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# Arc segmentation in linear time

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**Abstract.** A linear algorithm based on a discrete geometry approach is proposed for the detection of digital arcs and digital circles using a new representation of them. It is introduced by inspiring from the work of Latecki [1]. By utilizing this representation, we transform the problem of digital arc detection into a problem of digital straight line recognition. We then develop a linear method for arc segmentation of digital curves.

## 1 Introduction

The digital arcs and circles are basic geometric objects of which the recognition is an interesting topic. In the literature, some methods have been proposed for the recognition of digital circles. Nakamura et al. [2] proposed a recursive algorithm for determining the center of a digital circle, but its complexity is exponential in the general case. Kim [3, 4] proposed several results on digital disks. The first result [3] detects if a set of grid points in a  $N \times N$  image is a digital disk with complexity  $O(n^3)$ . The second result [4] reduces this task to  $O(n^2)$ . Based on the classical separating arc problem, Kovalevsky [5] (resp. Fisk [6]) proposed an algorithm for the recognition of a digital disk in  $O(n^2 \log n)$  (resp.  $O(n^2)$ ) time. Coeurjolly [7] transformed the problem of circle recognition into a problem of search a 2D point that belongs to the intersection of  $n^2$  half-plane. Sauer [8] (resp. Damaschke [9]) presented a linear algorithm to decide if a curve is a digital circle (resp. arc) based on Megiddo's algorithm [10]. Worring [11] introduced a digital circle segmentation method by using a fixed size window process. Roussillon [12] proposed a linear algorithm of circle recognition in 3 particular cases.

We present in this paper a linear method for the detection of digital circles or digital arcs based on a discrete geometry approach. Firstly, a polygonalization is applied in linear time on the input curve [13]. Secondly, we use a transform proposed by Latecki et al. [1] to represent the obtained polygon in a novel space called *tangent space*. We show that a sequence of chords of a circle will correspond to a sequence of collinear points in the tangent space. So the problem of arc/circle detection can be considered as a problem of digital straight line recognition.

This paper is organized as follows. Section 2 recalls some definitions concerning digital circles and blurred segments. The next section presents a technique to transform an arc into the tangent space and proposes some principal properties of the arc in this representation. Section 4 proposes a linear algorithm for the detection of digital arcs or digital circles. In Section 5, we present a linear method for the segmentation of a curve into arcs and some experimentations.

## 2 Discrete circle and blurred segment

**Discrete circle:** In the literature, there exist several definitions of discrete circle. They are proposed by considering a real circle on the grid digitization. The difference among

them is the process of discretization. Nakamura et al. [2] considered a discretization of a real circle by the points of  $\mathbb{Z}^2$  that are the nearest points of that circle. Kim [3] proposed a definition of discrete circle as a boundary of a digital disk superimposed by a real circle. Andres [14] used an arithmetic approach to define a digital circle as a sequence of points superimposed by a ring.

**Discrete line and blurred segment:** The notion of blurred segment [13] was introduced from the notion of arithmetical discrete line. An *arithmetical discrete line*, noted  $D(a, b, \mu, \omega)$ , is the set of points  $(x, y) \in \mathbb{Z}^2$  that verifies:  $\mu \leq ax - by < \mu + \omega$  with a main vector  $(b, a)$ , lower bound  $\mu$  and thickness  $\omega$ . A *width  $\nu$  blurred segment* (BS) is a set of points  $(x, y) \in \mathbb{R}^2$  that is optimally bounded (see [13] for more details) by a discrete line  $D(a, b, \mu, \omega)$  verifying  $\frac{\omega-1}{\max(|a|, |b|)} \leq \nu$ . Fig. 1 shows a BS of with 1.25 (the sequence of gray points) whose optimal bounding line is  $\mathcal{D}(5, 8, -8, 11)$ . A linear method for recognition of BS has been also proposed in [13].

### 3 Arc representation in tangent space

**Modified tangent space representation:** We recall in this section some notions concerning a representation of a polygon in the tangent space. Latecki et al. [1] proposed *the tangent space representation* as a tool of similarity measure for shape matching. Inspired from this representation, we propose a modified tangent space to represent a polygonal curve. The difference is that we do not normalize the axis  $Ox$  in the tangent space. Let  $C = \{C_i\}_{i=0}^n$  be a polygonal curve,  $\alpha_i = \angle(\overrightarrow{C_{i-1}C_i}, \overrightarrow{C_iC_{i+1}})$  and  $l_i$  - the length of the line segment  $C_iC_{i+1}$ ,  $i \in \{0, \dots, n-1\}$ . If  $C_{i+1}$  is on the right of  $\overrightarrow{C_{i-1}C_i}$  then  $\alpha_i > 0$ , otherwise  $\alpha_i < 0$ . From now, we denote  $P.x$  (resp.  $P.y$ ) to indicate the x (resp. y)-coordinate of point  $P$ . We consider a transformation that associates the polygon  $C$  of  $\mathbb{Z}^2$  to a polygon of  $\mathbb{R}^2$  that is constituted by line segments  $T_{i2}T_{(i+1)1}, T_{(i+1)1}T_{(i+1)2}$  for  $i$  from 0 to  $n-1$  with

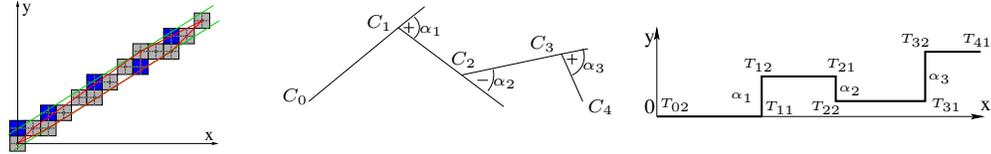
$$\begin{aligned} T_{02} &= (0, 0), \\ T_{i1} &= (T_{(i-1)2}.x + l_{i-1}, T_{(i-1)2}.y), \quad i \text{ from } 1 \text{ to } n, \\ T_{i2} &= (T_{i1}.x, T_{i1}.y + \alpha_i), \quad i \text{ from } 1 \text{ to } n-1. \end{aligned}$$


Fig. 1. A BS [13].

Fig. 2. Transformation to the tangent space: on the left, the input polygonal curve and on the right, its tangent space representation.

**Properties of arcs in the modified tangent space representation** The theorem below allows us to study the properties of a representation in the tangent space of a polygon that corresponds to an arc or a circle.

**Theorem 1.** Let  $C = \{C_i\}_{i=0}^n$  be a polygon,  $\alpha_i = \angle(\overrightarrow{C_{i-1}C_i}, \overrightarrow{C_iC_{i+1}})$ . The length of  $C_iC_{i+1}$  is  $l_i$ , for  $i \in \{0, \dots, n-1\}$ . The vertices of  $C$  are on a real arc of radius  $R$  and of center  $O$  such that  $\angle C_iOC_{i+1} \leq \frac{\pi}{4}$  for  $i \in \{1, \dots, n-1\}$ . This results below is obtained.

$$\frac{1}{R} < \frac{\alpha_i}{\frac{l_i + l_{i+1}}{2}} < \frac{1}{0.9742979R}$$



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**Algorithm 1:** Detection of a digital arc/circle

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**Data:**  $P = \{P_i\}_{i=0}^n$  digital curve,  $\alpha_{max}$  - maximal admissible angle,  $\nu_1$ - width of BS for polygonalization,  $\nu_2$ - width of BS for collinear test <sup>1</sup>

**Result:** ARC (resp. CIRCLE):  $C$  is a digital arc (resp. circle), FALSE if not.

**begin**

- Use algorithm [13] to polygonalize  $P$  into BS of width  $\nu_1$ :  $C = \{C\}_{i=0}^m$ ;
- Represent  $C$  in the modified tangent space by  $T(C)$ ;  $BS = \emptyset$ ;
- if** there exists  $i$  such that  $|T_{i2}.y - T_{i1}.y| > \alpha_{max}$  **then return** FALSE;
- Determine midpoint set  $MpC = \{M_i\}_{i=0}^{m-1}$  of  $\{T_{i2}T_{(i+1)1}\}_{i=0}^{m-1}$ ;  $i = 1$ ;
- while**  $|M_i.y - M_0.y| \leq 2\pi$  **do**  $BS = BS \cup M_i$ ;  $i++$ ;
- Use algorithm [13] to verify if  $BS$  is a blurred segment of width  $\nu_2$ ;
- if**  $BS$  is a blurred segment of width  $\nu_2$  **then**
  - if**  $|M_{m-1}.y - M_0.y| == 2 * \pi$  **then return** CIRCLE;
  - else return** ARC;
- else return** FALSE;

**end**

---

evident that we can lightly change Algorithm 1 to check the condition of Corollary 1 and Proposition 2 in linear time also.

The algorithms of polygonalization and of blurred segment recognition are linear. The complexity of the tangent space transform, and the construction of midpoint curve  $MpC = \{M_i\}_{i=1}^{m-1}$  is in  $O(m)$ . Because  $m \ll n$ , the total complexity of our detection method is then in  $O(n)$ .

**Segmentation of curves into digital arcs** Based on the above idea for detection of an arc, we then develop a linear method for the segmentation of digital arcs by using a width  $\nu$  blurred segment [13] polygonalization on the curve of midpoints. Its main idea, illustrated in figure 5, is based on the polygonalization of the midpoint curve (Fig. 5.e). Contrariwise to Algorithm 1, we polygonalize the midpoint curve  $MpC$  in spite of recognizing if  $MpC$  is a BS and then each line segment corresponds to a circular arc.

**Experimental results and application to real images** We have implemented this linear method. An example of a curve segmented into arcs is presented in Fig. 5. Firstly, the approximating polygon (see Fig. 5.b) is constructed from the input curve in Fig. 5.a. After that, we transform it into the modified tangent space representation (see Fig. 5.c). Then, by polygonalizing the curve of midpoints in this tangent space (see Fig. 5.d), the corresponding arcs can be detected (see Fig. 5.e).

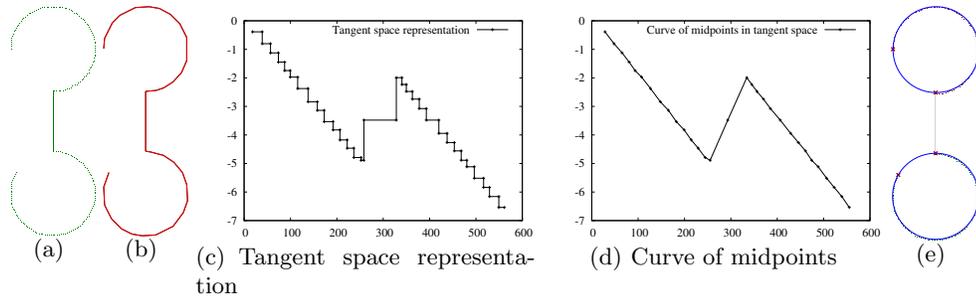
Fig. 6 shows an experimentation on technical drawing images. Figs. 6.a, 6.d are input images. Figs. 6.c, 6.f present the extracted arcs from the borders presented in 6.b, 6.e. Our method gives good results on this type of images which frequently contain arc and circle primitives. Fig. 7 presents our obtained result with a real image.

## 5 Study of quasi collinearity property of midpoint curve

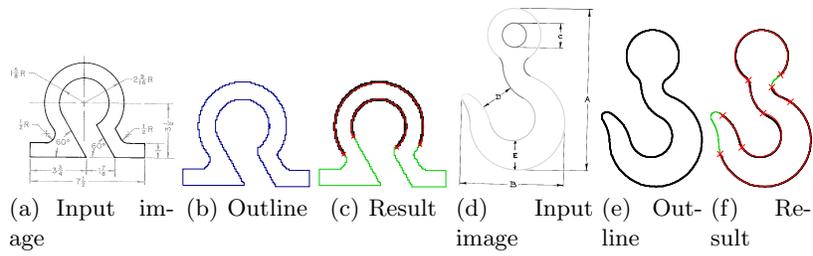
Algorithm 1 works well in practice. However, we have a problem to estimate error approximation in general case when the value of  $\nu_2$  is fixed. In this section, we present a first study concerning the utilization of this algorithm where  $\nu_2$  is chosen adaptively.

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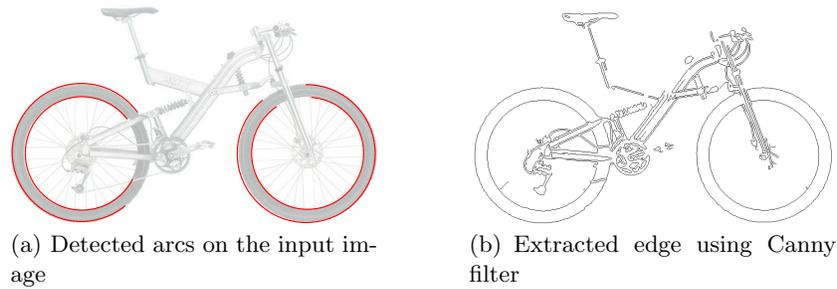
<sup>1</sup> By default,  $\alpha_{max} = \frac{\pi}{4}$ ,  $\nu_1 = 1$  for normal curves and  $\nu_2 = 0.2$  (see algo. 1).



**Fig. 5.** Arc segmentation results on a digital curve: (a) Input curve, (b) Approximated polygon, (c) Results of arc segmentation.



**Fig. 6.** Experimentation on technical drawing images.



**Fig. 7.** Experimentation on a real image at width 2.

Let us suppose that  $\alpha_{max} = \max\{\alpha_i\}_{i=1}^n$ . Let us suppose that  $R_i$  is the radius of the approximating circle that passes through 3 points  $C_{i-1}, C_i, C_{i+1}$ ;  $\alpha_{i1} = \angle H_{i-1}OC_i$ ,  $\alpha_{i2} = \angle H_iOC_i$  (see Fig. 3). We suppose that  $\alpha_{i1}, \alpha_{i2} \leq \frac{\pi}{8}$  for  $i = 1, \dots, n-1$  to guarantee the condition  $\sin x \simeq x$  in Theorem 1. It means that we consider the condition  $\sin x \simeq x$  with  $x \in [0, \frac{\pi}{8}]$ . Therefore, we have  $\alpha_i \leq \frac{\pi}{4}$ .

### Comparison of radius of local circumcircles to the global radius

**Proposition 1.** Let  $C = \{C_i\}_{i=0}^n$  be a polygon,  $\alpha_i = \angle(\overrightarrow{C_{i-1}C_i}, \overrightarrow{C_iC_{i+1}})$ . The length of  $C_iC_{i+1}$  is  $l_i$ , for  $i \in \{0, \dots, n-1\}$ . We denote  $O_i$  (resp.  $R_i$ ) respectively the center (resp. the radius) of circumcircle that passes through 3 points  $C_{i-1}, C_i, C_{i+1}$ ,  $H_i$  the projection of  $O_i$  on  $C_iC_{i+1}$ . Suppose that  $R_i - OH_i \leq h$  for  $i \in \{1, \dots, n-1\}$ . This results below is obtained.  $R_i\alpha_i \geq \frac{l_{i-1}+l_i}{2} \geq R_i\alpha_i - 0.3377h\alpha_i$

*Proof.* We denote  $\alpha_{i1} = \angle H_{i-1}O_iC_i$ ,  $\alpha_{i2} = \angle H_iO_iC_i$  (see Fig. 3). Firstly,  $\cos \alpha_{i1} = 1 - 2 \sin^2 \frac{\alpha_{i1}}{2} = \frac{OH_i}{R_i} \geq \frac{R_i-h}{R_i} = 1 - \frac{h}{R_i}$ . In addition, thanks to  $\alpha_{i1} \leq \frac{\pi}{8}$  and  $\frac{\sin(x)}{x}$  is decreasing in  $[0, \frac{\pi}{16}]$ , we have  $\sin x \geq x \frac{\sin \frac{\pi}{16}}{\frac{\pi}{16}}$ . Therefore  $\frac{h}{R_i} \geq 2 \sin^2 \frac{\alpha_{i1}}{2} \geq 2 \cdot \left(\frac{\sin \frac{\pi}{16}}{\frac{\pi}{16}}\right)^2 \left(\frac{\alpha_{i1}}{2}\right)^2 > \frac{0.9872}{2} \alpha_{i1}^2$ . Similarly, we have  $\frac{h}{R_i} > \frac{0.9872}{2} \alpha_{i2}^2$ , for  $i \in \{1, \dots, n-1\}$  (1).

In addition, we have this remark  $x \geq \sin x \geq x - \frac{x^3}{6}$  with  $\frac{\pi}{4} \geq x \geq 0$ . So,  $\alpha_i \geq \sin \alpha_{i1} + \sin \alpha_{i2} > \alpha_{i1} + \alpha_{i2} - \frac{1}{6}(\alpha_{i1}^3 + \alpha_{i2}^3) = (\alpha_{i1} + \alpha_{i2})(1 - \frac{1}{6}(\alpha_{i1}^2 - \alpha_{i1}\alpha_{i2} + \alpha_{i2}^2)) = \alpha_i(1 - \frac{1}{6}(\alpha_{i1}^2 - \alpha_{i1}\alpha_{i2} + \alpha_{i2}^2)) \geq \alpha_i(1 - \frac{1}{6}(\alpha_{i1}^2 + \alpha_{i2}^2))$ , for  $i \in \{1, \dots, n-1\}$  (2).

Thanks to (1) and (2), we obtain  $\sin \alpha_{i1} + \sin \alpha_{i2} > \alpha_i(1 - \frac{1}{3 \cdot 0.9872} \frac{h}{R_i})$ , for  $i \in \{1, \dots, n-1\}$  (3).

Moreover, we have  $\alpha_i = \alpha_{i1} + \alpha_{i2}$  and  $\alpha_{i1}, \alpha_{i2} \leq \frac{\pi}{8}$ . In addition, we have  $\sin \alpha_{i1} = \frac{l_{i-1}}{2R}$ ,  $\sin \alpha_{i2} = \frac{l_i}{2R}$ . Therefore, we have  $\frac{l_{i-1}+l_i}{2R} = \sin \alpha_{i1} + \sin \alpha_{i2}$ , for  $1 \leq i < n$  (4).

Thanks to (3) and (4), we obtain  $R_i\alpha_i \geq \frac{l_{i-1}+l_i}{2} \geq R_i\alpha_i(1 - \frac{1}{3 \cdot 0.9872} \frac{h}{R_i}) = \alpha_i(R_i - \frac{h}{3 \cdot 0.9872}) \Leftrightarrow R_i\alpha_i \geq \frac{l_{i-1}+l_i}{2} \geq R_i\alpha_i - 0.3377h\alpha_i$ .

Now we consider a set of midpoints  $MpC$  is a blurred segment whose horizontal width is  $\epsilon$ . Let us suppose that the slope of this blurred segment is  $\frac{1}{R}$ ,  $R \in \mathbb{R}$ .

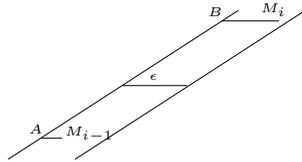


Fig. 8.

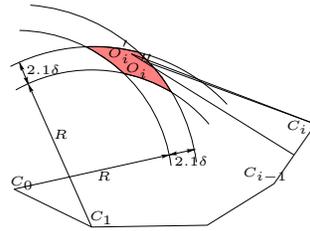


Fig. 9.  $\{O_i, O_i'\}_{i=1}^{n-1}$  is in a compact zone.

Let us consider Fig. 8, A and B respectively are horizontal projection of  $M_{i-1}$  and  $M_i$  on the left leaning line. We have:  $M_i.x - M_{i-1}.x = (B.x - A.x) + M_iB - AM_{i-1} = R\alpha_i + M_iB - AM_{i-1}$ . Because of  $M_i$  and  $M_{i-1}$  are limited by 2 leaning lines, we have  $M_iB \leq \epsilon$  and  $AM_{i-1} \leq \epsilon$ , so  $-\epsilon \leq M_iB - AM_{i-1} \leq \epsilon$ . Therefore,

we have  $R\alpha_i + \epsilon \geq \frac{l_{i-1}+l_i}{2} \geq R\alpha_i - \epsilon$ . This double inequations can be rewritten as  $\frac{l_{i-1}+l_i}{2} = R\alpha_i + \epsilon_i$ , where  $\epsilon_i \in [-\epsilon, \epsilon]$ .

Thank to proposition 1, we have:  $R_i\alpha_i > R\alpha_i + \epsilon_i > R_i\alpha_i - 0.3377h\alpha_i \Leftrightarrow R_i > R + \frac{\epsilon_i}{\alpha_i} > R_i - 0.3377h$ . So,  $\frac{\epsilon}{\alpha_i} < R_i - R < \frac{\epsilon}{\alpha_i} + 0.3377h$ . So, we have corollary 1.

**Corollary 1** *Let  $C = \{C_i\}_{i=0}^n$  be a polygon,  $\alpha_i = \angle(\overrightarrow{C_{i-1}C_i}, \overrightarrow{C_iC_{i+1}})$ . The length of  $C_iC_{i+1}$  is  $l_i$ , for  $i \in \{0, \dots, n-1\}$ . The set of midpoints  $\{M_i\}_{i=0}^{n-1}$  is a blurred segment whose horizontal width is  $\epsilon$ . We denote  $O_i$  (resp.  $R_i$ ) respectively the center (resp. the radius) of circumcircle that passes through 3 points  $C_{i-1}, C_i, C_{i+1}$ ,  $H_i$  the projection of  $O_i$  on  $C_iC_{i+1}$ . Suppose that  $R_i - OH_i \leq h$  for  $i \in \{1, \dots, n-1\}$ . This results below is obtained.  $0 < R_i - R < \frac{\epsilon}{\alpha_i} + 0.3377h \leq \frac{\epsilon}{\min\{\alpha_i\}_{i=1}^n} + 0.3377h$ .*

**Localization of centers of local circumcircles** In this section, we consider the convergence of local circumcircle centers if this condition below is satisfied:  $|R_i - R| \leq \delta$ ,  $R \in \mathbb{R}$ ,  $1 \leq i, j \leq n-1$ . Let us consider Fig. 10. To prove the convergence of centers of local circumcircle, we show this property for an approximation of a half of circle.

**Proposition 2.** *Let us consider a sequence of points  $\{C\}_{i=0}^n$ . There exist  $R$  and  $\delta$  such that  $R, \delta \in \mathcal{R}$ ,  $0 \leq R_i - R \leq \delta$ ,  $i = 1, \dots, n-1$ . Suppose that  $\angle C_k C_j C_{j+1} > \frac{\pi}{2}$  for  $k \in \{0, 1\}$ ,  $k < j < n$ . Therefore, we have this property  $0 \leq R'_i - R \leq \delta$ ,  $0 \leq R''_i - R \leq \delta$ , for  $1 \leq i \leq n-1$ .*

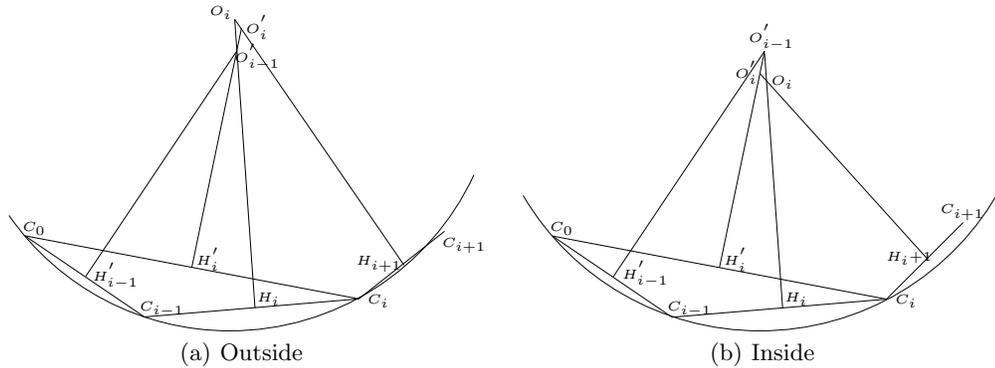
*Proof.* We denote  $O'_i$  (resp.  $O''_i$ ) and  $R'_i$  (resp.  $R''_i$ ) the centers and radius of circumcircles that passes through 3 points  $C_0$  (resp.  $C_1$ ),  $C_i$ ,  $C_{i+1}$ . Firstly, we have a trivial remark: the perpendicular bisector of  $C_iC_k$  is between that of  $C_iC_j$  and  $C_jC_k$ . Now, we prove this proposition by induction.

Because of  $R'_1 = R_1$ , the proposition is true with  $i = 1$ . Suppose that  $|R'_{i-1} - R| \leq \delta$ . Let us consider Fig. 10. We denote  $H'_i$ ,  $H_i$  are respectively the midpoint of  $C_0C_i$ ,  $C_{i-1}C_i$ . Thank to the above remark, we have  $O'_{i-1}H'_i$  is between  $O'_{i-1}H'_{i-1}$  and  $O'_{i-1}H_i$ . We consider now the position of  $C_{i+1}$  with the circle of center  $O'_{i-1}$ , of radius  $R'_{i-1}$ . The proposition is trivial if  $C_{i+1}$  is on this circle because of  $R'_i = R'_{i-1}$ . If  $C_{i+1}$  is outside of this circle (see Fig. 10.a), we then deduce  $O'_{i-1} \in [O'_iH'_i]$ ,  $O'_i \in [O_iH_{i+1}]$ . Therefore, we have  $O_{i+1}C_i > O'_iC_i$ ,  $O'_iC_i > O'_{i-1}C_i$ . It means that  $R_{i+1} > R'_i > R'_{i-1}$ . Thank to  $0 \leq R_{i+1} - R \leq \delta$  and  $0 \leq R'_{i-1} - R \leq \delta$ , we have  $0 \leq R'_i - R \leq \delta$ . In other case (see Fig. 10.b), by applying the same arguments, we obtain  $R_{i+1} < R'_i < R'_{i-1}$ . Therefore, we have  $0 \leq R'_i - R \leq \delta$ , for  $1 \leq i \leq n-1$ . By replacing  $C_0$  by  $C_1$  and using the same argument, we have  $0 \leq R''_i - R \leq \delta$ , for  $1 \leq i \leq n-1$ .

We have a simple remark: a triangle ABC that satisfies  $\angle ABC \geq \frac{7\pi}{8}$  have :  $AC > BC + \cos \frac{\pi}{8} \cdot AB$ . It is trivial because of  $AC^2 = BC^2 + AB^2 - 2BC \cdot AB \cos \angle ABC > BC^2 + \cos^2 \frac{\pi}{8} AB^2 + 2BC \cdot AB \cos \frac{\pi}{8}$ . Therefore, we have:  $O'_iO''_i \leq \frac{|R'_i - R''_i|}{\cos \frac{\pi}{8}} \leq \frac{\delta}{\cos \frac{\pi}{8}} \leq 1.1\delta$ , for  $i = 1, \dots, n-1$ . Thanks to this result and proposition 2, it is trivial now to show that set of center  $O'_i, O''_i$  is in a compact zone (see Fig. 9).

## 6 Conclusions

We have presented a linear method for the detection of digital circles or digital arcs. A linear method for the segmentation of a curve into digital arcs is also proposed.



**Fig. 10.** Position between  $C_{i+1}$  and circumcircle of  $C_0 C_{i-1} C_i$ .

This method is based on a discrete geometry approach. It is simple, easy and robust to implement. A more complete demonstration of the algorithm is under process.

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