^\varepsilon(t,x), \qquad (t,x)\in]0,T[\times\mathbb{R}, \quad (18)$$

$$\varepsilon^2\delta^2\partial_t B^\varepsilon + \partial_x E_2^\varepsilon = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}, \tag{19}$$

$$\partial_x E_1^\varepsilon = \rho^\varepsilon(t,x) - D(t,x), \qquad (t,x) \in [0,T]\times\mathbb{R}, \tag{20}$$

where $D, J : [0,T]\times\mathbb{R} \to \mathbb{R}$ are given functions satisfying $D \geq 0$ and the continuity equation

$$\partial_t D + \partial_x J = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}. \tag{21}$$

We prescribe initial conditions for the particle distribution and the electro-magnetic field

$$f^\varepsilon(0,x,p) = f_0^\varepsilon(x,p), \qquad (x,p) \in \mathbb{R}\times\mathbb{R}^2, \tag{22}$$

$$E^\varepsilon(0,x) = E_0^\varepsilon(x), \qquad B^\varepsilon(0,x) = B_0^\varepsilon(x), \qquad x \in \mathbb{R}, \tag{23}$$

satisfying

$$\frac{d}{dx}E_{0,1}^\varepsilon = \int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\,dp - D(0,x), \qquad x \in \mathbb{R}. \tag{24}$$

After integration of (16) with respect to $p \in \mathbb{R}^2$ we deduce that the charge and the current densities verify the continuity equation

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon}\partial_x j_1^\varepsilon = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}.$$

By using the continuity equations for positive/negative charges and by taking the derivative of (17) with respect to x we deduce that (20) is a consequence of (21) and (24). Notice that if initially the neutrality condition is satisfied *i.e.*, $\int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon\,dp\,dx = \int_{\mathbb{R}} D(0,x)\,dx$, then we have $\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\,dp\,dx = \int_{\mathbb{R}} D(t,x)\,dx$ for any $t \in]0,T]$. In what follows we consider only smooth solutions. Unfortunately, to our knowledge, there are no mathematical results concerning the existence and uniqueness of strong solution for the VMFP system. For the VPFP system the situation is better : results concerning the existence of weak solutions can be found in [14], [36] while for existence and uniqueness results of strong solution we refer to [11], [12], [18], [32]. The existence of classical solutions in the collisionless case has been investigated by different approaches, see [22], [13], [27]. Recently global existence and uniqueness results have been obtained for reduced model for laser-plasma interaction, cf. [15], [9]. In this paper, we restrict our purpose to the asymptotic problem. As $\varepsilon \to 0$, we derive a limit system very similar to (9), which was obtained when analyzing the VPFP system. Our proofs rely on compactness arguments. One of the crucial point is to obtain L^∞ bounds for the electro-magnetic field, uniformly with respect to the small parameter $\varepsilon > 0$. This is why we restrict our analysis to solutions depending on one space variable only. Besides, the relativistic framework provides better estimates, related to the bound (3) on the velocity.

The paper is organized as follows. In Section 2, we set up the working assumptions and state precisely our convergence result. In Section 3 we establish a priori estimates, uniformly with respect to the small parameter $\varepsilon > 0$. These bounds are obtained by performing classical computations involving the energy and the entropy of the VMFP system and by using also the hyperbolic structure of the Maxwell equations. We combine the dissipation properties induced by the collisional term – in the spirit of [34] – to the specific use of the finite speed of propagation that is reminiscent to [21]. In Section 4 we detail the passage to the limit, while Section 5 is devoted to some comments and precisions. The dimensional analysis can be found in the Appendix.

2 Assumptions and Main Result

Throughout the paper, we make use of the following hypotheses

$H1)$ $f_0^\varepsilon \geq 0, \qquad D \geq 0, \qquad \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\, dp\, dx = \int_{\mathbb{R}} D(0,x)\, dx, \qquad \forall \varepsilon > 0$;

$H2)$ $$\sup_{\varepsilon>0}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(1+|\ln f_0^\varepsilon|+|x|+\mathcal{E}(p))f_0^\varepsilon\, dp\, dx + \frac{1}{2}\int_{\mathbb{R}}\left(|E_0^\varepsilon|^2+\varepsilon^2\delta^2|B_0^\varepsilon|^2\right)\, dx\right) < \infty \ ;$$

$H3)$ D, J are given integrable smooth functions satisfying $\partial_t D + \partial_x J = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}$;

$H4)$ $J \in L^1(]0,T[;L^2(\mathbb{R})) \cap L^1(]0,T[;L^\infty(\mathbb{R}))$;

$H5)$ $\sup_{\varepsilon>0}\left(\|E_0^\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})}\right) < +\infty$;

$H6)$ There is $r > 1$ such that
$$\sup_{\varepsilon>0}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon(x,p))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx < +\infty.$$

We introduce the notations

$$\begin{aligned}
M_0^\varepsilon &:= \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\, dp\, dx,\\
W_0^\varepsilon &:= \int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{E}(p)f_0^\varepsilon(x,p)\, dp\, dx + \frac{1}{2}\int_{\mathbb{R}}\{|E_0^\varepsilon(x)|^2+\varepsilon^2\delta^2|B_0^\varepsilon(x)|^2\}\, dx,\\
H_0^\varepsilon &:= \int_{\mathbb{R}}\int_{\mathbb{R}^2} |\ln f_0^\varepsilon|\ f_0^\varepsilon(x,p)\, dp\, dx,\\
L_0^\varepsilon &:= \int_{\mathbb{R}}\int_{\mathbb{R}^2} |x| f_0^\varepsilon(x,p)\, dp\, dx,\\
R_0^\varepsilon &:= \|E_0^\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

which are thus uniformly bounded with respect to ε. Our main result states as follows.

Theorem 2.1 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ be smooth solutions of* $(16)-(23)$. *Assume that H1-H6 hold. Then, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to 0 such that*

$$\rho^{\varepsilon_k}(t,x) = \int_{\mathbb{R}^2} f^{\varepsilon_k}(t,x,p)\,dp \rightharpoonup \rho \geq 0 \quad \textit{weakly in } L^1(]0,T[\times\mathbb{R}),$$

$$(E_1^{\varepsilon_k}, E_2^{\varepsilon_k}, \delta\varepsilon_k B^{\varepsilon_k}) \rightharpoonup (E_1,0,0) \quad \textit{weakly in } L^2(]0,T[\times\mathbb{R})^3 \\ \textit{and weakly} \star \textit{ in } L^\infty(]0,T[\times\mathbb{R})^3,$$

$$E_1^{\varepsilon_k} \to E_1, \qquad \textit{strongly in } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}).$$

The limits ρ, E_1 satisfy in the distribution sense

$$\begin{cases} \theta\partial_t E_1 + \rho(t,x)E_1(t,x) - \partial_x^2 E_1 = \partial_x D + \theta J(t,x), & (t,x)\in]0,T[\times\mathbb{R}, \\ \partial_x E_1 = \rho(t,x) - D(t,x), & (t,x)\in[0,T]\times\mathbb{R}, \\ E_1(0,x) = \lim_{k\to+\infty} E_{0,1}^k(x), & \textit{uniformly on compact sets of } \mathbb{R}. \end{cases}$$

The limit equation is nothing but the convection-diffusion model (9) obtained when dealing with the low field regime of the VPFP system. Indeed, taking the derivative with respect to x of the evolution equation for E_1 and using H3 yield

$$\theta\partial_t\rho + \partial_x(\rho E_1) - \partial_x{}^2\rho = 0.$$

Let Φ verify $\partial_x\Phi = -E_1$; since $\partial_x E_1 = \rho - D = -\partial_x{}^2\Phi$, we recover (9). This is not so surprising since our scaling hypothesis assume that the speed of light is large compared to the reference unit of velocity, see the Appendix; it is well known that in such a regime relativistic Vlasov-Maxwell and Vlasov-Poisson systems are connected, see [3, 19, 38]. Here we are coupling the classical limit to the hydrodynamic and low field regime.

3 A Priori Estimates

In this section we establish a priori estimates for smooth solutions $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ of the relativistic VMFP system in one and one half dimension. We split the discussion into three steps: first, we describe the usual energy and entropy dissipation estimates, second, we obtain a refined dissipation property and finally we justify a uniform L^∞ estimate on the electro-magnetic field.

3.1 Moments and Entropy Dissipation

Our analysis is based on the moment equations associated to (16). Integrating (16)) with respect to $p \in \mathbb{R}^2$ yields the continuity equation

$$\partial_t \int_{\mathbb{R}^2} f^\varepsilon \, dp + \frac{1}{\varepsilon}\partial_x \int_{\mathbb{R}^2} v_1(p) f^\varepsilon \, dp = 0. \tag{25}$$

Multiplying now by $p = (p_1, p_2)$ and integrating with respect to $p \in \mathbb{R}^2$ yield

$$\varepsilon\partial_t \int_{\mathbb{R}^2} p_1 f^\varepsilon \, dp + \partial_x \int_{\mathbb{R}^2} v_1(p) p_1 f^\varepsilon \, dp - E_1^\varepsilon \rho^\varepsilon - \varepsilon\delta^2 B^\varepsilon j_2^\varepsilon = \theta(\partial_t E_1^\varepsilon - J), \tag{26}$$

$$\varepsilon\partial_t \int_{\mathbb{R}^2} p_2 f^\varepsilon \, dp + \partial_x \int_{\mathbb{R}^2} v_1(p) p_2 f^\varepsilon \, dp - E_2^\varepsilon \rho^\varepsilon + \varepsilon\delta^2 B^\varepsilon j_1^\varepsilon = -\theta\, \frac{j_2^\varepsilon}{\varepsilon}. \tag{27}$$

We aim at passing to the limit $\varepsilon \to 0$ in these relations. In order to obtain useful estimates, it is also convenient to multiply (16) by $1 + \ln f^\varepsilon + |x| + \mathcal{E}(p)$. We are led to

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^2} (\ln f^\varepsilon + |x| + \mathcal{E}(p)) f^\varepsilon dp \; &+ \; \frac{1}{\varepsilon}\partial_x \int_{\mathbb{R}^2} v_1(p)(\ln f^\varepsilon + |x| + \mathcal{E}(p)) f^\varepsilon dp \\
&+ \; \theta \int_{\mathbb{R}^2} \left| \frac{v(p)\sqrt{f^\varepsilon} + 2\nabla_p \sqrt{f^\varepsilon}}{\varepsilon} \right|^2 dp \\
&= \; \frac{1}{\varepsilon}\int_{\mathbb{R}^2} E^\varepsilon \cdot v(p) f^\varepsilon \, dp + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \frac{x}{|x|} v_1(p) f^\varepsilon \, dp.
\end{aligned} \tag{28}$$

This is completed by looking at the evolution of the electro-magnetic energy

$$\frac{1}{2}\partial_t(|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2) + \partial_x(E_2^\varepsilon B^\varepsilon) = -\frac{1}{\varepsilon}E^\varepsilon(t,x)\cdot j^\varepsilon(t,x) + E_1^\varepsilon(t,x)J(t,x). \tag{29}$$

Proposition 3.1 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem $(16)-(23)$. Let us set*

$$\begin{aligned}
h^\varepsilon(t,x,p) &= \frac{1}{\varepsilon}(v(p)\sqrt{f^\varepsilon} + 2\nabla_p\sqrt{f^\varepsilon}), \\
e^\varepsilon(t,x) &= \int_{\mathbb{R}^2} \Big(\frac{1/\delta^2 - 1}{2} + \ln f^\varepsilon + |x| + \mathcal{E}(p) \Big) f^\varepsilon(t,x,p)\, dp \\
&\qquad\qquad + \frac{1}{2}\left(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2 |B^\varepsilon(t,x)|^2 \right), \\
\pi^\varepsilon(t,x) &= \int_{\mathbb{R}^2} v_1(p) \Big(\frac{1/\delta^2 - 1}{2} + \ln f^\varepsilon + |x| + \mathcal{E}(p) \Big) f^\varepsilon \, dp + \varepsilon E_2^\varepsilon(t,x) B^\varepsilon(t,x), \\
r^\varepsilon(t,x) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \frac{x}{|x|} v_1(p) f^\varepsilon \, dp + E_1^\varepsilon(t,x) J(t,x) \\
&= \frac{x}{|x|}\int_{\mathbb{R}^2} \sqrt{f^\varepsilon} h_1^\varepsilon \, dp + E_1^\varepsilon(t,x) J(t,x).
\end{aligned}$$

Then, we have

$$\partial_t e^\varepsilon + \frac{1}{\varepsilon}\partial_x \pi^\varepsilon + \theta \int_{\mathbb{R}^2} |h^\varepsilon(t,x,p)|^2 \, dp = r^\varepsilon(t,x), \qquad (t,x) \in]0,T[\times\mathbb{R}. \tag{30}$$

As a consequence of these local properties, we can justify uniform estimates on the total energy and entropy. To this end, we make use of the following classical claim.

Lemma 3.1 *Assume that $f = f(x,p)$ satisfies $f \geq 0$, $(|x| + \mathcal{E}(p) + |\ln f|)f \in L^1(\mathbb{R} \times \mathbb{R}^2)$, where $\mathcal{E}(p)$ is given by (1). Then for all $k > 0$ we have*

$$f|\ln f| \leq f \ln f + 2k(|x| + \mathcal{E}(p))f + \frac{4}{e}\ e^{-\frac{k}{2}(|x|+\mathcal{E}(p))},$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f|\ln f|\, dp\, dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} f \ln f\ dp\, dx + 2k \int_{\mathbb{R}}\int_{\mathbb{R}^2} (|x| + \mathcal{E}(p))f\ dp\, dx + C_k,$$

with $C_k = (4/e) \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{-\frac{k}{2}(|x|+\mathcal{E}(p))}\ dp\, dx$.

Proof. Since $f|\ln f| = f \ln f + 2f(\ln f)_-$, it is sufficient to estimate $f(\ln f)_-$. Take $k > 0$ and remark that $2/e = \sup_{0<y<1}\{-\sqrt{y} \ln y\}$. For any $(x,p) \in \mathbb{R} \times \mathbb{R}^2$, we have

$$\begin{aligned} f(\ln f)_-(x,p) &= -f \ln f \cdot \mathbf{1}_{\{0<f<e^{-k(|x|+\mathcal{E}(p))}\}} - f \ln f \cdot \mathbf{1}_{\{e^{-k(|x|+\mathcal{E}(p))} \leq f<1\}} \\ &\leq \frac{2}{e}\ e^{-\frac{k}{2}(|x|+\mathcal{E}(p))} + k(|x| + \mathcal{E}(p))f. \end{aligned}$$

Therefore, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(\ln f)_-\ dp\, dx \leq k \int_{\mathbb{R}}\int_{\mathbb{R}^2} (|x| + \mathcal{E}(p))f\ dp\, dx + \frac{2}{e} \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{-\frac{k}{2}(|x|+\mathcal{E}(p))}\ dp\, dx,$$

and the conclusion follows easily. $\square$

Then, the starting point of our analysis relies on the following statement.

Proposition 3.2 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem (16) − (23). Assume that the initial conditions satisfy H1, H2 and that H3, H4 hold. Then we have for any $t \in [0,T]$*

i)
$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\ dp\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\ dp\, dx < +\infty,$$

ii)
$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}^2} (|\ln f^\varepsilon| + |x| + \mathcal{E}(p))f^\varepsilon(t,x,p)\ dp\, dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} (|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\ dx \\ &\qquad \leq 4\left(\frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4}\right) + 2\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}, \end{aligned}$$

iii)
$$\begin{aligned} \theta \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}^2} |h^\varepsilon(t,x,p)|^2\ dp\, dx\, dt &\leq 4\left(\frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4}\right) \\ &\quad + 2\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}, \end{aligned}$$

iv)
$$\left\|\frac{j^\varepsilon}{\varepsilon}\right\|^2_{L^2(]0,T[;L^1(\mathbb{R}))} \leq M_0^\varepsilon\ \|h^\varepsilon\|^2_{L^2(]0,T[\times\mathbb{R}\times\mathbb{R}^2)}.$$

Proof. Integrating (16) with respect to $(x,p) \in \mathbb{R}\times\mathbb{R}^2$ yields the charge conservation

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\,dp\,dx = 0, \qquad t\in]0,T[,$$

which implies that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\,dp\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\,dp\,dx = M_0^\varepsilon, \qquad t\in[0,T].$$

Similarly by H3 one gets $\int_{\mathbb{R}} D(t,x)\,dx = \int_{\mathbb{R}} D(0,x)\,dx$. Integration with respect to the space variable of the local property (30) leads to

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(\ln f^\varepsilon + |x| + \mathcal{E}(p))f^\varepsilon\,dp\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\,dx$$
$$+\theta\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx = \int_{\mathbb{R}} JE_1^\varepsilon\,dx + \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\frac{x}{|x|}v_1(p)f^\varepsilon(t,x,p)\,dp\,dx. \tag{31}$$

The last term in the right hand side can be rewritten as follows

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\frac{x}{|x|}(v_1(p)f^\varepsilon + \partial_{p_1}f^\varepsilon)\,dp\,dx &\le \int_{\mathbb{R}}\int_{\mathbb{R}^2}\sqrt{f^\varepsilon(t,x,p)}\;|h_1^\varepsilon(t,x,p)|\,dp\,dx\\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon\,dp\,dx\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx\right)^{1/2}\\
&\le \frac{1}{2\theta}M_0^\varepsilon + \frac{\theta}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx.
\end{aligned}$$

Integrating (31) with respect to time and using Lemma 3.1 with $k = 1/4$ yield for any $t\in[0,T]$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^2}(|\ln f^\varepsilon| + |x| + \mathcal{E}(p))f^\varepsilon(t,x,p)\,dp\,dx &+ \frac{1}{2}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\,dx\\
&+ \theta\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx\,ds\\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^2}(\ln f_0^\varepsilon + |x| + \mathcal{E}(p))f_0^\varepsilon\,dp\,dx + \frac{1}{2}\int_{\mathbb{R}}(|E_0^\varepsilon|^2 + \varepsilon^2\delta^2|B_0^\varepsilon|^2)\,dx\\
&+ \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(|x| + \mathcal{E}(p))f^\varepsilon(t,x,p)\,dp\,dx + \frac{\theta}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx\,ds\\
&+ \frac{t}{2\theta}M_0^\varepsilon + C_{1/4} + \int_0^t \|J(s)\|_{L^2(\mathbb{R})}\|E^\varepsilon(s)\|_{L^2(\mathbb{R})}ds,
\end{aligned} \tag{32}$$

which implies

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^2}(|\ln f^\varepsilon| + \frac{1}{2}(|x| + \mathcal{E}(p)))f^\varepsilon(t,x,p)\,dp\,dx\\
&+\frac{1}{2}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\,dx + \frac{\theta}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\,dp\,dx\,ds\\
&\leq \frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4} + \int_0^t \|J(s)\|_{L^2(\mathbb{R})}\|E^\varepsilon(s)\|_{L^2(\mathbb{R})}\,ds.
\end{aligned}
\tag{33}
$$

Hence ii)-iii) follow easily by using Bellman's lemma. For proving iv), we write

$$
\int_0^T \left\| \frac{j^\varepsilon(t)}{\varepsilon}\right\|^2_{L^1(\mathbb{R})} dt \;=\; \int_0^T \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}^2}\sqrt{f^\varepsilon(t,x,p)}h^\varepsilon(t,x,p)\,dp\right|\,dx\right)^2 dt,
$$

and we apply the Cauchy-Schwartz inequality. $\square$

This statement shows that we can expect a relaxation effect since the estimate on h^ε indicates that $f^\varepsilon(t,x,p) \approx \rho(t,x)\frac{e^{-\mathcal{E}(p)}}{K}$ where $K = \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp$. For the time being, let us focus on the discussion of further useful estimates. In particular, for the macroscopic density we get

Corollary 3.1 *Under the hypotheses of Proposition* 3.2 *we have for any* $t \in [0,T]$

$$
\int_{\mathbb{R}}\rho^\varepsilon(t,x)|\ln\rho^\varepsilon|\,dx \leq C_T(1 + M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}),
$$

for some constant C_T *depending on* T *but not on* ε.

The proof is an immediate consequence of the following standard result.

Lemma 3.2 *Assume that* f *is a non negative function satisfying*

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^2}(1 + |\ln f(x,p)| + |x| + \mathcal{E}(p))f(x,p)\,dp\,dx < +\infty,
$$

and denote by $\rho(x) = \int_{\mathbb{R}^2} f(x,p)\,dp,\ x \in \mathbb{R}$. *Then we have*

$$
\int_{\mathbb{R}}|\ln\rho(x)|\,\rho(x)\,dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}^2}(\ln K + |\ln f| + |x| + \mathcal{E}(p))f(x,p)\,dp\,dx + \frac{4}{e}\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\,dx,
$$

where $K = \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp$.

Proof. Consider the convex function $\varphi : [0,+\infty[\to \mathbb{R}$, $\varphi(s) = s\ln s$ for $s > 0$, $\varphi(0) = 0$ and the measure $d\nu = \frac{e^{-\mathcal{E}(p)}}{K}dp$. By applying the Jensen inequality

$$
\varphi\left(\int_{\mathbb{R}^2} g(p)d\nu\right) \leq \int_{\mathbb{R}^2}\varphi(g(p))d\nu,
$$

with the function $g(\cdot) = Kf(x,\cdot)e^{\mathcal{E}(\cdot)}$ one gets

$$\rho(x)\ln\rho(x) \leq \int_{\mathbb{R}^2} (\ln K + \ln f(x,p) + \mathcal{E}(p)) f(x,p)\, dp.$$

As in the proof of Lemma 3.1 one has

$$\rho(x)|\ln\rho(x)| \leq \rho(x)\ln\rho(x) + 2k|x|\rho(x) + \frac{4}{e}\, e^{-\frac{k|x|}{2}},$$

and therefore, by taking $k = 1/2$ one deduces

$$\begin{aligned}
\int_{\mathbb{R}} \rho(x)|\ln\rho(x)|\, dx &\leq \int_{\mathbb{R}} (\ln\rho(x) + |x|)\rho(x)\, dx + \frac{4}{e}\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\, dx \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} (\ln K + |\ln f| + |x| + \mathcal{E}(p)) f\, dp\, dx + \frac{4}{e}\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\, dx.
\end{aligned}$$

□

3.2 Further Dissipation Properties

Another way of estimating the solutions of the Fokker-Planck equation can be obtained by adapting the strategy of Poupaud-Soler [34]: we multiply (16) by $H'(fe^{\mathcal{E}(p)})$, where H is a convex function.

Proposition 3.3 *Assume that $E^\varepsilon, B^\varepsilon$ are bounded smooth functions and that f^ε is a smooth solution of* (16), (22) *with a non negative initial condition f_0^ε satisfying*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx < +\infty,$$

for some convex non negative function H. Then we have for any $t \in [0,T]$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon(t)e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx &+ \frac{\theta}{2\varepsilon^2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)} H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v f^\varepsilon|^2\, dp\, dx\, ds \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx \\
&+ \frac{1}{2\theta}(\|E_1^\varepsilon\|_{L^\infty} + \|E_2^\varepsilon\|_{L^\infty} + 2\varepsilon\delta\|B^\varepsilon\|_{L^\infty})^2 \\
&\times \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon)^2 e^{\mathcal{E}(p)} H''(f^\varepsilon(s)e^{\mathcal{E}(p)})\, dp\, dx\, ds.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\partial_t H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)} &+ \frac{v_1}{\varepsilon}\partial_x H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)} + \partial_{p_1}\left\{\left(\frac{E_1^\varepsilon}{\varepsilon} + \delta^2 v_2 B^\varepsilon\right) f^\varepsilon\right\} H'(f^\varepsilon e^{\mathcal{E}(p)}) \\
&+ \partial_{p_2}\left\{\left(\frac{E_2^\varepsilon}{\varepsilon} - \delta^2 v_1 B^\varepsilon\right) f^\varepsilon\right\} H'(f^\varepsilon e^{\mathcal{E}(p)}) \\
&= \frac{\theta}{\varepsilon^2}\mathrm{div}_p(\nabla_p f^\varepsilon + v(p) f^\varepsilon) H'(f^\varepsilon e^{\mathcal{E}(p)}). \qquad (34)
\end{aligned}$$

After integration with respect to $(x,p) \in \mathbb{R} \times \mathbb{R}^2$ we get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\,dp\,dx - \int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left(\frac{E_1^\varepsilon}{\varepsilon} + \delta^2 v_2 B^\varepsilon\right)\partial_{p_1}(e^{\mathcal{E}(p)}f^\varepsilon)\,dp\,dx \\
- \int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left(\frac{E_2^\varepsilon}{\varepsilon} - \delta^2 v_1 B^\varepsilon\right)\partial_{p_2}(e^{\mathcal{E}(p)}f^\varepsilon)\,dp\,dx \\
= -\frac{\theta}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v f^\varepsilon|^2\,dp\,dx. \quad (35)
\end{aligned}$$

We introduce the notation $R^\varepsilon(t) = \|E_1^\varepsilon(t)\|_{L^\infty(\mathbb{R})} + \|E_2^\varepsilon(t)\|_{L^\infty(\mathbb{R})} + 2\varepsilon\delta\|B^\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ and

$$q_H^\varepsilon(t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v(p)f^\varepsilon|^2\,dp\,dx.$$

By the Cauchy-Schwartz inequality and by taking into account that $|v(p)| < 1/\delta$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left\{\left(\frac{E_1^\varepsilon}{\varepsilon}+\delta^2 v_2 B^\varepsilon\right)\partial_{p_1}(e^{\mathcal{E}(p)}f^\varepsilon)+\left(\frac{E_2^\varepsilon}{\varepsilon}-\delta^2 v_1 B^\varepsilon\right)\partial_{p_2}(e^{\mathcal{E}(p)}f^\varepsilon)\right\}dp\,dx \\
&\quad\le\ R^\varepsilon(t)\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v(p)f^\varepsilon|\,dp\,dx \\
&\quad\le\ R^\varepsilon(t)\ (q_H^\varepsilon(t))^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\,dp\,dx\right)^{1/2}. \quad (36)
\end{aligned}$$

Combining (35), (36) yields

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\,dp\,dx + \theta\, q_H^\varepsilon(t) \le R^\varepsilon(t)\ (q_H^\varepsilon(t))^{1/2} \quad (37) \\
\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\,dp\,dx\right)^{1/2} \\
\le \theta\frac{q_H^\varepsilon(t)}{2} + \frac{R^\varepsilon(t)^2}{2\theta}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\,dp\,dx.
\end{aligned}$$

Finally one gets for any $t \in [0,T]$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\,dp\,dx + \frac{\theta}{2}\int_0^t q_H^\varepsilon(s)\,ds \le \int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\,dp\,dx \\
+\ \frac{\|R^\varepsilon\|_{L^\infty}^2}{2\theta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon(s))^2 e^{\mathcal{E}(p)}H''(f^\varepsilon(s)e^{\mathcal{E}(p)})\,dp\,dx\,ds.
\end{aligned}$$

$\square$

Corollary 3.2 *Assume that $E^\varepsilon, B^\varepsilon$ are bounded smooth functions and that f^ε is a smooth solution of* (16), (22) *with a non negative initial condition f_0^ε satisfying*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\,dp\,dx < +\infty,$$

for some $r > 1$. Then for any $t \in [0, T]$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon(t))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx \le e^{C^\varepsilon(t)} \int_{\mathbb{R}}\int_{\mathbb{R}^2} (f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx,$$

and

$$\begin{aligned}\frac{\theta}{\varepsilon^2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon)^{r-2} e^{(r-1)\mathcal{E}(p)} |\nabla_p f^\varepsilon + v(p) f^\varepsilon|^2\, dp\, dx\, dt \quad &\le \quad \frac{2e^{C^\varepsilon(T)}}{r(r-1)} \\ &\times \quad \int_{\mathbb{R}}\int_{\mathbb{R}^2} (f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx,\end{aligned}$$

where $C^\varepsilon(t) = \frac{t\, r(r-1)}{2\theta}(\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} + \|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} + 2\varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})})^2$.

Proof. By applying the previous proposition with the convex function $H(s) = s^r$, $s \ge 0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon(t))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx \; + \; \frac{\theta}{2}\int_0^t q_H^\varepsilon(s)\, ds &\le \int_{\mathbb{R}}\int_{\mathbb{R}^2} (f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx \\ &+ \; r(r-1)\frac{\|R^\varepsilon\|_{L^\infty}^2}{2\theta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon(s))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx\, ds.\end{aligned}$$

We conclude by applying the Gronwall lemma. $\square$

3.3 L^∞ Estimates on the Electro-Magnetic Field

We are looking now for L^∞ bounds of the electro-magnetic field. We exploit the hyperbolic structure of the Maxwell equations and the entropy dissipation of the Fokker-Planck collision operator. We adapt the method used by Glassey-Schaeffer [21], who obtained L^∞ bounds of the electro-magnetic field for the collisionless relativistic Vlasov-Maxwell system in one and one half dimension. Here, we wish to justify the following statement.

Proposition 3.4 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem* (16) – (23) *and assume that H1-H6 hold. Then we have*

$$\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} \le \|E_{0,1}^\varepsilon\|_{L^\infty(\mathbb{R})} + M_0^\varepsilon + \|J\|_{L^1(]0,T[;L^\infty(\mathbb{R}))},$$

and

$$\begin{aligned}&\max(\|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}, \varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}) \\ &\quad \le R_0^\varepsilon + \frac{2}{1/\delta^2 - 1}\Bigg(\Big(\frac{1/\delta^2 - 1}{2} + \frac{5T}{2\theta}\Big)M_0^\varepsilon + 6C_{1/4} \\ &\qquad + 5(W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon) + \frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}\Bigg).\end{aligned}$$

Notice that the Maxwell equations (17), (18), (19) can be written in the following diagonal form, for $(t,x) \in]0,T[\times\mathbb{R}$,

$$\begin{cases} \partial_t E_1^\varepsilon = -\dfrac{j_1^\varepsilon(t,x)}{\varepsilon} + J(t,x), \\ \partial_t(E_2^\varepsilon + \varepsilon\delta B^\varepsilon) + \dfrac{1}{\varepsilon\delta}\partial_x(E_2^\varepsilon + \varepsilon\delta B^\varepsilon) = -\dfrac{j_2^\varepsilon(t,x)}{\varepsilon}, \\ \partial_t(E_2^\varepsilon - \varepsilon\delta B^\varepsilon) - \dfrac{1}{\varepsilon\delta}\partial_x(E_2^\varepsilon - \varepsilon\delta B^\varepsilon) = -\dfrac{j_2^\varepsilon(t,x)}{\varepsilon}. \end{cases}$$

Therefore the electro-magnetic field is given by

$$E_1^\varepsilon(t,x) = E_{0,1}^\varepsilon(x) - U^\varepsilon(t,x) + \int_0^t J(s,x)\,ds, \qquad (t,x) \in [0,T]\times\mathbb{R}, \tag{38}$$

$$\begin{aligned} E_2^\varepsilon(t,x) &= \frac{1}{2}(E_{0,2}^\varepsilon + \varepsilon\delta B_0^\varepsilon)(x - \frac{t}{\varepsilon\delta}) + \frac{1}{2}(E_{0,2}^\varepsilon - \varepsilon\delta B_0^\varepsilon)(x + \frac{t}{\varepsilon\delta}) \\ &- \frac{1}{2}V_+^\varepsilon(t,x) - \frac{1}{2}V_-^\varepsilon(t,x), \qquad (t,x) \in [0,T]\times\mathbb{R}, \end{aligned} \tag{39}$$

$$\begin{aligned} \varepsilon\delta B^\varepsilon(t,x) &= \frac{1}{2}(E_{0,2}^\varepsilon + \varepsilon\delta B_0^\varepsilon)(x - \frac{t}{\varepsilon\delta}) - \frac{1}{2}(E_{0,2}^\varepsilon - \varepsilon\delta B_0^\varepsilon)(x + \frac{t}{\varepsilon\delta}) \\ &- \frac{1}{2}V_+^\varepsilon(t,x) + \frac{1}{2}V_-^\varepsilon(t,x), \qquad (t,x) \in [0,T]\times\mathbb{R}, \end{aligned} \tag{40}$$

where

$$U^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t j_1^\varepsilon(s,x)\,ds, \qquad V_\pm^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t j_2^\varepsilon(s, x \mp \frac{t-s}{\varepsilon\delta})\,ds.$$

Finally, the question reduces to estimate in L^∞ norm the functions $U^\varepsilon, V_\pm^\varepsilon$. This can be done by using the local energy conservation and entropy dissipation, namely, we go back to Proposition 3.1. The proof makes use of the following claims.

Lemma 3.3 *Let $u, z, w : [0,T]\times\mathbb{R} \to \mathbb{R}$ be smooth functions satisfying*

$$\partial_t u + \frac{1}{\varepsilon}\partial_x z = w(t,x), \qquad (t,x) \in]0,T[\times\mathbb{R}. \tag{41}$$

Then for any $\delta > 0$ and $(t,x) \in [0,T]\times\mathbb{R}$ we have

$$\begin{aligned} \frac{1}{\varepsilon}\int_0^t (\delta^{-1}u \mp z)\left(s, x \mp \frac{t-s}{\varepsilon\delta}\right)\,ds \pm \frac{1}{\varepsilon}\int_0^t z(s,x)\,ds \\ = \pm\int_{x\mp\frac{t}{\varepsilon\delta}}^x u(0,y)\,dy \pm \int_0^t \int_{x\mp\frac{t-s}{\varepsilon\delta}}^x w(s,y)\,dy\,ds, \end{aligned} \tag{42}$$

and

$$\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u - z)\left(s, x - \frac{t-s}{\varepsilon\delta}\right)\, ds + \frac{1}{\varepsilon}\int_0^t (\delta^{-1}u + z)\left(s, x + \frac{t-s}{\varepsilon\delta}\right)\, ds$$
$$= \int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}} u(0,y)\, dy + \int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} w(s,y)\, dy\, ds. \qquad (43)$$

Lemma 3.4 *For any $p \in \mathbb{R}^2$, the following inequality holds*

$$\left(\mathcal{E}(p) + \frac{1/\delta^2 - 1}{2}\right)\left(\frac{1}{\delta} - |v_1(p)|\right) \geq \frac{1/\delta^2 - 1}{2}\ |v_2(p)|.$$

These results allow us to prove the following statement which in turn, coming back to (38), (39), (40) justifies Proposition 3.4.

Proposition 3.5 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem (16) − (23), assume that the initial conditions satisfy H1, H2 and that H3, H4 hold. Then we have*

$$|U^\varepsilon(t,x)| \leq M_0^\varepsilon, \qquad (t,x) \in [0,T] \times \mathbb{R},$$

$$|V_+^\varepsilon| + |V_-^\varepsilon| \leq \frac{4}{\delta^{-2}-1}\left[\left(\frac{\delta^{-2}-1}{2} + \frac{5T}{2\theta}\right)M_0^\varepsilon + 5(W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon) + \frac{5}{2}\|J\|^2_{L_t^1(L_x^2)} + 6C_{1/4}\right].$$

Proof. Let us start by estimating $V_\pm^\varepsilon$. Combining Proposition 3.1, and Lemma 3.3, (43) we obtain

$$\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2} (\delta^{-1} - v_1)\left(\ln f^\varepsilon + \left|x - \frac{t-s}{\varepsilon\delta}\right| + \mathcal{E}(p) + \frac{1/\delta^2 - 1}{2}\right) f^\varepsilon(s, x - \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds$$
$$+ \frac{1}{2\varepsilon\delta}\int_0^t (|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2 - 2\varepsilon\delta E_2^\varepsilon B^\varepsilon)(s, x - \frac{t-s}{\varepsilon\delta})\, ds$$
$$+ \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2} (\delta^{-1} + v_1)\left(\ln f^\varepsilon + \left|x + \frac{t-s}{\varepsilon\delta}\right| + \mathcal{E}(p) + \frac{1/\delta^2 - 1}{2}\right) f^\varepsilon(s, x + \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds$$
$$+ \frac{1}{2\varepsilon\delta}\int_0^t (|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2 + 2\varepsilon\delta E_2^\varepsilon B^\varepsilon)(s, x + \frac{t-s}{\varepsilon\delta})\, ds$$
$$+ \theta \int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \int_{\mathbb{R}^2} |h^\varepsilon(s,y,p)|^2\, dp\, dy\, ds$$
$$= \int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}}\int_{\mathbb{R}^2}\left(\ln f_0^\varepsilon + |y| + \mathcal{E}(p) + \frac{1/\delta^2 - 1}{2}\right) f_0^\varepsilon\, dp\, dy + \frac{1}{2}\int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}} (|E_0^\varepsilon|^2 + \varepsilon^2\delta^2|B_0^\varepsilon|^2)\, dy$$
$$+ \int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \left(\int_{\mathbb{R}^2} \frac{y}{|y|}\sqrt{f^\varepsilon(s,y,p)}\, h_1^\varepsilon(s,y,p)\, dp + E_1^\varepsilon(s,y)J(s,y)\right) dy\, ds$$
$$\leq \frac{1/\delta^2 - 1}{2}M_0^\varepsilon + W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon + \frac{t}{2\theta}M_0^\varepsilon + \frac{\theta}{2}\int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \int_{\mathbb{R}^2} |h^\varepsilon(s,y,p)|^2\, dp\, dy\, ds$$
$$+ \|E^\varepsilon\|_{L^\infty(]0,T[;L^2(\mathbb{R}))}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}))}. \qquad (44)$$

Then, we can reproduce the tricks of Lemma 3.1 so that for any fixed $(t,x) \in [0,T] \times \mathbb{R}$ and $s \in [0,t]$ we get

$$\begin{aligned} f^\varepsilon |\ln f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)| \quad &\leq \quad \left(\ln f^\varepsilon + \frac{1}{2}\left(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p)\right)\right) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p) \\ &\quad + \frac{4}{e}\ e^{-\frac{1}{8}(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p))}. \end{aligned} \tag{45}$$

Reminding (3), we obtain the inequalities

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p)) |\ln f^\varepsilon| f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds \qquad (46) \\ \leq \quad & \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p))(\ln f^\varepsilon) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds \\ + \quad & \frac{1}{2\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p))(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p)) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds + C_{\varepsilon\delta}, \end{aligned}$$

where the crucial fact consists in remarking that the last term can actually be bounded uniformly with respect to $\varepsilon, \delta > 0$ since

$$C_{\varepsilon\delta} = \frac{4}{e\varepsilon\delta}\int_0^t \int_{\mathbb{R}^2} e^{-(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p))/8}\ dp\ ds \leq \frac{4}{e}\int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-(|y| + \mathcal{E}(p))/8}\ dp\ dy = C_{1/4}.$$

Combining (44), (46) yields

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2} (\delta^{-1} - v_1)\Big(|\ln f^\varepsilon| + \frac{1}{2}|x - \frac{t-s}{\varepsilon\delta}| + \frac{\mathcal{E}}{2} + \frac{\delta^{-2}-1}{2}\Big) f^\varepsilon(s, x - \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds \\ & + \frac{1}{2\varepsilon\delta}\int_0^t (|E_1^\varepsilon|^2 + |E_2^\varepsilon - \varepsilon\delta B^\varepsilon|^2)(s, x - \frac{t-s}{\varepsilon\delta})\ ds \\ & + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2} (\delta^{-1} + v_1)\Big(|\ln f^\varepsilon| + \frac{1}{2}|x + \frac{t-s}{\varepsilon\delta}| + \frac{\mathcal{E}}{2} + \frac{\delta^{-2}-1}{2}\Big) f^\varepsilon(s, x + \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds \\ & + \frac{1}{2\varepsilon\delta}\int_0^t (|E_1^\varepsilon|^2 + |E_2^\varepsilon + \varepsilon\delta B^\varepsilon|^2)(s, x + \frac{t-s}{\varepsilon\delta})\ ds \\ & + \frac{\theta}{2}\int_0^t \int_{x - \frac{t-s}{\varepsilon\delta}}^{x + \frac{t-s}{\varepsilon\delta}} \int_{\mathbb{R}^2} |h^\varepsilon(s,y,p)|^2\ dp\ dy\ ds \\ & \leq \frac{\delta^{-2}-1}{2} M_0^\varepsilon + W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon + \frac{tM_0^\varepsilon}{2\theta} + 2C_{1/4} + \frac{1}{2}\|E^\varepsilon\|^2_{L^\infty(]0,T[;L^2)} + \frac{1}{2}\|J\|^2_{L^1(]0,T[;L^2)} \\ & \leq \left(\frac{\delta^{-2}-1}{2} + \frac{5T}{2\theta}\right) M_0^\varepsilon + 5(L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon) + 6C_{1/4} + \tfrac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))} =: C_0^\varepsilon. \end{aligned}$$

Since $0 < \delta < 1$ is kept fixed, notice that H1-H5 guarantees that C_0^ε remains bounded with respect to $\varepsilon > 0$. We deduce that

$$\sum_{k=0}^{1} \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^2} (\delta^{-1} + (-1)^k v_1(p))\Big(\mathcal{E}(p) + \frac{\delta^{-2}-1}{2}\Big) f^\varepsilon(s, x + (-1)^k \frac{t-s}{\varepsilon\delta}, p)\ dp\ ds \leq C_0^\varepsilon,$$

and finally by Lemma 3.4 we get

$$\begin{aligned}
&\frac{1/\delta^2-1}{2}(|V_+^\varepsilon|+|V_-^\varepsilon|)\\
&\le \frac{1/\delta^2-1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^2}|v_2|\Big(f^\varepsilon(s,x-\frac{t-s}{\varepsilon\delta},p)+f^\varepsilon(s,x+\frac{t-s}{\varepsilon\delta},p)\Big)\,dp\,ds\\
&\le \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}(\delta^{-1}-v_1(p))(\mathcal{E}(p)+\frac{1/\delta^2-1}{2})f^\varepsilon(s,x-\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
&\quad+\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}(\delta^{-1}+v_1(p))(\mathcal{E}(p)+\frac{1/\delta^2-1}{2})f^\varepsilon(s,x+\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
&\le 2C_0^\varepsilon.
\end{aligned}$$

The estimate of U^ε follows by applying Lemma 3.3 to the continuity equation (25). Indeed, by (42) we have for any $(t,x)\in[0,T]\times\mathbb{R}$

$$\begin{aligned}
\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}(\delta^{-1}\mp v_1(p))f^\varepsilon(s,x &\mp \frac{t-s}{\varepsilon\delta},p)\,dp\,ds \pm \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}v_1(p)f^\varepsilon(s,x,p)\,dp\,ds\\
&= \pm\int_{x\mp\frac{t}{\varepsilon\delta}}^x\int_{\mathbb{R}^2}f_0^\varepsilon(y,p)\,dp\,dy,
\end{aligned}$$

and thus we deduce that $\pm U^\varepsilon(t,x)\le M_0^\varepsilon$. $\square$

We can end the proof of Proposition 3.4. By (38) and Proposition 3.5 we get

$$\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}\le\|E_{0,1}^\varepsilon\|_{L^\infty(\mathbb{R})}+M_0^\varepsilon+\|J\|_{L^1(]0,T[;L^\infty(\mathbb{R}))}.$$

Similarly, combining (39), (40), Proposition 3.5 implies

$$\begin{aligned}
&\max(\|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})},\varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})})\\
&\le\|E_{0,2}^\varepsilon\|_{L^\infty(\mathbb{R})}+\varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})}+\frac{2}{1/\delta^2-1}\Big[\Big(\frac{1/\delta^2-1}{2}+\frac{5T}{2\theta}\Big)M_0^\varepsilon+6C_{1/4}\\
&\qquad\qquad+5(W_0^\varepsilon+L_0^\varepsilon+H_0^\varepsilon)+\frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}\Big].
\end{aligned}$$

$\square$

It remains to justify Lemma 3.3 and Lemma 3.4.

Proof of Lemma 3.3. For any $(t,x)\in[0,T]\times\mathbb{R}$ consider the sets $\Delta_\pm^\varepsilon$ given by

$$\Delta_+^\varepsilon=\{(s,y)\in]0,T[\times\mathbb{R}\;:\;x-\frac{t-s}{\varepsilon\delta}<y<x\},$$

$$\Delta_-^\varepsilon=\{(s,y)\in]0,T[\times\mathbb{R}\;:\;x<y<x+\frac{t-s}{\varepsilon\delta}\}.$$

Integrating (41) with respect to $(s,y)\in\Delta_+^\varepsilon$ yields

$$\begin{aligned}
\int_{x-\frac{t}{\varepsilon\delta}}^x\Big(u(t-\varepsilon\delta(x-y),y)-u(0,y)\Big)\,dy &+ \frac{1}{\varepsilon}\int_0^t\Big(z(s,x)-z(s,x-\frac{t-s}{\varepsilon\delta})\Big)\,ds\\
&= \int_0^t\int_{x-\frac{t-s}{\varepsilon\delta}}^x w(s,y)\,dy\,ds,
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u - z)\left(s, x - \frac{t-s}{\varepsilon\delta}\right)\, ds + \frac{1}{\varepsilon}\int_0^t z(s,x)\, ds &= \int_{x-\frac{t}{\varepsilon\delta}}^{x} u(0,y)\, dy \qquad (47)\\ &+ \int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x} w(s,y)\, dy\, ds.\end{aligned}$$

Similarly, integrating (41) over Δ_-^ε implies

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u + z)\left(s, x + \frac{t-s}{\varepsilon\delta}\right)\, ds - \frac{1}{\varepsilon}\int_0^t z(s,x)\, ds &= \int_{x}^{x+\frac{t}{\varepsilon\delta}} u(0,y)\, dy \qquad (48)\\ &+ \int_0^t \int_{x}^{x+\frac{t-s}{\varepsilon\delta}} w(s,y)\, dy\, ds.\end{aligned}$$

The equality (43) follows by adding (47), (48). $\square$

Proof of Lemma 3.4. For any $p \in \mathbb{R}^2$, we set

$$\begin{aligned}Q(p) &:= \left(\mathcal{E}(p) + \frac{1/\delta^2 - 1}{2}\right)\left(\frac{1}{\delta} - |v_1(p)|\right)\\ &= \frac{1/\delta^2-1}{2}\sqrt{1+\frac{4\,|p|^2}{\delta^2(1/\delta^2-1)^2}} \times \frac{1}{\delta}\left(1 - \frac{\frac{2}{\delta(1/\delta^2-1)}|p_1|}{\sqrt{1+\frac{4\,|p|^2}{\delta^2(1/\delta^2-1)^2}}}\right)\\ &= \frac{1/\delta^2-1}{2\delta}\left(\sqrt{1+\frac{4\,|p|^2}{\delta^2(1/\delta^2-1)^2}} - \frac{2}{\delta(1/\delta^2-1)}|p_1|\right)\end{aligned}$$

However, for any $q \in \mathbb{R}^2$ $(= 2p/(\delta(1/\delta^2-1)))$, we have

$$\sqrt{1+|q|^2} - |q_1| = \frac{1+|q_2|^2}{\sqrt{1+|q|^2}+|q_1|} \geq \frac{|q_2|}{\sqrt{1+|q|^2}}.$$

It follows that

$$Q(p) \geq \frac{1}{\delta^2}\frac{|p_2|}{\sqrt{1+\frac{4\,|p|^2}{\delta^2(1/\delta^2-1)^2}}} = \frac{1/\delta^2-1}{2}\;|v_2(p)|.$$

$\square$

4 Asymptotic Analysis

We are now in position to perform the asymptotic analysis when ε goes to zero. The uniform estimates obtained in the previous section allow us to extract converging sequences as follows.

Proposition 4.1 *Assume that H1-H5 hold. Suppose that for any* $\varepsilon > 0$ $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ *is a smooth solution of* $(16)-(23)$. *Then there is a sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *decreasing to zero such that the sequences*

$$(f_0^k, E_0^k, B_0^k)_{k\in\mathbb{N}} := (f_0^{\varepsilon_k}, E_0^{\varepsilon_k}, B_0^{\varepsilon_k})_{k\in\mathbb{N}}, \qquad (f^k, E^k, B^k)_{k\in\mathbb{N}} := (f^{\varepsilon_k}, E^{\varepsilon_k}, B^{\varepsilon_k})_{k\in\mathbb{N}},$$

satisfy

$$f_0^k \rightharpoonup f_0 \text{ weakly in } L^1(\mathbb{R}^2), \tag{49}$$

$$\rho_0^k := \int_{\mathbb{R}^2} f_0^k \, dp \rightharpoonup \rho_0 := \int_{\mathbb{R}^2} f_0 \, dp \text{ weakly in } L^1(\mathbb{R}), \tag{50}$$

$$E_{0,1}^k \to E_{0,1} \text{ uniformly on compact sets of } \mathbb{R}, \tag{51}$$

$$E_{0,2}^k \rightharpoonup E_{0,2} \text{ weakly in } L^2(\mathbb{R}), \tag{52}$$

$$f^k \rightharpoonup f \text{ weakly in } L^1(]0,T[\times\mathbb{R}\times\mathbb{R}^2), \tag{53}$$

$$\rho^k := \int_{\mathbb{R}^2} f^k \, dp \rightharpoonup \rho := \int_{\mathbb{R}^2} f \, dp \text{ weakly in } L^1(]0,T[\times\mathbb{R}), \tag{54}$$

$$E_1^k \to E_1 \text{ strongly in } L^1_{\text{loc}}([0,T]\times\mathbb{R}), \text{ weakly in } L^2(]0,T[\times\mathbb{R}), \text{ weakly} \star \text{ in } L^\infty, \tag{55}$$

$$(E_2^k, \varepsilon_k \delta B^k) \rightharpoonup (0,0) \text{ weakly in } L^2(]0,T[\times\mathbb{R})^2, \text{ weakly } \star \text{ in } L^\infty(]0,T[\times\mathbb{R})^2. \tag{56}$$

Proof. We split the proof into three steps.
Step 1. Proof of (49)*,* (50)*,* (53)*,* (54)*.* By Proposition 3.2 i) and ii) and hypotheses H1-H5, we can apply the Dunford-Pettis theorem, see e. g. [20] (Th. 4.21.2, p. 274), which justifies (49) and (53). Moreover the limit f is non negative and satisfies

$$\sup_{0\le t\le T} \int_{\mathbb{R}}\int_{\mathbb{R}^2} (1+|x|+\mathcal{E}(p)) f(t,x,p) \, dp \, dx < +\infty.$$

Similarly by Corollary 3.1 we deduce

$$\sup_{\varepsilon>0, t\in[0,T]} \int_{\mathbb{R}} (1+|x|+|\ln \rho^\varepsilon|)\rho^\varepsilon(t,x) \, dx < +\infty,$$

which implies (50), (54).

Step 2. Proof of (51) *and* (52)*.* Since $\sup_{\varepsilon>0}\int_{\mathbb{R}} |\ln \rho_0^\varepsilon|\rho_0^\varepsilon(x) \, dx < +\infty$ we deduce that for any $\eta > 0$ there is $h = h(\eta) > 0$ such that $\int_x^{x+h} \rho_0^\varepsilon(y) \, dy < \eta$ for any $\varepsilon > 0$ and $x \in \mathbb{R}$. Taking h small enough, since $D(0,\cdot)$ belongs to $L^1(\mathbb{R})$, we also have

$\int_x^{x+h} D(0,y)\,dy < \eta$ for any $x \in \mathbb{R}$. Therefore, by (24) we have for any $\varepsilon > 0$ and $x \in \mathbb{R}$

$$|E^\varepsilon_{0,1}(x+h) - E^\varepsilon_{0,1}(x)| = \left| \int_x^{x+h} \{\rho^\varepsilon_0(y) - D(0,y)\}\,dy \right| < 2\eta,$$

and since $(E^\varepsilon_{0,1})_\varepsilon$ is bounded in $L^\infty(\mathbb{R})$, by using the Arzela-Ascoli theorem we deduce (51). The convergence (52) is a direct consequence of H2. Moreover we check easily that $E_{0,1}, E_{0,2} \in L^\infty(\mathbb{R})$ and $\frac{d}{dx}E_{0,1} = \rho_0 - D(0,\cdot)$.

Step 3. Proof of (55) *and* (56). We claim that $(E^\varepsilon_1)_{\varepsilon>0}$ is bounded in $W^{1,1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Indeed, $(E^\varepsilon_1)_{\varepsilon>0}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and thus in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Moreover, $(\partial_x E^\varepsilon_1)_{\varepsilon>0} = (\rho^\varepsilon - D)_{\varepsilon>0}$ is bounded in $L^1(]0,T[\times\mathbb{R})$ and $(\partial_t E^\varepsilon_1)_{\varepsilon>0} = (-\frac{1}{\varepsilon}j^\varepsilon_1 + J)_{\varepsilon>0}$ is bounded in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ by Proposition 3.2-iv). We deduce that $(E^\varepsilon_1)_{\varepsilon>0}$ is relatively compact in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$.

Observe also that $(E^\varepsilon_1, E^\varepsilon_2, \varepsilon\delta B^\varepsilon)_{\varepsilon>0}$ is weakly relatively compact in $(L^2(]0,T[\times\mathbb{R}))^3$ and weakly $\star$ relatively compact in $(L^\infty(]0,T[\times\mathbb{R}))^3$. Thus we obtain (55) and $(E^k_2, \varepsilon_k\delta B^k) \rightharpoonup (E_2, B)$ weakly in $L^2(]0,T[\times\mathbb{R})^2$, weakly $\star$ in $L^\infty(]0,T[\times\mathbb{R})^2$. Moreover the limits E_1, E_2, B belong to $L^\infty(]0,T[;L^2(\mathbb{R}))$ and we have $\partial_x E_1 = \rho - D$. Let us now prove that $(E_2, B) = (0,0)$.

By (19) we have for any $\varphi \in C^1_c(]0,T[\times\mathbb{R})$

$$\begin{aligned} \left|\int_0^T\int_{\mathbb{R}} E^\varepsilon_2 \partial_x\varphi\,dx\,dt\right| &= \varepsilon^2\delta^2 \left|\int_0^T\int_{\mathbb{R}} B^\varepsilon \partial_t\varphi\,dx\,dt\right| \\ &\le \varepsilon\delta\|\varepsilon\delta B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} \int_0^T\int_{\mathbb{R}} |\partial_t\varphi|\,dx\,dt. \end{aligned}$$

Since $\sup_{\varepsilon>0}\|\varepsilon\delta B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ we obtain $\partial_x E_2 = 0$. Taking into account that $E_2 \in L^\infty(]0,T[;L^2(\mathbb{R}))$ we deduce that $E_2 = 0$. Similarly for any $\varphi \in C^1_c(]0,T[\times\mathbb{R})$ we have by (18)

$$\begin{aligned} \left|\int_0^T\int_{\mathbb{R}} \varepsilon\delta B^\varepsilon \partial_x\varphi\,dx\,dt\right| &= \varepsilon\delta \left|\int_0^T\int_{\mathbb{R}} \left(\frac{j^\varepsilon_2}{\varepsilon}\varphi - E^\varepsilon_2\partial_t\varphi\right)\,dx\,dt\right| \\ &\le \varepsilon\delta\left(\|\varphi\|_{C^0}\left\|\frac{j^\varepsilon_2}{\varepsilon}\right\|_{L^1(]0,T[\times\mathbb{R})} + \|E^\varepsilon_2\|_{L^\infty}\int_0^T\int_{\mathbb{R}} |\partial_t\varphi|\,dx\,dt\right). \end{aligned}$$

By using the uniform bounds in Proposition 3.2-iv) and Proposition 3.4, we obtain $\partial_x B = 0$ and since we know that $B \in L^\infty(]0,T[;L^2(\mathbb{R}))$ we conclude that $B = 0$. $\square$

We focus our attention to the moment equations of (16); namely, let us go back to (25), (26) and (27). As mentioned above, we guess from the entropy estimate that $f^\varepsilon(t,x,p) \approx \rho(t,x)\frac{e^{-\mathcal{E}(p)}}{K}$ where $K = \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp$. In this case we obtain

$$\int_{\mathbb{R}^2} v_1(p)p_1 f^\varepsilon\,dp \approx -\frac{\rho(t,x)}{K}\int_{\mathbb{R}^2} p_1\partial_{p_1}(e^{-\mathcal{E}(p)})\,dp = \rho(t,x),$$

and
$$\int_{\mathbb{R}^2} v_1(p) p_2 f^\varepsilon \, dp \approx -\frac{\rho(t,x)}{K} \int_{\mathbb{R}^2} p_2 \partial_{p_1}(e^{-\mathcal{E}(p)}) \, dp = 0.$$
Let us make this statement rigorous.

Proposition 4.2 *Assume that H1-H6 hold. Suppose that $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ are smooth solutions of $(16)-(23)$ and consider $(\varepsilon_k)_{k\in\mathbb{N}}$ the sequence constructed in Proposition* 4.1. *Then we have*
$$\lim_{k\to+\infty} \varepsilon_k \left(\partial_t \int_{\mathbb{R}^2} p_1 f^k \, dp, \partial_t \int_{\mathbb{R}^2} p_2 f^k \, dp \right) = (0,0),$$
$$\lim_{k\to+\infty} \left(\partial_x \int_{\mathbb{R}^2} v_1(p) p_1 f^k \, dp, \partial_x \int_{\mathbb{R}^2} v_1(p) p_2 f^k \, dp \right) = (\partial_x \rho, 0)$$
in $\mathcal{D}'([0,T]\times\mathbb{R})^2$.

Proof. We shall use the estimates in Proposition 3.2, Corollary 3.2 and Corollary 3.4. Let $\varphi \in C^1_c([0,T]\times\mathbb{R})$. For $l \in \{1,2\}$ we have
$$\sup_{k\in\mathbb{N}} \left| \langle \partial_t \int_{\mathbb{R}^2} p_l f^k \, dp, \varphi \rangle \right| \leq \|\partial_t \varphi\|_{L^\infty} \sup_{k\in\mathbb{N}} \left(2 \sup_{t\in[0,T]} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^k \, dp \, dx + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^k dp \, dx \, dt \right),$$
and therefore $\lim_{k\to+\infty} \varepsilon_k \partial_t \int_{\mathbb{R}^2} p_l f^k \, dp = 0$ in $\mathcal{D}'([0,T]\times\mathbb{R})$. Next, observe that
$$\langle \partial_x \int_{\mathbb{R}^2} v_1(p) p_l f^k \, dp, \varphi \rangle = -\varepsilon_k \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} \partial_x \varphi \, p_l \sqrt{f^k} \, h_1^k \, dp \, dx \, dt - \int_0^T \int_{\mathbb{R}} \partial_x \varphi \delta_{1l} \rho^k \, dx \, dt,$$
and we can conclude provided that for any $R > 0$ we have
$$\sup_{k\in\mathbb{N}} \int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p| \sqrt{f^k} \, |h^k| \, dp \, dx \, dt < +\infty. \tag{57}$$
By using the Cauchy-Schwartz inequality we deduce that
$$\begin{aligned} \int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p| \sqrt{f^k} \, |h^k| \, dp \, dx \, dt \leq & \left(\int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k \, dp \, dx \, dt \right)^{1/2} \\ & \times \left(\int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |h^k|^2 \, dp \, dx \, dt \right)^{1/2}, \end{aligned}$$
and therefore we are done if we prove that $\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k(t,x,p) \, dp \, dx < +\infty$. The Hölder inequality yields
$$\begin{aligned} \sup_{k\in\mathbb{N}, t\in[0,T]} \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k(t,x,p) \, dp \leq & \sup_{k\in\mathbb{N}, t\in[0,T]} \left(\int_{-R}^R \int_{\mathbb{R}^2} (f^k(t))^r e^{(r-1)\mathcal{E}(p)} \, dp \, dx \right)^{1/r} \\ & \times \left(\int_{-R}^R \int_{\mathbb{R}^2} |p|^{2r'} e^{-\mathcal{E}(p)} \, dp \right)^{1/r'} < +\infty \end{aligned}$$
where r' is the conjugate exponent of r, *i.e.*, $1/r + 1/r' = 1$. This ends the proof of Proposition 4.2 $\square$

Having identified the limit of higher moments involved in (26), (27), the difficulty relies in the non linear terms.

Proposition 4.3 *Assume that H1-H6 hold. Suppose that* $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ *are smooth solutions of* $(16)-(23)$ *and consider* $(\varepsilon_k)_{k\in\mathbb{N}}$ *the sequence constructed in Proposition 4.1. Then we have*

$$\lim_{k\to+\infty}(E_1^k\rho^k, E_2^k\rho^k) = (E_1\rho, 0) \text{ in } \mathcal{D}'([0,T]\times\mathbb{R})^2, \tag{58}$$

$$\lim_{k\to+\infty}(\varepsilon_k\delta^2 B^k j_1^k, \varepsilon_k\delta^2 B^k j_2^k) = (0,0) \text{ in } L^1(]0,T[\times\mathbb{R})^2. \tag{59}$$

Proof. We write $E_1^k\rho^k = E_1^k(\partial_x E_1^k + D)$. Since $(E_1^k)_{k\in\mathbb{N}}$ converges towards E_1 weakly $\star$ in $L^\infty(]0,T[\times\mathbb{R})$ we have for any $\varphi\in C_c^1([0,T]\times\mathbb{R})$

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}} E_1^k(t,x)D(t,x)\varphi(t,x)\;dx\,dt = \int_0^T\int_{\mathbb{R}} E_1(t,x)D(t,x)\varphi(t,x)\;dx\,dt.$$

It remains to analyze the term $E_1^k\partial_x E_1^k$

$$\begin{aligned}
|\langle E_1^k\partial_x E_1^k - E_1\partial_x E_1, \varphi\rangle| &= \left|\int_0^T\int_{\mathbb{R}}\frac{1}{2}\partial_x|E_1^k|^2\varphi\;dx\,dt - \frac{1}{2}\langle\partial_x|E_1|^2,\varphi\rangle\right| \\
&= \left|-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(E_1^k-E_1)E_1^k\partial_x\varphi\;dx\,dt + \frac{1}{2}\int_0^T\int_{\mathbb{R}}E_1(E_1-E_1^k)\partial_x\varphi\;dx\,dt\right| \\
&\leq \|\varphi\|_{C^1}\sup_{k'\in\mathbb{N}}\|E_1^{k'}\|_{L^\infty(]0,T[\times\mathbb{R})}\|E_1^k-E_1\|_{L^1(\mathrm{supp}\varphi)},
\end{aligned}$$

and therefore $\lim_{k\to+\infty}E_1^k\rho^k = E_1\rho$ in $\mathcal{D}'([0,T]\times\mathbb{R})$ by using (55).

Consider now the term $E_2^k\rho^k$. Since $(E_2^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and $(\rho^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[;L^1(\mathbb{R}))$ it is sufficient to prove that $E_2^k\rho^k = E_2^k(\partial_x E_1^k + D)\to 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R})$. Take $\varphi\in C_c^1(]0,T[\times\mathbb{R})$. As before we have

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}} E_2^k(t,x)D(t,x)\varphi(t,x)\;dx\,dt = \int_0^T\int_{\mathbb{R}} E_2(t,x)D(t,x)\varphi(t,x)\;dx\,dt,$$

and for the term $E_2^k\partial_x E_1^k$ we write

$$\begin{aligned}
|\langle E_2^k\partial_x E_1^k,\varphi\rangle| &= \left|\int_0^T\int_{\mathbb{R}}(\partial_x(E_2^kE_1^k) - \partial_x E_2^k E_1^k)\varphi\;dx\,dt\right| \\
&\leq Q_1^k + Q_2^k, \qquad (60)
\end{aligned}$$

where $Q_1^k := \left|\int_0^T\int_{\mathbb{R}} E_2^kE_1^k\partial_x\varphi\;dx\,dt\right|$ and $Q_2^k := \left|\int_0^T\int_{\mathbb{R}}\partial_x E_2^k E_1^k\varphi\;dx\,dt\right|$. Observe that

$$Q_1^k \leq \left|\int_0^T\int_{\mathbb{R}}E_2^k(E_1^k-E_1)\partial_x\varphi\;dx\,dt\right| + \left|\int_0^T\int_{\mathbb{R}}E_1E_2^k\partial_x\varphi\;dx\,dt\right|, \tag{61}$$

and therefore, by using the strong convergence of $(E_1^k)_{k\in\mathbb{N}}$ in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ and the weak convergence of $(E_2^k)_{k\in\mathbb{N}}$ in $L^2(]0,T[\times\mathbb{R})$ we deduce that $\lim_{k\to+\infty} Q_1^k = 0$. By using (19), (17) we have

$$\begin{aligned}
Q_2^k &= \varepsilon_k\delta \left|\int_0^T\int_{\mathbb{R}} \varepsilon_k\delta B^k E_1^k \partial_t\varphi \, dx\, dt\right| + \varepsilon_k\delta\left|\int_0^T\int_{\mathbb{R}} \varepsilon_k\delta B^k \partial_t E_1^k \varphi\, dx\, dt\right| \\
&\le \varepsilon_k\delta \|\varepsilon_k\delta B^k\|_{L^\infty(]0,T[\times\mathbb{R})} \|E_1^k\|_{L^\infty(]0,T[\times\mathbb{R})} \int_0^T\int_{\mathbb{R}} |\partial_t\varphi|\, dx\, dt \\
&+ \varepsilon_k\delta \|\varepsilon_k\delta B^k\|_{L^\infty(]0,T[\times\mathbb{R})} \left|\int_0^T\int_{\mathbb{R}} \left(J - \frac{j_1^k}{\varepsilon_k}\right)\varphi\, dx\, dt\right|.
\end{aligned}$$

Since $(E_1^k)_{k\in\mathbb{N}}$, $(\varepsilon_k\delta B^k)_{k\in\mathbb{N}}$ are bounded in $L^\infty(]0,T[\times\mathbb{R})$, $(\frac{j_1^k}{\varepsilon_k})_{k\in\mathbb{N}}$ is bounded in $L^2(]0,T[;L^1(\mathbb{R}))$ and J belongs to $L^1(]0,T[;L^\infty(\mathbb{R}))$ we deduce that $\lim_{k\to+\infty} Q_2^k = 0$. Thus we proved that $\lim_{k\to+\infty} E_2^k\ \partial_x E_1^k = 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R})$ and therefore the second convergence in (58) holds. The convergence (59) follows easily since $(\varepsilon_k\delta B^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and $(\frac{j^k}{\varepsilon_k})_{k\in\mathbb{N}}$ is bounded in $L^2(]0,T[;L^1(\mathbb{R}))$. We have

$$\|\varepsilon_k\delta^2 B^k j^k\|_{L^1(]0,T[\times\mathbb{R})} \le \varepsilon_k\delta \sup_{k'\in\mathbb{N}}\left(\|\varepsilon_{k'}\delta B^{k'}\|_{L^\infty(]0,T[\times\mathbb{R})}\left\|\frac{j^{k'}}{\varepsilon_{k'}}\right\|_{L^1(]0,T[\times\mathbb{R})}\right).$$

□

Remark 4.1 *By easy density arguments we deduce that* $\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}} E_2^k\rho^k\varphi\, dx\, dt = 0$ *for any continuous bounded function* $\varphi\in C^0([0,T]\times\mathbb{R})$ *(use the uniform bounds* $\sup_{k\in\mathbb{N}}\|E_2^k\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ *and* $\sup_{k\in\mathbb{N},t\in[0,T]}\int_{\mathbb{R}}(1+|x|)\rho^k(t,x)\, dx < +\infty$*).*

Remark 4.2 *Notice that assuming H6 with* $r = 2$ *we have the uniform bound* $\sup_{\varepsilon>0}\|\frac{j^\varepsilon}{\varepsilon}\|_{L^2(]0,T[\times\mathbb{R})} < +\infty$*. Indeed, by Corollary* 3.2 *we know that*

$$\sup_{\varepsilon>0}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\, dp\, dx\, dt = \sup_{\varepsilon>0}\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}|\nabla_p f^\varepsilon + v(p) f^\varepsilon|^2\, dp\, dx\, dt < \infty,$$

and thus we obtain for any $(t,x)\in[0,T]\times\mathbb{R}$

$$\left|\frac{j^\varepsilon(t,x)}{\varepsilon}\right|^2 = \left|\int_{\mathbb{R}^2}\sqrt{f^\varepsilon}h^\varepsilon(t,x,p)\, dp\right|^2 \le \int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\, dp \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\, dp,$$

implying that

$$\sup_{\varepsilon>0}\left\|\frac{j^\varepsilon}{\varepsilon}\right\|^2_{L^2(]0,T[\times\mathbb{R})} \le \left(\int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\, dp\right)\sup_{\varepsilon>0}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\, dp\, dx\, dt.$$

The convergences of Propositions 4.2 and 4.3 are sufficient for passing to the limit with respect to k in (26). We obtain the equations

$$\theta\partial_t E_1 + \rho E_1 - \partial_x^2 E_1 = \partial_x D + \theta J, \qquad (t,x) \in]0,T[\times\mathbb{R},$$

$$\partial_x E_1 = \rho - D, \qquad (t,x) \in [0,T]\times\mathbb{R},$$
$$E_1(0,x) = E_{0,1}(x), \qquad x \in \mathbb{R}.$$

This finishes the proof of Theorem 2.1.

Let us only check that the initial data is preserved for the limit equation. We rewrite (26) as follows

$$\partial_t\Big(\theta E_1^\varepsilon - \varepsilon\int_{\mathbb{R}^2} p_1 f^\varepsilon\,dp\Big) = \partial_x\int_{\mathbb{R}^2} p_1 v_1(p) f^\varepsilon\,dp - E_1^\varepsilon\rho^\varepsilon - \varepsilon\delta^2 B^\varepsilon j_2^\varepsilon + \theta J.$$

For any $\varphi \in C_c^\infty(\mathbb{R})$, by virtue of the estimates established above, the set

$$\Big\{\int_{\mathbb{R}}\Big(\theta E_1^\varepsilon(\cdot,x) - \varepsilon\int_{\mathbb{R}^2} p_1 f^\varepsilon(\cdot,x,p)\,dp\Big)\varphi(x)\,dx, \quad \varepsilon > 0\Big\}$$

is relatively compact in $C^0([0,T])$, as a consequence of the Arzela-Ascoli theorem. Since $\int_{\mathbb{R}^2} p_1 f^\varepsilon\,dp$ is bounded in $L^\infty(]0,T[;L^1(\mathbb{R}))$, the conclusion applies to $\big\{\int_{\mathbb{R}} E_1^\varepsilon(t,x)\varphi(x)\,dx, \quad \varepsilon > 0\big\}$ as well. Using an approximation argument we can consider a trial function $\varphi \in L^1(\mathbb{R})$. Finally, by separability, we use a diagonal argument and we conclude that we can extract a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to 0 such that for any $\varphi \in L^1(\mathbb{R})$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}} E_1^{\varepsilon_k}(t,x)\varphi(x)\,dx = \int_{\mathbb{R}} E_1(t,x)\varphi(x)\,dx \quad \text{uniformly on } [0,T].$$

□

5 Comments

5.1 Rate of Convergence for the Electro-Magnetic Field

Let us now show that the behavior of E_2^k and B^k can be made precise with the strengthened assumption

H8) $\lim_{R\to+\infty}\sup_{\varepsilon>0}\int_{|x|>R}\Big(\frac{|E_{0,2}^\varepsilon(x)|}{\varepsilon} + \delta|B_0^\varepsilon(x)|\Big)\,dx = 0.$

Proposition 5.1 *If in addition to the hypotheses of Theorem 2.1, H8 holds, then we have*

$$\lim_{k\to+\infty}\Big(\frac{E_2^k}{\varepsilon_k}, \delta B^k\Big) = (0,0), \qquad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R})^2.$$

Proof. For any $\varphi \in C^1_c(]0,T[\times\mathbb{R})$ we have by (40)

$$\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}} \delta B^k \varphi \, dx\, dt\right| &\leq \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}} (\frac{E^k_{0,2}}{\varepsilon_k} + \delta B^k_0)(x)\varphi(t, x + \frac{t}{\varepsilon_k \delta}) \, dx\, dt\right| \\
&+ \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}} (\frac{E^k_{0,2}}{\varepsilon_k} - \delta B^k_0)(x)\varphi(t, x - \frac{t}{\varepsilon_k \delta}) \, dx\, dt\right| \\
&+ \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}} V^k_+ \frac{\varphi}{\varepsilon_k} \, dx\, dt\right| + \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}} V^k_- \frac{\varphi}{\varepsilon_k} \, dx\, dt\right|. \quad (62)
\end{aligned}$$

Take R large enough such that $\int_{|x|>R} (\frac{E^k_{0,2}(x)}{\varepsilon_k} + \delta |B^k_0(x)|) \, dx \leq \eta$ for any k. Take $d > 0$ large enough such that supp $\varphi \subset [\frac{1}{d}, T - \frac{1}{d}] \times [-d, d]$. Then for any $(t,x) \in [\frac{1}{d}, T] \times [-R, R]$ and k satisfying $\varepsilon_k < \frac{1}{\delta d(R+d)}$ we have $|x \pm \frac{t}{\varepsilon_k \delta}| \geq \frac{1}{d\varepsilon_k \delta} - R > d$ saying that

$$\int_0^T \int_{-R}^R (E^k_{0,2} \pm \varepsilon_k \delta B^k_0)(x)\varphi(t, x \pm \frac{t}{\varepsilon_k \delta}) \, dx\ dt = 0,$$

and thus we have

$$\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}} (\frac{E^k_{0,2}}{\varepsilon_k} \pm \delta B^k_0)(x)\varphi(t, x \pm \frac{t}{\varepsilon_k \delta}) \, dx\, dt\right| &\leq \|\varphi\|_{L^\infty} \int_0^T\int_{|x|>R} (\frac{|E^k_{0,2}|}{\varepsilon_k} + \delta |B^k_0|) \, dxdt \\
&\leq \eta \|\varphi\|_{L^\infty}.
\end{aligned}$$

Therefore the first and second term in the right hand side of (62) vanish as $k \to +\infty$. For the last two terms observe that we have

$$\int_0^T\int_{\mathbb{R}} V^k_\pm(t,x) \frac{\varphi(t,x)}{\varepsilon_k} \, dx\, dt = \int_0^T\int_{\mathbb{R}} \frac{j^k_2(t,x)}{\varepsilon_k} \psi^k_\pm(t,x) \, dx\, dt, \quad (63)$$

where for any $(t,x) \in [0,T] \times \mathbb{R}$

$$\psi^k_\pm(t,x) = \frac{1}{\varepsilon_k}\int_t^T \varphi(s, x \pm \frac{s-t}{\varepsilon_k \delta}) \, ds = \pm\delta \int_x^{x \pm \frac{T-t}{\varepsilon_k \delta}} \varphi(t \pm \varepsilon_k \delta (y - x), y) \, dy.$$

By using (27) we can write

$$\begin{aligned}
-\theta \int_0^T\int_{\mathbb{R}} V^k_\pm \frac{\varphi}{\varepsilon_k} \, dx\, dt &= \int_0^T\int_{\mathbb{R}} \varepsilon_k \partial_t \left(\int_{\mathbb{R}^2} p_2 f^k \, dp\right) \psi^k_\pm \, dx\, dt \\
&+ \int_0^T\int_{\mathbb{R}} \partial_x \left(\int_{\mathbb{R}^2} v_1(p) p_2 f^k \, dp\right) \psi^k_\pm \, dx\, dt \\
&- \int_0^T\int_{\mathbb{R}} E^k_2 \rho^k \psi^k_\pm \, dx\, dt + \int_0^T\int_{\mathbb{R}} \varepsilon_k \delta^2 B^k j^k_1 \psi^k_\pm \, dx\, dt \\
&= T^k_{\pm,1} + T^k_{\pm,2} + T^k_{\pm,3} + T^k_{\pm,4}. \quad (64)
\end{aligned}$$

We are done if we prove that $\lim_{k\to+\infty} T^k_{\pm,l} = 0$, $l \in \{1,2,3,4\}$. Observe that

$$\partial_t \psi^k_\pm(t,x) = \pm\delta \int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \partial_t\varphi(t\pm\varepsilon_k\delta(y-x),y)\,dy, \tag{65}$$

and

$$\begin{aligned}\partial_x \psi^k_\pm(t,x) &= -\varepsilon_k\delta^2 \int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \partial_t\varphi(t\pm\varepsilon_k\delta(y-x),y)\,dy \mp \delta\varphi(t,x) \\ &= \mp\varepsilon_k\delta\partial_t\psi^k_\pm \mp \delta\varphi(t,x). \end{aligned}\tag{66}$$

Notice that $\psi^k_\pm$, $\partial_t\psi^k_\pm$, $\partial_x\psi^k_\pm$ are uniformly bounded for $k \geq 1$

$$\|\psi^k_\pm\|_{L^\infty(]0,T[\times\mathbb{R})} \leq \delta \int_{\mathbb{R}} \sup_{t\in[0,T]} |\varphi(t,x)|\,dx,$$

$$\|\partial_t\psi^k_\pm\|_{L^\infty(]0,T[\times\mathbb{R})} \leq \delta \int_{\mathbb{R}} \sup_{t\in[0,T]} |\partial_t\varphi(t,x)|\,dx,$$

$$\|\partial_x\psi^k_\pm\|_{L^\infty(]0,T[\times\mathbb{R})} \leq \delta\|\varphi\|_{L^\infty(]0,T[\times\mathbb{R})} + \varepsilon_1\delta^2 \int_{\mathbb{R}} \sup_{t\in[0,T]} |\partial_t\varphi(t,x)|\,dx.$$

After integration by parts, by taking into account that $\psi^k_\pm(T,\cdot) = 0$ we find

$$\begin{aligned}|T^k_{\pm,1}| &\leq \left|\varepsilon_k \int_{\mathbb{R}}\int_{\mathbb{R}^2} p_2 f^k_0(x,p)\psi^k_\pm(0,x)\,dp\,dx\right| \\ &+ \left|\varepsilon_k \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} p_2 f^k(t,x,p)\partial_t\psi^k_\pm\,dp\,dx\,dt\right| \\ &\leq C\varepsilon_k\left(\sup_{k'}\int_{\mathbb{R}}\int_{\mathbb{R}^2}|p|f^{k'}_0\,dp\,dx + \sup_{k'}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2}|p|f^{k'}\,dp\,dx\,dt\right),\end{aligned}$$

implying that $\lim_{k\to+\infty} T^k_{\pm,1} = 0$. Similarly one gets by using (66)

$$\begin{aligned}|T^k_{\pm,2}| &\leq \left|\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} v_1(p)p_2 f^k\varepsilon_k\delta\partial_t\psi^k_\pm\,dp\,dx\,dt\right| \\ &+ \left|\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} v_1(p)p_2 f^k\delta\varphi\,dp\,dx\,dt\right| \\ &\leq \varepsilon_k\|\partial_t\psi^k_\pm\|_{L^\infty}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2}|p|f^k\,dp\,dx\,dt + \varepsilon_k\delta\left|\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} p_2\sqrt{f^k}\,h^k_1\varphi\,dp\,dx\,dt\right|.\end{aligned}$$

Notice that φ has compact support and then we deduce by (57) that $\lim_{k\to+\infty} T^k_{\pm,2} = 0$. The convergence $\lim_{k\to+\infty} T^k_{\pm,4} = 0$ follows by (59). Let us concentrate our attention on the convergence of $(T^k_{\pm,3})_{k\in\mathbb{N}}$. Consider the functions

$$\tilde\psi_\pm(t,x) = \pm\delta\int_x^{\pm\infty}\varphi(t,y)\,dy, \qquad (t,x)\in[0,T]\times\mathbb{R}.$$

Here $\varphi \in C^1_c(]0,T[\times\mathbb{R})$ with $\mathrm{supp} \subset [\frac{1}{d}, T-\frac{1}{d}] \times [-d,d]$ with $d>0$ large enough. By Remark 4.1 we know that

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} E_2^k \rho^k \tilde{\psi}_\pm \, dx\, dt = 0.$$

Since $(E_2^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$, $\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{\mathbb{R}} (1+|x|)\rho^k(t,x)\, dx < +\infty$ and $\sup_{k\in\mathbb{N}} \|\psi^k_\pm\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ for any $\eta>0$ there is $R=R(\eta)$ large enough such that

$$\left|\int_0^T \int_{|x|>R} E_2^k \rho^k \psi^k_\pm \, dx\, dt\right| < \eta, \qquad \left|\int_0^T \int_{|x|>R} E_2^k \rho^k \tilde{\psi}_\pm \, dx\, dt\right| < \eta, \qquad k \geq 1.$$

Take $k_1(\eta)$ such that

$$\left|\int_0^T \int_{\mathbb{R}} E_2^k \rho^k \tilde{\psi}_\pm \, dx\, dt\right| < \eta, \qquad k \geq k_1(\eta).$$

Therefore we can write for any $k \geq k_1(\eta)$

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{R}} E_2^k \rho^k \psi^k_\pm \, dx\, dt\right| &< \eta + \left|\int_0^T \int_{\mathbb{R}} E_2^k \rho^k (\psi^k_\pm - \tilde{\psi}_\pm)\, dx\, dt\right| \\ &< 3\eta + \left|\int_0^T \int_{-R}^R E_2^k \rho^k (\psi^k_\pm - \tilde{\psi}_\pm)\, dx\ dt\right| \\ &= 3\eta + \left|\int_0^{T-\frac{1}{d}} \int_{-R}^R E_2^k \rho^k (\psi^k_\pm - \tilde{\psi}_\pm)\, dx\ dt\right|. \end{aligned} \qquad (67)$$

Take now k_2 large enough such that $\frac{1}{d\varepsilon_k \delta} > R + d$ for any $k \geq k_2$. Observe that for all $(t,x) \in [0, T-\frac{1}{d}] \times [-R,R]$ and $k \geq k_2$ we have

$$x + \frac{T-t}{\varepsilon_k \delta} \geq \frac{1}{d\varepsilon_k\delta} - R > d, \qquad x - \frac{T-t}{\varepsilon_k\delta} \leq R - \frac{1}{d\varepsilon_k\delta} < -d,$$

saying that for any $(t,x) \in [0,T-\frac{1}{d}] \times [-R,R]$ and $k \geq k_2$ we have

$$\tilde{\psi}_\pm(t,x) = \pm\delta \int_x^{\pm\infty} \varphi(t,y)\, dy = \pm\delta \int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \varphi(t,y)\, dy.$$

Thus for any $(t,x) \in [0,T-\frac{1}{d}] \times [-R,R]$ and $k \geq k_2$ we have

$$\begin{aligned} |\psi^k_\pm(t,x) - \tilde{\psi}_\pm(t,x)| &= \delta \left|\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \{\varphi(t \pm \varepsilon_k\delta(y-x), y) - \varphi(t,y)\}\, dy\right| \\ &\leq \delta \left|\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \|\partial_t\varphi\|_{L^\infty}\ \varepsilon_k\delta\ |y-x| \mathbf{1}_{\{y\in[-d,d]\}}\, dy\right| \\ &\leq 2\|\partial_t\varphi\|_{L^\infty}\ \varepsilon_k \delta^2 (d+R) d. \end{aligned} \qquad (68)$$

Combining (67), (68) yields for any $k \geq \max\{k_1(\eta), k_2\}$

$$\left| \int_0^T \int_{\mathbb{R}} E_2^k \rho^k \psi_\pm^k \, dx\, dt \right| \ < \ 3\eta + 2\|\partial_t \varphi\|_{L^\infty} \varepsilon_k \delta^2 (d+R) d \sup_{k'} \|E_2^{k'}\|_{L^\infty} T \sup_{k'} \|\rho_0^{k'}\|_{L^1},$$

and we deduce that $\lim_{k\to+\infty} \int_0^T\int_{\mathbb{R}} E_2^k \rho^k \psi_\pm^k \, dx\, dt = 0$. The above computations show also that $(\frac{E_2^k}{\varepsilon_k})_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{D}'(]0,T[\times\mathbb{R})$ (use (39)). $\square$

5.2 Convergence to the Equilibrium Function

It is possible to show that $(f^k)_{k\in\mathbb{N}}$ converges towards $\rho(t,x)\ e^{-\mathcal{E}(p)}/\int_{\mathbb{R}^2} e^{-\mathcal{E}(q)} dq$ in some sense. We need to establish first that $(\rho^k)_{k\in\mathbb{N}}$ converges towards ρ in $C^0([0,T]; \mathrm{w}{-}L^1(\mathbb{R}))$.

Lemma 5.1 *Assume that $(\rho^\varepsilon)_{\varepsilon>0}$, $(j_1^\varepsilon)_{\varepsilon>0}$ satisfy $\rho^\varepsilon \geq 0$,*

$$\partial_t \rho^\varepsilon + \partial_x \frac{j_1^\varepsilon}{\varepsilon} = 0, \qquad \text{in} \qquad \mathcal{D}'(]0,T[\times\mathbb{R}),$$

$$\sup_{\varepsilon>0, t\in[0,T]} \int_{\mathbb{R}} (1 + |x| + |\ln \rho^\varepsilon|) \rho^\varepsilon(t,x)\, dx < +\infty,$$

and

$$\sup_{\varepsilon>0} \int_0^T \left(\int_{\mathbb{R}} \frac{|j_1^\varepsilon(t,x)|}{\varepsilon}\, dx \right)^2 dt < +\infty.$$

Then $(\rho^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T]; \mathrm{w}{-}L^1(\mathbb{R}))$.

Proof. Following the ideas in [25] we can extract a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing towards 0 such that for any $\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}} \rho^k(t,x)\varphi(x)\, dx = \int_{\mathbb{R}} \rho(t,x)\varphi(x)\, dx, \qquad \text{uniformly in} \qquad t \in [0,T]. \quad (69)$$

Actually (69) holds for any $\varphi \in L^\infty(\mathbb{R})$. Indeed, for any $\eta > 0$ take $R > 0$ large enough such that

$$\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{|x|>R} \rho^k(t,x)\, dx < \eta, \qquad \sup_{t\in[0,T]} \int_{|x|>R} \rho(t,x)\, dx < \eta. \quad (70)$$

By the hypotheses we can find $\mu = \mu(\eta) > 0$ such that for any $t \in [0,T]$ and measurable set A satisfying $\mathrm{meas}(A) < \mu$ we have

$$\sup_{\varepsilon>0, t\in[0,T]} \int_A \rho^\varepsilon(t,x)\, dx < \eta, \qquad \sup_{t\in[0,T]} \int_A \rho(t,x)\, dx < \eta. \quad (71)$$

By Lusin theorem (cf. [35], p. 52) there is a function $\varphi_\eta \in C_c^0(\mathbb{R})$, $\|\varphi_\eta\|_{L^\infty} \le \|\varphi\|_{L^\infty}$ such that

$$\operatorname{meas}\left(\{x \in [-R,R] \ : \ \varphi_\eta(x) \neq \varphi(x)\}\right) < \mu. \tag{72}$$

Combining (70), (71), (72) yields

$$\begin{aligned}
\left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi(x)\ dx\right| &\le \left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi_\eta(x)\ dx\right| \\
&+ \left|\int_{-R}^{R}(\rho^k(t,x) - \rho(t,x))(\varphi(x) - \varphi_\eta(x))\ dx\right| \\
&+ \left|\int_{|x|>R}(\rho^k(t,x) - \rho(t,x))(\varphi(x) - \varphi_\eta(x))\ dx\right| \\
&\le \left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi_\eta(x)\ dx\right| + 8\eta\|\varphi\|_{L^\infty}.
\end{aligned}$$

Since we know that $\lim_{k\to+\infty}\int_{\mathbb{R}} \rho^k(t,x)\varphi_\eta(x)\ dx = \int_{\mathbb{R}} \rho(t,x)\varphi_\eta(x)\ dx$ uniformly in $t \in [0,T]$ we conclude that $\lim_{k\to+\infty}\int_{\mathbb{R}} \rho^k(t,x)\varphi(x)\ dx = \int_{\mathbb{R}} \rho(t,x)\varphi(x)\ dx$ uniformly in $t \in [0,T]$. $\square$

Corollary 5.1 *Let us set $M(p) = e^{-\mathcal{E}(p)}/\int_{\mathbb{R}^2} e^{-\mathcal{E}(q)}dq$ for any $p \in \mathbb{R}^2$. Under the assumptions of Theorem 2.1, $(f^k)_{k\in\mathbb{N}}$ converges towards $\rho(t,x)M(p)$ in the following sense*

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(f^k(t,x,p) - \rho(t,x)M(p))\varphi(x)\ dx\right|\ dp\ dt = 0,$$

for any test function $\varphi \in L^\infty(\mathbb{R})$.

Proof. We write $f^k - \rho(t,x)M(p) = f^k - \rho^k(t,x)M(p) + (\rho^k(t,x) - \rho(t,x))M(p)$. Consider now $\varphi \in L^\infty(\mathbb{R})$. By Lemma 5.1 we have

$$\lim_{k\to+\infty}\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi(x)\ dx = 0, \qquad \text{uniformly in} \qquad t \in [0,T].$$

By using the dominated convergence theorem we have

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))M(p)\varphi(x)\ dx\right|\ dp\ dt = 0.$$

It remains to discuss $f^k - \rho^k M$. By logarithmic Sobolev inequality (see [4], [2]) we obtain

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^2} \left\{ \frac{f^k}{\rho^k M(p)} \ln\left(\frac{f^k}{\rho^k M(p)}\right) - \frac{f^k}{\rho^k M(p)} + 1 \right\} \rho^k M(p)\, dp \\ &= \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M(p)}\right)\, dp \\ &\leq \lambda \int_{\mathbb{R}^2} \left| \nabla_p \sqrt{\frac{f^k}{M(p)}} \right|^2 M(p)\, dp \\ &= \frac{\lambda \varepsilon_k^2}{4} \int_{\mathbb{R}^2} |h^k(t,x,p)|^2\, dp, \end{aligned}$$

for some $\lambda > 0$. We conclude by using the Csiszar-Kullback-Pinsker inequality, see [17], [28]

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |f^k - \rho^k M(p)|\, dp\, dx \right)^2 \leq \mu \int_{\mathbb{R}} \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M(p)}\right)\, dp\, dx,$$

for some $\mu > 0$ which implies that

$$\begin{aligned} \int_0^T\!\!\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} (f^k - \rho^k M)\varphi\, dx \right| dpdt &\leq \|\varphi\|_{L^\infty} \sqrt{T} \left(\int_0^T \left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |f^k - \rho^k M|\, dp\, dx \right)^2 dt \right)^{1/2} \\ &\leq \|\varphi\|_{L^\infty} \sqrt{\mu T} \left(\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M}\right)\, dp\, dx\, dt \right)^{1/2} \\ &\leq \|\varphi\|_{L^\infty} \sqrt{\lambda \mu T} \frac{\varepsilon_k}{2} \left(\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |h^k|^2\, dp\, dx\, dt \right)^{1/2} \end{aligned} \tag{73}$$

tends to 0 as $k \to \infty$. □

6 Appendix

We detail here the dimensional analysis of the equations and the physical meaning of the different parameters. We introduce the following physical constants
- ε_0 the vacuum permittivity,
- c_0 the vacuum light speed,
- q the charge of particles,
- m the mass of particles,
- τ the relaxation time which characterizes the interactions of the particles with the thermal bath,
- K_B the Boltzmann constant,
- T_{th} the temperature of the thermal bath.

We will also need the total number of particles involved in the physical system under consideration, denoted by $\mathcal{N}$. We define the thermal potential by

$$U_{\text{th}} = \frac{K_B T_{\text{th}}}{q}.$$

The thermal impulsion $p_{\text{th}} > 0$ is given by the relation

$$mc_0^2 \left(\sqrt{1 + \frac{p_{\text{th}}^2}{m^2 c_0^2}} - 1 \right) = K_B T_{\text{th}},$$

which leads to

$$p_{\text{th}} = \sqrt{(K_B T_{\text{th}})^2 / c_0^2 + 2 K_B T_{\text{th}} m}.$$

Eventually, we denote by $v_{\text{th}} > 0$ the thermal velocity given by

$$v_{\text{th}} = \frac{p_{\text{th}}}{m \sqrt{1 + \frac{p_{\text{th}}^2}{m^2 c_0^2}}} = c_0 \sqrt{1 - \frac{1}{\left(1 + \frac{K_B T_{\text{th}}}{m c_0^2}\right)^2}}.$$

Remark that

$$\frac{p_{\text{th}} v_{\text{th}}}{K_B T_{\text{th}}} = \frac{K_B T_{\text{th}} + 2 m c_0^2}{K_B T_{\text{th}} + m c_0^2} = \vartheta \in]1, 2[.$$

We are interested in the evolution of the distribution function $f(t, x, p)$ of the charged particles; it depends on time $t > 0$, space $x \in \mathbb{R}^3$ and impulsion $p \in \mathbb{R}^3$. Given a momentum p, the associated energy reads

$$\mathcal{E}(p) = mc_0^2 \big(\sqrt{1 + |p|^2 / (m^2 c_0^2)} - 1 \big)$$

and the velocity is given by

$$v(p) = \nabla_p \mathcal{E}(p) = \frac{p}{m \sqrt{1 + |p|^2 / (m^2 c_0^2)}}.$$

Then, the evolution of f obeys the Fokker-Planck equation

$$\partial_t f + v(p) \cdot \nabla_x f + q(E(t, x) + v(p) \wedge B(t, x)) \cdot \nabla_p f = L_{FP}(f),$$

for $(t, x, p) \in]0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3$, where the relativistic Fokker-Planck collision operator reads

$$L_{FP}(f) = \frac{p_{\text{th}}^2}{\tau} \text{div}_p \left(\frac{v(p)}{K_B T_{\text{th}}} f + \nabla_p f \right) = \frac{p_{\text{th}}^2}{\tau} \text{div}_p \left(\mathcal{M}(p) \nabla_p \left(\frac{f}{\mathcal{M}(p)} \right) \right).$$

Here $\mathcal{M}(p) = e^{-\mathcal{E}(p)/(K_B T_{\rm th})}$ is the relativistic Maxwellian. The force acting on the particles depends on the electro-magnetic field (E, B) the evolution of which is driven by the Maxwell equations

$$\left\{\begin{array}{lll} \partial_t E - c_0^2 \,\mathrm{curl}_x B = -q\dfrac{j(t,x)}{\varepsilon_0}, & \partial_t B + \mathrm{curl}_x E = 0, & (t,x) \in]0,+\infty[\times\mathbb{R}^3, \\ \mathrm{div}_x E = q\dfrac{\rho(t,x)}{\varepsilon_0}, & \mathrm{div}_x B = 0, & (t,x) \in]0,+\infty[\times\mathbb{R}^3, \end{array}\right.$$

where $\rho = \int_{\mathbb{R}^3} f\, dp$, $j = \int_{\mathbb{R}^3} v(p) f\, dp$ are respectively the charge and current densities.

Let us write the equations in dimensionless form. To this end, we introduce a length unit L, and a time unit T. As impulsion unit we set $P = p_{\rm th}$. We define dimensionless variables and unknowns by the relations

$$\begin{array}{lll} t = Tt', & x = Lx', & p = p_{\rm th} p', \\ f(t,x,p) = \dfrac{\mathcal{N}}{L^3 p_{\rm th}^3} f'\Big(\dfrac{t}{T}, \dfrac{x}{L}, \dfrac{p}{P}\Big), & E(t,x) = \dfrac{U_{\rm th}}{L} E'\Big(\dfrac{t}{T}, \dfrac{x}{L}\Big), & B(t,x) = \dfrac{U_{\rm th}}{Tc_0^2} B'\Big(\dfrac{t}{T}, \dfrac{x}{L}\Big). \end{array}$$

We set

$$\mathcal{E}'(p') = \frac{\mathcal{E}(p_{\rm th} p')}{K_B T_{\rm th}}, \qquad v'(p') = \nabla_{p'} \mathcal{E}'(p') = \frac{p_{\rm th}^2}{m K_B T_{\rm th}} \frac{p'}{\sqrt{1 + \dfrac{p_{\rm th}^2}{m^2 c_0^2} |p'|^2}}.$$

As a matter of fact, note that $v(p) = \frac{K_B T_{\rm th}}{p_{\rm th}} v'(p/p_{\rm th})$. We also introduce the Debye length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 K_B T_{\rm th} L^3}{q^2 \mathcal{N}}}.$$

Then, the equation becomes (having dropped the primes)

$$\begin{aligned} \partial_t f + \frac{K_B T_{\rm th}}{p_{\rm th}} \frac{T}{L} v(p) \cdot \nabla_x f + \frac{q U_{\rm th}}{p_{\rm th}} \frac{T}{L} \left(E(t,x) + \frac{K_B T_{\rm th}}{p_{\rm th}} \frac{L}{Tc_0^2}\, v(p) \wedge B(t,x) \right) \cdot \nabla_p f \\ = \frac{T}{\tau} \mathrm{div}_p (f v(p) + \nabla_p f) \end{aligned}$$

coupled to

$$\left\{\begin{array}{ll} \partial_t E - \mathrm{curl}_x B = -\Big(\dfrac{L}{\lambda_D}\Big)^2 \dfrac{K_B T_{\rm th}}{p_{\rm th}} \dfrac{T}{L}\, j(t,x), & \partial_t B + \Big(\dfrac{c_0 T}{L}\Big)^2 \mathrm{curl}_x E = 0, \\ \mathrm{div}_x E = \Big(\dfrac{L}{\lambda_D}\Big)^2 \rho(t,x), & \mathrm{div}_x B = 0, \end{array}\right.$$

with

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\, dp, \qquad j(t,x) = \int_{\mathbb{R}^3} v(p) f(t,x,p)\, dp.$$

We are concerned with the physical situation where

$$\begin{cases} \lambda_D = L, \\ 1 \ll \dfrac{K_B T_{\text{th}}}{p_{\text{th}}} \dfrac{T}{L} = \dfrac{1}{\varepsilon} \ll \dfrac{T}{\tau} = \dfrac{\theta}{\varepsilon^2}, \end{cases}$$

where $\theta > 0$ is a fixed dimensionless parameter. The first relation determines the length unit $L = q^2\mathcal{N}/(\varepsilon_0 K_B T_{\text{th}})$. The second says that the time unit is large compared to $Lp_{\text{th}}/(K_B T_{\text{th}})$ which is itself larger than the relaxation time. Accordingly, it means that the thermal velocity v_{th} is large compared to the reference velocity L/T (the ratio being of order $\mathcal{O}(1/\varepsilon)$). Besides, the mean free path $\ell = v_{\text{th}}\tau$ is small compared to both the Debye length λ_D and the length unit L (the ratio being of order $\mathcal{O}(\varepsilon)$). Next, we check that

$$\frac{c_0 T}{L} = \frac{c_0}{v_{\text{th}}} \frac{v_{\text{th}} p_{\text{th}}}{K_B T_{\text{th}}} \frac{K_B T_{\text{th}} T}{p_{\text{th}} L} = \frac{1}{\sqrt{1 - \left(1 + \dfrac{K_B T_{\text{th}}}{m c_0^2}\right)^{-2}}} \frac{\vartheta}{\varepsilon}.$$

We denote

$$\frac{c_0 T}{L} = \frac{1}{\delta\varepsilon}$$

where, by definition, the parameter $\delta = (1 + 2mc_0^2/(K_B T_{\text{th}}))^{-1/2}$ belongs to $(0,1)$.

As a matter of fact, we remark that the magnetic effects are always dominated by the electric forces. Eventually, let us go back to the expression of the rescaled velocity which involves the dimensionless quantities

$$\frac{p_{\text{th}}^2}{m K_B T_{\text{th}}} \qquad \text{and} \qquad \frac{p_{\text{th}}^2}{m^2 c_0^2}.$$

It turns out that we can rewrite these quantities by means of the previously defined coefficient and we get

$$\begin{cases} v(p) = \dfrac{2}{1-\delta^2} \dfrac{p}{\sqrt{1 + \dfrac{4}{\delta^2(1/\delta^2 - 1)^2}|p|^2}}, \\ \mathcal{E}(p) = \dfrac{1/\delta^2 - 1}{2}\left(\sqrt{1 + \dfrac{4}{\delta^2(1/\delta^2 - 1)^2}|p|^2} - 1\right). \end{cases}$$

We recap the asymptotic problem as follows

$$\begin{cases} \partial_t f + \dfrac{1}{\varepsilon} v(p) \cdot \nabla_x f + \left(\dfrac{1}{\varepsilon} E(t,x) + \delta^2 v(p) \wedge B(t,x)\right) \cdot \nabla_p f = \dfrac{\theta}{\varepsilon^2} \mathrm{div}_p(v(p) f + \nabla_p f), \\ \partial_t E - \mathrm{curl}_x B = -\dfrac{1}{\varepsilon} j(t,x), \qquad \varepsilon^2\delta^2 \partial_t B + \mathrm{curl}_x E = 0, \\ \mathrm{div}_x E = \rho(t,x), \qquad \mathrm{div}_x B = 0. \end{cases}$$

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