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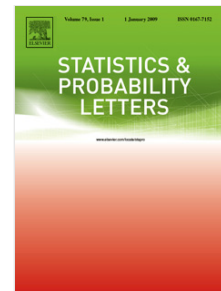
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Centered and non-centered principal component analysis in the frequency domain

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Abstract

Given a stationary multidimensional process and the process which is deduced after centering, we wish to study possible links between the principal component analyses, in the frequency domain, of these two processes. It is well-known that there is, a priori, no obvious relationship between the centered and non-centered principal component analyses in the temporal domain. Furthermore, we also know that principal component analysis in the frequency domain is reduced to principal component analysis of each spectral component. In this paper, we show the remarkable result that the centered and non-centered principal component analyses in the frequency domain are equal except for a given frequency.

Key words: principal component analysis, stationary random function, stationary process, frequency domain, temporal domain, random measure

1. Introduction

Principal Component Analysis (PCA) of order q , in the frequency domain, of a p -dimensional stationary random function ($q < p$) can be seen as a search for a q -dimensional stationary random function which best summarizes the

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p -dimensional function. This method, which combines harmonic analysis and classical PCA, was introduced by Brillinger (1975, 2001) in a study of PCA of a p -vector-valued stationary time series. Under absolute summability of the autocovariance function, Brillinger obtained a q -vector-valued stationary summary in the form of a moving average. Some applications of Brillinger's PCA method can be found in Shumway and Stoffer (2000), and Boudou and Dauxois (1994) have extended the method to a random function defined on a locally compact Abelian group by using spectral representation.

Nowadays, PCA in the frequency domain has found diverse important applications. It has been widely adopted as the most promising face recognition algorithm in pattern detection and image processing (Feng *et al.*, 2000; Mitra *et al.*, 2005; El-Bakry, 2007; Rama *et al.*, 2008), signal processing (e.g. Jung *et al.*, 2000; Winter *et al.*, 2006), biomedical research (Lagerlund *et al.*, 2004, 2005; Bernat *et al.*, 2005), the multidimensional acoustic or electrical data processing problem and other areas (see Brillinger, 1975, 2001).

In this paper, we study centered and non-centered PCA in the frequency domain of a stationary random function (here we only use weak stationarity). In the temporal domain, we know that, generally, there is no equivalence between centered and non-centered PCA (as it is well-known that there is no obvious relationship between the spectral analyses of a Gram matrix and its associated covariance matrix). So, in many articles on the subject, the data are assumed to be centered. In this paper, given a stationary random function $(X_g)_{g \in G}$ and its centered version $(X_g^c)_{g \in G} = (X_g - \mathbb{E}(X_g))_{g \in G}$, we show that the PCA's of $(X_g)_{g \in G}$ and $(X_g^c)_{g \in G}$ in the frequency domain can be obtained, respectively, by the PCA of another centered random function $(X'_g)_{g \in G}$ deduced from the decomposition of $(X_g)_{g \in G}$ and $(X_g^c)_{g \in G}$ by a filter of *pass-band* type.

In section 2, we introduce our notation and recall some mathematical tools useful for solving our problem. In section 3, we give a brief survey of PCA in the frequency domain. In section 4, we present the first main result of this paper, rewriting the function $(\mathbb{E}(X_g))_{g \in G}$ as $((\gamma_{\mathbb{E}}, g)_{\widehat{G}} \mathbb{E}(X_0))_{g \in G}$ (where \widehat{G} is the dual space of G , $(\cdot, \cdot)_{\widehat{G}}$ denotes the duality bracket and $\gamma_{\mathbb{E}}$ is a character). That result leads to some relationships between the associated spectral tools of $(X_g)_{g \in G}$ and $(X_g^c)_{g \in G}$ such as the spectral and random measures. In section 5, we give the second main result : with $(Y'_g)_{g \in G}$ the q -dimensional summary resulting from the PCA of $(X'_g)_{g \in G}$, we obtain the q -dimensional summary $(Y_g)_{g \in G}$ of the PCA of the non-centered random function $(X_g)_{g \in G}$ by the relation $Y_g = Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}} W$ (where W is a q -dimensional vector which is given in Proposition 5.3). In an analogous way, we obtain the q -dimensional summary $(Y_g^c)_{g \in G}$ of the centered random function $(X_g^c)_{g \in G}$ by the relation $Y_g^c = Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}} W_c$ (where W_c is also given in Proposition 5.3). This result legitimizes the aforementioned decomposition, and some particular cases are presented in the last section.

2. Notations and preliminaries

In this section, we introduce the main mathematical tools used throughout this paper. We emphasize the representation of a vector-valued stationary continuous random function (CRF) as a stochastic integral (see Boudou and Dauxois, 1994); tools mentioned here are developed further by Rosenberg (1974), Boudou and Dauxois (1994) and Boudou and Romain (2002).

Let denote $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space such that the complex Hilbert space $L_p^2 = L_{\mathbb{C}^p}^2(\Omega, \mathcal{A}, \mathbb{P})$ is separable. When X is an element of L_p^2 , the mapping $y \in L_{\mathbb{C}}^2(\mathbb{P}) \mapsto \mathbb{E}(yX) \in \mathbb{C}^p$, denoted by \widetilde{X} , is a Hilbert-Schmidt operator, and its adjoint \widetilde{X}^* is the mapping $u \in \mathbb{C}^p \mapsto \langle u, X \rangle_{\mathbb{C}^p} \in L_{\mathbb{C}}^2(\mathbb{P})$.

By G we denote a locally compact Abelian group whose dual \widehat{G} admits a countable basis, and by \mathcal{B} the Borel σ -field of \widehat{G} .

A p -dimensional *random measure* (p -RM) Z is a vector measure defined on \mathcal{B} with values in L_p^2 such that $\widetilde{Z(A)} \circ \widetilde{Z(B)}^* = 0$ for each pair (A, B) of disjoint elements of \mathcal{B} .

Let $\sigma_2(p, q)$ (resp. $\sigma_2(p)$) be the complex Hilbert space of all Hilbert-Schmidt operators mapping \mathbb{C}^p into \mathbb{C}^q (resp. \mathbb{C}^p) with the inner product defined by

$$\langle \cdot, \cdot \rangle_2 : (L, K) \mapsto \text{tr} L \circ K^*, \quad (1)$$

and M_Z the vector-valued measure in $\sigma_2(p)$ defined by $M_Z(A) = \widetilde{Z(A)} \circ \widetilde{Z(A)}^*$ for any $A \in \mathcal{B}$.

For any σ -finite measure ν defined on $(\widehat{G}, \mathcal{B})$ which dominates M_Z , the Radon-Nikodym derivative $\frac{dM_Z}{d\nu}$ of M_Z with respect to ν admits (e.g. Rosenberg, 1974, p.174) a measurable Schmidt decomposition:

$$\frac{dM_Z}{d\nu}(\cdot) = \sum_{j=1}^{j=p} \mu_j(\cdot) a_j(\cdot) \otimes a_j(\cdot). \quad (2)$$

More precisely, there exists two families of measurable applications, $(\mu_j(\gamma))_{j=1, \dots, p}$ and $(a_j(\gamma))_{j=1, \dots, p}$, defined on \widehat{G} with values in \mathbb{R}^+ and \mathbb{C}^p , respectively. Furthermore, for each γ of \widehat{G} , $\sum_{j=1}^{j=p} \mu_j(\gamma) a_j(\gamma) \otimes a_j(\gamma)$ is a Schmidt-decomposition of $\frac{dM_Z}{d\nu}(\gamma)$ which belongs to $\sigma_2(p)$.

By $L_{pq}^2(M_Z)$ we denote the space of functions whose values are operators belonging to $\sigma_2(p, q)$ (see Boudou and Dauxois, 1994, for more details). This space can be regarded as a quotient space of a linear subspace of $[\sigma_2(p, q)]^{\widehat{G}}$, by a relation of equivalence defined from M_Z . Each element φ , i.e. a coset in $L_{pq}^2(M_Z)$, contains a \mathcal{B} -measurable function which is also denoted by φ . The

space $L_{pq}^2(M_Z)$ associated with the inner product

$$\langle \varphi, \psi \rangle = \int \text{tr}[\varphi(\lambda) \circ \left(\frac{dM_Z}{d\nu}(\lambda)\right) \circ \psi^*(\lambda)] d\nu(\lambda) \quad (3)$$

is a Hilbert space.

Let $H_Z(p, q)$ be the closure of $\text{span}\{K \circ Z(A); A \in \mathcal{B}, K \in \sigma_2(p, q)\}$.

Since $\text{span}\{\mathbf{1}_A K; A \in \mathcal{B}, K \in \sigma_2(p, q)\}$ is dense in $L_{pq}^2(M_Z)$, the *stochastic integral* with respect to Z is the unique isometry from $L_{pq}^2(M_Z)$ onto $H_Z(p, q)$ which associates $K \circ Z(A) = \int \mathbf{1}_A K dZ$ with $\mathbf{1}_A K$.

Given a p -RM Z defined on \mathcal{B} , it can be shown (see Azencott and Dacunha-Castelle, 1984; Boudou and Dauxois, 1994) that $(X_g)_{g \in G}$, where

$$X_g = \int (\gamma, g)_{\widehat{G}G} I_{\mathbb{C}^p} dZ(\gamma),$$

is a p -stationary CRF; that is, the mapping $g \in G \mapsto X_g \in L_p^2$ is continuous and, for all $(g, g') \in G \times G$,

$$\langle X_g, X_{g'} \rangle = \langle X_{g-g'}, X_0 \rangle.$$

Conversely, given a p -stationary CRF $(X_g)_{g \in G}$, there exists a unique p -RM Z , called the *associated p -RM*, such that $X_g = \int (\cdot, g)_{\widehat{G}G} I_{\mathbb{C}^p} dZ$ for any g in G .

If Z is the p -RM associated with the p -stationary CRF $(X_g)_{g \in G}$, one has

$$H_Z(p, q) = \overline{\text{span}}\{K \circ X_g; g \in G, K \in \sigma_2(p, q)\}.$$

Two stationary CRF's $(X_g)_{g \in G}$ and $(Y_g)_{g \in G}$, respectively p and q dimensional, are said to be *stationarily correlated* if, for any $(g, g') \in G \times G$, one has

$$\widetilde{X}_g \circ \widetilde{Y}_{g'}^* = \widetilde{X}_{g-g'} \circ \widetilde{Y}_0^*.$$

It can be shown (see Boudou and Dauxois, 1994) that this condition holds if and only if the associated p -RM Z_X and q -RM Z_Y , respectively, satisfy

$$\widetilde{Z_X(A)} \circ \widetilde{Z_Y(B)}^* = 0,$$

for all pairs (A, B) of disjoint elements of \mathcal{B} . Then, we say that Z_X and Z_Y are stationarily correlated.

Let $(X_g)_{g \in G}$ be a p -stationary CRF, Z its associated p -RM. The image of $(X_g)_{g \in G}$ by any filter (whose transfer function is) φ belonging to $L_{pq}^2(M_Z)$, is the q -stationary CRF

$$(Y_g)_{g \in G} = \left(\int_{g \in G} (\gamma, g)_{\widehat{G}} \varphi(\gamma) dZ(\gamma) \right)_{g \in G}$$

with associated q -RM $Z_\varphi : A \in \mathcal{B} \mapsto \int \mathbf{1}_A \varphi dZ \in L_q^2$ which is stationarily correlated with Z .

If ψ belongs to $L_{ql}^2(M_{Z_\varphi})$, then $\psi(\cdot)\varphi(\cdot)$ belongs to $L_{pl}^2(M_Z)$ and

$$\int \psi dZ_\varphi = \int \psi(\cdot)\varphi(\cdot) dZ.$$

3. Principal Component Analysis

Given a p -stationary CRF $(X_g)_{g \in G}$, we want to summarize it by a q -stationary CRF $(X'_g)_{g \in G}$ ($q < p$), with associated q -RM Z' , which is stationarily correlated with $(X_g)_{g \in G}$. The stationarity properties permit us to affirm that any p -dimensional filter $(W_g)_{g \in G}$ of $(X'_g)_{g \in G}$ is such that

$$\|X_g - W_g\|_{L_p^2} = \|X_0 - W_0\|_{L_p^2}$$

for all g in G . So we will measure the quality of the q -dimensional summary $(X'_g)_{g \in G}$ of $(X_g)_{g \in G}$ by the quantity

$$\inf \left\{ \|X_0 - \int \varphi dZ'\|_{L_p^2} ; \varphi \in L_{qp}^2(M_{Z'}) \right\}.$$

Of course, between all the possible summaries, we will choose the best one from this point of view. So this leads to the following

Definition 3.1. *The PCA of order q of the p -stationary CRF $(X_g)_{g \in G}$, with associated p -RM Z , is a q -stationary CRF $(X'_g)_{g \in G}$, with associated q -RM Z' , stationarily correlated with $(X_g)_{g \in G}$, and an element φ of $L^2_{qp}(M_{Z'})$ such that $\|X_0 - \int \varphi dZ'\|_{L^2_p}$ is minimum.*

If we see that $X_0 = \int I_{\mathbb{C}^p} dZ$, we can remark that this analysis is equivalent to the PCA of order q of Z (see Boudou and Dauxois, 1989).

Let $\sum_{j=1}^{j=p} \mu_j(\cdot) a_j(\cdot) \otimes a_j(\cdot)$ be a measurable Schmidt decomposition of M'_Z , the derivative of M_Z with respect to ν , a σ -finite measure which dominates M_Z , and let $\{f_1, \dots, f_q\}$ be the canonical basis of \mathbb{C}^q . Following Boudou and Dauxois (1994), we can give the

Proposition 3.1. *A solution of the PCA of order q of the p -RM Z is given by $Z' = Z_\alpha$, where $\alpha(\cdot) = \sum_{j=1}^{j=q} a_j(\cdot) \otimes f_j$, $\varphi(\cdot) = \alpha^*(\cdot) = \sum_{j=1}^{j=q} f_j \otimes a_j(\cdot)$.*

4. Spectral tools associated to centered and non-centered stationary CRF

From now on, we consider a p -stationary CRF $(X_g)_{g \in G}$ defined in G with values on L^2_p . In order to study the PCA's of $(X_g)_{g \in G}$ and its centered version $(X_g^c)_{g \in G}$, we present in this section the relationships between their associated p -RM, on one hand, and the derivatives of their respective spectral elements on other hand.

We assume that $\mathbb{E}(X_0) \neq 0_{\mathbb{C}^p}$, otherwise $X_g = X_g^c$ for all g in G .

For this, let us consider at first the orthogonal projectors

$$D : x \in L^2(\mathbb{P}) \mapsto \mathbf{1}_\Omega \mathbb{E}(x) \in L^2(\mathbb{P}) \quad (4)$$

$$P_p : X \in L^2_p \mapsto \mathbf{1}_\Omega \mathbb{E}(X) \in L^2_p, \quad (5)$$

with $D^\perp = I_{L^2(\mathbb{P})} - D$ and $P_p^\perp = I_{L_p^2} - P_p$, and define $(X_g^c)_{g \in G}$ by $(P_p^\perp X_g)_{g \in G}$.

The following property is easily proved if we point out that P_p (resp. P_p^\perp) is such that $\widetilde{P_p X} = \widetilde{X} \circ D$ (resp. $\widetilde{P_p^\perp X} = \widetilde{X} \circ D^\perp$), for any $X \in L_p^2$.

Lemma 4.1. *If $(X_g)_{g \in G}$ is a p -stationary CRF defined on G with values on L_p^2 , then for all $(g, g') \in G \times G$*

$$(i) \quad \widetilde{X}_g \circ \widetilde{X}_{g'}^{c*} = \widetilde{X}_g^c \circ \widetilde{X}_{g'}^{*};$$

$$(ii) \quad \widetilde{X}_g \circ \widetilde{P_p X_{g'}}^{*} = \widetilde{P_p X_g} \circ \widetilde{P_p X_{g'}}^{*};$$

$$(iii) \quad \widetilde{X}_g^c \circ \widetilde{P_p X_{g'}}^{*} = 0.$$

Let Q be a bounded endomorphism of L_p^2 . Then the mapping

$$g \in G \mapsto Q(X_g) \in L_p^2$$

is continuous and we can state the following

Proposition 4.1. *The CRF $(X_g^c)_{g \in G}$ is p -stationary if and only if $(P_p X_g)_{g \in G}$ is a p -stationary CRF. In this case, the pair $((X_g)_{g \in G}, (X_g^c)_{g \in G})$ (resp. $((X_g)_{g \in G}, (P_p X_g)_{g \in G}); ((X_g^c)_{g \in G}, (P_p X_g)_{g \in G})$) is stationarily correlated.*

Now, for preserving the continuity and p -stationarity of the centered random function $(X_g^c)_{g \in G}$, we assume that $(P_p X_g)_{g \in G}$ is a p -stationary CRF. Note that the case of strict stationary series verifies such assumption. One can affirm the

Proposition 4.2. *If $(\mathbb{E}(X_g))_{g \in G}$ is a p -stationary CRF with values in \mathbb{C}^p then there exists a character $\gamma_{\mathbb{E}}$ such that for any g in G ,*

$$\mathbb{E}(X_g) = (\gamma_{\mathbb{E}}, g)_{\widehat{G}} \mathbb{E}(X_0). \quad (6)$$

Proof. Let us consider the mapping

$$t : g \in G \mapsto \|\mathbb{E}(X_0)\|^{-2} \langle \mathbb{E}(X_g), \mathbb{E}(X_0) \rangle \in \mathbb{C}.$$

We verify that $t(g)\mathbb{E}(X_0) = \mathbb{E}(X_g)$, for each element g of G , and then $|t(g)| = 1$. Obviously t is continuous and for each pair (g, g') of elements of G , we have $t(g - g') = \overline{t(g')}t(g)$. So $t(0) = 1$ and $t(g + g') = t(g)t(g')$. So $t \in \widehat{G}$ and there exists an element $\gamma_{\mathbb{E}}$ of \widehat{G} such that, for all g in G , $\mathbb{E}(X_g) = (\gamma_{\mathbb{E}}, g)_{\widehat{G}} \mathbb{E}(X_0)$. \square

Then, it is easy to verify the following

Corollary 4.1. *The p -RM associated to the p -stationary CRF $(P_p X_g)_{g \in G}$ is $\delta_{\gamma_{\mathbb{E}}}(\cdot) \mathbf{I}_{\Omega} \mathbb{E}(X_0)$ and is denoted by $Z_{\mathbb{E}}$.*

By the one-to-one correspondence (e.g. Boudou and Dauxois, 1994) between a p -stationary process and its p -RM, we propose now to find some relationships between the associated p -RM's of $(X_g)_{g \in G}$, $(X_g^c)_{g \in G}$ and $(P_p X_g)_{g \in G} = (\mathbf{I}_{\Omega} \mathbb{E}(X_g))_{g \in G}$.

For all (K, g) of $\sigma_2(p) \times G$, one has

$$P_p(\widetilde{K \circ X_g^c}) = \widetilde{K \circ X_g^c} \circ D = K \circ \widetilde{X_g^c} \circ D = K \circ \widetilde{P_p X_g^c} = 0.$$

If we denote by Z_c the p -RM associated to the p -stationary CRF $(X_g^c)_{g \in G} = (P_p^{\perp} X_g)_{g \in G}$, we know that $Z_c(A) \in \overline{\text{span}}\{K \circ X_g^c; g \in G, K \in \sigma_2(p)\}$ for all A in \mathcal{B} and so we can establish

Proposition 4.3. *With the previous notation we have $P_p \circ Z_c = 0$.*

And consequently

Proposition 4.4. *For all pairs (A, B) of elements of \mathcal{B} ,*

$$\widetilde{Z_c(A)} \circ \widetilde{Z_{\mathbb{E}}(B)}^* = 0.$$

Proof. Indeed

$$\begin{aligned}\widetilde{Z_c(A)} \circ \widetilde{Z_{\mathbb{E}}(B)}^* &= \widetilde{Z_c(A)} \circ \delta_{\gamma_{\mathbb{E}}(B)} \widetilde{P_p X_0}^* = \delta_{\gamma_{\mathbb{E}}(B)} \widetilde{Z_c(A)} \circ D \circ \widetilde{X_0}^* \\ &= \delta_{\gamma_{\mathbb{E}}(B)} \widetilde{P_p Z_c(A)} \circ \widetilde{X_0}^* = 0.\end{aligned}$$

□

So, it is easy to verify that

Proposition 4.5. *One has*

(i) *The mapping $Z : A \in \mathcal{B} \mapsto Z_c(A) + Z_{\mathbb{E}}(A) \in L_p^2$ is a p -RM.*

And for all A in \mathcal{B} , one has

(ii) $\mu_Z(A) = \mu_{Z_c}(A) + \mu_{Z_{\mathbb{E}}}(A) = \mu_{Z_c}(A) + \delta_{\gamma_{\mathbb{E}}(A)} \|\mathbb{E}(X_0)\|^2$

(iii) $M_Z(A) = M_{Z_c}(A) + M_{Z_{\mathbb{E}}}(A) = M_{Z_c}(A) + \delta_{\gamma_{\mathbb{E}}(A)} \mathbb{E}(X_0) \otimes \mathbb{E}(X_0)$.

Since $(\cdot, g)_{\widehat{G}}$ belongs to $L^2(\mu_Z) = \overline{\text{span}}\{\mathbf{1}_A; A \in \mathcal{B}\}$, it can be put in the form $(\cdot, g)_{\widehat{G}} = \lim_n \sum_{j \in J_n} \alpha_{jn} \mathbf{1}_{A_{jn}}$ in $L^2(\mu_Z)$ where $A_{jn} \in \mathcal{B}$ and $|J_n| < +\infty$. This equality is also true in $L^2(\mu_{Z_c})$ and $L^2(\mu_{Z_{\mathbb{E}}})$ (because $\|\cdot\|_{L^2(\mu_Z)}^2 = \|\cdot\|_{L^2(\mu_{Z_c})}^2 + \|\cdot\|_{L^2(\mu_{Z_{\mathbb{E}}})}^2$). We can now write

$$(\cdot, g)_{\widehat{G}} I_{\mathbb{C}^p} = \lim_n \sum_{j \in J_n} \alpha_{jn} \mathbf{1}_{A_{jn}} I_{\mathbb{C}^p} \quad (7)$$

in $L_{pp}^2(M_Z)$, $L_{pp}^2(M_{Z_c})$ and $L_{pp}^2(M_{Z_{\mathbb{E}}})$. By integrating equality (7) with respect to Z , Z_c and $Z_{\mathbb{E}}$ successively, we obtain

$$\begin{aligned}\int (\cdot, g)_{\widehat{G}} I_{\mathbb{C}^p} dZ &= \lim_n \sum_{j \in J_n} \alpha_{jn} Z(A_{jn}) \\ X_g^c &= \lim_n \sum_{j \in J_n} \alpha_{jn} Z_c(A_{jn}) \\ P_p X_g &= \lim_n \sum_{j \in J_n} \alpha_{jn} Z_{\mathbb{E}}(A_{jn}).\end{aligned}$$

So, since $Z(A_{jn}) = Z_c(A_{jn}) + Z_{\mathbb{E}}(A_{jn})$, we have

$$\int (\cdot, g)_{\widehat{G}} I_{\mathbb{C}^p} dZ = X_g^c + P_p X_g = X_g$$

and this leads to the following

Proposition 4.6. *Z is the p -RM associated to the p -stationary CRF $(X_g)_{g \in G}$.*

From this point of view, one can easily establish the

Proposition 4.7. *With the previous notations*

- (i) $P_p \circ Z = Z_{\mathbb{E}}$;
- (ii) $P_p^\perp \circ Z = Z_c$;
- (iii) for all A in \mathcal{B} , $\mathbb{E}(Z(A)) = \delta_{\gamma_{\mathbb{E}}}(A) \mathbb{E}(X_0)$.

Now, let us show that the images of the centered and non-centered processes by the pass-band filter $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}$ are equal. For all A in \mathcal{B} , we have

$$\begin{aligned} Z_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}}(A) &= Z(A \cap (\widehat{G} - \{\gamma_{\mathbb{E}}\})) \\ &= Z_c(A \cap (\widehat{G} - \{\gamma_{\mathbb{E}}\})) + Z_{\mathbb{E}}(A \cap (\widehat{G} - \{\gamma_{\mathbb{E}}\})) \\ &= Z_c(A \cap (\widehat{G} - \{\gamma_{\mathbb{E}}\})) = Z_{c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}}(A). \end{aligned}$$

So, we have

Proposition 4.8. $Z_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}} = Z_{c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}}$.

The image of $(X_g)_{g \in G}$ by the filter $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}$ and the image of $(X_g^c)_{g \in G}$ by $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}$ which belongs to $L_{pp}^2(M_{Z_c})$, admit, respectively, $Z_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}}$ and $Z_{c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p}}$, as associated p -RM's which are equal.

If we denote

$$\int (\cdot, g)_{\widehat{G}} \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p} dZ = \int (\cdot, g)_{\widehat{G}} \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^p} dZ_c$$

by X'_g , for all g in G , and taking into account that $\{\widehat{G} - \{\gamma_{\mathcal{E}}\}, \{\gamma_{\mathcal{E}}\}\}$ constitutes a partition of \widehat{G} , we can affirm that

Proposition 4.9. *For all g in G ,*

$$(i) \quad X_g = X'_g + (\gamma_{\mathcal{E}}, g)_{\widehat{G}G} Z(\{\gamma_{\mathcal{E}}\});$$

$$(ii) \quad X_g^c = X'_g + (\gamma_{\mathcal{E}}, g)_{\widehat{G}G} Z_c(\{\gamma_{\mathcal{E}}\});$$

$$(iii) \quad \mathbb{E}(X'_g) = 0.$$

The equality (iii) is due to the fact that $X'_g \in H_{Z_c}(p)$ and $P_p \circ Z_c = 0$.

5. Comparison of the centered PCA and its non-centered version

According to section 3, let us recall that the PCA of a p -stationary CRF with associated p -RM \mathfrak{Z} consists on the diagonalization of the derivative of $M_{\mathfrak{Z}}$ with respect to any σ -finite measure ν defined on \mathcal{B} which dominates $M_{\mathfrak{Z}}$. For this, we propose the following

Proposition 5.1. .

$$(i) \quad \mu_Z \text{ dominates } M_{Z_c};$$

(ii) *If M'_c is the derivative of M_{Z_c} with respect to μ_Z , then the mapping M' :*

$$\gamma \in \widehat{G} \mapsto M'_c(\gamma) + \mathbf{I}_{\{\gamma_{\mathcal{E}}\}}(\gamma) [\mu_Z(\{\gamma_{\mathcal{E}}\})]^{-1} \mathbb{E}(X_0) \otimes \mathbb{E}(X_0) \in \sigma_2(p) \quad (8)$$

is the derivative of M_Z with respect to μ_Z .

Proof. The relation (i) is true since $\mu_Z = \mu_{Z_c} + \mu_{Z_{\mathcal{E}}}$ and then $\mu_{Z_c} \ll \mu_Z$. Let M'_c be the derivative of M_{Z_c} with respect to μ_Z , it is easy to verify that the mapping M' is \mathcal{B} -measurable and μ_Z -integrable. Let $A \in \mathcal{B}$ such that

$\gamma_{\mathcal{E}} \notin A$, we have $\int \mathbf{1}_A(\cdot)M'(\cdot)d\mu_Z(\cdot) = M_{Z_c}(A) = M_Z(A)$. Let $A \in \mathcal{B}$ such that $\gamma_{\mathcal{E}} \in A$, as $A = (A - \{\gamma_{\mathcal{E}}\}) \cup \{\gamma_{\mathcal{E}}\}$ and $(A - \{\gamma_{\mathcal{E}}\}) \cap \{\gamma_{\mathcal{E}}\} = \emptyset$ we have

$$\begin{aligned} \int \mathbf{1}_A(\cdot)M'(\cdot)d\mu_Z(\cdot) &= \int \mathbf{1}_{A - \{\gamma_{\mathcal{E}}\}}(\cdot)M'(\cdot)d\mu_Z(\cdot) + \int \mathbf{1}_{\{\gamma_{\mathcal{E}}\}}(\cdot)M'(\cdot)d\mu_Z(\cdot) \\ &= M_Z(A - \{\gamma_{\mathcal{E}}\}) + M_Z(\{\gamma_{\mathcal{E}}\}) \\ &= M_Z(A). \end{aligned}$$

□

Now, suppose that $\sum_{j=1}^{j=p} \lambda_j(\cdot)a_j(\cdot) \otimes a_j(\cdot)$ is a measurable Schmidt decomposition of M'_c and $\sum_{j=1}^{j=p} \mu_j b_j \otimes b_j$ a Schmidt decomposition of $M'(\gamma_{\mathcal{E}})$. If we denote by $\alpha_j = \mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} \lambda_j + \mathbf{1}_{\{\gamma_{\mathcal{E}}\}} \mu_j$ and $u_j = \mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} a_j + \mathbf{1}_{\{\gamma_{\mathcal{E}}\}} b_j$, for any $j \in \{1, \dots, p\}$, it is clear that $\sum_{j=1}^{j=p} \alpha_j(\cdot)u_j(\cdot) \otimes u_j(\cdot)$ is a measurable Schmidt decomposition of M' .

Let $\varphi(\cdot) = \sum_{j=1}^{j=q} u_j(\cdot) \otimes f_j$ and $\varphi_c(\cdot) = \sum_{j=1}^{j=q} a_j(\cdot) \otimes f_j$. So a solution of the order q PCA of $(X_g)_{g \in G}$ (resp. $(X_g^c)_{g \in G}$) is the q -stationary CRF $(Y_g)_{g \in G}$ (resp. $(Y_g^c)_{g \in G}$) defined by $Y_g = \int (\cdot, g)_{\widehat{G}} \varphi dZ$ (resp. $Y_g^c = \int (\cdot, g)_{\widehat{G}} \varphi_c dZ_c$) and which associated q -r.m. is Z_φ (resp. $Z_{c\varphi_c}$). The link between Z_φ and $Z_{c\varphi_c}$ is given now.

Proposition 5.2. $(Z_\varphi)_{\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}} = (Z_{c\varphi_c})_{\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}}$.

Proof. When $\gamma \in \widehat{G} - \{\gamma_{\mathcal{E}}\}$, $\mathbf{1}_{\{\gamma_{\mathcal{E}}\}}(\gamma) = 0$; then we have $u_j(\gamma) = a_j(\gamma)$ and $\varphi(\gamma) = \sum_{j=1}^{j=q} a_j(\gamma) \otimes f_j = \varphi_c(\gamma)$. So, for all $A \in \mathcal{B}$ we can write

$$\begin{aligned} (Z_\varphi)_{\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}}(A) &= \int \mathbf{1}_A \mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} \varphi dZ = \int \mathbf{1}_A \varphi dZ_{\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}} \\ &= \int \mathbf{1}_A \varphi dZ_{c\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}} = \int \mathbf{1}_A \mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} \varphi dZ_c \\ &= \int \mathbf{1}_A \mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} \varphi_c dZ_c = (Z_{c\varphi_c})_{\mathbf{1}_{\widehat{G} - \{\gamma_{\mathcal{E}}\}} I_{C^q}}(A), \end{aligned}$$

which completes the proof. □

Taking into account what precedes, we can deduce that the q -stationary CRF image of $(Y_g)_{g \in G}$ by $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^q}$ is equal to the image of $(Y_g^c)_{g \in G}$ by $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^q}$ (since they admit the same associated q -RM $(Z_\varphi)_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^q}} = (Z_{c\varphi_c})_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}} I_{\mathbb{C}^q}}$). If we denote by $(Y'_g)_{g \in G}$ this q -stationary CRF, the following result stands

Proposition 5.3. *For any g in G , one has*

- (i) $Y_g = Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} W$, where $W = \varphi(\gamma_{\mathbb{E}}) \circ Z(\{\gamma_{\mathbb{E}}\})$;
- (ii) $Y_g^c = Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} W_c$, where $W_c = \varphi_c(\gamma_{\mathbb{E}}) \circ Z_c(\{\gamma_{\mathbb{E}}\})$;
- (iii) $\mathbb{E}(Y'_g) = 0$.

Proof. Let g in G , by using $\widehat{G} = (\widehat{G} - \{\gamma_{\mathbb{E}}\}) \cup \{\gamma_{\mathbb{E}}\}$ we have

$$\begin{aligned} Y_g &= \int (\cdot, g)_{\widehat{G}G} \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}(\cdot) I_{\mathbb{C}^q} dZ_\varphi + \int (\cdot, g)_{\widehat{G}G} \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot) I_{\mathbb{C}^q} dZ_\varphi \\ &= Y'_g + \int (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot) \varphi(\gamma_{\mathbb{E}}) dZ \\ &= Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} \varphi(\gamma_{\mathbb{E}}) \circ Z(\{\gamma_{\mathbb{E}}\}) \end{aligned}$$

and

$$\begin{aligned} Y_g^c &= \int (\cdot, g)_{\widehat{G}G} \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}(\cdot) I_{\mathbb{C}^q} dZ_{c\varphi_c} + \int (\cdot, g)_{\widehat{G}G} \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot) I_{\mathbb{C}^q} dZ_{c\varphi_c} \\ &= Y'_g + \int (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot) \varphi_c(\gamma_{\mathbb{E}}) dZ_c \\ &= Y'_g + (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} \varphi_c(\gamma_{\mathbb{E}}) \circ Z_c(\{\gamma_{\mathbb{E}}\}). \end{aligned}$$

The proof of (iii) is similar to the proof of (iii) in Proposition 4.9. \square

Remark : For any g in G , since $\mathbb{E}(Y'_g) = 0$, it is easy to verify that $\mathbb{E}(Y_g) = (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} [\varphi(\gamma_{\mathbb{E}}) \mathbb{E}(X_0)]$ and $(\mathbf{I}_\Omega \mathbb{E}(Y_g))_{g \in G}$ is a q -stationary CRF with associated q -RM $\delta_{\gamma_{\mathbb{E}}}(\cdot) \mathbf{I}_\Omega(\cdot) [\varphi(\gamma_{\mathbb{E}}) \mathbb{E}(X_0)]$. And moreover,

$$Y_g = Y_g^c + \mathbf{I}_\Omega \mathbb{E}(Y_g) + (\gamma_{\mathbb{E}}, g)_{\widehat{G}G} [\varphi(\gamma_{\mathbb{E}}) - \varphi_c(\gamma_{\mathbb{E}})] \circ Z_c(\{\gamma_{\mathbb{E}}\}).$$

In addition to Proposition 5.3 we can establish the following statement

Proposition 5.4. *The CRF $(Y'_g)_{g \in G}$ is the PCA of order q of $(X'_g)_{g \in G}$.*

Proof. By recalling that the associated p -RM of $(X'_g)_{g \in G}$ is $Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}}$, the derivative of $M_{Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}}}$ with respect to μ_Z is $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}(\cdot)M'_c(\cdot)$.

Let $\lambda'_j = \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}(\cdot)\lambda_j$, for any $j \in \{1, \dots, p\}$. Then $\sum_{j=1}^{j=p} \lambda'_j(\cdot)a_j(\cdot) \otimes a_j(\cdot)$ is a measurable Schmidt decomposition of $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}(\cdot)M'_c(\cdot)$.

Let $\varphi'(\cdot) = \sum_{j=1}^{j=q} a_j(\cdot) \otimes f_j$. A solution of the PCA (of order q) of $(X'_g)_{g \in G}$ is given by $\int (\cdot, g)_{\widehat{G}G} \varphi'(\cdot) dZ_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}} = \int (\cdot, g)_{\widehat{G}G} I_{C^q} d(Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}})_{\varphi'}$.

Since $(Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}})_{\varphi'} = Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{\varphi'} = Z_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{\varphi_c} = (Z_{c\varphi_c})_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{C^q}}}$ (because $\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{\varphi'} = \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{\varphi_c}$), we have

$$\int (\cdot, g)_{\widehat{G}G} \varphi'(\cdot) dZ_c \mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{CP}} = \int (\cdot, g)_{\widehat{G}G} I_{C^q} d(Z_{c\varphi_c})_{\mathbf{I}_{\widehat{G}-\{\gamma_{\mathbb{E}}\}}^{I_{C^q}}} = Y'_g. \quad \square$$

In Proposition 5.3, the random vectors W and W_c are obtained from the PCA's of order q of $Z(\{\gamma_{\mathbb{E}}\})$ and $Z_c(\{\gamma_{\mathbb{E}}\})$ respectively. More precisely, we have

$$\begin{aligned} M_Z(\{\gamma_{\mathbb{E}}\}) &= \int \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot)M'(\cdot) d\mu_Z = \int \mathbf{I}_{\{\gamma_{\mathbb{E}}\}}(\cdot)M'(\gamma_{\mathbb{E}}) d\mu_Z \\ &= \mu_Z(\{\gamma_{\mathbb{E}}\})M'(\gamma_{\mathbb{E}}) \end{aligned}$$

and so, $\sum_{j=1}^{j=p} \mu_Z(\{\gamma_{\mathbb{E}}\})\mu_j b_j \otimes b_j$ is the Schmidt decomposition of $Z(\{\gamma_{\mathbb{E}}\}) \circ Z(\{\gamma_{\mathbb{E}}\})^*$.

Therefore, the PCA (of order q) of $Z(\{\gamma_{\mathbb{E}}\})$ is given by $[\sum_{j=1}^{j=q} b_j \otimes f_j] \circ Z(\{\gamma_{\mathbb{E}}\})$.

Since $\varphi(\gamma_{\mathbb{E}}) = \sum_{j=1}^{j=q} u_j(\gamma_{\mathbb{E}}) \otimes f_j = \sum_{j=1}^{j=q} b_j \otimes f_j$, the PCA (of order q) of $Z(\{\gamma_{\mathbb{E}}\})$ is $\varphi(\gamma_{\mathbb{E}}) \circ Z(\{\gamma_{\mathbb{E}}\}) = W$.

In an analogous way, we show that $\sum_{j=1}^{j=p} \mu_Z(\{\gamma_{\mathbb{E}}\})\lambda_j(\gamma_{\mathbb{E}})a_j(\gamma_{\mathbb{E}}) \otimes a_j(\gamma_{\mathbb{E}})$ is the Schmidt decomposition of $Z_c(\{\gamma_{\mathbb{E}}\}) \circ Z_c(\{\gamma_{\mathbb{E}}\})^*$ and the PCA of $Z_c(\{\gamma_{\mathbb{E}}\})$ is $[\sum_{j=1}^{j=q} a_j(\gamma_{\mathbb{E}}) \otimes f_j] \circ Z_c(\{\gamma_{\mathbb{E}}\}) = \varphi_c(\gamma_{\mathbb{E}}) \circ Z_c(\{\gamma_{\mathbb{E}}\}) = W_c$.

So, Proposition 5.3 can be interpreted as follows: *to obtain the PCA of order q of $(X_g)_{g \in G}$ (resp. $(X_g^c)_{g \in G}$), we must add to the PCA of $(X'_g)_{g \in G}$*

the q -dimensional vector $(\gamma_{\mathcal{E}}, g)_{\widehat{G}}W$ (resp. $(\gamma_{\mathcal{E}}, g)_{\widehat{G}}W_c$) deduced from the PCA of $Z(\{\gamma_{\mathcal{E}}\})$ (resp. $Z_c(\{\gamma_{\mathcal{E}}\})$).

6. Some particular cases

The p -stationary CRF $(X_g)_{g \in G}$ is a real random function if for each $g \in G$, one has $\overline{X_g} = X_g$. Equivalently, that is (e.g. Boudou and Dauxois, 1994), the associated p -RM Z satisfies $\overline{Z}(-A) = Z(A)$, for each $A \in \mathcal{B}$. From this point of view, if $(X_g)_{g \in G}$ is a real CRF, it is clear that $(\mathbf{I}_{\Omega} \mathcal{E}(X_g))_{g \in G}$ is a real CRF and then, according to the Corollary 4.1 of Proposition 4.2, one has

$$\delta_{\gamma_{\mathcal{E}}}(-A) \mathbf{I}_{\Omega} \mathcal{E}(X_0) = \delta_{\gamma_{\mathcal{E}}}(A) \mathbf{I}_{\Omega} \mathcal{E}(X_0) \iff -\gamma_{\mathcal{E}} = \gamma_{\mathcal{E}}. \quad (9)$$

When $G = \mathbb{R}$, its dual \widehat{G} is identifiable to \mathbb{R} , so $\gamma_{\mathcal{E}} = 0$ is the unique solution of (9) and equality (6) is such that $\mathcal{E}(X_g) = \mathcal{E}(X_0)$.

When $G = \mathbb{Z}$, its dual $\widehat{\mathbb{Z}}$ is identifiable to $\Pi = [-\pi, \pi[$ which is a (locally compact) Abelian group for

$$\oplus : (\lambda_1, \lambda_2) \in \Pi \times \Pi \mapsto \lambda_1 + \lambda_2 + 2\pi \left[\frac{\lambda_1 + \lambda_2 + \pi}{2\pi} \right] \in \Pi,$$

where $[a]$ is the integer part of a . The equation (9) is equivalent to $\ominus \lambda = \lambda$ with $\lambda \in \Pi$, where $\ominus \lambda = -\lambda - 2\pi \left[\frac{-\lambda + \pi}{2\pi} \right]$, and admits two solutions $\{0, -\pi\}$. So the random function $(\mathcal{E}(X_n))_{n \in \mathbb{Z}}$ defined by (6) is such that $\mathcal{E}(X_n) = \varepsilon^n \mathcal{E}(X_0)$, with $\varepsilon \in \{0; 1\}$, for all $n \in \mathbb{Z}$.

When $G = \mathbb{Z}^k$, its dual $\widehat{\mathbb{Z}^k}$ is identifiable to Π^k by the homeomorphism

$$\mathcal{I} : (\lambda_1, \dots, \lambda_k) \in \Pi^k \mapsto e^{i(\lambda_1 \cdot + \dots + \lambda_k \cdot)} \in \widehat{\mathbb{Z}^k},$$

with $(\mathcal{I}(\lambda_1, \dots, \lambda_k); (n_1, \dots, n_k))_{\widehat{\mathbb{Z}^k}} = e^{i\lambda_1 n_1} e^{i\lambda_2 n_2} \dots e^{i\lambda_k n_k}$. The equation $\gamma_{\mathcal{E}} = -\gamma_{\mathcal{E}}$ in $\widehat{\mathbb{Z}^k}$ means that $\mathcal{I}(\lambda_1, \dots, \lambda_k) = \mathcal{I}(\ominus \lambda_1, \dots, \ominus \lambda_k)$ and so, λ_j

belongs to $\{0, -\pi\}$ for $j = 1, \dots, k$. And moreover,

$$\begin{aligned} (\gamma_{\mathbb{E}}; (n_1, \dots, n_k))_{\widehat{\mathbb{Z}^k \mathbb{Z}^k}} &= (\mathcal{I}(\lambda_1, \dots, \lambda_k); (n_1, \dots, n_k))_{\widehat{\mathbb{Z}^k \mathbb{Z}^k}} \\ &= e^{i\lambda_1 n_1} e^{i\lambda_2 n_2} \dots e^{i\lambda_k n_k} = \prod_{j=1}^{j=k} e^{i\lambda_j n_j}, \end{aligned}$$

with $\lambda_j \in \{0, -\pi\}$ for $j = 1, \dots, k$. If $\lambda_j = 0$ then $e^{i\lambda_j n_j} = e^0 = 1 = (-1)^{\varepsilon_j n_j}$, with $\varepsilon_j = 0$. If $\lambda_j = -\pi$ then $e^{i\lambda_j n_j} = e^{-i\pi n_j} = (-1)^{n_j} = (-1)^{\varepsilon_j n_j}$, with $\varepsilon_j = 1$. So, the equation (9) admits 2^k solutions, that is,

$$(\gamma_{\mathbb{E}}; (n_1, \dots, n_k))_{\widehat{\mathbb{Z}^k \mathbb{Z}^k}} = \prod_{j=1}^{j=k} (-1)^{\varepsilon_j n_j} = (-1)^{\sum_{j=1}^{j=k} \varepsilon_j n_j}$$

and $(\mathbb{E}(X_{n_1, \dots, n_k}))_{(n_1, \dots, n_k) \in \mathbb{Z}^k}$ defined by (6) is such that

$$\mathbb{E}(X_{n_1, \dots, n_k}) = (-1)^{\sum_{j=1}^{j=k} \varepsilon_j n_j} \mathbb{E}(X_{0, \dots, 0})$$

with $\varepsilon_j \in \{0, 1\}$, for all $(n_1, \dots, n_k) \in \mathbb{Z}^k$.

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