REGULARITY ACTION OF ABELIAN LINEAR GROUPS ON $\mathbb{C}^n$

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ABSTRACT. In this paper, we give a characterization of the action of any abelian subgroup G of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$. We prove that any orbit of G is regular with order $m \leq 2n$. Moreover, we give a method to determine this order. In the other hand, we specify the region of all orbits which are isomorphic. If G is finitely generated, this characterization is explicit.

1. Introduction

Let $GL(n,\mathbb{C})$ be the group of all reversible square matrix over $\mathbb{C}$ with order n. There is a natural linear action $GL(n,\mathbb{C})\times\mathbb{C}^n :\longrightarrow \mathbb{C}^n$. $(A,v) \longmapsto Av$. For a vector $v \in \mathbb{C}^n$, denote by $G(v) = \{Av,\ A \in G\} \subset \mathbb{C}^n$ the *orbit* of G through v. A subset $E \subset \mathbb{C}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$; that is E is a union of orbits. Denote by $\overline{E}$ (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E.

An orbit γ is called *regular* with order m if for every $v \in \gamma$ there exists an open set O containing v such that $\overline{\gamma} \cap O$ is a manifold with dimension m over $\mathbb{R}$. In particular, γ is locally dense in $\mathbb{C}^n$ if and only if $m = 2n$, and γ is discrete if and only if $m = 0$. Notice that, the closure of a regular orbit is not necessary a manifold (see example 8.4). Here, the question to investigate is the following:
(1) The orbits of G are they regular?
(2) If G has a regular orbit, how can we determine its order?

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of homeomorphism on a topological space X. A nonempty compact subset $Y \subset X$ is a minimal set if for every $y \in Y$ the orbit of y is dense in Y. In [10], Gottschalk discussed the question of what sets can be minimal sets. A minimal set which is a Cantor set is called an exceptional set. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

For a subset $E \subset \mathbb{C}^n$, denote by $vect(E)$ the vector subspace of $\mathbb{C}^n$ generated by all elements of E. For every $u \in \mathbb{C}^n$ denote by;

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- $E(u) = vect(G(u))$
- $r(u) = \dim(E(u))$ over $\mathbb{C}$.
- $rank(G(u)) = \dim(E(u))$.

S.Chihi proved in [12] that for an abelian linear action, every orbit γ is contained in a locally closed sub-manifold $V(\gamma)$ such that $\gamma \subset V(\gamma) \subset \overline{\gamma}$. Moreover, he shows that if $\overline{\gamma_1} = \overline{\gamma_2}$ then γ_1 and γ_2 are isomorphic and $rank(\gamma_1) = rank(\gamma_2)$.

The purpose of this paper is to give a complete answer to the above question for any abelian subgroup of $GL(n,\mathbb{C})$. In [1], the authors present a global dynamic of every abelian subgroup of $GL(n,\mathbb{C})$ and in [2], they gave a characterization of existence of dense orbit for any abelian subgroup of $GL(n,\mathbb{C})$. Our main result is viewed as continuation of work in [1] and [2]. We found similar result given in [12] as a consequence of Theorem 1.5, and we prove that every orbit is regular and we characterize its order m. If G is finitely generated, this characterization is explicit.
Denote by:
- $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.
- $e^{(k)} = [e_1^{(k)}, \ldots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } \ j \neq k \\ e_{k,1} & \text{if } \ j = k \end{cases} \qquad for \ \ every \ \ 1 \leq j,k \leq r.$$

- $M_n(\mathbb{C})$ the set of all square matrix of order $n \geq 1$ with coefficients in $\mathbb{C}$.
- $M_{p,q}(\mathbb{C})$ the set of all matrix having p lines and q colons with coefficients in $\mathbb{C}$.
- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \ldots & a_{m,m-1} & \mu \end{bmatrix} \qquad (1)$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}^*$ and $\eta = (n_1, \ldots, n_r) \in (\mathbb{N}^*)^r$ such that $\sum_{i=1}^{r} n_i = n$. Denote by:
- $\mathcal{K}_{\eta,r}(\mathbb{C}) = \mathbb{T}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathbb{T}_{n_r}(\mathbb{C})$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) = \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n,\mathbb{C})$.
- $\mathcal{C}_0 = (e_1, \ldots, e_n)$ the canonical basis of $\mathbb{C}^n$.

The author have proved in [2], that for every abelian subgroup of $GL(n,\mathbb{C})$ there exists $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ for some $r \in \mathbb{N}^*$ and $\eta = (n_1, \ldots, n_r) \in (\mathbb{N}^*)^r$ (see Proposition 2.8). We say that $\widetilde{G} = P^{-1}GP$ is *a normal form* of G. We let
- $\mathrm{g} = \exp^{-1}(G) \cap \left[P\left(\mathcal{K}_{\eta,r}(\mathbb{C})\right)P^{-1}\right]$

- $\mathrm{g}_u = \{Bu, \quad B \in \mathrm{g}\},\ u \in \mathbb{C}^n$.
One has $exp(\mathrm{g}) = G$ (see Lemma 2.7).

For any closed additive subgroup F of $\mathbb{C}^n$, we say that $\dim(F) = s$, if s is the bigger dimension of all vector spaces over $\mathbb{R}$ contained in F. Moreover, if F is considered as a manifold over $\mathbb{R}$, then $\dim(F) = s$. (See Proposition 2.4).

Finally, consider the following rank condition on a collection of vectors $u_1, \ldots, u_p \in \mathbb{R}^n$, $p > n$. Suppose that $(u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ and there exists $0 \leq m \leq n$ such that $u_k = \sum_{j=1}^{m} \alpha_{k,j} u_{n-m+j}$, for every $n+1 \leq k \leq p$, $\alpha_{k,j} \in \mathbb{R}^*$.

• We say that $u_1, \ldots, u_p \in \mathbb{R}^n$ satisfy *property* $\mathcal{D}(m)$ if and only if for every $(t_1, ..., t_m, s_1, ..., s_{p-n}) \in \mathbb{Z}^{p-n+m} - \{0\}$:

$$rank \left(\begin{bmatrix} 1 & 0 & \ldots & 0 & \alpha_{n+1,1} & \ldots & \ldots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & \alpha_{n+1,m} & \ldots & \ldots & \alpha_{p,m} \\ t_1 & \ldots & \ldots & t_m & s_1 & \ldots & \ldots & s_{p-n} \end{bmatrix} \right) = m+1.$$

and this is equivalent by Lemma 5.3 to say that $\mathbb{Z}u_{n-m+1} + \cdots + \mathbb{Z}u_p$ is dense in $\mathbb{R}u_{n-m+1} \oplus \cdots \oplus \mathbb{R}u_n$.

• For every permutation $\sigma \in \mathcal{S}_p$ we say that $u_{\sigma(1)}, \ldots, u_{\sigma(p)} \in \mathbb{R}^n$ satisfy *property* $\mathcal{D}(m)$ if $u_1, \ldots, u_p \in \mathbb{R}^n$ satisfy *property* $\mathcal{D}(m)$.

For a vector $v \in \mathbb{C}^n$, we write $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$ where $\mathrm{Re}(v),\ \mathrm{Im}(v) \in \mathbb{R}^n$.

Let $\theta : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ be the isomorphism, defined by

$$\theta(z_1, \ldots, z_n) = (Re(z_1), \ldots, Re(z_n); Im(z_1), \ldots, Im(z_n)).$$

• We say that $v_1, \ldots, v_p \in \mathbb{C}^n$ satisfy *property* $\mathcal{D}(m)$, if $\theta(v_1), \ldots, \theta(v_p) \in \mathbb{R}^{2n}$ satisfy *property* $\mathcal{D}(m)$.

Our principal results can be stated as follows:

Theorem 1.1. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. Then for every $u \in \mathbb{C}^n$, there exists a G-invariant dense open subset U_u of $E(u)$, containing u, such that:*

(i) *For every $v \in U_u$, we have $E(v) = E(u)$.*

(ii) *All orbit in* U_u *are isomorphic.*
(iii) $E(u)\backslash U_u$ *is a union of at most* $r(u)$ *G-invariant vector subspaces of* $E(u)$ *with dimension* $r(u)-1$ *over* $\mathbb{C}$.

By the following Theorem, the order of any orbit is equal to dimension of a closed additive subgroup of $\mathbb{C}^n$, over $\mathbb{R}$.

Theorem 1.2. *Let* G *be an abelian subgroup of* $GL(n,\ \mathbb{C})$ *and* $u \in \mathbb{C}^n$. *The following are equivalent:*

(i) $G(u)$ *is regular with order* m.
(ii) *For every* $v \in U_u$, $G(v)$ *is regular with order* m.
(iii) $dim(\overline{\mathrm{g}_u}) = m$ *over* $\mathbb{R}$.

As a consequence, from Theorem 1.2, we obtain the following corollary.

Corollary 1.3. *Let* G *be an abelian subgroup of* $GL(n,\mathbb{C})$*(resp.* $GL(n,\mathbb{R})$*). Then every orbit of* G *is regular with order* $0 \leq m \leq 2n$ *(resp.* $0 \leq m \leq n$*).*

Where $GL(n,\mathbb{R})$ denotes the group of all reversible square matrix over $\mathbb{R}$ with order n.

Corollary 1.4. *Let* G *be an abelian subgroup of* $GL(n,\mathbb{K})$ $(\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C})$.

(i) *If* $\mathbb{K} = \mathbb{R}$ *then an orbit* $G(u)$ *is regular with order* $m = n$ *if and only if it is locally dense.*
(ii) *If* $\mathbb{K} = \mathbb{C}$ *then an orbit* $G(u)$ *is regular with order* $m = 2n$ *if and only if it is dense in* $\mathbb{C}^n$.

Theorem 1.5. *Let* G *be an abelian subgroup of* $GL(n,\mathbb{C})$ *and* $u \in \mathbb{C}^n$. *Then the closure* $\overline{G(u)}$ *is a vector subspace of* $\mathbb{C}^n$ *if and only if* $G(u)$ *is regular with order* $2r(u)$.

For a finitely generated subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$, such that its normal form $\widetilde{G} = P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. We give an explicit condition to determine the order of any orbit.

Theorem 1.6. *If* G *is an abelian subgroup of* $GL(n,\mathbb{C})$ *generated by* $A_1,\ldots,A_p$ *and let* $B_1,\ldots,B_p \in \mathrm{g}$ *such that* $A_1 = e^{B_1},\ldots,A_p = e^{B_p}$. *Then for every* $u \in \mathbb{C}^n\backslash\{0\}$. *The following are equivalent:*

(i) $G(u)$ *is regular with order* m.
(ii) $B_1u,\ldots,B_pu,\ 2i\pi P(e^{(1)}),\ldots,2i\pi P(e^{(r)})$ *satisfy property* $\mathcal{D}(m)$.
(iii) $\mathrm{g}_u = \sum_{k=1}^{p}\mathbb{Z}(B_ku) + \sum_{k=1}^{r}2i\pi\mathbb{Z}P(e^{(k)})$ *and* $dim(\overline{\mathrm{g}_u}) = m$.

This paper is organized as follows: In Section 2, we give preliminary results. In Section 3, we prove the Theorem 1.1. A parametrization of an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ is given in section 4. The proof of Theorems 1.2, 1.5 and 1.6 and Corollaries 1.3 and 1.4 are done in Section 5. Section 6 is devoted to give some examples.

2. **Preliminary results**

We present some results for abelian subgroup of $GL(n,\mathbb{C})$.

Proposition 2.1. ([8], *Proposition 7'*) *Let $A \in M_n(\mathbb{C})$. Then if no two eigenvalues of A have a difference of the form $2ik\pi$, $k \in Z\backslash\{0\}$, then $exp:\ M_n(\mathbb{C}) \longrightarrow GL(n,\mathbb{C})$ is a local diffeomorphism at A.*

Corollary 2.2. *The restriction $exp/_{\mathcal{K}_{\eta,r}(\mathbb{C})}:\ \mathcal{K}_{\eta,r}(\mathbb{C}) \longrightarrow \mathcal{K}^*_{\eta,r}(\mathbb{C})$ is a local diffeomorphism.*

Proof. The proof results from Proposition 2.1 and the fact that $exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} = exp_{/\mathbb{T}_{n_1}(\mathbb{C})} \oplus \cdots \oplus exp_{/\mathbb{T}_{n_r}(\mathbb{C})}$. □

Proposition 2.3. ([11], *Theorem 2.1*) *Let H be a discrete additive subgroup of $\mathbb{C}^n$. Then there exist a basis $(u_1,\ldots,u_n)$ of $\mathbb{C}^n$ and $1 \leq r \leq n$ such that $H = \sum_{k=1}^{r} \mathbb{Z}u_k$.*

Proposition 2.4. ([11], *Theorem 3.1*) *Let F be a closed additive subgroup of $\mathbb{C}^n$. Then there exist a vector subspace V of $\mathbb{C}^n$, over $\mathbb{R}$ contained in F and a vector subspace W of $\mathbb{C}^n$, over $\mathbb{R}$, such that:*

(i) $W \oplus V = \mathbb{C}^n$.
(ii) $F \cap W$ *is a discrete subgroup of* $\mathbb{C}^n$ *and* $F = (F \cap W) \oplus V$.

For any closed additive subgroup F of $\mathbb{C}^n$, we have $\dim(F) = \dim(V)$.

Corollary 2.5. (*Under above notations*) *For every $y \in F \cap W$, there exist an open subset O_y of $\mathbb{C}^n$ such that $O_y \cap F = \{y\} + V$. In particular $O_0 \cap F = V$.*

Proof. We have $F = (F \cap W) \oplus V$, where $F \cap W$ is a discrete subgroup of W. By Proposition 2.3 there exists a basis $(u_1,\ldots,u_p)$ of W and $1 \leq s \leq p$ such that $F \cap W = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_s$. Let $y = m_1u_1 + \ \ldots\ + m_su_s \in F \cap W$, take

$$O_y = \Big]m_1 - \frac{1}{2},\ m_1 + \frac{1}{2}\Big[\ u_1 \oplus \cdots \oplus\ \Big]m_p - \frac{1}{2},\ \frac{1}{2} + m_p\Big[\ u_p \oplus V.$$

It follows that O_y is an open subset of $\mathbb{C}^n$ such that $O_y \cap F = V + \{y\}$. The proof is complete. □

Proposition 2.6. *For every $u \in \mathbb{C}^n$, there exist two vector subspaces V and W of $\mathbb{C}^n$ over $\mathbb{R}$, such that V is contained in $\overline{\mathrm{g}_u}$ and $W \oplus V = \mathbb{C}^n$, satisfying:*

i) $\overline{\mathrm{g}_u} = (\overline{\mathrm{g}_u} \cap W) \oplus V$, *with* $\overline{\mathrm{g}_u} \cap W$ *is discrete.*
ii) *For every $\lambda \in \mathbb{R}$, we have $\overline{\mathrm{g}_{\lambda u}} = (\overline{\mathrm{g}_{\lambda u}} \cap W) \oplus V$, with $\overline{\mathrm{g}_{\lambda u}} \cap W$ is discrete subgroup of W.*

The proof uses the following lemma.

Lemma 2.7. ([2], *Lemmas 4.1 and 4.2*)

(i) *If G is an abelian subgroup of $GL(n, \mathbb{C})$ then for every $u \in \mathbb{C}^n$, g_u is an additive subgroup of $\mathbb{C}^n$.*
(ii) *$exp(\mathrm{g}) = G$.*

Proof of Proposition 2.6. The proof of (i) results from Lemma 2.7.(i) and Proposition 2.4.
The proof of (ii) follows from the fact that $\mathrm{g}_{\lambda u} = \lambda \mathrm{g}_u$, $\lambda W = W$ and $\lambda V = V$, for every $\lambda \in \mathbb{R}$. □

Let the fundamental result proved in [2]:

Proposition 2.8. ([2], *Proposition* 2.3) *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. Then there exists $P \in GL(n, \mathbb{C})$ such that $P^{-1}GP$ is an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $r \in \{1, \dots, n\}$ and $\eta \in (\mathbb{N}^*)^r$.*

3. **Proof of Theorem 1.1**

Throughout this section, suppose that G is an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ fore some $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$.
Every $A \in G$ has the form $A = \mathrm{diag}(A_1, \dots, A_r)$ with $A_k \in \mathbb{T}_{n_k}(\mathbb{C})$ $k = 1, \dots, r$. Denote by:
- $G_k = \{A_k, \quad A \in G\}$, $k = 1, \dots, r$.
- $E(u_k) = \mathrm{vect}(G_k(u_k))$, for every $u = [u_1, \dots, u_r]^T \in \mathbb{C}^n$.
- $\mathcal{G} = \mathrm{vect}(G)$, the vector subspace of $\mathcal{K}_{\eta,r}(\mathbb{C})$ generated by all elements of G.
- $U = \prod_{k=1}^{r} \mathbb{C}^* \times \mathbb{C}^{n_k - 1}$.

Lemma 3.1. *Let G be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Then for every $u \in \mathbb{C}^n$, $E(u)$ is G-invariant.*

Proof. Suppose that $E(u)$ is generated by $A_1 u, \dots, A_p u$, with $A_k \in G$, $1 \leq k \leq p$. Let $w = \sum_{k=1}^{p} \alpha_k A_k u \in E(u)$ and $B \in G$, then $Bw = \sum_{k=1}^{p} \alpha_k B A_k u$.

Since $BA_ku \in G(u) \subset E(u)$ then it has the form $BA_ku = \sum_{j=1}^{p} \beta_{k,j}A_ju$, $\beta_{k,j} \in \mathbb{C}$, so $Bw = \sum_{1\leq k,j\leq p} \alpha_k\beta_{k,j}A_ju \in E(u)$. □

Lemma 3.2. ([12], *Proposition 3.1*) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$, $u \in \mathbb{C}^n$ and $v \in E(u)$.*

(i) *Then there exist $B \in \mathcal{G}$ such that $Bu = v$.*
(ii) *If $E(u) = E(v)$, then $G(u)$ and $G(v)$ are isomorphic.*

Proposition 3.3. *Let G be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ and $u \in U$ such that $E(u) = \mathbb{C}^n$. Then for every $v \in U$ there exists $B \in \mathcal{G} \cap GL(n,\mathbb{C})$ such that $Bu = v$. In particular, $E(v) = \mathbb{C}^n$.*

Lemma 3.4. *Let G be an abelian subgroup of $\mathbb{T}^*_n(\mathbb{C})$ and $u \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. Then for every $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ there exist $B \in \mathcal{G} \cap GL(n,\mathbb{C})$ such that $Bu = v$.*

Proof. Let $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ since $E(u) = \mathbb{C}^n$, by lemma 3.2, there exist $B \in \mathcal{G}$ such that $Bu = v$. Since $\mathbb{T}_n(\mathbb{C})$ is a vector space so $\mathcal{G} \subset \mathbb{T}_n(\mathbb{C})$. Write $u = [x_1,\ldots,x_n]^T$, $v = [y_1,\ldots,y_n]^T$ and

$$B = \begin{bmatrix} \mu_B & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n,1} & \ldots & a_{n,n-1} & \mu_B \end{bmatrix},$$

then $\mu_B = \frac{y_1}{x_1} \neq 0$, hence $B \in GL(n,\mathbb{C})$. As $\mathcal{G} \subset \mathcal{C}(G)$, $B(E(u)) = E(v)$. □

Proof of Proposition 3.3. Write $u = [u_1,\ldots,u_r]^T$ and let $v = [v_1,\ldots,v_r]^T \in U$, with $u_k, v_k \in \mathbb{C}^{n_k}$. Since $E(u) = \mathbb{C}^n$, then $E(u_k) = \mathbb{C}^{n_k}$ for every $k = 1,\ldots,r$. As $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_r$ and $v_k \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, where $\mathcal{G}_k = Vect(G_k)$. Hence the proof results from Lemma 3.4. In particular, $E(v) = B(E(u)) = \mathbb{C}^n$. □

3.1. **Proof of Theorem 1.1.** Let $u \in \mathbb{C}^n$. By Proposition 2.8, we can assume that $G \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$ and $E(u) = \mathbb{C}^n$, otherwise, by Lemma 3.1 we replace G by the restriction $G_{/E(u)}$. By construction, $\mathbb{C}^n \backslash U$ is union of r G-invariant vector subspaces of $\mathbb{C}^n$ with dimension $n-1$, then we proves (iii) and we deduce that $u \in U$. Now, for every $v \in U$, there exists by Proposition 3.3, $B \in \mathcal{G} \cap GL(n,\mathbb{C})$ such that $Bu = v$. Hence $E(v) = B(E(u)) = \mathbb{C}^n$ and $G(v) = B(G(u))$, this proves (i) and (ii). □

4. **Parametrization**

Let G be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ and $u \in \mathbb{C}^n$, by Lemma 3.1, $E(u)$ is G-invariant, so consider the linear map

$$\begin{aligned} \Phi_u : vect\left(G_{/E(u)}\right) &\longrightarrow E(u) \\ A &\mapsto Au \end{aligned}$$

Proposition 4.1. *For every $u \in \mathbb{C}^n \backslash \{0\}$, Φ_u is a linear isomorphism.*

Proof. By construction, Φ_u is surjective, since $\Phi_u\left(\text{Vect}\left(G_{/E(u)}\right)\right) = E(u)$. - Φ_u is injective: let $A \in \text{Ker}(\Phi_u)$, so $Au = 0$. Let $x \in E(u)$, then by above there exists $B \in \text{Vect}\left(G_{/E(u)}\right)$ such that $x = Bu$. As $A \in \text{Ker}(\Phi_u) \subset vect\left(G_{/E(u)}\right)$ then $AB = BA$. Therefore $Ax = ABu = BAu = B(0) = 0$. It follows that $A = 0$ and hence $\text{Ker}(\Phi_u) = \{0\}$. □

Corollary 4.2. *We have $\Phi_u^{-1}(G(u)) = G_{/E(u)}$ and $\Phi_u^{-1}(\mathrm{g}_u) = \mathrm{g}_{/E(u)}$.*

Under the above notation, we have:

Proposition 4.3. *Let G be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Then*

$$exp(\Phi_u^{-1}(E(u))) \subset \Phi_u^{-1}(U_u).$$

To prove Proposition 4.3, we need the following Lemma:

Lemma 4.4. *Let G be an abelian subgroup of $\mathbb{T}^*_n(\mathbb{C})$. If $E(u) = \mathbb{C}^n$, then*

$$exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

Proof. Here $G_{/E(u)} = G$ and $U_u = U = \mathbb{C}^* \times \mathbb{C}^{n-1}$. First, one has $exp(\mathcal{G}) \subset \mathcal{G}$: indeed; for every $A \in \mathcal{G}$ we have $A^k \in \mathcal{G}$, for every $k \in \mathbb{N}$ and so $e^A = \sum\limits_{k\in\mathbb{N}} \frac{A^k}{k!} \in \mathcal{G}$. Moreover, we can check that $\mathcal{G}$ is the subalgebra of $M_n(\mathbb{C})$ generated by G.

Second, by corollary 4.2, $\Phi_u^{-1}(G(u)) = G$. Since $E(u) = \mathbb{C}^n$ and by Proposition 4.1, Φ_u is an isomorphism, then

$$\begin{aligned} exp(\Phi_u^{-1}(\mathbb{C}^n)) &= exp(\Phi_u^{-1}(E(u))) \\ &= exp(\mathcal{G}) \\ &\subset \mathcal{G} = \Phi_u^{-1}(\mathbb{C}^n) \end{aligned}$$

Therefore

$$exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^n) \quad (1)$$

On the other hand, by constraction of Φ_u and U, we have $u \in U$ and so $\Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1})$. As $exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \mathbb{T}_n^*(\mathbb{C})$ then by (1) we obtain

$$exp\left(\Phi_u^{-1}(\mathbb{C}^n)\right) \subset \Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

□

Proof of Proposition 4.3. Write $u = [u_1, \ldots, u_r]^T \in \mathbb{C}^n$. Suppose that $E(u) = \mathbb{C}^n$, otherwise we replace G by $G_{/E(u)}$. So $G_{/E(u)} = G$ and $U_u = U$. Since $exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} = exp_{/\mathbb{T}_{n_1}(\mathbb{C})} \oplus \cdots \oplus exp_{/\mathbb{T}_{n_r}(\mathbb{C})}$ and $\Phi_u^{-1} = \Phi_{u_1}^{-1} \oplus \cdots \oplus \Phi_{u_r}^{-1}$. Then by Lemma 4.4

$$exp(\Phi_u^{-1}(U)) = \prod_{k=1}^{r} exp_{/\mathbb{T}_{n_k}(\mathbb{C})}\left(\Phi_{u_k}^{-1}(\mathbb{C}^{n_k})\right) \subset \prod_{k=1}^{r} \Phi_{u_k}^{-1}(\mathbb{C}^* \times \mathbb{C}^{n_k - 1}) = \Phi^{-1}(U).$$

□

As consequence of Proposition 4.3, we have the following results:

Corollary 4.5. *The map $f = \Phi_u \circ exp_{/\Phi(\mathbb{C}^n)} \circ \Phi_u^{-1} : E(u) \longrightarrow U_u$ is well defined and continuous.*

5. Proof of Theorems 1.2, 1.5, 1.6 and Corollaries 1.3 and 1.4

By Proposition 2.8, suppose that G is an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ and then $\mathrm{g} = exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$.

In all this section fixed $u \in \mathbb{C}^n$ and suppose that $E(u) = \mathbb{C}^n$ and $G_{/E(u)} = G$, leaving to replace G by $G_{/E(u)}$. Denote by $\Phi = \Phi_u^{-1} : \mathbb{C}^n \longrightarrow \mathcal{G}$. By Corollary 2.2, $exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \longrightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$ is a local difeomorphism and by Proposition 4.1, Φ is an open map. Then we introduce the following Lemma which will be used in the proof of Theorem 1.2.

Lemma 5.1. *Let O' be an open subset of $\mathbb{C}^n$ such that $exp_{/\Phi(O')} : \Phi(O') \longrightarrow exp(\Phi(O'))$ is a diffeomorphism. Then*

$$exp\left(\Phi(O')\right) \cap \overline{exp(\mathrm{g})} = exp\left(\Phi(O') \cap \overline{\mathrm{g}}\right).$$

Proof. Let $A \in exp(\Phi(O')) \cap \overline{exp(\mathrm{g})}$, then $A = e^{\Phi(x)}$ for some $x \in O'$. By Lemma 2.7.(ii), $exp(\mathrm{g}) = G$, so $A \in G$. Since $A \in \overline{exp(\mathrm{g})}$, there exists a sequence $(B_m)_{m\in\mathbb{N}}$ in g such that $\lim\limits_{m\to+\infty} e^{B_m} = A = e^{\Phi(x)}$. Since $exp_{/\Phi(O')} : \Phi(O') \longrightarrow exp(\Phi(O'))$ is a diffeomorphism, $exp(\Phi(O'))$ is an open set containing e^A, so $e^{B_m} \in exp(\Phi(O'))$, $\forall m \geq p$, for some $p \in \mathbb{N}$. Then $B'_m = exp^{-1}_{/\Phi(O')}(e^{B_m}) \in \Phi(O')$, $\forall m \geq p$. Since $e^{B_m} \in exp(\mathrm{g}) = G$ and $\Phi(O') \subset \mathcal{K}_{\eta,r}(\mathbb{C})$, then $B'_m \in exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) = \mathrm{g}$, for every $m \geq p$. Therefore $\lim\limits_{m\to+\infty} B'_m = \Phi(x)$, so $\Phi(x) \in \Phi(O') \cap \overline{\mathrm{g}}$ and hence $A \in exp\left(\Phi(O')\right) \cap \overline{\mathrm{g}}$.

Conversely, by continuity of $exp_{/\Phi(O')} : \Phi(O') \longrightarrow exp(\Phi(O'))$, one has

$$exp\left(\Phi(O')\right) \cap \overline{\mathrm{g}} \subset exp\left(\Phi(O')\right) \cap exp\left(\overline{\mathrm{g}}\right) \subset exp\left(\Phi(O')\right) \cap \overline{exp(\mathrm{g})}.$$

The prove is completed. □

5.1. **Proof of Theorem 1.5.** The equivalence $(i) \iff (ii)$ follows from Theorem 1.1.(i).
$(i) \iff (iii)$: By Lemma 3.1, suppose that $E(u) = \mathbb{C}^n$and $U_u = U$ (leaving to replace G by $G_{/E(u)}$). Then by Proposition 4.1 and Corollary 4.2, $\Phi = \Phi_u^{-1} : \mathbb{C}^n \longrightarrow \mathcal{G}$ is an isomorphism satisfying $\Phi(G(u)) = G$ and $\Phi(\mathrm{g}_u) = \mathrm{g}$.

By Proposition 2.6.(i), there exist a vector space V, contained in $\overline{\mathrm{g}_u}$, a vector space W such that $V \oplus W = \mathbb{C}^n$ and $\overline{\mathrm{g}_u} = (\overline{\mathrm{g}_u} \cap W) \oplus V$, with $\overline{\mathrm{g}_u} \cap W$ is discrete.

By corollary 2.5 and Proposition 2.6, there exists an open subset O of $\mathbb{C}^n$ such that $O \cap \overline{\mathrm{g}_u} = V$. By Corollary 2.2, the exponential map $exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \longrightarrow \mathcal{K}^*_{\eta,r}(\mathbb{C})$ is a locally diffeomorphism, then there exists an open subset $O' \subset O$, of $\mathbb{C}^n$ such that the restriction $exp_{/\Phi(O')} : \Phi(O') \longrightarrow exp\left(\Phi(O')\right)$ of the exponential map on $\Phi(O')$ is a diffeomorphism. Since $O' \subset O$, then

$$O' \cap \overline{\mathrm{g}_u} = O' \cap V. \qquad (1)$$

Since $O' \subset U_u$ then by Corollary 4.5 the map $f_{/O'} = \Phi^{-1} \circ exp_{/\Phi(O')} \circ \Phi : O' \longrightarrow O'$ is well defined and as $exp_{/\Phi(O')}$ is a diffeomorphism then $f_{/O'}$ is a diffeomorphism. By Lemma 2.7, one has $exp(\mathrm{g}) = G$ and then:

$$\begin{aligned} f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ exp(\Phi(O')) \cap \overline{G(u)} \\ &= \Phi^{-1}\left(exp\left(\Phi(O')\right) \cap \overline{\Phi(G(u))}\right) \\ &= \Phi^{-1}\left(exp\left(\Phi(O')\right) \cap \overline{G}\right) \\ &= \Phi^{-1}\left(exp\left(\Phi(O')\right) \cap \overline{exp(\mathrm{g})}\right) \end{aligned}$$

By Lemma 5.1 and by (1), we obtain

$$\begin{aligned} f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ exp\left(\Phi(O') \cap \overline{\mathfrak{g}}\right) \\ &= \Phi^{-1} \circ exp\left(\Phi(O') \cap \Phi\left(\overline{\mathfrak{g}_u}\right)\right) \\ &= \Phi^{-1} \circ exp \circ \Phi\left(O' \cap \overline{\mathfrak{g}_u}\right) \\ &= \Phi^{-1} \circ exp \circ \Phi\left(O' \cap V\right) \\ &= f(O' \cap V). \end{aligned}$$

As $f(O')$ is an open subset of $\mathbb{C}^n$ and $O' \cap V$ is an open subset of the real vector space V, then $f(O') \cap \overline{G(u)}$ is a manifold with dimension $m = \dim(V) = \dim(\overline{\mathfrak{g}_u})$. We conclude that $G(u)$ is regular with order m. Since $\mathbb{C}^n$ is a real vector space with dimension $2n$ and V is a real subspace of $\mathbb{C}^n$ with dimension m, so $m \leq 2n$. We conclude the equivalence $(i) \Longleftrightarrow (iii)$. $\square$

5.2. **Proof of Corollary 1.3.**
$\diamond$ *Complex Case:* Suppose that $\mathbb{K} = \mathbb{C}$. Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $u \in \mathbb{C}^n$. By Theorem 1.5, the orbit $G(u)$ is regular with order m if and only if $\dim(\overline{\mathfrak{g}_u}) = m$. Since $\overline{\mathfrak{g}_u}$ is a closed additive subgroup of $\mathbb{C}^n$ (Lemma 2.7.(i)). Then $\dim(\overline{\mathfrak{g}_u}) \geq 0$, so $G(u)$ is regular with order $m \geq 0$. This proves the complex case.

$\diamond$ *Real Case:* Suppose that $\mathbb{K} = \mathbb{R}$. Let G be an abelian subgroup of $GL(n, \mathbb{R})$ and $x \in \mathbb{R}^n$. So G is considered as an abelian subgroup of $GL(n, \mathbb{C})$. By the above case, $G(x)$ is regular with some order m, with $m \leq 2n$. Then there exists an open subset $O = O_1 + iO_2$ of $\mathbb{C}^n$ with O_1, O_2 are open subsets of $\mathbb{R}^n$, such that $\overline{G(x)} \cap O$ is a manifold with dimension m. One has $0 \in O_2$, since $G(x) \subset \mathbb{R}^n$. Then $\overline{G(x)} \cap O = \overline{G(x)} \cap O_1$ and $m \leq n$. It follows that $G(x)$ is a regular orbit in $\mathbb{R}^n$ with order m. The proof is completed. $\square$

5.3. **Proof of Corollary 1.4.**

Lemma 5.2. ([1] *Corollary 1.3). If G has a locally dense orbit γ in $\mathbb{C}^n$ then γ is dense in $\mathbb{C}^n$.*

Proof of Corollary 1.4.
(i) If $\overline{G(u)} = \mathbb{R}^n$ then $\overline{G(u)}$ is a manifold with dimension n, so $G(u)$ is regular with order $m = n$.

Conversely, if $G(u)$ is regular with order $m = n$, then $\overline{G(u)} \cap O$ is a manifold with order $m = n$, for some open subset O of $\mathbb{R}^n$. Hence $\overline{G(u)} \cap O$ is an open subset of $\mathbb{R}^n$. Therefore $G(u)$ is locally dense.

(ii) We use the same proof of (i) and by Lemma 5.2 we have $\overline{G(u)} = \mathbb{C}^n$. $\square$

5.4. **Proof of theorem 1.6.** If $\overline{G(u)}$ is a vector subspace then $\overline{G(u)} = E(u)$, so $G(u)$ is regular with order $2r(u)$.
Conversely, if $G(u)$ is regular with order $2r(u)$, then $\overline{G(u)} \cap O$ is a manifold with dimension $2r(u)$, for some open set O. Since $\dim(E(u)) = 2r(u)$ over $\mathbb{R}$, then $\overline{G(u)} \cap O$ is an open subset of $E(u)$. So $G(u)$ is locally dense in $E(u)$. By lemma 5.2 applied on $G_{/E(u)}$ we have $\overline{G(u)} = E(u)$. □

5.5. **Algebric Lemmas.**

Lemma 5.3. ([11], *Proposition 4.3*). *Let $H = \mathbb{Z}u_1 + ... + \mathbb{Z}u_p$ with $u_k = [u_{k,1}, \dots, u_{k,n}]^T \in \mathbb{R}^n$, $k = 1, ..., p$. Then H is dense in $\mathbb{R}^n$ if and only if for every $(s_1, ..., s_p) \in \mathbb{Z}^p - \{0\}$:*

$$rank\left(\begin{bmatrix} u_{1,1} & \dots & \dots & u_{p,1} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1,n} & \dots & \dots & u_{p,n} \\ s_1 & \dots & \dots & s_p \end{bmatrix}\right) = n + 1.$$

Corollary 5.4. *Let $p \geq n+1$ and $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$, $u_k \in \mathbb{R}^n$, $1 \leq k \leq p$, such that $(u_1, \dots, u_n)$ is a basis of $\mathbb{R}^n$. If there exists $0 \leq m \leq n$ such that $u_k = \sum_{j=1}^{m} \alpha_{k,j} u_{j+n-m}$, for every $n+1 \leq k \leq p$. Then the following assertions are equivalent:*

(i) *$dim(\overline{H}) = m$.*
(ii) *$u_1, \dots, u_p$ satisfies property $\mathcal{D}(m)$.*

Proof. Let $E = \mathbb{R}u_{n-m+1} \oplus \cdots \oplus \mathbb{R}u_n$. We replace $\mathbb{R}^n$ by E in Lemma 5.3 and we obtain: $u_1, \dots, u_p$ satisfies property $\mathcal{D}(m)$ if and only if $K = \mathbb{Z}u_{n-m+1} + ... + \mathbb{Z}u_p$ is dense in E and this is equivalent to $dim(\overline{H}) = m$ since $\overline{H} = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_{n-m} + \overline{K}$. □

5.6. **Proof of Theorem 1.6.** Denote by‘:
- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T$ and $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $k = 1, \dots, r$.
- $v_0 = Pu_0$, where $P \in GL(n, \mathbb{C})$ is defined in Proposition 4.1 so that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Proposition 5.5. ([2], *Theorem 1.3*) *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ generated by $A_1, \dots, A_p$. Let $B_1, \dots, B_p \in \mathrm{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$. Then $\mathrm{g}_{v_0} = \sum_{k=1}^{p} \mathbb{Z}B_k v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}Pe^{(k)}$.*

By construction of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, remark that for every $u \in U$ there exists $Q \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$ such that $Qu_0 = u$. Then as a consequence of Proposition 5.5 we get the following Corollary:

Corollary 5.6. *Let G be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ generated by $A_1, \ldots, A_p$. Let $B_1, \ldots, B_p \in \mathrm{g}$ such that $A_k = e^{B_k}$, $k = 1, \ldots, p$. Then for every $u \in U$, $\mathrm{g}_u = \sum_{k=1}^{p} \mathbb{Z} B_k u + \sum_{k=1}^{r} 2i\pi\mathbb{Z} Q e^{(k)}$, where $Q \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$ such that $Qu_0 = u$.*

Proof of Theorem 1.6. The proof of Theorem 1.6 results from Theorem 1.2, Corollary 5.4 and Corollary 5.6. □

6. **Examples**

Let $\mathbb{D}_n(\mathbb{C}) = \{A = \mathrm{diag}(a_1, \ldots, a_n) \ : \ a_k \in \mathbb{C}^*, \ 1 \leq k \leq n \ \}$ and let G be an abelian subgroup of $\mathbb{D}_n(\mathbb{C})$. In this case we have $\widetilde{G} = G$ is a subgroup of $\mathcal{K}_{(1,\ldots,1),n}(\mathbb{C})$ and $r_G = n$.

Example 6.1. Let G be the group generated by $A_k = \mathrm{diag}(\lambda_{k,1}e^{i\alpha_{k,1}}, \ldots, \lambda_{k,n}e^{i\alpha_{k,n}})$, $k = 1, \ldots, p$, where $\lambda_{k,j} \in \mathbb{R}^*_+$, $\alpha_{k,j} \in \mathbb{R}$, $1 \leq j \leq n$. Let $u = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$, then the following assertions are equivalent:

(i) $G(u)$ is regular with order m.
(ii) $u_k = [(log\lambda_{k,1} + i\alpha_{k,1})x_1, \ldots, (log\lambda_{k,n} + i\alpha_{k,n})x_n]^T$, $1 \leq k \leq p$ with $2i\pi e_1, \ldots, 2i\pi e_n$, satisfies $\mathcal{D}(m)$.

Proof. We let $B_k = \mathrm{diag}(log\lambda_{k,1} + i\alpha_{k,1}, \ldots, log\lambda_{k,n} + i\alpha_{k,n})$,

One has $e^{B_k} = A_k$ and $B_k \in \mathbb{D}_n(\mathbb{C})$, $1 \leq k \leq p$. Then $B_k \in \mathrm{g}$. The result follows then from Theorem 1.6. □

Example 6.2. ([1], Example 6.2) Let G be the subgroup of $GL(4, \mathbb{R})$ generated by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then:

i) if $u \in (\mathbb{Q}^*)^2 \times \mathbb{R}^2$, $G(u)$ is discrete. So $G(u)$ is regular with order 0.
ii) if $u \in \mathbb{Q}^* \times (\mathbb{R}\backslash\mathbb{Q}) \times \mathbb{R}^2$, $G(u)$ is dense in a straight line. So $G(u)$ is regular with order 1.

In this example, G is considered as a subgroup of $GL(4,\mathbb{C})$. We have $\widetilde{G} = G$ is a subgroup of $\mathbb{T}_4^*(\mathbb{C})$ and $G(e_1) = \big\{[1,0,0,n+m]^T, \quad n,m \in \mathbb{Z}\big\}$, then $E(e_1) = \mathbb{C}e_1 + \mathbb{C}e_4$ and $U_{e_1} = \mathbb{C}^*e_1 + \mathbb{C}e_4$. So $E(e_1) \neq \mathbb{C}^n$.
Let $B_1 = A_1 - I_4$ and $B_2 = A_2 - I_4$. Since $B_1^2 = B_2^2 = 0$, so $e^{B_1} = A_1$ and $e^{B_2} = A_2$. By Theorem 1.6, $\mathrm{g}_u = \mathbb{Z}B_1u + \mathbb{Z}B_2u + 2i\pi\mathbb{Z}e_1$.
For every $u = [x,y,z,t]^T \in \mathbb{R}^* \times \mathbb{R}^3$, we have $\mathrm{g}_u = (\mathbb{Z}x + \mathbb{Z}y)e_4 + 2i\pi\mathbb{Z}e_1$, then:
-if $\frac{y}{x} \notin \mathbb{Q}$, then $\overline{\mathrm{g}_u} = \mathbb{R}e_4 + \mathbb{Z}e_1$, so $\dim(\overline{\mathrm{g}_u}) = 1$. By Theorem 1.6, $G(u)$ is regular with order 1.
-if $\frac{y}{x} \in \mathbb{Q}$, then $\overline{\mathrm{g}_u} = \mathbb{Z}ae_4 + \mathbb{Z}e_1$, for some $a \in \mathbb{R}$, so $\dim(\overline{\mathrm{g}_u}) = 0$. By Theorem 1.6, $G(u)$ is regular with order 0. □

A simple example for $n = 1$ and $\mathbb{K} = \mathbb{R}$ is given in the following, to show that the closure of a regular orbit is not necessarily a manifold.

Example 6.3. Let $\lambda > 1$ and G be the group generated by $\lambda.id_{\mathbb{R}}$, then for every $x \in \mathbb{R}^*$, we have $G(x) = \{\lambda^n,\ n \in \mathbb{Z}\}$ and $\overline{G(x)} = \{\lambda^n,\ n \in \mathbb{Z}\} \cup \{0\}$. Thus $G(x)$ is discrete, so it is regular with order 0, but $\overline{G(x)}$ is not a manifold.

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