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Complex Plateau problem: old and new results and prospects

Pierre Dolbeault

Abstract. The Plateau problem is the research of a surface of minimal area, in the 3-dimensional Euclidean space, whose boundary is a given continuous closed curve. The complex Plateau problem is analogous in a Hermitian complex manifold: it is a geometrical problem of extension of a closed real curve or manifold into a complex analytic subvariety, or into a Levi-flat subvariety. Wirtinger's inequality in $\mathbb{C}^n$ is recalled. Minimatity of complex analytic subvarieties and analogous properties of Levi-flat subvarieties, in Kähler manifolds, are given. Known results in $\mathbb{C}^n$ and $\mathbb{C}P^n$ are recalled. Extensions to real parametric problems are solved or proposed, leading to the construction of Levi-flat hypersurfaces with prescribed boundary in some complex manifolds.

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1. Introduction.

Given a Hermitian manifold X, the *complex Plateau problem* is the research of an even dimensional subvariety with negligible singularities, with given boundary, and of minimal volume in X. We will call *mixed Plateau problem* the research of a real hypersurface with given boundary, and of minimal volume in X. More briefly, both problems will be called complex Plateau problem.

First we shall recall or show that complex analytic subvarieties, resp. Levi-flat hypersurfaces are solutions of the Plateau problem when X is Kähler (section 2).

Then we will consider the complex Plateau problem as the research of the extension of an odd dimensional, compact, oriented, connected submanifold into a complex analytic subvariety, and recall known solutions (section 3, 4).

To solve the mixed Plateau problem as the research of the extension of an oriented, compact, connected, 2-codimensional submanifold into a Levi-flat hypersurface, we will need solutions of the complex Plateau problem with real parameter, in $\mathbb{C}^n$ and $\mathbb{C}\mathrm{P}^n$; in $\mathbb{C}\mathrm{P}^n$, it is an open problem to explicit satisfactory conditions on the boundary. In this way, we get very peculiar solutions of mixed Plateau problems (section 5).

Finally, the mixed Plateau problem is solved in $\mathbb{C}^n$, in particular cases, as a projection of a Levi-flat variety, and set up in $\mathbb{C}\mathrm{P}^n$ (sections 6,7): known solutions are recalled in $\mathbb{C}^n$ when the complex points of the boundary are elliptic; special elliptic and hyperbolic points of the boundary are defined, and a solution when the boundary is a "horned sphere" is described; this will be the opportunity to precise and complete results announced in ([D 08], section 4). Problems when the boundary has general hyperbolic points are still open.

Proofs of the results in sections 6 and 7 will appear in detail elsewhere [D 09].

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2. Volume minimality of complex analytic subvarieties and of Levi-flat hypersurfaces in Kähler manifolds.

2.1. *Wirtinger's inequality* (1936) [H 77].

In $\mathbb{C}^n$,with complex coordinates $(z_1, \dots, z_n)$, we have the Hermitian metric $H = \sum_{j=1}^{n} dz_j \otimes d\overline{z}_j$ and the exterior form (standard Kähler) $\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$.

From the real vector space $I\!R^{2n} \cong \mathbb{C}^n$, we consider the real vector space $\Lambda_{2p} I\!R^{2n}$ of the $2p$-vectors with the associated norm $|.|$; every decomposable vector (exterior product of elements of $I\!R^{2n}$) defines a real $2p$-plane of $\mathbb{C}^n$ i.e. an element of the Grassmannian G_{2n}^{2p}. We define the norm $\| \zeta \| = \inf \sum_j | \zeta_j |$ where $\zeta = \sum_j \zeta_j$, ζ_j is decomposable.

Let $P_{pp} = \{\sum_{j=1}^{N} \lambda_j \zeta_j ; \zeta_j$ decomposable defining a complex p-plane of $\mathbb{C}^n$; $\lambda_j \geq 0; N \in I\!N^*\}$.

2.1.1. **Theorem.** *For every $\zeta \in \Lambda_{2p}\mathbb{C}^n$, we have:*

$$\frac{1}{p!}\omega^p(\zeta) \leq \| \zeta \|;$$

equality uniquely for $\zeta \in P_{pp}$ [W 36].

2.1.2. **Corollary.** *Let V be a smooth real oriented $2p$-dimensional submanifold of a Hermitian manifold $X = (X, \omega)$ of complex dimension n. Then $\int_V \omega^p/p! \leq \mathrm{vol}_{2p}(V)$ with equality iff V is complex.*

2.2. *Currents with measure coefficients.* [H 77]

2.2.1. *Comass of an r-form; mass of a current with measure coefficients.*

Let $\varphi \in \Lambda^r I\!R^{2n}$, the comass of φ is defined as

$$\| \varphi \|^* = \sup\{\varphi(\zeta) : \zeta \in G_{2n}^r \subset \Lambda_{2p} I\!R^{2n}\}$$

Let Ω be an open subset of $\mathbb{C}^n$, for every differential form φ of degree r on Ω, let

$$\| \varphi \|^* = \sup\{\| \varphi(z) \|^* : z \in \Omega\}$$

where $\| \varphi(z) \|^*$ is the comass of $\varphi(z)$.

Let T be a current with measure coefficients on Ω, K be any compact subset of Ω and χ_K the characteristic function of K,

$$M_K(T) = \sup_{\|\varphi\|^* \leq 1} |\chi_K T(\varphi)|$$

is, by definition, the *mass* of T on K.

The measure which assigns the number $M_K(T)$ to each compact set $K \subset \Omega$ is called the *mass* or *volume measure* of T and denoted $\| T \|$, so that $M_K(T) = \| T \| (K)$.

2.3. *Complex Plateau problem.*

2.3.1. [H 77] On $\Omega \subset \mathbb{C}^n$, or more generally, on a Hermitian manifold (X, ω), let B be a d-closed current of dimension $2p-1$ with compact support, and let T be a $(2p)$-current with compact support and measure coefficients such that $dT = B$. The complex Plateau problem is to find such a T with minimal mass, i.e. for every compactly supported current S, with measure coefficients such that $dS = B$, to have $M(T) \leq M(S)$, or equivalently, for every compactly supported, d-closed $(2p)$-current with measure coefficients R,

$$M(T) \leq M(T + R)$$

Such a T is said *absolutely volume minimizing* on X.

Let T be a d-closed $(2p)$-current with measure coefficients on X. If, for each compact subset K of X,

$$M_K(T) \leq M(\chi_K T + R)$$

for all compactly supported d-closed $(2p)$-current R with measure coefficients on X, then T is said to be *absolutely volume minimizing* on X.

2.3.2. **Theorem.** [H 77] *Let T be a $2p$-current with measure coefficients on a Hermitian manifold (X, ω) and K be a compact subset of X. Then*

$$(\chi_K T)(\omega^p/p!) \leq M_K(T).$$

and equality holds iff $\chi_K T$ is strongly positive. $\square$

2.3.3. [H 77] *Volume minimality of complex analytic sets in a Kähler manifold.*

2.3.4. **Corollary to Theorem 2.3.2.** *Assume that $X = (X, \omega)$ is a Käkler manifold and does not contain compact p-dimensional complex subvarieties. Let V be a p-dimensional complex subvatiety, and $T = [V]$, then T is absolutely volume minimizing* on X.

Proof. T is strongly positive. Let K be a compact subset of X and R be a compactly supported d-closed $(2p)$-current with measure coefficients. From Theorem 2.3.2, $M_K(T) = (\chi_K T)(\omega^p/p!)$. But locally $\omega = dd^c\psi$, then $\omega^p = \omega^{p-1} \wedge dd^c\psi = d(\omega^{p-1} \wedge d^c\psi)$, so in the neighborhood of any point of X, $R(\omega^p) = R(d(\omega^{p-1} \wedge d^c\psi))$. Let $(\alpha_j)_{j\in J}$ be a partition C^∞ of unity subordinate to a locally finite open covering $(U_j)_{j\in J}$ of X such that for every j, $\omega|_{U_j} = dd^c\psi_j$. Then

$$R(\omega^p) = \sum_j \alpha_j R(d(\omega^{p-1} \wedge d^c\psi_j)) = \pm \sum_j d(\alpha_j R)(\omega^{p-1} \wedge d^c\psi_j) = 0,$$

because:

$$\sum_j d(\alpha_j R) = \sum_j d\alpha_j \wedge R + \sum_j \alpha_j \wedge dR = 0$$

and, as in the proof of ([H 77], Corollary 1.25), in an open set of the Hermitian $\mathbb{C}^n$,

$$M_K(T) = (\chi_K T)(\omega^p/p!) = (\chi_K T + R)(\omega^p/p!) \leq M_K(T + R). \qquad \square$$

2.3.5. **Remark.** If X contains a compact p-dimensional complex subvariety W, $d[V] = 0$, but $M_K([V] > 0$; then T is *relatively volume minimizing* on X.

2.4. *Volume minimality of Levi-flat hypersurfaces in Kähler manifolds.*

We suppose to be in the category of currents with measure coefficients.

Recall the definition: A *Levi-flat subvariety* (with negligible singularities), of odd dimension, is, outside of the singularities, a submanifold with Levi form $\equiv 0$, or, equivalently, is foliated by complex analytic hypersurfaces.

Let M be a C^∞ Levi-flat hypersurface of a C^∞ Kähler manifold $X = (X, \omega)$ bearing a foliation $\mathcal{L}$ by complex hypersurfaces M_l and let L be the space of the foliation $\mathcal{L}$ assumed to be a C^∞ real curve.

Let M' be a C^∞ hypersurface of X bearing a foliation $\mathcal{L}'$ with the same space L; the leaves of $\mathcal{L}'$ being C^∞ subvarieties with negligible singularities.

Let S be a C^∞ compact submanifold of codimension 2 of X. We denote by the same notation the hypersurfaces and submanifold and the integration currents they define.

2.5. *Mixed Plateau problem.*

Given S to find a C^∞ hypersurface in $X \setminus S$ whose boundary is S in the category $\mathcal{H}$ of foliated hypersurfaces with the same space of foliation, a real curve. If M' is such a hypersurface whose space of foliation is L and the leaves $(M'_l, l \in L)$, then $\mathrm{vol}(M') = \int_L \mathrm{vol}(M'_l)dl$.

From section 2, for every $l \in L$, $\mathrm{vol}(M'_l) \geq \mathrm{vol}(M_l)$ then M *is relatively volume minimizing in the category* $\mathcal{H}$ and, by definition, M *is solution of the mixed Plateau problem.*

2.6. *Research of solutions of the complex Plateau problem.*

The present method of resolution consists in finding complex analytic, resp. Levi-flat subvarieties, in $X \setminus S$, whose boundary S (in the sense of currents) is a submanifold of X with convenient properties. .

3. Possible origin: holomorphic extension; polynomial envelope of a real curve.

3.1. The extension theorem of Hartogs, obtained at the beginning of the 20th century, has been completely proved by Bochner and Martinelli, independently, in 1943. The simplest version is:

Let Ω be a bounded open set of $\mathbb{C}^n$, $n \geq 2$. Suppose that $\partial\Omega$ be of class C^k $(1 \leq k \leq \infty)$ or of class C^ω (i.e. real analytic). Let f be a function in $C^l(\partial\Omega)$, $1 \leq l \leq k$.

Then the two conditions are equivalent:

(i) f is a CR function, i.e. the differential of f restricted to the complex subspaces of the tangent space to $\partial\Omega$, at every point, is $\mathbb{C}$-linear;

(ii) there exists $F \in C^l(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ *such that* $F \mid_{\partial\Omega} = f$.

Then the graph of f is the boundary of the complex analytic submanifold defined by the graph of F in $\mathbb{C}^{n+1}$.

3.2. Let M be a compact submanifold of dimension 1 of $\mathbb{C}^n$, we call polynomial envelope of M, the compact set $\{z \in \mathbb{C}^n; \mid P(z) \mid \leq \max_{\zeta \in M} \mid P(\zeta) \mid; P \in \mathbb{C}[z]$, the polynomial ring with complex coefficients $\}$.

Then (J. Wermer (1958)), the polynomial envelope of M is either M, or the union of M with the support of a complex analytic variety T, of complex dimension 1, whose boundary is M [We 58].

4. Solutions of the complex Plateau problem (or boundary problem) in different spaces.

4.1. The first result has been obtained in 1958, by J. Wermer, in $\mathbb{C}^n$, for $p = 1$ and M holomorphic image of the unit circle in $\mathbb{C}$ [We 58]; this result has been generalized to the case where M is a union of C^1 real connected curves by Bishop, Stolzenberg (1966), looking for the polynomial envelope of M according to section 3.2.

In $\mathbb{C}^n$, after preliminary results by Rothstein (1959) [Rs 59] , the boundary problem has been solved by Harvey and Lawson (1975), for $p \geq 2$, under the necessary and sufficient condition: M is compact, maximally complex and, for $p = 1$, under the moment condition: $\int_M \varphi = 0$, for every holomorphic 1-form φ on $\mathbb{C}^n$ [HL 75]. For $n = p + 1$, the method, inspired by the Hartogs' theorem consists in building T as the divisor of a meromorphic function the *defining function* R; this function itself is constructed, step by step, from solutions of $\overline{\partial}$-problems with compact support. T can also be viewed as graph (with multiplicities on the irreducible components) of an analytic function with a finite number of determinations. For any p, we come back to the particular case using projections.

In $\mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^{n-r}$, $1 \leq r \leq n$, for compact M, the problem has a une solution if and only if, for $p \geq r+1$, M is maximally complex and if, for $p = r$, M satisfies the moment condition: $\int_M \varphi = 0$, for every $\overline{\partial}$-closed $(p, p-1)$-forme φ . The method consists in solving the boundary problem, in $\mathbb{C}^{n+1} \setminus \mathbb{C}^{n-r+1}$, for the inverse image of M by the canonical projection [HL 77].

In both cases, the solution is unique.

Harvey et Lawson assume the given M to be, except for a closed set of Hausdorff $(2p-1)$-dimensional measure zero, an oriented manifold of class C^1; we will say: M is a *variety* C^1 *with negligible singularities.*

The boundary problem in $\mathbb{C}\mathrm{P}^n$ has been set up, for the first time, by J. King [Ki 79]; uniqueness of the solution is no more possible, since two solutions differ by an algebraic p-chain.

4.2. In $\mathbb{C}\mathrm{P}^n$, a solution of the boundary problem has been obtained by P. Dolbeault et G. Henkin for $p = 1$, (1994), then for every p (1997) and more generally, in a q-linearly concave domain X of $\mathbb{C}\mathrm{P}^n$, i.e. a union of projective subspaces of dimension q [DH 97].

The necessary and sufficient condition for the existence of a solution is an extension of the moment condition: it uses a Cauchy residue formula in one variable and a non linear differential condition which appears in many questions of Geometry or mathematical Physics. In the simplest case: $p = 1$, $n = 2$, this is the shock wave equation for a local holomorphic function in 2 variables ξ, η, $\quad f\dfrac{\partial f}{\partial \xi} = \dfrac{\partial f}{\partial \eta}$.

From a local condition, the above relation allows to construct, by extension ot the coefficients, a meromorphic function playing, in $\mathbb{C}^n$, the same part as the Harvey-Lawson defining function described above; it defines a holomorphic p-chain extendable to $\mathbb{C}\mathrm{P}^n$ using the classical Bishop-Stoll theorem.

4.2.1. The conditions of regularity of M have been weakened, first in $\mathbb{C}^n$, and for $p = 1$, to a condition, a little stronger than the rectifiability, by H. Alexander [Al 88] who, moreover, has given an essential counter-example [Al 87], then by Lawrence [Lce 95] and finally, and for any p, in $\mathbb{C}^n$ and $\mathbb{C}\mathrm{P}^n$, by T.C. Dinh [Di 98]: M is a rectifiable current whose tangent cone is a vector subspace almost everywhere. Moreover, Dinh has obtained the reduction of the boundary problem in $\mathbb{C}\mathrm{P}^n$ to the case $p = 1$, with weaker conditions than above and by an elementary analytic procedure [Di 98].

All the previous results are obtained as Corollaries.

New progress by Harvey and Lawson [HL 04].

5. Extension to real parametric problems.

5.1. *In a real hyperplane of* $\mathbb{C}^n$.

5.1.1. Let $E \cong \mathbf{R}\times\mathbb{C}^{n-1}$, and $k : \mathbf{R}\times\mathbb{C}^{n-1} \to \mathbf{R}$ be the projection. Let $N \subset E$ be a compact, (oriented) CR subvariety of $\mathbb{C}^n$ of real dimension $2n-4$ and CR dimension $n-3$, $(n \geq 4)$, of class C^∞, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n-4)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold). Let τ' be the set of all points $z \in N$ such that either $z \in \tau$ or $z \in N \setminus \tau$ and N is not transversal to the complex hyperplane $k^{-1}(k(z))$ at z. Assume that N, as a current of integration, is d-closed and satisfies:

(H) there exists a closed subset $L \subset \mathbf{R}_{x_1}$ with $H^1(L) = 0$ such that for every $x \in k(N) \setminus L$, the fiber $k^{-1}(x) \cap N$ is connected and does not intersect τ'.

5.1.2. **Theorem** [DTZ 09] (see also [DTZ 05]). *Let N satisfy* (H) *with L chosen accordingly. Then, there exists, in $E' = E \setminus k^{-1}(L)$, a unique C^∞ Levi-flat $(2n-3)$-subvariety M with negligible singularities in $E' \setminus N$, foliated by complex $(n-2)$-subvarieties, with the properties that M simply (or trivially) extends to E' as a $(2n-3)$-current (still denoted M) such that $dM = N$ in E'. The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N) \setminus L$, and are the solutions of the "Harvey-Lawson problem" for finding a holomorphic subvariety in $E_{x_1^0} \cong \mathbb{C}^{n-1}$ with prescribed boundary $N \cap E_{x_1^0}$.*

5.2. *In a real hyperplane of $\mathbb{C}\mathrm{P}^{n+1}$.*

5.2.1. The simplest significant case is the boundary problem in $\mathbb{C}P^3$. For the boundary problem with real parameter in $\mathbb{C}^3$, we considered a boundary problem in $I\!R \times \mathbb{C}^3$, i.e. in the subspace of $\mathbb{C}^4$, in which the first coordinate is real. In the same way, we will consider in $\mathbb{C}P^4$, with homogeneous coordinates $(w_0, w_1, \ldots, w_4)$, a boundary problem in the subspace E defined by $w_1 = \lambda w_0$, with $\lambda \in I\!R$. Then, for personal convenience, we will follow, step by step, the known construction in $\mathbb{C}P^3$ in the oldest version [DH 97].

Particularly, the coefficients C_m of the defining function R of the solution are estimated as for the problem in $\mathbb{C}P^3$.

The end of the proof of the main theorem seems analogous to the known case in $I\!R \times \mathbb{C}^3$.

5.2.2. The projective space $\mathbb{C}P^3$ has homogeneous coordinates $w' = (w_0, w_2, \ldots, w_4)$; denote $Q = \{w_0 = 0\}$ the hyperplane at infinity of $\mathbb{C}P^3$.

For $w_0 \neq 0$, let k be the projection: $E \to I\!R_\lambda$, $(w_0, w_1 = \lambda w_0, w_2, w_3, w_4) \mapsto \lambda$; for $w_0 = 0$, λ is indeterminate.. We also have the projection: $\pi : E \to \mathbb{C}P^3$, $(w_0, \lambda w_0, w_2, w_3, w_4) \mapsto (w_0, w_2, w_3, w_4)$. In the same way, $(E \setminus \{w_0 = 0\}) \cong I\!R \times \mathbb{C}^3$.

5.2.3. Let $N \subset E \subset \mathbb{C}P^4$ be a submanifold of class C^∞, CR, oriented, compact of E, of dimension 4, of CR dimension 1, with negligible singularities. N being compact in E, $k(N)$ is compact in $I\!R$, i.e. in N, the parameter λ varies in a closed, bounded interval Λ of $I\!R$.

Assume that N satisfies the same properties as in subsection 5.1.1.

5.2.4. Consider the complex hyperplanes of $\mathbb{C}P^4$, whose equation is $\tilde{h}(w) = w_4 - \xi_2 w_0 - \eta_2' w_1 - \eta_2 w_2 = 0$ and, in E, the subspaces $P_{\nu_\lambda'}$ whose equation is

$$\tilde{h}_1(w', \lambda) = w_4 - \xi_2 w_0 - \eta_2' \lambda w_0 - \eta_2 w_2 = w_4 - (\xi_2 + \eta_2' \lambda) w_0 - \eta_2 w_2 = 0,$$

of real dimension 5. Their restrictions to $(E \setminus I\!R \times Q) \cong I\!R \times \mathbb{C}^3$ are real affine sub-spaces of dimension 5.

We note ν_λ' the 1×2-matrix $\big((\xi_2 + \eta_2' \lambda)\ \eta_2\big)$.

Generically, $\Gamma_{\nu_\lambda'} = N \cap P_{\nu_\lambda'}$ is of dimension 2.

For $z \in N, \lambda = k(z)$. let $E_\lambda = k^{-1}k(z)$; for $\lambda \notin L$, $N \cap E_\lambda$ is of dimension 3 and is contained in $E_\lambda \cong \mathbb{C}P^3$.

Consider the linear forms

$$\underline{h}(w) = w_3 - \xi_1 w_0 - \eta_1' w_1 - \eta_1 w_2$$

$$\tilde{h}_0(w', \lambda) = w_3 - \xi_1 w_0 - \eta_1' \lambda w_0 - \eta_1 w_2 = w_3 - (\xi_1 + \eta_1' \lambda) w_0 - \eta_1 w_2$$

Denote $\nu_\lambda = (\xi_\lambda, \eta)$ the 2×2-matrix

$$\begin{pmatrix} (\xi_1 + \eta_1' \lambda) & \eta_1 \\ (\xi_2 + \eta_2' \lambda) & \eta_2 \end{pmatrix}$$

For fixed λ, ν_λ is a coordinate system of a chart of the Grassmannian $G_{\mathbb{C}}(2,4)$, i.e. ν_λ is a coordinate system of a chart of $G_{\mathbb{C}}(2,4) \times I\!R$. and we identify ν_λ with the point of $G_{\mathbb{C}}(2,4) \times I\!R$ having these coordinates.

Let $\xi_\lambda =^t \big((\xi_1+\eta'_1\lambda)\ (\xi_2+\eta'_2\lambda)\big)$; $\eta =^t (\eta_1\ \eta_2)$. Remark that ξ_λ depends on $(\xi_1,\xi_2,\eta'_1,\eta'_2)$; to get effective dependance on the parameter λ, it suffices to fix $\eta'_1\neq 0$, $.\eta'_2\neq 0$.

Recall: $\xi_\lambda =^t (\xi_{\lambda 1}\ \xi_{\lambda 2})$, $\xi_{\lambda l}=\xi_l+\eta'_l\lambda$, $l=1,2$, $\eta =^t (\eta_1\ \eta_2)$.

Let $z_j=w_j/w_0$, $j=2,\dots,4$, be the non homogeneous coordinates ; $\tilde h_0$ defines the affine function:

$$h=z_3-(\xi_1+\eta'_1\lambda)-\eta_1 z_2$$

The two forms $\tilde h_0$ et $\tilde h_1$ are linearly independent, then the set of their common zeros D_{ν_λ} is of real dimension 3, is contained in $P_{\nu'_\lambda}$; in general, $D_{\nu_\lambda}\cap N$ is a finite set Z_{ν_λ} ; then, for general enough fixed λ and ν_λ, $Z_{\nu_\lambda}=\emptyset$. For every fixed real number $\lambda\notin L$, the situation in E_λ is the classical situation in $\mathbb{C}P^3$.

5.2.5. *Boundary problem.* Given N, find a complex analytic subvariety M depending on the real parameter λ such that $dM=N$ in the sense of currents, under a necessary and sufficient condition on N.

To do this, we can check, step by step, the solution of the boundary problem in $\mathbb{C}P^3$ [HL 97], introducing the parameter λ.

For $\lambda\notin L$, $\gamma_{\nu'_\lambda}=N\cap P_{\nu'_\lambda}\cap E_\lambda$ is of dimension 1. Under the notations of the sub-section 5.2.4, consider the function

$$G(\nu_\lambda)=\frac{1}{2\pi i}\int_{\gamma_{\nu'_\lambda}} z_2\frac{dh}{h} \tag{1}$$

5.2.6. **Tentative statement.** *The following two conditions are equivalent:*

(i) There exists, in $E'=E\setminus k^{-1}L$, a C^∞ Levi-flat subvariety M, (with negligible singularities), of dimension 5, foliated by complex analytic subvarieties M_λ of complex dimension 2, such that M extends simply (or trivially) to E' as a current of dimension 5 (still denoted M) such that $dM=N$ in E'. The leaves are the sections by the subspaces $E_\lambda,\lambda\in k(N)\setminus L$, and are the solutions of the boundary problem for finding complex analytic subvarieties in $E_\lambda\cong\mathbb{C}P^3$ with given boundary $N\cap E_\lambda$.

(ii) N is a submanifold CR, oriented, of CR dimension 1 outside a closed set of 4-dimensional Hausdorff measure 0.

There exists a matrix $\nu^*_{\lambda^*}$ in the neighborhood of which

$$D^2_{\xi_\lambda}G(\nu_\lambda)=D^2_{\xi_\lambda}\sum_{j=1}^N f_j(\nu_\lambda)$$

where f_j, $j=1,\dots,N$, is a holomorphic function in ν_λ, C^∞ en λ, and satisfies the system of P.D.E.

$$f_j\frac{\partial f_j}{\partial\xi_{\lambda l}}=\frac{\partial f_j}{\partial\eta_l},\quad l=2,3 \tag{2}$$

5.2.7. *Remark.* This result is not satisfactory because the relation of the analytic conditions with the geometry of the submanifold N is not explicit.

5.3. *Boundary problem in a real hyperplane of $\mathbb{C}^{n+1}$ or $\mathbb{C}P^{n+1}$.*

$\mathbb{C}^{n+1}$ and $\mathbb{C}P^{n+1}$ are both Kähler. *The solutions of the above boundary problems are* both Levi flat, hence, from a plain extension of section 2.5, volume minimal, i.e. *solution, of codimension 3, of mixed Plateau problems.*

6. Levi-flat hypersurfaces with prescribed boundary: preliminaries.

6.1. *Introduction.*

Let $S\subset\mathbb{C}^n$ be a compact connected 2-codimensional submanifold. Find a Levi-flat hypersurface $M\subset\mathbb{C}^n\setminus S$ such that $dM=S$ (i.e. whose boundary is S, possibly as a current).

For $n=2$, near an elliptic complex point $p\in S$, $S\setminus\{p\}$ is foliated by smooth compact real curves which bound analytic discs (Bishop [Bi 65]). The family of these discs fills a smooth Levi-flat hypersurface.

In 1983, Bedford-Gaveau considered the case of a particular sphere with two elliptic complex points. If S is contained in the boundary of a strictly pseudoconvex bounded domain, then the families of analytic discs in the neighborhood of each elliptic point extend to a global family filling a 3-dimensional ball M bounded by S. In 1991, Bedford-Klingenberg [BeK 91] and Kruzhilin extended the result when there exist hyperbolic complex points on S with the same global condition.

Results of increasing generality have been obtained by Chirka, Shcherbina, Slodowski, G. Tomassini until 1999. The global sufficient condition of embedding of S in the boundary of a strictly pseudoconvex domain is still required in these papers.

A first result for $n \geq 3$ (in the sense of currents), and for elliptic points only, has been obtained four years ago ([DTZ 05] and [DTZ 09] in detailed form); we got new results when S is homeomorphic to a sphere, with three elliptic and one hyperbolic special points (see [D 08] for a first draft), or a torus, with two elliptic and two hyperbolic special points and, more generally, a manifold which is obtained by *gluing together elementary models*.

A local condition is required because, in general, S is not locally the boundary of a Levi-flat hypersurface. The proof uses the construction of a foliation of S by CR orbits, Thurston's stability theorem for foliations on S, and a parametric version of the Harvey-Lawson theorem on boundaries of complex analytic varieties. There is no global condition.

6.2. *Preliminaries and definitions.*

6.2.1. A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called *minimal* at a point p if there does not exist a submanifold N of M of lower dimension through p such that $HN = HM|_N$. By a theorem of Sussman, all possible submanifolds N such that $HN = HM|_N$ contain, at p, one of the minimal possible dimension, called a CR *orbit* of p in M whose germ at p is uniquely determined.

6.2.2. S is said to be a *locally flat boundary* at a point p if it locally bounds a Levi-flat hypersurface near p. Assume that S is CR in a small enough neighborhood U of $p \in S$. If all CR orbits of S are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ 05]:

(i) S is a locally flat boundary on U;

(ii) S is nowhere minimal on U.

6.2.3. *Complex points of* S [DTZ 05].

At such a point $p \in S$, T_pS is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p, S satisfies

$$w = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \leq i,j \leq n-1} (a_{ij} z_i z_j + b_{ij} z_i \overline{z}_j + c_{ij} \overline{z}_i \overline{z}_j) \tag{1}$$

S is said *flat* at a complex point $p \in S$ if $\sum b_{ij} z_i \overline{z}_j \in \lambda \mathbf{R}, \lambda \in \mathbb{C}$. We also say that p is flat.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point p. Then p is flat.

By making the change of coordinates $(z, w) \mapsto (z, \lambda^{-1} w)$, we make $\sum b_{ij} z_i \overline{z}_j \in I\!R$ for all z. By a change of coordinates $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form Q real-valued.

We say that S is in a *flat normal form* at p if the coordinates (z, w) as in (1) are chosen such that $Q(z) \in \mathbf{R}$ for all $z \in \mathbb{C}^{n-1}$.

6.2.4. *Properties of* Q.

Assume that S is in a flat normal form; then, the quadratic form Q is real valued. Only holomorphic linear changes of coordinates are allowed. If Q is positive definite or negative definite, the point $p \in S$ is said to be *elliptic*; if the point $p \in S$ is is not elliptic, and if Q is non degenerate, p is said to be *hyperbolic*. From section 6.4, we will only consider particular cases of the quadratic form Q.

From [Bi 65], for $n = 2$, in suitable holomorphic coordinates, $Q(z) = (z\overline{z} + \lambda Re\ z^2)$, $\lambda \geq 0$, under the notations of [BeK 91]; for $0 \leq \lambda < 1$, p is said to be *elliptic*, and for $1 < \lambda$, it is said to be *hyperbolic*. The parabolic case $\lambda = 1$, not generic, is omitted [BeK 91]. When $n \geq 3$, the Bishop's result is not valid in general.

6.3. *Elliptic points.*

6.3.2. Proposition ([DTZ 05], [DTZ 09]). *Assume that $S \subset \mathbb{C}^n$, $(n \geq 3)$ is nowhere minimal at all its CR points and has an elliptic flat complex point p. Then there exists a neighborhood V of p such that $V \setminus \{p\}$ is foliated by compact real $(2n-3)$-dimensional CR orbits diffeomorphic to the sphere $\mathbf{S}^{2n-3}$ and there exists a smooth function ν, having the CR orbits as the level surfaces.*

Sketch of Proof (see [DTZ 09]).

In the case of a quadric S_0 $(w = Q(z))$, the CR orbits are defined by $w_0 = Q(z)$, where w_0 is constant. Using (1), we approximate the tangent space to S by the tangent space to S_0 at a point with the same coordinate z; the same is done for the tangent spaces to the CR orbits on S and S_0; then we construct the global CR orbit on S through any given point close enough to p.

6.4. *Special flat complex points.* We say that the flat complex point $p \in S$ is *special* if in convenient holomorphic coordinates,

$$Q(z) = \sum_{j=1}^{n-1} (z_j \overline{z}_j + \lambda_j Re\ z_j^2), \quad , \lambda_j \geq 0 \tag{2}$$

Let $z_j = x_j + iy_j,\ x_j, y_j$ real, $j = 1, \ldots, n-1$, then:

$$Q(z) = \sum_{l=1}^{n-1} \left((1+\lambda_l)x_l^2 + (1-\lambda_l)y_l^2\right) + O(|z|^3). \tag{3}$$

A flat point $p \in S$ is said to be *special elliptic* if $0 \leq \lambda_j < 1$ for any j.

A flat point $p \in S$ is said to be *special k-hyperbolic* if $1 < \lambda_j$ for $j \in J \subset \{1, \ldots, n-1\}$ and $0 \leq \lambda_j < 1$ for $j \in \{1, \ldots, n-1\} \setminus J \neq \emptyset$, where k denotes the number of elements of J.

Special elliptic (resp. k-hyperbolic) points are elliptic (resp. hyperbolic).

6.5. *Special hyperbolic points.*

6.5.1. We will not consider *special parabolic points* (one $\lambda_j = 1$ at least) which don't appear generically.

S being given by (1), let S_0 be the quadric of equation $w = Q(z)$. *Suppose that S_0 is flat at* 0 *and that* 0 *is a special k-hyperbolic point. Then, in a neighborhood of* 0, *and with the above local coordinates, it is CR and nowhere minimal outside* 0, *and the CR orbits of S_0 are the $(2n-3)$-dimensional submanifolds given by $w = const. \neq 0$.*

The section $w = 0$ of S_0 is a real quadratic cone Σ_0' in $\mathbf{R}^{2n}$ whose vertex is 0 and, outside 0, it is a CR orbit Σ_0 in the neighborhood of 0.

6.6. *Foliation by* CR*-orbits in the neighborhood of a special* 1*-hyperbolic point.*

We mimic the begining of the proof of 2.4.2. in ([DTZ 05], [DTZ 09]).

6.6.1. *Local* 2*-codimensional submanifolds.*

In $\mathbb{C}^3$, consider the 4-dimensional submanifold S locally defined by the equation

$$w = \varphi(z) = Q(z) + O(|z|^3) \tag{1}$$

and the 4-dimensional submanifold S_0 of equation

$$w = Q(z) \tag{4}$$

with

$$Q = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1+\lambda_2)x_2^2 + (1-\lambda_2)y_2^2$$

having a special 1-hyperbolic point at 0, $(\lambda_1 > 1, 0 \leq \lambda_2 < 1)$, and the cone Σ_0' whose equation is: $Q = 0$. On S_0, a CR orbit is the 3-dimensional submanifold $\mathcal{K}_{w_0}$ whose equation is $w_0 = Q(z)$. If $w_0 > 0$, $\mathcal{K}_{w_0}$ does not cut the line $L = \{x_1 = x_2 = y_2 = 0\}$; if $w_0 < 0$, $\mathcal{K}_{w_0}$ cuts L at two points.

6.6.2. *Remark.* $\Sigma_0 = \Sigma_0' \setminus 0$ *has two connected components in a neighborhood of* 0.

Proof. The equation of $\Sigma_0' \cap \{y_1 = 0\}$ is

$(\lambda_1 + 1)x_1^2 + (1+\lambda_2)x_2^2 + (1-\lambda_2)y_2^2 = 0$ whose only zero , in the neighborhood of 0, is $\{0\}$: the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively. □

6.6.3. *Behaviour of local CR orbits.*

Under the notations of [DTZ 09], follow the construction of the complex tangent space $E(z, \varphi(z))$ to the CR orbit at z; compare with $E_0(z, Q(z))$. We know the integral manifold, the orbit of $E_0(z, Q(z))$; deduce an evaluation of the integral manifold of $E(z, \varphi(z))$.

6.6.4. Lemma. *Under the above hypotheses, if $k = 1$, the local orbit Σ corresponding to Σ_0 has two connected components in the neighborhood of* 0.

Proof. Use Remark 6.6.2 and the adaptation of the technique of [DTZ 09]. □

6.7. CR-*orbits near a subvariety containing a special* 1-*hyperbolic point.*

6.7.2. Proposition. *Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$, is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique spcial* 1-*hyperbolic flat complex point p, and such that:*

(i) the orbit Σ whose closure Σ' contains p is compact;

(ii) Σ has two connected components σ_1, σ_2, whose closures are homeomorphic to spheres of dimension $2n-3$.

Then, there exists a neighborhood V of Σ' such that $V \setminus \Sigma'$ is foliated by compact real $(2n-3)$-dimensional CR orbits whose equation, in a neighborhood of p is (3), *and, the $w(= x_n)$-axis being assumed to be vertical, each orbit being diffeomorphic to*

the sphere $\mathbf{S}^{2n-3}$ above Σ',

the union of two spheres $\mathbf{S}^{2n-3}$ under Σ',

and there exists a smooth function ν, having the CR orbits as the level surfaces.

6.8. *Geometry of the complex points of S.*

6.8.1. Let G be the manifold of the oriented real linear $(2n-2)$-subspaces of $\mathbb{C}^n$. The submanifold S of $\mathbb{C}^n$ has a given orientation which defines an orientation of the tangent space to S at any point $p \in S$. By mapping each point of S into its oriented tangent space, we get a smooth Gauss map

$$t : S \to G$$

6.8..2. *Dimension of G.* $\dim G = 2(2n-2)$.

6.8..3. **Proposition**. *For $n \geq 2$, in general, S has isolated complex points.*

Proof. Let $\pi \in G$ be a complex hyperplane of $\mathbb{C}^n$ whose orientation is induced by its complex structure; the set of such π is $H = G^{\mathbb{C}}_{n-1,n} = \mathbb{C}\mathrm{P}^{n-1*} \subset G$, as real submanifold. If p is a complex point of S, then $t(p) \in H$ or $-t(p) \in H$. The set of complex points of S is the inverse image by t of the intersections $t(S) \cap H$ and $-t(S) \cap H$ in G. Since $\dim t(S) = 2n-2$, $\dim H = 2(n-1)$, $\dim G = 2(2n-2)$, the intersection is 0-dimensional, in general.

6.8.4. *Homology of G.* (cf [P 08]). G has the structure of a complex quadric; let S_1, S_2 be generators of $H_{2n-2}(G, \mathbb{Z})$; we assume that S_1 and S_2 are fundamental cycles of complex projective subspaces of complex dimension $(n-1)$ of G. Then, denoting also S, the fundamental cycle of the submanifold S and t_* the homomorphism defined by t, we have:

$$t_*(S) \sim u_1 S_1 + u_2 S_2$$

where $\sim$ means *homologous to.*

6.8.5. **Lemma** (proved for $n = 2$ in [CS 51]). *With the notations of* 6.8.1′ *, we have:* $u_1 = u_2$*;* $u_1 + u_2 = \chi(S)$*, Euler-Poincaré characteristic of S.*

The proof for $n = 2$ works for any $n \geq 3$.

6.8.6. *Local intersection numbers of H and $t(S)$ when all complex points are flat.*

Proposition (known for $n = 2$ [Bi 65], *here for $n \geq 3$). Let S be a smooth, oriented, compact, 2-codimensional, real submanifold of $\mathbb{C}^n$ whose all complex points are flat and special. Then, on S, ♯ (special elliptic points) + ♯ (special k-hyperbolic points, with k even) - ♯ (special k-hyperbolic points, with k odd) $= \chi(S)$. If S is a sphere, this number is 2.*

7. Levi-flat hypersurfaces with prescribed boundary: particular cases.

7.1. To solve the boundary problem by Levi-fllat hypersurfaces, S has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.

7.2. *Sphere with elliptic points.*

7.2.1. *Example.* In $\mathbb{C}^3$, Let S be defined by the equations:

$$(S) \qquad \begin{cases} z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3 & = & 1 \\ z_3 & = & \overline{z}_3 \end{cases}$$

We have CR-*dim* $S = 1$ except at the points $z_1 = z_2 = 0; z_3 = \pm 1$ where CR-*dim* $S = 2$. S is the unit sphere in $\mathbb{C}^2 \times I\!R$; it bounds the unit ball M in $\mathbb{C}^2 \times I\!R$, which is foliated by the complex balls $\mathbb{C}^2 \times \{x_3\} \cap M$. The leaves are relatively compact of real dimension 4 and are bounded by compact leaves (3-spheres) of a foliation of M.

7.2.2. **Theorem** [DTZ 05]. *Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the conditions*

(i) S is nonminimal at every CR point;

(ii) every complex point of S is flat and elliptic and there exists at least one such point;

(iii) S does not contain complex manifold of dimension $(n-2)$.

Then S is a topological sphere, and there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{S}$ and S outside the complex points of S.

7.3. *Sphere with one special 1-hyperbolic point (sphere with two horns).*

7.3.1. *Example.* In $\mathbb{C}^3$, let (z_j), $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbf{R}^6 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) S defined by: $y_3 = 0$

$0 \leq x_3 \leq 1; \quad x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0$

$-1 \leq x_3 \leq 0; \quad x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2$

The singular set of S is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined.

S being in the real hyperplane $\{y_3 = 0\}$, the complex tangent spaces to S are $\{x_3 = x^0\}$ for convenient x^0.

The set S will be smoothed along the complement of 0 (origin of $\mathbb{C}^3$) in its section by the hyperplane $\{x_3 = 0\}$ by a small deformation leaving h unchanged. *In the following S will denote this smooth submanifold.*

From elementary analytic geometry, complex points of S are defined by their coordinates:

e_3: $x_j = 0, y_j = 0$, $(j = 1, 2)$, $x_3 = 1$.

h: $x_j = 0, y_j = 0$, $(j = 1, 2)$, $x_3 = 0$;

e_1, e_2: $x_1 = 0, y_1 = \pm 1, x_2 = 0, y_2 = 0, x_3 = -1$.

Lemma. *The complex points are flat and special. The points e_1, e_2, e_3 are special elliptic; the point h is special 1-hyperbolic.*

Remark that the numbers of special elliptic and special hyperbolic points satisfy the conclusion of Proposition 6.8.6.

7.3.1'. *Shape of $\Sigma' = S \cap \{x_3 = 0\}$ in the neighborhood of the origin 0 of $\mathbb{C}^3$.*

Lemma. *Under the above hypotheses and notations,*

(i) $\Sigma = \Sigma' \setminus 0$ has two connected components σ_1, σ_2.

(ii) The closures of the three connected components of $S \setminus \Sigma'$ are submanifolds with boundaries and corners.

Proof. (i) The only singular point of Σ' is 0. We work in the ball $B(0, A)$ of $\mathbb{C}^2$ (x_1, y_1, x_2, y_2) for small A and in the 3-space $\pi_\lambda = \{y_2 = \lambda x_2\}$, $\lambda \in I\!R$. For λ fixed, $\pi_\lambda \cong I\!R^3(x_1, y_1, x_2)$, and $\Sigma' \cap \pi_\lambda$ is the cone of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ with vertex 0 and basis in the plane $x_2 = x_2^0$ the hyperboloid H_λ of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^{02} + O(|z|^3) = 0$; the curves H_λ have no common point outside 0. So, when λ varies, the surfaces $\Sigma' \cap \pi_\lambda$ are disjoint outside 0. The set Σ' is clearly connected; $\Sigma' \cap \{y_1 = 0\} = \{0\}$, the origin of $\mathbb{C}^3$; from above: $\sigma_1 = \Sigma \cap \{y_1 > 0\}$; $\sigma_2 = \Sigma \cap \{y_1 < 0\}$.

(ii) The three connected components of $S \setminus \Sigma'$ are the components which contain, respectively e_1, e_2, e_3 and whose boundaries are $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_1 \cup \overline{\sigma}_2$; these boundaries have corners as shown in the first part of the proof. $\square$

The connected component of $\mathbb{C}^2 \times I\!R \setminus S$ containing the point $(0,0,0,0,1/2)$ is the Levi-flat solution, the complex leaves being the sections by the hyperplanes $x_3 = x_3^0$, $-1 < x_3^0 < 1$.

The sections by the hyperplanes $x_3 = x_3^0$ are diffeomorphic to a 3-sphere for $0 < x_3^0 < 1$ and to the union of two disjoint 3-spheres for $-1 < x_3^0 < 0$, as can be shown intersecting S by lines through the origin in the hyperplane $x_3 = x_3^0$; Σ' is homeomorphic to the union of two 3-spheres with a common point.

7.3.2. **Proposition** (cf [D 08], Proposition 2.6.1). *Let $S \subset \mathbb{C}^n$ be a compact connected real 2-codimensional manifold such that the following holds:*

(i) S is a topological sphere; S is nonminimal at every CR point;

(ii) every complex point of S is flat; there exist three special elliptic points $e_j, j = 1,2,3$ and one special 1-hyperbolic point h;

(iii) S does not contain complex manifolds of dimension $(n-2)$;

(iv) the singular CR *orbit Σ' through h on S is compact and $\Sigma' \setminus \{h\}$ has two connected components σ_1 and σ_2 whose closures are homeomorphic to spheres of dimension $2n-3$;*

(v) the closures S_1, S_2, S_3 of the three connected components S_1', S_2', S_3' of $S \setminus \Sigma'$ are submanifolds with (singular) boundary.

Then each $S_j \setminus \{e_j \cup \Sigma'\}$, $j = 1,2,3$ carries a foliation $\mathcal{F}_j$ of class C^∞ with 1-codimensional CR *orbits as compact leaves.*

Proof. From conditions (i) and (ii), S satisfying the hypotheses of Proposition 6.3.2, near any elliptic flat point e_j, and of Proposition 6.7.2 near Σ', all CR orbits are diffeomorphic to the sphere $\mathbf{S}^{2n-3}$. The assumption (iii) guarantees that all CR orbits in S must be of real dimension $2n-3$. Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of Σ', we obtain, from $S \setminus \Sigma'$, three compact manifolds S_j", $j = 1,2,3$, with boundary and with the foliation $\mathcal{F}_j$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbf{R}$ is 0, near e_j. It is easy to show that this foliation is transversely oriented.

7.3.2'. Recall the Thurston's Stability Theorem ([CaC], Theorem 6.2.1). *Let $(M, \mathcal{F})$ be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class C^1. If there is a compact leaf L with $H^1(L, \mathbf{R}) = 0$, then every leaf is homeomorphic to L and M is homeomorphic to $L \times [0,1]$, foliated as a product,*

Then, from the above theorem, S_j" is homeomorphic to $\mathbf{S}^{2n-3} \times [0,1]$ with CR orbits being of the form $\mathbf{S}^{2n-3} \times \{x\}$ for $x \in [0,1]$. Then the full manifold S_j is homeomorphic to a half-sphere supported by $\mathbf{S}^{2n-2}$ and $\mathcal{F}_j$ extends to S_j; S_3 having its boundary pinched at the point h.

7.3.3. **Theorem.** *Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the conditions (i) to (v) of Proposition 7.3.2. Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{S}$ and S outside the complex points of S.*

Proof. By Proposition 6.3.2 , for every e_j, a continuous function ν_j', C^∞ outside e_j, can be constructed in a neighborhood U_j of e_j, $j = 1,2,3$, and by Proposition 6.7.2, we have an analogous result in a neighborhood of Σ'.

Furthermore, from section 7.3.2', a smooth function $\nu"_j$ whose level sets are the leaves of $\mathcal{F}_j$ can be obtained globally on $S_j' \setminus \{e_j \cup \Sigma'\}$. With the functions ν_j' and $\nu"_j$, and analogous functions near Σ', then using a partition of unity, we obtain a global smooth function $\nu_j \colon S_j \to \mathbf{R}$ without critical points away from the complex points e_j and from Σ'.

Let σ_1, resp. σ_2 the two connected, relatively compact components of $\Sigma \setminus \{h\}$, according to condition (iv); $\overline{\sigma}_1$, resp. $\overline{\sigma}_2$ are the boundary of S_1, resp. S_2, and $\overline{\sigma}_1 \cup \overline{\sigma}_2$ the boundary of S_3. We can assume that the three functions ν_j are finite valued and get the same values on $\overline{\sigma}_1$ and $\overline{\sigma}_2$. Hence a function $\nu : S \to \mathbf{R}$.

The submanifold S being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{S} = N = \operatorname{gr} \nu = \{(\nu(z), z) : z \in S\}$. Let $S_s = \{e_1, e_2, e_3, \overline{\sigma_1 \cup \sigma_2}\}$.

$\lambda : S \to \tilde{S}$ $\big(z \mapsto \nu((z), z)\big)$ is bicontinuous; $\lambda|_{S \setminus S_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on S. Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(S_s)$.

At every point of $S \setminus S_s$, $d_{x_1}\nu \neq 0$, then condition (H) (section 5.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 5.1.2 being satisfied by $N = \tilde{S}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbf{R} \times \mathbf{C}^n$.

Taking $\pi : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ to be the standard projection, we obtain the conclusion.

7.4. *Case of a torus.*

7.4.1. *Euler-Poincaré characteristic of a torus is* $\chi(\mathbf{T}^k) = 0$.

7.4.2. *Example.* In C^3, let (z_j), $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbf{R}^6 \cong \mathbf{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) S defined by:

$y_3 = 0$

$0 \leq x_3 \leq 1; \quad x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0$

$-\frac{1}{2} \leq x_3 \leq 0; \quad x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2$

glue it with the symmetric with respect to the real hyperplane $x_3 = -\frac{1}{2}$, and *and smooth along* $\{x_3 = 0\}$, $\{x_3 = \pm\frac{1}{2}\}$. The complex points are flat and special.

7.4.3. **Theorem.** *Let $S \subset \mathbf{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the following conditions:*

(i) S is a topological torus; S is nonminimal at every CR point;

(ii) every complex point of S is flat; there exist two special elliptic points e_1, e_2 and two special 1-hyperbolic points h_1, h_2;

(iii) S does not contain complex manifolds of dimension $(n-2)$;

(iv) the singular CR *orbits Σ_1', Σ_2' through h_1 and h_2 on S are compact and, for $j = 1, 2$, $\Sigma_j' \setminus \{h_j\}$ have two connected components σ_j^1 and σ_j^2;*

(v) the closures S_1, S_2, S_3, S_4 of the four connected components S_1', S_2', S_3', S_4' of $S \setminus \Sigma_1' \cup \Sigma_2'$ are submanifolds with (singular) boundary.

Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbf{C} \times \mathbf{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{S}$ and S outside the complex points.

7.5. *Generalizations.*

7.5.1. *Elementary models and their gluing.* The examples and the proofs of the theorems when S is homeomorphic to a sphere (sections 7.3) or a torus (section 7.4) suggest the following definitions.

7.5.2. *Definitions.*

Let T' be a smooth, locally closed (i.e. closed in an open set), connected submanifold of $\mathbf{C}^n$, $n \geq 3$. We assume that T' has the following properties:

(i) T' is relatively compact, non necessarily compact, and of codimension 2.

(ii) T' is nonminimal at every CR point;

(iii) T' has exactly 2 complex points which are flat and either special elliptic or special 1-hyperbolic.

(iv) If $p \in T'$ is 1-hyperbolic, the singular orbit Σ' through p is compact, $\Sigma' \setminus p$ has two connected components σ_1, σ_2, whose closures are homeomorphic to spheres of dimension $2n-3$.

(v) If $p \in T'$ is 1-hyperbolic, in the neighborhood of p, with convenient coordinates, the equation of T', up to third order terms is

$$z_n = \sum_{j=1}^{n-1} (z_j \overline{z}_j + \lambda_j \mathcal{R}e\ z_j^2); \ \lambda_1 > 1; \ 0 \leq \lambda_j < 1 \ \text{ for } \ j \neq 1$$

or in real coordinates x_j, y_j with $z_j = x_j + iy_j$,

$$x_n = \big((\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2\big) + \sum_{j=2}^{n-1} \big((1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2\big) + O(|z|^3)$$

Other configurations are easily imagined.

up- and down- 1-hyperbolic points. Let T be the $(2n-2)$-submanifold with (singular) boundary contained into T' such that either $\overline{\sigma}_1$ (resp. $\overline{\sigma}_2$) is the boundary of T near p, or Σ' is the boundary of T near p. In

the first case, we say that p is 1-*up*, (resp. 2-*up*), in the second that p is *down*. Such a T will be called an *elementary model*.

For instance, T is 1-up and has one special elliptic point, we solve the boundary problem as in S_1 in the proof of Theorem 7.3.3.

7.5.3. The gluing (to be precised) happens between two compatible elementary models along boundaries, for instance down and 1-up.

7.6. *Other possible generalizations.*

The mixed Plateau problem can be set up in projective space $\mathbb{C}\mathrm{P}^n$ and in subspaces of $\mathbb{C}\mathrm{P}^n$ on which the complex Plateau problem can be solved, using Statement 5.2.6, its gemeralisation to any $n \geq 3$ and a better geometric condition on the given boundary.

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