

# REGULARITY ACTION OF ABELIAN LINEAR GROUPS ON $\mathbb{C}^n$

ADLENE AYADI AND EZZEDDINE SALHI

ABSTRACT. In this paper, we give a characterization of the action of any abelian subgroup  $G$  of  $GL(n, \mathbb{C})$  on  $\mathbb{C}^n$ . We prove that any orbit of  $G$  is regular with order  $m \leq 2n$ . Moreover, we give a method to determine this order. In the other hand, we specify the region of all orbits which are isomorphic. If  $G$  is finitely generated, this characterization is explicit.

## 1. Introduction

Let  $GL(n, \mathbb{C})$  be the group of all reversible square matrix over  $\mathbb{C}$  with order  $n$  and let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$ . There is a natural linear action  $GL(n, \mathbb{C}) \times \mathbb{C}^n : \rightarrow \mathbb{C}^n$ .  $(A, v) \mapsto Av$ . For a vector  $v \in \mathbb{C}^n$ , denote by  $G(v) = \{Av, A \in G\} \subset \mathbb{C}^n$  the *orbit* of  $G$  through  $v$ . A subset  $E \subset \mathbb{C}^n$  is called  *$G$ -invariant* if  $A(E) \subset E$  for any  $A \in G$ ; that is  $E$  is a union of orbits. Denote by  $\overline{E}$  (resp.  $\overset{\circ}{E}$ ) the closure (resp. interior) of  $E$ .

An orbit  $\gamma$  is called *regular* with order  $m$  if for every  $v \in \gamma$  there exists an open set  $O$  containing  $v$  such that  $\overline{\gamma} \cap O$  is a manifold with dimension  $m$  over  $\mathbb{R}$ . In particular,  $\gamma$  is locally dense in  $\mathbb{C}^n$  if and only if  $m = 2n$ , and  $\gamma$  is discrete if and only if  $m = 0$ . Notice that, the closure of a regular orbit is not necessary a manifold (see example 8.4). We say that the action of  $G$  is *regular* on  $\mathbb{C}^n$  if every orbit of  $G$  is regular. Here, the question to investigate is the following:

- (1) *The orbits of  $G$  are they regular?*
- (2) *If  $G$  has a regular orbit, how can we determine its order?*

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of homeomorphism on a topological space  $X$ . A nonempty compact subset  $Y \subset X$  is a minimal set if for every  $y \in Y$  the orbit of  $y$  is dense in  $Y$ . In [10], Gottschalk discussed the question of what sets can be minimal sets. A minimal set which is a Cantor set is called an exceptional set. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

---

2000 *Mathematics Subject Classification.* 37C85.

*Key words and phrases.* orbit, regular, dense orbit, minimal, subgroup, . . .

This work is supported by the research unit: systèmes dynamiques et combinatoire: 99UR15-15.

For a subset  $E \subset \mathbb{C}^n$ , denote by  $\text{vect}(E)$  the vector subspace of  $\mathbb{C}^n$  generated by all elements of  $E$ . For every  $u \in \mathbb{C}^n$  denote by;

- $E(u) = \text{vect}(G(u))$
- $r(u) = \dim(E(u))$  over  $\mathbb{C}$ .
- $\text{rank}(G(u)) = \dim(E(u))$ .

S.Chihi proved in [12] that for an abelian linear action, every orbit  $\gamma$  is contained in a locally closed sub-manifold  $V(\gamma)$  such that  $\gamma \subset V(\gamma) \subset \bar{\gamma}$ . Moreover, he shows that if  $\bar{\gamma}_1 = \bar{\gamma}_2$  then  $\gamma_1$  and  $\gamma_2$  are isomorphic and  $\text{rank}(\gamma_1) = \text{rank}(\gamma_2)$ .

The purpose of this paper is to give a complete answer to the above question for any abelian subgroup of  $GL(n, \mathbb{C})$ . In [1], the authors present a global dynamic of every abelian subgroup of  $GL(n, \mathbb{C})$  and in [2], they gave a characterization of existence of dense orbit for any abelian subgroup of  $GL(n, \mathbb{C})$ . Our main result is viewed as continuation of work in [1] and [2]. We found similar result given in [12] as a consequence of Theorem 1.5, and we prove that every orbit is regular and we characterize its order  $m$ . If  $G$  is finitely generated, this characterization is explicit.

Denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ,  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .
- $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$  where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r.$$

- $M_n(\mathbb{C})$  the set of all square matrix of order  $n \geq 1$  with coefficients in  $\mathbb{C}$ .
- $M_{p,q}(\mathbb{C})$  the set of all matrix having  $p$  lines and  $q$  colons with coefficients in  $\mathbb{C}$ .
- $\mathbb{T}_m(\mathbb{C})$  the set of matrices over  $\mathbb{C}$  of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \quad (1)$$

- $\mathbb{T}_m^*(\mathbb{C})$  the group of matrices of the form (1) with  $\mu \neq 0$ .

Let  $r \in \mathbb{N}^*$  and  $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$  such that  $\sum_{i=1}^r n_i = n$ . Denote by:

- $\mathcal{K}_{\eta,r}(\mathbb{C}) = \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$ .
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) = \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n, \mathbb{C})$ .
- $\mathcal{C}_0 = (e_1, \dots, e_n)$  the canonical basis of  $\mathbb{C}^n$ .

The author have proved in [2], that for every abelian subgroup of  $GL(n, \mathbb{C})$  there exists  $P \in GL(n, \mathbb{C})$  such that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$  for some  $r \in \mathbb{N}^*$  and  $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$  (see Proposition 2.8). We say

that  $\tilde{G} = P^{-1}GP$  is a normal form of  $G$ . We let

-  $\mathfrak{g} = \exp^{-1}(G) \cap [P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}]$

-  $\mathfrak{g}_u = \{Bu, B \in \mathfrak{g}\}, u \in \mathbb{C}^n$ .

One has  $\exp(\mathfrak{g}) = G$  (see Lemma 2.7).

For any closed additive subgroup  $F$  of  $\mathbb{C}^n$ , we say that  $\dim(F) = s$ , if  $s$  is the bigger dimension of all vector spaces over  $\mathbb{R}$  contained in  $F$ . Moreover, if  $F$  is considered as a manifold over  $\mathbb{R}$ , then  $\dim(F) = s$ . (See Proposition 2.4).

Finally, consider the following rank condition on a collection of vectors  $u_1, \dots, u_p \in \mathbb{R}^n$ ,  $p > n$ . Suppose that  $(u_1, \dots, u_n)$  is a basis of  $\mathbb{R}^n$  and there exists  $0 \leq m \leq n$  such that  $u_k = \sum_{j=1}^m \alpha_{k,j} u_{n-m+j}$ , for every  $n+1 \leq k \leq p$ ,  $\alpha_{k,j} \in \mathbb{R}^*$ .

• We say that  $u_1, \dots, u_p \in \mathbb{R}^n$  satisfy *property  $\mathcal{D}(m)$*  if and only if for every  $(t_1, \dots, t_m, s_1, \dots, s_{p-n}) \in \mathbb{Z}^{p-n+m} - \{0\}$  :

$$\text{rank} \left( \begin{bmatrix} 1 & 0 & \dots & 0 & \alpha_{n+1,1} & \dots & \dots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \alpha_{n+1,m} & \dots & \dots & \alpha_{p,m} \\ t_1 & \dots & \dots & t_m & s_1 & \dots & \dots & s_{p-n} \end{bmatrix} \right) = m+1.$$

and this is equivalent by Lemma 5.3 to say that  $\mathbb{Z}u_{n-m+1} + \dots + \mathbb{Z}u_p$  is dense in  $\mathbb{R}u_{n-m+1} \oplus \dots \oplus \mathbb{R}u_n$ .

• For every permutation  $\sigma \in \mathcal{S}_p$  we say that  $u_{\sigma(1)}, \dots, u_{\sigma(p)} \in \mathbb{R}^n$  satisfy *property  $\mathcal{D}(m)$*  if  $u_1, \dots, u_p \in \mathbb{R}^n$  satisfy *property  $\mathcal{D}(m)$* .

For a vector  $v \in \mathbb{C}^n$ , we write  $v = \text{Re}(v) + i\text{Im}(v)$  where  $\text{Re}(v), \text{Im}(v) \in \mathbb{R}^n$ .

Let  $\theta : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$  be the isomorphism, defined by

$$\theta(z_1, \dots, z_n) = (\text{Re}(z_1), \dots, \text{Re}(z_n); \text{Im}(z_1), \dots, \text{Im}(z_n)).$$

• We say that  $v_1, \dots, v_p \in \mathbb{C}^n$  satisfy *property  $\mathcal{D}(m)$* , if  $\theta(v_1), \dots, \theta(v_p) \in \mathbb{R}^{2n}$  satisfy *property  $\mathcal{D}(m)$* .

Our principal results can be stated as follows:

**Theorem 1.1.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$ . Then for every  $u \in \mathbb{C}^n$ , there exists a  $G$ -invariant dense open subset  $U_u$  of  $E(u)$ , containing  $u$ , such that:*

- (i) *For every  $v \in U_u$ , we have  $E(v) = E(u)$ .*
- (ii) *All orbit in  $U_u$  are isomorphic.*
- (iii)  *$E(u) \setminus U_u$  is a union of at most  $r(u)$   $G$ -invariant vector subspaces of  $E(u)$  with dimension  $r(u) - 1$  over  $\mathbb{C}$ .*

By the following Theorem, the order of any orbit is equal to dimension of a closed additive subgroup of  $\mathbb{C}^n$ , over  $\mathbb{R}$ .

**Theorem 1.2.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  and  $u \in \mathbb{C}^n$ . The following are equivalent:*

- (i)  *$G(u)$  is regular with order  $m$ .*
- (ii) *For every  $v \in U_u$ ,  $G(v)$  is regular with order  $m$ .*
- (iii)  *$\dim(\overline{g_u}) = m$  over  $\mathbb{R}$ .*

As a consequence from Theorem 1.2 we prove, by the following corollary, that the action of any abelian subgroup of  $GL(n, \mathbb{K})$  on  $\mathbb{K}^n$  is regular ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ).

**Corollary 1.3.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  (resp.  $GL(n, \mathbb{R})$ ). Then every orbit of  $G$  is regular with order  $0 \leq m \leq 2n$  (resp.  $0 \leq m \leq n$ ).*

Where  $GL(n, \mathbb{R})$  denotes the group of all reversible square matrix over  $\mathbb{R}$  with order  $n$ .

**Corollary 1.4.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{K})$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ).*

- (i) *If  $\mathbb{K} = \mathbb{R}$  then an orbit  $G(u)$  is regular with order  $m = n$  if and only if it is locally dense.*
- (ii) *If  $\mathbb{K} = \mathbb{C}$  then an orbit  $G(u)$  is regular with order  $m = 2n$  if and only if it is dense in  $\mathbb{C}^n$ .*

**Theorem 1.5.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  and  $u \in \mathbb{C}^n$ . Then the closure  $\overline{G(u)}$  is a vector subspace of  $\mathbb{C}^n$  if and only if  $G(u)$  is regular with order  $2r(u)$ .*

For a finitely generated subgroup  $G \subset GL(n, \mathbb{C})$ , such that its normal form  $\tilde{G} = P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ . We give an explicit condition to determine the order of any orbit.

**Theorem 1.6.** *If  $G$  is an abelian subgroup of  $GL(n, \mathbb{C})$  generated by  $A_1, \dots, A_p$  and let  $B_1, \dots, B_p \in \mathfrak{g}$  such that  $A_1 = e^{B_1}, \dots, A_p = e^{B_p}$ . Then for every  $u \in \mathbb{C}^n \setminus \{0\}$ . The following are equivalent:*

- (i)  *$G(u)$  is regular with order  $m$ .*
- (ii)  *$B_1u, \dots, B_pu, 2i\pi P(e^{(1)}), \dots, 2i\pi P(e^{(r)})$  satisfy property  $\mathcal{D}(m)$ .*

$$(iii) \ g_u = \sum_{k=1}^p \mathbb{Z}(B_k u) + \sum_{k=1}^r 2i\pi \mathbb{Z}P(e^{(k)}) \text{ and } \dim(\overline{g_u}) = m.$$

This paper is organized as follows: In Section 2, we give preliminary results. In Section 3, we prove the Theorem 1.1. A parametrization of an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$  is given in section 4. The proof of Theorems 1.2, 1.5 and 1.6 and Corollaries 1.3 and 1.4 are done in Section 5. Section 6 is devoted to give some examples.

## 2. Preliminary results

We present some results for abelian subgroup of  $GL(n, \mathbb{C})$ .

**Proposition 2.1.** ([8], Proposition 7) *Let  $A \in M_n(\mathbb{C})$ . Then if no two eigenvalues of  $A$  have a difference of the form  $2ik\pi$ ,  $k \in \mathbb{Z} \setminus \{0\}$ , then  $\exp : M_n(\mathbb{C}) \rightarrow GL(n, \mathbb{C})$  is a local diffeomorphism at  $A$ .*

**Corollary 2.2.** *The restriction  $\exp/\mathcal{K}_{\eta,r}(\mathbb{C}) : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$  is a local diffeomorphism.*

*Proof.* The proof results from Proposition 2.1 and the fact that  $\exp/\mathcal{K}_{\eta,r}(\mathbb{C}) = \exp/\mathbb{T}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \exp/\mathbb{T}_{n_r}(\mathbb{C})$ .  $\square$

**Proposition 2.3.** ([11], Theorem 2.1) *Let  $H$  be a discrete additive subgroup of  $\mathbb{C}^n$ . Then there exist a basis  $(u_1, \dots, u_n)$  of  $\mathbb{C}^n$  and  $1 \leq r \leq n$  such that  $H = \sum_{k=1}^r \mathbb{Z}u_k$ .*

**Proposition 2.4.** ([11], Theorem 3.1) *Let  $F$  be a closed additive subgroup of  $\mathbb{C}^n$ . Then there exist a vector subspace  $V$  of  $\mathbb{C}^n$ , over  $\mathbb{R}$  contained in  $F$  and a vector subspace  $W$  of  $\mathbb{C}^n$ , over  $\mathbb{R}$ , such that:*

- (i)  $W \oplus V = \mathbb{C}^n$ .
- (ii)  $F \cap W$  is a discrete subgroup of  $\mathbb{C}^n$  and  $F = (F \cap W) \oplus V$ .

For any closed additive subgroup  $F$  of  $\mathbb{C}^n$ , we have  $\dim(F) = \dim(V)$ .

**Corollary 2.5.** (Under above notations) *For every  $y \in F \cap W$ , there exist an open subset  $O_y$  of  $\mathbb{C}^n$  such that  $O_y \cap F = \{y\} + V$ . In particular  $O_0 \cap F = V$ .*

*Proof.* We have  $F = (F \cap W) \oplus V$ , where  $F \cap W$  is a discrete subgroup of  $W$ . By Proposition 2.3 there exists a basis  $(u_1, \dots, u_p)$  of  $W$  and  $1 \leq s \leq p$  such that  $F \cap W = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_s$ . Let  $y = m_1 u_1 + \dots + m_s u_s \in F \cap W$ , take

$$O_y = \left] m_1 - \frac{1}{2}, m_1 + \frac{1}{2} \left[ u_1 \oplus \cdots \oplus \left] m_p - \frac{1}{2}, \frac{1}{2} + m_p \left[ u_p \oplus V.$$

It follows that  $O_y$  is an open subset of  $\mathbb{C}^n$  such that  $O_y \cap F = V + \{y\}$ . The proof is complete.  $\square$

**Proposition 2.6.** *For every  $u \in \mathbb{C}^n$ , there exist two vector subspaces  $V$  and  $W$  of  $\mathbb{C}^n$  over  $\mathbb{R}$ , such that  $V$  is contained in  $\overline{g_u}$  and  $W \oplus V = \mathbb{C}^n$ , satisfying:*

- i)  $\overline{g_u} = (\overline{g_u} \cap W) \oplus V$ , with  $\overline{g_u} \cap W$  is discrete.
- ii) For every  $\lambda \in \mathbb{R}$ , we have  $\overline{g_{\lambda u}} = (\overline{g_{\lambda u}} \cap W) \oplus V$ , with  $\overline{g_{\lambda u}} \cap W$  is discrete subgroup of  $W$ .

The proof uses the following lemma.

**Lemma 2.7.** ([2], Lemmas 4.1 and 4.2)

- (i) If  $G$  is an abelian subgroup of  $GL(n, \mathbb{C})$  then for every  $u \in \mathbb{C}^n$ ,  $g_u$  is an additive subgroup of  $\mathbb{C}^n$ .
- (ii)  $\exp(g) = G$ .

*Proof of Proposition 2.6.* The proof of (i) results from Lemma 2.7.(i) and Proposition 2.4.

The proof of (ii) follows from the fact that  $g_{\lambda u} = \lambda g_u$ ,  $\lambda W = W$  and  $\lambda V = V$ , for every  $\lambda \in \mathbb{R}$ .  $\square$

Let the fundamental result proved in [2]:

**Proposition 2.8.** ([2], Proposition 2.3) *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$ . Then there exists  $P \in GL(n, \mathbb{C})$  such that  $P^{-1}GP$  is an abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ , for some  $r \in \{1, \dots, n\}$  and  $\eta \in (\mathbb{N}^*)^r$ .*

### 3. Proof of Theorem 1.1

Throughout this section, suppose that  $G$  is an abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$  for some  $r \in \mathbb{N}^*$  and  $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$ .

Every  $A \in G$  has the form  $A = \text{diag}(A_1, \dots, A_r)$  with  $A_k \in \mathbb{T}_{n_k}(\mathbb{C})$   $k = 1, \dots, r$ . Denote by:

- $G_k = \{A_k, A \in G\}$ ,  $k = 1, \dots, r$ .
- $E(u_k) = \text{vect}(G_k(u_k))$ , for every  $u = [u_1, \dots, u_r]^T \in \mathbb{C}^n$ .
- $\mathcal{G} = \text{vect}(G)$ , the vector subspace of  $\mathcal{K}_{\eta, r}(\mathbb{C})$  generated by all elements of  $G$ .
- $U = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$ .

**Lemma 3.1.** *Let  $G$  be an abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ . Then for every  $u \in \mathbb{C}^n$ ,  $E(u)$  is  $G$ -invariant.*

*Proof.* Suppose that  $E(u)$  is generated by  $A_1u, \dots, A_pu$ , with  $A_k \in G$ ,  $1 \leq k \leq p$ . Let  $w = \sum_{k=1}^p \alpha_k A_k u \in E(u)$  and  $B \in G$ , then  $Bw = \sum_{k=1}^p \alpha_k B A_k u$ . Since  $B A_k u \in G(u) \subset E(u)$  then it has the form  $B A_k u = \sum_{j=1}^p \beta_{k,j} A_j u$ ,  $\beta_{k,j} \in \mathbb{C}$ , so  $Bw = \sum_{1 \leq k, j \leq p} \alpha_k \beta_{k,j} A_j u \in E(u)$ .  $\square$

**Lemma 3.2.** ([12], Proposition 3.1) *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$ ,  $u \in \mathbb{C}^n$  and  $v \in E(u)$ .*

- (i) *Then there exist  $B \in \mathcal{G}$  such that  $Bu = v$ .*
- (ii) *If  $E(u) = E(v)$ , then  $G(u)$  and  $G(v)$  are isomorphic.*

**Proposition 3.3.** *Let  $G$  be an abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$  and  $u \in U$  such that  $E(u) = \mathbb{C}^n$ . Then for every  $v \in U$  there exists  $B \in \mathcal{G} \cap GL(n, \mathbb{C})$  such that  $Bu = v$ . In particular,  $E(v) = \mathbb{C}^n$ .*

**Lemma 3.4.** *Let  $G$  be an abelian subgroup of  $\mathbb{T}_n^*(\mathbb{C})$  and  $u \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ . Then for every  $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$  there exist  $B \in \mathcal{G} \cap GL(n, \mathbb{C})$  such that  $Bu = v$ .*

*Proof.* Let  $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$  since  $E(u) = \mathbb{C}^n$ , by lemma 3.2, there exist  $B \in \mathcal{G}$  such that  $Bu = v$ . Since  $\mathbb{T}_n(\mathbb{C})$  is a vector space so  $\mathcal{G} \subset \mathbb{T}_n(\mathbb{C})$ . Write  $u = [x_1, \dots, x_n]^T$ ,  $v = [y_1, \dots, y_n]^T$  and

$$B = \begin{bmatrix} \mu_B & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n,1} & \dots & a_{n,n-1} & \mu_B \end{bmatrix},$$

then  $\mu_B = \frac{y_1}{x_1} \neq 0$ , hence  $B \in GL(n, \mathbb{C})$ . As  $\mathcal{G} \subset \mathcal{C}(G)$ ,  $B(E(u)) = E(v)$ .  $\square$

*Proof of Proposition 3.3.* Write  $u = [u_1, \dots, u_r]^T$  and let  $v = [v_1, \dots, v_r]^T \in U$ , with  $u_k, v_k \in \mathbb{C}^{n_k}$ . Since  $E(u) = \mathbb{C}^n$ , then  $E(u_k) = \mathbb{C}^{n_k}$  for every  $k = 1, \dots, r$ . As  $\mathcal{G} = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_r$  and  $v_k \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$ , where  $\mathcal{G}_k = \text{Vect}(G_k)$ . Hence the proof results from Lemma 3.4. In particular,  $E(v) = B(E(u)) = \mathbb{C}^n$ .  $\square$

**3.1. Proof of Theorem 1.1.** Let  $u \in \mathbb{C}^n$ . By Proposition 2.8, we can assume that  $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$  and  $E(u) = \mathbb{C}^n$ , otherwise, by Lemma 3.1 we replace  $G$  by the restriction  $G_{/E(u)}$ . By construction,  $\mathbb{C}^n \setminus U$  is union of  $r$   $G$ -invariant vector subspaces of  $\mathbb{C}^n$  with dimension  $n - 1$ , then we proves (iii) and we deduce that  $u \in U$ . Now, for every  $v \in U$ , there exists by Proposition 3.3,  $B \in \mathcal{G} \cap GL(n, \mathbb{C})$  such that  $Bu = v$ . Hence  $E(v) = B(E(u)) = \mathbb{C}^n$  and  $G(v) = B(G(u))$ , this proves (i) and (ii).  $\square$

#### 4. Parametrization

Let  $G$  be an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$  and  $u \in \mathbb{C}^n$ , by Lemma 3.1,  $E(u)$  is  $G$ -invariant, so consider the linear map

$$\begin{aligned} \Phi_u : \text{vect}(G_{/E(u)}) &\longrightarrow E(u) \\ A &\mapsto Au \end{aligned}$$

**Proposition 4.1.** *For every  $u \in \mathbb{C}^n \setminus \{0\}$ ,  $\Phi_u$  is a linear isomorphism.*

*Proof.* By construction,  $\Phi_u$  is surjective, since  $\Phi_u(\text{Vect}(G_{/E(u)})) = E(u)$ .  
-  $\Phi_u$  is injective: let  $A \in \text{Ker}(\Phi_u)$ , so  $Au = 0$ . Let  $x \in E(u)$ , then by above there exists  $B \in \text{Vect}(G_{/E(u)})$  such that  $x = Bu$ . As  $A \in \text{Ker}(\Phi_u) \subset \text{vect}(G_{/E(u)})$  then  $AB = BA$ . Therefore  $Ax = ABu = BAu = B(0) = 0$ . It follows that  $A = 0$  and hence  $\text{Ker}(\Phi_u) = \{0\}$ .  $\square$

**Corollary 4.2.** *We have  $\Phi_u^{-1}(G(u)) = G_{/E(u)}$  and  $\Phi_u^{-1}(g_u) = g_{/E(u)}$ .*

Under the above notation, we have:

**Proposition 4.3.** *Let  $G$  be an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ . Then*

$$\exp(\Phi_u^{-1}(E(u))) \subset \Phi_u^{-1}(U_u).$$

To prove Proposition 4.3, we need the following Lemma:

**Lemma 4.4.** *Let  $G$  be an abelian subgroup of  $\mathbb{T}_n^*(\mathbb{C})$ . If  $E(u) = \mathbb{C}^n$ , then*

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

*Proof.* Here  $G_{/E(u)} = G$  and  $U_u = U = \mathbb{C}^* \times \mathbb{C}^{n-1}$ . First, one has  $\exp(\mathcal{G}) \subset \mathcal{G}$ : indeed; for every  $A \in \mathcal{G}$  we have  $A^k \in \mathcal{G}$ , for every  $k \in \mathbb{N}$  and so  $e^A = \sum_{k \in \mathbb{N}} \frac{A^k}{k!} \in \mathcal{G}$ . Moreover, we can check that  $\mathcal{G}$  is the subalgebra of  $M_n(\mathbb{C})$  generated by  $G$ .



Second, by corollary 4.2,  $\Phi_u^{-1}(G(u)) = G$ . Since  $E(u) = \mathbb{C}^n$  and by Proposition 4.1,  $\Phi_u$  is an isomorphism, then

$$\begin{aligned} \exp(\Phi_u^{-1}(\mathbb{C}^n)) &= \exp(\Phi_u^{-1}(E(u))) \\ &= \exp(\mathcal{G}) \\ &\subset \mathcal{G} = \Phi_u^{-1}(\mathbb{C}^n) \end{aligned}$$

Therefore

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^n) \quad (1)$$

On the other hand, by construction of  $\Phi_u$  and  $U$ , we have  $u \in U$  and so  $\Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1})$ . As  $\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \mathbb{T}_n^*(\mathbb{C})$  then by (1) we obtain

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

□

*Proof of Proposition 4.3.* Write  $u = [u_1, \dots, u_r]^T \in \mathbb{C}^n$ . Suppose that  $E(u) = \mathbb{C}^n$ , otherwise we replace  $G$  by  $G_{/E(u)}$ . So  $G_{/E(u)} = G$  and  $U_u = U$ . Since  $\exp/\mathcal{K}_{\eta,r}(\mathbb{C}) = \exp/\mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \exp/\mathbb{T}_{n_r}(\mathbb{C})$  and  $\Phi_u^{-1} = \Phi_{u_1}^{-1} \oplus \dots \oplus \Phi_{u_r}^{-1}$ . Then by Lemma 4.4

$$\exp(\Phi_u^{-1}(U)) = \prod_{k=1}^r \exp/\mathbb{T}_{n_k}(\mathbb{C}) (\Phi_{u_k}^{-1}(\mathbb{C}^{n_k})) \subset \prod_{k=1}^r \Phi_{u_k}^{-1}(\mathbb{C}^* \times \mathbb{C}^{n_k-1}) = \Phi^{-1}(U).$$

□

As consequence of Proposition 4.3, we have the following results:

**Corollary 4.5.** *The map  $f = \Phi_u \circ \exp/\Phi(\mathbb{C}^n) \circ \Phi_u^{-1} : E(u) \rightarrow U_u$  is well defined and continuous.*

## 5. Proof of Theorems 1.2, 1.5, 1.6 and Corollaries 1.3 and 1.4

By Proposition 2.8, suppose that  $G$  is an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$  and then  $\mathfrak{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$ .

In all this section fixed  $u \in \mathbb{C}^n$  and suppose that  $E(u) = \mathbb{C}^n$  and  $G_{/E(u)} = G$ , leaving to replace  $G$  by  $G_{/E(u)}$ . Denote by  $\Phi = \Phi_u^{-1} : \mathbb{C}^n \rightarrow \mathcal{G}$ . By Corollary 2.2,  $\exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$  is a local diffeomorphism and by Proposition 4.1,  $\Phi$  is an open map. Then we introduce the following Lemma which will be used in the proof of Theorem 1.2.

**Lemma 5.1.** *Let  $O'$  be an open subset of  $\mathbb{C}^n$  such that  $\exp/\Phi(O') : \Phi(O') \rightarrow \exp(\Phi(O'))$  is a diffeomorphism. Then*

$$\exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})} = \exp(\Phi(O') \cap \overline{\mathfrak{g}}).$$

*Proof.* Let  $A \in \exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})}$ , then  $A = e^{\Phi(x)}$  for some  $x \in O'$ . By Lemma 2.7.(ii),  $\exp(\mathfrak{g}) = G$ , so  $A \in G$ . Since  $A \in \overline{\exp(\mathfrak{g})}$ , there exists a sequence  $(B_m)_{m \in \mathbb{N}}$  in  $\mathfrak{g}$  such that  $\lim_{m \rightarrow +\infty} e^{B_m} = A = e^{\Phi(x)}$ . Since  $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$  is a diffeomorphism,  $\exp(\Phi(O'))$  is an open set containing  $e^A$ , so  $e^{B_m} \in \exp(\Phi(O'))$ ,  $\forall m \geq p$ , for some  $p \in \mathbb{N}$ . Then  $B'_m = \exp_{/\Phi(O')}^{-1}(e^{B_m}) \in \Phi(O')$ ,  $\forall m \geq p$ . Since  $e^{B_m} \in \exp(\mathfrak{g}) = G$  and  $\Phi(O') \subset \mathcal{K}_{\eta,r}(\mathbb{C})$ , then  $B'_m \in \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) = \mathfrak{g}$ , for every  $m \geq p$ . Therefore  $\lim_{m \rightarrow +\infty} B'_m = \Phi(x)$ , so  $\Phi(x) \in \Phi(O') \cap \overline{\mathfrak{g}}$  and hence  $A \in \exp(\Phi(O')) \cap \overline{\mathfrak{g}}$ .

Conversely, by continuity of  $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$ , one has

$$\exp(\Phi(O')) \cap \overline{\mathfrak{g}} \subset \exp(\Phi(O')) \cap \exp(\overline{\mathfrak{g}}) \subset \exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})}.$$

The prove is completed.  $\square$

**5.1. Proof of Theorem 1.5.** The equivalence (i)  $\iff$  (ii) follows from Theorem 1.1.(i).

(i)  $\iff$  (iii): By Lemma 3.1, suppose that  $E(u) = \mathbb{C}^n$  and  $U_u = U$  (leaving to replace  $G$  by  $G_{/E(u)}$ ). Then by Proposition 4.1 and Corollary 4.2,  $\Phi = \Phi_u^{-1} : \mathbb{C}^n \rightarrow \mathcal{G}$  is an isomorphism satisfying  $\Phi(G(u)) = G$  and  $\Phi(\mathfrak{g}_u) = \mathfrak{g}$ .

By Proposition 2.6.(i), there exist a vector space  $V$ , contained in  $\overline{\mathfrak{g}_u}$ , a vector space  $W$  such that  $V \oplus W = \mathbb{C}^n$  and  $\overline{\mathfrak{g}_u} = (\overline{\mathfrak{g}_u} \cap W) \oplus V$ , with  $\overline{\mathfrak{g}_u} \cap W$  is discrete.

By corollary 2.5 and Proposition 2.6, there exists an open subset  $O$  of  $\mathbb{C}^n$  such that  $O \cap \overline{\mathfrak{g}_u} = V$ . By Corollary 2.2, the exponential map  $\exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$  is a locally diffeomorphism, then there exists an open subset  $O' \subset O$ , of  $\mathbb{C}^n$  such that the restriction  $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$  of the exponential map on  $\Phi(O')$  is a diffeomorphism. Since  $O' \subset O$ , then

$$O' \cap \overline{\mathfrak{g}_u} = O' \cap V. \quad (1)$$

Since  $O' \subset U_u$  then by Corollary 4.5 the map  $f_{/O'} = \Phi^{-1} \circ \exp_{/\Phi(O')} \circ \Phi : O' \rightarrow O'$  is well defined and as  $\exp_{/\Phi(O')}$  is a diffeomorphism then  $f_{/O'}$  is a diffeomorphism. By Lemma 2.7, one has  $\exp(\mathfrak{g}) = G$  and then:

$$\begin{aligned} f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ \exp(\Phi(O')) \cap \overline{G(u)} \\ &= \Phi^{-1} \left( \exp(\Phi(O')) \cap \overline{\Phi(G(u))} \right) \\ &= \Phi^{-1} \left( \exp(\Phi(O')) \cap \overline{G} \right) \\ &= \Phi^{-1} \left( \exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})} \right) \end{aligned}$$

By Lemma 5.1 and by (1), we obtain

$$\begin{aligned}
f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ \exp(\Phi(O') \cap \overline{\mathfrak{g}}) \\
&= \Phi^{-1} \circ \exp(\Phi(O') \cap \Phi(\overline{\mathfrak{g}_u})) \\
&= \Phi^{-1} \circ \exp \circ \Phi(O' \cap \overline{\mathfrak{g}_u}) \\
&= \Phi^{-1} \circ \exp \circ \Phi(O' \cap V) \\
&= f(O' \cap V).
\end{aligned}$$

As  $f(O')$  is an open subset of  $\mathbb{C}^n$  and  $O' \cap V$  is an open subset of the real vector space  $V$ , then  $f(O') \cap \overline{G(u)}$  is a manifold with dimension  $m = \dim(V) = \dim(\overline{\mathfrak{g}_u})$ . We conclude that  $G(u)$  is regular with order  $m$ . Since  $\mathbb{C}^n$  is a real vector space with dimension  $2n$  and  $V$  is a real subspace of  $\mathbb{C}^n$  with dimension  $m$ , so  $m \leq 2n$ . We conclude the equivalence (i)  $\iff$  (iii).  $\square$

### 5.2. Proof of Corollary 1.3.

$\diamond$  *Complex Case:* Suppose that  $\mathbb{K} = \mathbb{C}$ . Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  and  $u \in \mathbb{C}^n$ . By Theorem 1.5, the orbit  $G(u)$  is regular with order  $m$  if and only if  $\dim(\overline{\mathfrak{g}_u}) = m$ . Since  $\overline{\mathfrak{g}_u}$  is a closed additive subgroup of  $\mathbb{C}^n$  (Lemma 2.7.(i)). Then  $\dim(\overline{\mathfrak{g}_u}) \geq 0$ , so  $G(u)$  is regular with order  $m \geq 0$ . This proves the complex case.

$\diamond$  *Real Case:* Suppose that  $\mathbb{K} = \mathbb{R}$ . Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{R})$  and  $x \in \mathbb{R}^n$ . So  $G$  is considered as an abelian subgroup of  $GL(n, \mathbb{C})$ . By the above case,  $G(x)$  is regular with some order  $m$ , with  $m \leq 2n$ . Then there exists an open subset  $O = O_1 + iO_2$  of  $\mathbb{C}^n$  with  $O_1, O_2$  are open subsets of  $\mathbb{R}^n$ , such that  $\overline{G(x)} \cap O$  is a manifold with dimension  $m$ . One has  $0 \in O_2$ , since  $G(x) \subset \mathbb{R}^n$ . Then  $\overline{G(x)} \cap O = \overline{G(x)} \cap O_1$  and  $m \leq n$ . It follows that  $G(x)$  is a regular orbit in  $\mathbb{R}^n$  with order  $m$ . The proof is completed.  $\square$

### 5.3. Proof of Corollary 1.4.

**Lemma 5.2.** ([1] Corollary 1.3). *If  $G$  has a locally dense orbit  $\gamma$  in  $\mathbb{C}^n$  then  $\gamma$  is dense in  $\mathbb{C}^n$ .*

*Proof of Corollary 1.4.*

(i) If  $\overline{G(u)} = \mathbb{R}^n$  then  $\overline{G(u)}$  is a manifold with dimension  $n$ , so  $G(u)$  is regular with order  $m = n$ .

Conversely, if  $G(u)$  is regular with order  $m = n$ , then  $\overline{G(u)} \cap O$  is a manifold with order  $m = n$ , for some open subset  $O$  of  $\mathbb{R}^n$ . Hence  $\overline{G(u)} \cap O$  is an open subset of  $\mathbb{R}^n$ . Therefore  $G(u)$  is locally dense.

(ii) We use the same proof of (i) and by Lemma 5.2 we have  $\overline{G(u)} = \mathbb{C}^n$ .  $\square$

**5.4. Proof of theorem 1.6.** If  $\overline{G(u)}$  is a vector subspace then  $\overline{G(u)} = E(u)$ , so  $G(u)$  is regular with order  $2r(u)$ .

Conversely, if  $G(u)$  is regular with order  $2r(u)$ , then  $\overline{G(u)} \cap O$  is a manifold with dimension  $2r(u)$ , for some open set  $O$ . Since  $\dim(E(u)) = 2r(u)$  over  $\mathbb{R}$ , then  $\overline{G(u)} \cap O$  is an open subset of  $E(u)$ . So  $G(u)$  is locally dense in  $E(u)$ . By lemma 5.2 applied on  $G/E(u)$  we have  $\overline{G(u)} = E(u)$ .  $\square$

### 5.5. Algebraic Lemmas.

**Lemma 5.3.** ([11], Proposition 4.3). Let  $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$  with  $u_k = [u_{k,1}, \dots, u_{k,n}]^T \in \mathbb{R}^n$ ,  $k = 1, \dots, p$ . Then  $H$  is dense in  $\mathbb{R}^n$  if and only if for every  $(s_1, \dots, s_p) \in \mathbb{Z}^p - \{0\}$  :

$$\text{rank} \left( \begin{bmatrix} u_{1,1} & \dots & \dots & u_{p,1} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1,n} & \dots & \dots & u_{p,n} \\ s_1 & \dots & \dots & s_p \end{bmatrix} \right) = n + 1.$$

**Corollary 5.4.** Let  $p \geq n+1$  and  $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$ ,  $u_k \in \mathbb{R}^n$ ,  $1 \leq k \leq p$ , such that  $(u_1, \dots, u_n)$  is a basis of  $\mathbb{R}^n$ . If there exists  $0 \leq m \leq n$  such that  $u_k = \sum_{j=1}^m \alpha_{k,j} u_{j+n-m}$ , for every  $n+1 \leq k \leq p$ . Then the following assertions are equivalent:

- (i)  $\dim(\overline{H}) = m$ .
- (ii)  $u_1, \dots, u_p$  satisfies property  $\mathcal{D}(m)$ .

*Proof.* Let  $E = \mathbb{R}u_{n-m+1} \oplus \dots \oplus \mathbb{R}u_n$ . We replace  $\mathbb{R}^n$  by  $E$  in Lemma 5.3 and we obtain:  $u_1, \dots, u_p$  satisfies property  $\mathcal{D}(m)$  if and only if  $K = \mathbb{Z}u_{n-m+1} + \dots + \mathbb{Z}u_p$  is dense in  $E$  and this is equivalent to  $\dim(\overline{H}) = m$  since  $\overline{H} = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_{n-m} + \overline{K}$ .  $\square$

### 5.6. Proof of Theorem 1.6.

Denote by:

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T$  and  $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$ ,  $k = 1, \dots, r$ .
- $v_0 = Pu_0$ , where  $P \in GL(n, \mathbb{C})$  is defined in Proposition 4.1 so that  $P^{-1}GP \subset \mathcal{K}_{\eta, r}^*(\mathbb{C})$ .

**Proposition 5.5.** ([2], Theorem 1.3) Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  generated by  $A_1, \dots, A_p$ . Let  $B_1, \dots, B_p \in \mathfrak{g}$  such that  $A_k = e^{B_k}$ ,  $k = 1, \dots, p$ . Then  $\mathfrak{g}_{v_0} = \sum_{k=1}^p \mathbb{Z}B_k v_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}P e^{(k)}$ .

By construction of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ , remark that for every  $u \in U$  there exists  $Q \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$  such that  $Qu_0 = u$ . Then as a consequence of Proposition 5.5 we get the following Corollary:

**Corollary 5.6.** *Let  $G$  be an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$  generated by  $A_1, \dots, A_p$ . Let  $B_1, \dots, B_p \in \mathfrak{g}$  such that  $A_k = e^{B_k}$ ,  $k = 1, \dots, p$ . Then for every  $u \in U$ ,  $\mathfrak{g}u = \sum_{k=1}^p \mathbb{Z}B_k u + \sum_{k=1}^r 2i\pi\mathbb{Z}Qe^{(k)}$ , where  $Q \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$  such that  $Qu_0 = u$ .*

*Proof of Theorem 1.6.* The proof of Theorem 1.6 results from Theorem 1.2, Corollary 5.4 and Corollary 5.6.  $\square$

## 6. Examples

Let  $\mathbb{D}_n(\mathbb{C}) = \{A = \text{diag}(a_1, \dots, a_n) : a_k \in \mathbb{C}^*, 1 \leq k \leq n\}$  and let  $G$  be an abelian subgroup of  $\mathbb{D}_n(\mathbb{C})$ . In this case we have  $\tilde{G} = G$  is a subgroup of  $\mathcal{K}_{(1,\dots,1),n}(\mathbb{C})$  and  $r_G = n$ .

**Example 6.1.** Let  $G$  be the group generated by  $A_k = \text{diag}(\lambda_{k,1}e^{i\alpha_{k,1}}, \dots, \lambda_{k,n}e^{i\alpha_{k,n}})$ ,  $k = 1, \dots, p$ , where  $\lambda_{k,j} \in \mathbb{R}_+$ ,  $\alpha_{k,j} \in \mathbb{R}$ ,  $1 \leq j \leq n$ . Let  $u = [x_1, \dots, x_n]^T \in \mathbb{C}^n$ , then the following assertions are equivalent:

- (i)  $G(u)$  is regular with order  $m$ .
- (ii)  $u_k = [(\log\lambda_{k,1} + i\alpha_{k,1})x_1, \dots, (\log\lambda_{k,n} + i\alpha_{k,n})x_n]^T$ ,  $1 \leq k \leq p$  with  $2i\pi e_1, \dots, 2i\pi e_n$ , satisfies  $\mathcal{D}(m)$ .

*Proof.* We let  $B_k = \text{diag}(\log\lambda_{k,1} + i\alpha_{k,1}, \dots, \log\lambda_{k,n} + i\alpha_{k,n})$ ,

One has  $e^{B_k} = A_k$  and  $B_k \in \mathbb{D}_n(\mathbb{C})$ ,  $1 \leq k \leq p$ . Then  $B_k \in \mathfrak{g}$ . The result follows then from Theorem 1.6.  $\square$

**Example 6.2.** ([1], Example 6.2) Let  $G$  be the subgroup of  $GL(4, \mathbb{R})$  generated by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then:

- i) if  $u \in (\mathbb{Q}^*)^2 \times \mathbb{R}^2$ ,  $G(u)$  is discrete. So  $G(u)$  is regular with order 0.
- ii) if  $u \in \mathbb{Q}^* \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}^2$ ,  $G(u)$  is dense in a straight line. So  $G(u)$  is regular with order 1.

In this example,  $G$  is considered as a subgroup of  $GL(4, \mathbb{C})$ . We have  $\tilde{G} = G$  is a subgroup of  $\mathbb{T}_4^*(\mathbb{C})$  and  $G(e_1) = \{[1, 0, 0, n + m]^T, n, m \in \mathbb{Z}\}$ , then  $E(e_1) = \mathbb{C}e_1 + \mathbb{C}e_4$  and  $U_{e_1} = \mathbb{C}^*e_1 + \mathbb{C}e_4$ . So  $E(e_1) \neq \mathbb{C}^n$ .

Let  $B_1 = A_1 - I_4$  and  $B_2 = A_2 - I_4$ . Since  $B_1^2 = B_2^2 = 0$ , so  $e^{B_1} = A_1$  and  $e^{B_2} = A_2$ . By Theorem 1.6,  $g_u = \mathbb{Z}B_1u + \mathbb{Z}B_2u + 2i\pi\mathbb{Z}e_1$ .

For every  $u = [x, y, z, t]^T \in \mathbb{R}^* \times \mathbb{R}^3$ , we have  $g_u = (\mathbb{Z}x + \mathbb{Z}y)e_4 + 2i\pi\mathbb{Z}e_1$ , then:

-if  $\frac{y}{x} \notin \mathbb{Q}$ , then  $\overline{g_u} = \mathbb{R}e_4 + \mathbb{Z}e_1$ , so  $\dim(\overline{g_u}) = 1$ . By Theorem 1.6,  $G(u)$  is regular with order 1.

-if  $\frac{y}{x} \in \mathbb{Q}$ , then  $\overline{g_u} = \mathbb{Z}ae_4 + \mathbb{Z}e_1$ , for some  $a \in \mathbb{R}$ , so  $\dim(\overline{g_u}) = 0$ . By Theorem 1.6,  $G(u)$  is regular with order 0.  $\square$

A simple example for  $n = 1$  and  $\mathbb{K} = \mathbb{R}$  is given in the following, to show that the closure of a regular orbit is not necessarily a manifold.

**Example 6.3.** Let  $\lambda > 1$  and  $G$  be the group generated by  $\lambda.id_{\mathbb{R}}$ , then for every  $x \in \mathbb{R}^*$ , we have  $G(x) = \{\lambda^n, n \in \mathbb{Z}\}$  and  $\overline{G(x)} = \{\lambda^n, n \in \mathbb{Z}\} \cup \{0\}$ . Thus  $G(x)$  is discrete, so it is regular with order 0, but  $\overline{G(x)}$  is not a manifold.

#### REFERENCES

1. Ayadi.A and Marzougui.H, *Dynamic of abelian subgroups of  $GL(n, \mathbb{C})$ : a structure Theorem*, Geometria Dedicata, 116(2005)111-127.
2. Ayadi.A and Marzougui.H, *Dense orbits for abelian subgroups of  $GL(n, \mathbb{C})$* , Foliations 2005: 47-69. World Scientific,Hackensack,NJ, 2006.
3. H.Poincaré, *Mémoire sur les courbes définies par une équation différentielle*,Princeton Univ. Press, 1970.
4. A.Denjoy, *Sur les courbes définies par les quations différentiables à la surface de tore*, J.Math.Pures appl.11(1932)333-375.
5. Y.Katznelson, *Sigma-finite invariant measures for smooth mapping of circle* J. Analyse Math.31(1977)1-18.
6. P.Schweitzer, *Counterexample to seifert conjecture and opening closed leaves of foliations* Ann.of Math.(2)100(1974)386-400.
7. V.L.Ginzburg, *On the existence and non-existence of closed trajectories for some Hamiltonian flows*, Math.Zeit.223(1996)397-409.
8. W. Rossmann, *Lie groups: an introduction through linear groups*, Oxford, University Press, 2002.
9. G.Kuperberg and K.Kuperberg,*Generalized counterexample to Seifert conjecture*, Ann.of Math.144(1996)239-268.
10. W. H. Gottschalk, *Minimal Sets: An Introduction to Topological Dynamics*, Bull. Amer. Math. Soc. 64 (1958) 336-351.
11. Waldschmidt.M, *Topologie des points rationnels.*, Cours de troisième Cycle, Université P. et M. Curie (Paris VI), 1994/95.
12. S.Chihi, *On the minimal orbits of an abelian linear action* , Differential Geometry-Dynamical System, vol.12,(2010),61-72.

ADLENE AYADI<sup>1</sup>, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF GAFFSA, TUNISIA; EZZEDDINE SALHI<sup>3</sup>, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF SFAX B.P. 802, 3018 SFAX, TUNISIA

*E-mail address:* adleneso@yahoo.fr; Ezzeddine.Salhi@fss.rnu.tn.