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Marcin Dudziński. The almost sure central limit theorems in the joint version for the maxima and sums of certain stationary Gaussian sequences. *Statistics and Probability Letters*, 2009, 78 (4), pp.347. 10.1016/j.spl.2007.07.007 . hal-00594454

**HAL Id: hal-00594454**

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Submitted on 20 May 2011

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## Author's Accepted Manuscript

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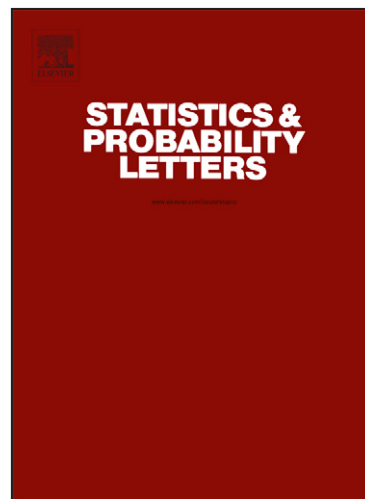
PII: S0167-7152(07)00247-7  
DOI: doi:10.1016/j.spl.2007.07.007  
Reference: STAPRO 4713

To appear in: *Statistics & Probability Letters*

Received date: 3 November 2004  
Revised date: 30 March 2007  
Accepted date: 25 July 2007

Cite this article as: Marcin Dudziński, The almost sure central limit theorems in the joint version for the maxima and sums of certain stationary Gaussian sequences, *Statistics & Probability Letters* (2007), doi:[10.1016/j.spl.2007.07.007](https://doi.org/10.1016/j.spl.2007.07.007)

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The almost sure central limit theorems in the joint version for the maxima and sums of certain stationary Gaussian sequences

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**Abstract**

Suppose that  $X_1, X_2, \dots$  is a standardized stationary Gaussian sequence. Let:  $M_n := \max(X_1, \dots, X_n)$ ,  $S_n := \sum_{i=1}^n X_i$ ,  $\sigma_n := \sqrt{\text{Var}(S_n)}$ , and  $a_n > 0$ ,  $b_n$  denote suitable normalizing constants. Our goal is to prove the almost sure central limit theorem for the sequence  $\{a_n(M_n - b_n), S_n/\sigma_n\}$ , under certain additional assumptions on the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$ .

MSC: primary 60F15, secondary 60F05.

Key words and phrases: Almost sure central limit theorem; Extreme values; Partial sums; Stationary Gaussian sequences; Associated random variables; Slowly varying functions at infinity; Karamata's theorem.

**1. Introduction**

The almost sure central limit theorem (ASCLT) has become an intensively studied subject in recent research. In our paper, we are concerned with the ASCLT in its two-dimensional version. A general outline of the investigations concerning the ASCLT in such a form may be described as follows. Suppose that  $X_1, X_2, \dots$  is a sequence of r.v.'s,  $f_1, f_2, \dots, f_k, \dots$  and  $g_1, g_2, \dots, g_k, \dots$  are some real-valued measurable functions, defined on  $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^k, \dots$ , respectively. We seek conditions under which, for some nondegenerate d.f.  $H$ , defined on  $\mathbb{R}^2$ ,

$$\frac{1}{D_N} \sum_{n=1}^N d_n I \{f_n(X_1, X_2, \dots, X_n) \leq x, g_n(X_1, X_2, \dots, X_n) \leq y\} \xrightarrow{\text{a.s.}} H(x, y) \quad (1)$$

for all  $(x, y) \in C_H$ , where:  $\{d_n\}$  is some sequence of weights,  $D_N = \sum_{n=1}^N d_n$ ,  $I$  stands for the indicator function,  $C_H$  denotes the set of continuity points of  $H$ .

In our investigations, we will restrict ourselves to the situation, when (1) is satisfied with:  $d_n = 1/n$ ,  $D_N \sim \log N$ ,  $f_n(X_1, \dots, X_n) := a_n(M_n - b_n)$ ,  $g_n(X_1, \dots, X_n) := S_n/\sigma_n$ , where:  $M_n := \max(X_1, \dots, X_n)$ ,  $S_n := X_1 + \dots + X_n$ ,  $\sigma_n := \sqrt{\text{Var}(S_n)}$ , and  $a_n > 0$ ,  $b_n$  are certain normalizing constants. The purpose of our paper is to prove that if:  $X_1, X_2, \dots$  is a standardized stationary Gaussian sequence, the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies

$$r(t) = L(t)/t^\alpha, \quad t = 1, 2, \dots, \quad (2)$$

where  $\alpha > 0$  and  $L(\cdot)$  is a positive, slowly varying function at infinity, and if moreover, the numerical sequence  $\{u_n\}$  fulfills the relation

$$\lim_{n \rightarrow \infty} n(1 - \Phi(u_n)) = \tau \text{ for some } \tau, 0 \leq \tau < \infty, \quad (3)$$

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where  $\Phi$  is the standard normal d.f., then

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \{M_n \leq u_n, S_n/\sigma_n \leq y\} = e^{-\tau} \Phi(y) \text{ a.s. for any } y \in \mathbb{R}. \quad (4)$$

As a direct conclusion, we will also show that if, for  $n \geq 3$ :

$$a_n := (2 \log n)^{1/2}, \quad b_n := (2 \log n)^{1/2} - \frac{1}{2} (2 \log n)^{-1/2} (\log \log n + \log 4\pi), \quad (5)$$

then for all  $x, y \in \mathbb{R}$

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \{a_n (M_n - b_n) \leq x, S_n/\sigma_n \leq y\} = \exp(-e^{-x}) \Phi(y) \text{ a.s.} \quad (6)$$

It should be mentioned that, although  $r(t) = L(t)/t^\alpha$  in (2) is defined only for positive integers, the slowly varying functions  $L(\cdot)$  are in general defined on the real line or on the  $\mathbb{R}_+$  semi-line.

The earlier given notations -  $M_n, S_n, \sigma_n, r(t), \Phi$  - will be intensively used throughout the paper. For our convenience, we will also introduce the following one  $M_{m,n} := \max(X_{m+1}, \dots, X_n)$ . Furthermore,  $f(n) \ll g(n)$  and  $f(n) \sim g(n)$  will stand for  $f(n) = \mathcal{O}(g(n))$  and  $f(n)/g(n) \rightarrow 1$ , as  $n \rightarrow \infty$ , respectively.

## 2. Main result

Our main result is the following almost sure central limit theorem in the joint version for the maxima and sums of some standardized stationary Gaussian sequences.

**Theorem 1** . *Let  $X_1, X_2, \dots$  be a standardized stationary Gaussian sequence. Assume that the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies (2) for some  $\alpha > 0$ . Then:*

- (i) *If the numerical sequence  $\{u_n\}$  fulfills (3), then (4) holds,*
- (ii) *If  $\{a_n\}, \{b_n\}$  are such as in (5), then (6) holds.*

## 3. Auxiliary results

In this section, we state and prove some lemmas, which will be needed for the proof of our main result. Here, we present the first one.

**Lemma 1** . *Let  $X_1, X_2, \dots$  be a standardized stationary Gaussian sequence. Assume that the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies assumption (2) with some  $\alpha > 0$ . Suppose moreover that the sequence  $\{u_n\}$  fulfills (3), and that  $m < n$ . Then for any  $y \in \mathbb{R}$  and some  $\gamma > 0$*

$$E |I \{M_n \leq u_n, S_n/\sigma_n \leq y\} - I \{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\}| \ll 1/n^\gamma + m/n. \quad (7)$$

**Proof.** Assume first that (2) holds with some  $0 < \alpha < 1$ . Let  $1 \leq i \leq n$ . We have

$$0 < \text{Cov}(X_i, S_n/\sigma_n) < \frac{2}{\sigma_n} \sum_{t=0}^{n-1} r(t) = \frac{2}{\sigma_n} + \frac{2}{\sigma_n} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \text{ for some } 0 < \alpha < 1. \quad (8)$$

As  $0 < \alpha < 1$ , then, by the two times application of Karamata's theorem, we get (see also Mielniczuk (2002), p. 394)

$$\sigma_n \sim C(\alpha) L(n)^{1/2} n^{1-\alpha/2}, \text{ where } C(\alpha) = \{2/(1-\alpha)(2-\alpha)\}^{1/2}. \quad (9)$$

Due to (8), (9), we have for some  $0 < \alpha < 1$  that

$$\text{Cov}(X_i, S_n/\sigma_n) \ll \frac{1}{L(n)^{1/2} n^{1-\alpha/2}} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \ll \frac{L(n) n^{1-\alpha}}{L(n)^{1/2} n^{1-\alpha/2}} = \frac{L(n)^{1/2}}{n^{\alpha/2}}.$$

Since  $L(n)$  is a slowly varying function at infinity, then  $L(n)^{1/2} \ll n^\epsilon$  for arbitrary  $\epsilon > 0$ . Consequently

$$\text{Cov}(X_i, S_n/\sigma_n) \ll \frac{n^\epsilon}{n^{\alpha/2}} \text{ for any } \epsilon > 0 \text{ and some } 0 < \alpha < 1.$$

Therefore

$$0 < \sup_{1 \leq i \leq n} \text{Cov}(X_i, S_n/\sigma_n) \ll 1/n^\beta \text{ for some } 0 < \beta < 1/2. \quad (10)$$

Since  $\lim_{n \rightarrow \infty} 1/n^\beta = 0$ , there exist numbers  $\lambda, n_0$ , such that

$$0 < \sup_{1 \leq i \leq n} \text{Cov}(X_i, S_n/\sigma_n) < \lambda < \beta/(1-\beta) < 1 \text{ for any } n > n_0. \quad (11)$$

In turn, by assumption (2), we have  $\lim_{t \rightarrow \infty} r(t) = \lim_{t \rightarrow \infty} L(t)/t^\alpha = 0$ . Thus, there exist numbers  $\delta, n_1$ , such that

$$0 < \sup_{t > n_1} r(t) = \delta < \alpha/(2-\alpha) < 1. \quad (12)$$

Relations (10)-(12) will be intensively used in the next stages of our proof.

Let  $y$  be an arbitrary real number and  $m < n$ . We have

$$\begin{aligned} & E |I\{M_n \leq u_n, S_n/\sigma_n \leq y\} - I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\}| \\ &= P\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\} - P\{M_n \leq u_n, S_n/\sigma_n \leq y\}. \end{aligned}$$

Suppose moreover that  $Y_n$  is a random variable, which has the same distribution as  $S_n/\sigma_n$ , but is independent of  $(X_1, \dots, X_n)$ . We can write that

$$\begin{aligned} & E |I\{M_n \leq u_n, S_n/\sigma_n \leq y\} - I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\}| \\ &\leq |P\{M_n \leq u_n, S_n/\sigma_n \leq y\} - P\{M_n \leq u_n\} P\{Y_n \leq y\}| \\ &\quad + |P\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\} - P\{M_{m,n} \leq u_n\} P\{Y_n \leq y\}| \\ &\quad + (P\{M_{m,n} \leq u_n\} - P\{M_n \leq u_n\}) =: A_1 + A_2 + A_3. \end{aligned} \quad (13)$$

Since  $(X_1, \dots, X_n, S_n/\sigma_n), (X_1, \dots, X_n, Y_n)$  are standard normal random vectors and (11), (12) hold, then, by Theorem 4.2.1 in Leadbetter et al. (1983), the fact that  $Y_n$  is independent of  $(X_1, \dots, X_n)$  and relation (10), we obtain

$$A_1 + A_2 \ll \sum_{i=1}^n \text{Cov}(X_i, S_n/\sigma_n) \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \ll n \frac{1}{n^\beta} \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \quad (14)$$

for some  $0 < \beta < 1/2$ , where  $\lambda$  is such as in (11).

As  $\{u_n\}$  satisfies (3), then, by (4.3.4(i)), (4.3.4(ii)) in Leadbetter et al. (1983),

$$\exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \sim L \frac{(\log n)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}} \text{ for some } L, \text{ not depending on } n. \quad (15)$$

From (14) and (15), we have

$$A_1 + A_2 \ll n \frac{1}{n^\beta} \frac{(\log n)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}} = \frac{(\log n)^{\frac{1}{2(1+\lambda)}}}{n^{\beta + \frac{1}{1+\lambda} - 1}}. \quad (16)$$

As, by (11),  $\lambda < \beta/(1-\beta)$ , we have  $\beta + \frac{1}{1+\lambda} - 1 > 0$ . Thus, by (16),

$$A_1 + A_2 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (17)$$

It remains only to estimate the term  $A_3$  in (13). By going through the first lines in the proof of Lemma 2.4 from Csaki and Gonchigdanzan (2002), we see that

$$\begin{aligned} A_3 &\leq |P\{M_n \leq u_n\} - \Phi^n(u_n)| + |P\{M_{m,n} \leq u_n\} - \Phi^{n-m}(u_n)| \\ &\quad + \{\Phi^{n-m}(u_n) - \Phi^n(u_n)\} =: B_1 + B_2 + B_3. \end{aligned} \quad (18)$$

Our goal now is to give the bounds for  $B_1$ - $B_3$  in (18). Let  $\delta$  be such as in (12). By Theorem 4.2.1 in Leadbetter et al. (1983), relation (15) and assumption (2),  $B_1 + B_2 \ll n \exp\left(-\frac{u_n^2}{1+\delta}\right) \sum_{t=1}^{n-1} r(t) \ll \frac{(\log n)^{\frac{1}{1+\delta}}}{n^{\frac{2}{1+\delta}-1}} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \ll$

$$\frac{(\log n)^{\frac{1}{1+\delta}} L(n) n^{1-\alpha}}{n^{\frac{2}{1+\delta}-1}} = \frac{(\log n)^{\frac{1}{1+\delta}} L(n)}{n^{\frac{2}{1+\delta} + \alpha - 2}}. \text{ As, by (12), } \delta < \alpha/(2-\alpha), \text{ we have}$$

$$\frac{2}{1+\delta} + \alpha - 2 > 0. \text{ Since } (\log n)^{\frac{1}{1+\delta}} L(n) \ll n^\epsilon \text{ for arbitrary } \epsilon > 0, \text{ we obtain}$$

$$B_1 + B_2 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (19)$$

Moreover, the bound

$$B_3 \leq m/n \quad (20)$$

follows from the fact that  $x^{n-m} - x^n \leq m/n$ , if  $0 \leq x \leq 1$ . By (18)-(20)

$$A_3 \ll 1/n^\gamma + m/n \text{ for some } \gamma > 0. \quad (21)$$

Relations (13), (17), (21) yield (7), provided (2) holds with some  $0 < \alpha < 1$ .

Assume now that (2) holds with some  $\alpha \geq 1$ . Since  $r(\cdot)$  is positive, we get  $\sigma_n = \sqrt{n + 2 \sum_{t=1}^{n-1} (n-t)r(t)} \geq n^{1/2}$ . This and the fact that  $\alpha \geq 1$  imply  $0 < Cov(X_i, S_n/\sigma_n) < \frac{2}{n^{1/2}} + \frac{2}{n^{1/2}} \sum_{t=1}^{n-1} \frac{L(t)}{t} \ll \frac{L(n) \log n}{n^{1/2}}$ . As  $L(n) \log n \ll n^\epsilon$

for any  $\epsilon > 0$ , we obtain that (10), (11) hold also in the case, when (2) is satisfied for some  $\alpha \geq 1$ . Let  $\mathbb{A}_1$ - $\mathbb{A}_3$  be defined as  $A_1$ - $A_3$  in (13), but for the case, when (2) holds with some  $\alpha \geq 1$ . By using Theorem 4.2.1 in Leadbetter et al. (1983) and reasoning as in (14)-(17), we get  $\mathbb{A}_1 + \mathbb{A}_2 \ll 1/n^\gamma$  for some  $\gamma > 0$ . Let moreover,  $\mathbb{A}_3$ ,  $\mathbb{B}_1$ - $\mathbb{B}_3$  be defined as  $A_3$ ,  $B_1$ - $B_3$  in (13), (18), but for the case, when (2) holds with some  $\alpha \geq 1$ . Since  $\lim_{t \rightarrow \infty} r(t) = 0$ , then (see the remark before Lemma 4.3.2 in Leadbetter et al. (1983))  $0 < \sup_{t \geq 1} r(t) = \delta < 1$ . By Theorem

4.2.1 in Leadbetter et al. (1983), we have  $\mathbb{B}_1 + \mathbb{B}_2 \ll n \exp\left(-\frac{u_n^2}{1+\delta}\right) \sum_{t=1}^{n-1} r(t) \ll \frac{(\log n)^{\frac{1}{1+\delta}} \sum_{t=1}^{n-1} L(t)}{n^{\frac{2}{1+\delta}-1}} \ll \frac{(\log n)^{\frac{1}{1+\delta}+1} L(n)}{n^{\frac{2}{1+\delta}-1}}$ . As  $(\log n)^{\frac{1}{1+\delta}+1} L(n) \ll n^\epsilon$  for any  $\epsilon > 0$ , we obtain that  $\mathbb{B}_1 + \mathbb{B}_2 \ll 1/n^\gamma$  for some  $\gamma > 0$ . As moreover,  $\mathbb{B}_3 \leq m/n$ , we get  $\mathbb{A}_3 \leq \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3 \ll 1/n^\gamma + m/n$  for some  $\gamma > 0$ . Consequently  $\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 \ll 1/n^\gamma + m/n$  for some  $\gamma > 0$ , which yields (7) in the case, when (2) holds with some  $\alpha \geq 1$ . This completes the whole proof of Lemma 1. ■

We also need to prove the following statement.

**Lemma 2 .** *Suppose that  $X_1, X_2, \dots$  is a standardized stationary Gaussian sequence, the sequence  $\{u_n\}$  fulfills condition (3), and  $m < n$ . We have:*

(a) *If the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies assumption (2) with some  $0 < \alpha < 1$ , then for any  $y \in \mathbb{R}$  and some  $\gamma > 0$*

$$\begin{aligned} & |\text{Cov}(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})| \\ & \ll \frac{1}{n^\gamma} + \frac{m(n+2m)^{1-\alpha} L(n+2m)}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} + \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}}, \end{aligned} \quad (22)$$

(b) *If the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies assumption (2) with some  $\alpha > 1$ , then for any  $y \in \mathbb{R}$  and some  $\gamma > 0$*

$$|\text{Cov}(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})| \ll \frac{1}{n^\gamma} + \frac{m^{1/2}}{n^{1/2}}, \quad (23)$$

(c) *If the covariance function  $r(t) := \text{Cov}(X_1, X_{1+t})$  satisfies assumption (2) with  $\alpha = 1$ , then for any  $y \in \mathbb{R}$  and some  $\gamma > 0$*

$$\begin{aligned} & |\text{Cov}(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})| \\ & \ll \frac{1}{n^\gamma} + \frac{m^{1/2} \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right)}{n^{1/2} \tilde{L}(m)^{1/2} \tilde{L}(n)^{1/2}} + \frac{m^{1/2} \tilde{L}(m)^{1/2}}{n^{1/2} \tilde{L}(n)^{1/2}}, \end{aligned} \quad (24)$$

where  $\tilde{L}(n) := 1 + 2 \sum_{t=1}^{n-1} r(t)$ .

**Proof of Lemma 2(a).** We assume that (2) holds with some  $0 < \alpha < 1$ . Let  $i \geq m + 1$ . By assumption (2) and relation (9), we have

$$0 < \text{Cov}(X_i, S_m/\sigma_m) \leq \frac{1}{\sigma_m} \sum_{t=i-m}^{i-1} \frac{L(t)}{t^\alpha} \sim \frac{1}{C(\alpha) L(m)^{1/2} m^{1-\alpha/2}} \sum_{t=i-m}^{i-1} \frac{L(t)}{t^\alpha}.$$

Since  $L(t) \ll t^\epsilon$  for arbitrary  $\epsilon > 0$ , we can write that

$$\begin{aligned} \text{Cov}(X_i, S_m/\sigma_m) &\ll \frac{1}{L(m)^{1/2} m^{1-\alpha/2}} \sum_{t=i-m}^{i-1} \frac{t^{\alpha/4}}{t^\alpha} \leq \frac{1}{L(m)^{1/2} m^{1-\alpha/2}} \sum_{t=1}^m \frac{1}{t^{3\alpha/4}} \\ &= \frac{1}{L(m)^{1/2} m^{\alpha/4} m^{1-3\alpha/4}} \sum_{t=1}^m \frac{1}{t^{3\alpha/4}} \leq \frac{1}{L(m)^{1/2} m^{\alpha/4}} \sum_{t=1}^m \frac{1}{t^{1-3\alpha/4} t^{3\alpha/4}} \\ &\ll \frac{\log m}{L(m)^{1/2} m^{\alpha/4}} \ll \frac{m^\epsilon}{m^{\alpha/4}} \text{ for any } \epsilon > 0. \end{aligned}$$

Hence, we get that  $0 < \sup_{i \geq m+1} \text{Cov}(X_i, S_m/\sigma_m) \ll 1/m^\eta$  for some  $0 < \eta < 1/4$ .

Consequently, there exist numbers  $\mu, m_0$ , such that

$$0 < \sup_{i \geq m+1} \text{Cov}(X_i, S_m/\sigma_m) < \mu < \alpha/(2-\alpha) < 1 \text{ for all } m > m_0. \quad (25)$$

We will use (25) later in our proof. Let:  $y$  be an arbitrary real number,  $m < n$ . We know that:  $(-X_1, \dots, -X_m, -S_m/\sigma_m, -X_{m+1}, \dots, -X_n, -S_n/\sigma_n)$  is a standard Gaussian vector, and (since  $r(\cdot)$  is positive) that:  $\text{Cov}(-X_i, -X_j) > 0$ ,  $\text{Cov}(-X_i, -S_j/\sigma_j) > 0$ ,  $\text{Cov}(-S_i/\sigma_i, -S_j/\sigma_j) > 0$  for all  $i, j$ . Therefore, by Theorem in Pitt (1991),  $-X_1, \dots, -X_m, -S_m/\sigma_m, -X_{m+1}, \dots, -X_n, -S_n/\sigma_n$  are the associated r.v.'s. Hence, due to Theorem 2 in Newman (1984),

$$\begin{aligned} &0 \leq \text{Cov}(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\}) \\ &\leq \sum_{i=1}^m \sum_{j=m+1}^n (P\{-X_i > -u_m, -X_j > -u_n\} - P\{-X_i > -u_m\} P\{-X_j > -u_n\}) \\ &\quad + \sum_{i=1}^m (P\{-X_i > -u_m, -S_n/\sigma_n > -y\} - P\{-X_i > -u_m\} P\{-S_n/\sigma_n > -y\}) \\ &\quad + \sum_{i=m+1}^n (P\{-X_i > -u_n, -S_m/\sigma_m > -y\} - P\{-X_i > -u_n\} P\{-S_m/\sigma_m > -y\}) \\ &\quad + (P\{-S_m/\sigma_m > -y, -S_n/\sigma_n > -y\} - P\{-S_m/\sigma_m > -y\} P\{-S_n/\sigma_n > -y\}) \\ &=: D_1 + D_2 + D_3 + D_4. \end{aligned} \quad (26)$$

Our task now is to estimate all the components  $D_1$ - $D_4$  in (26).

Let  $\{\hat{X}_n\}$  denote an i.i.d. standard normal sequence. Obviously, we have

$$D_1 = \sum_{i=1}^m \sum_{j=m+1}^n \left( P\{X_i \leq u_m, X_j \leq u_n\} - P\left\{ \hat{X}_i \leq u_m, \hat{X}_j \leq u_n \right\} \right).$$

As the random vectors  $(X_i, X_j)$ ,  $(\hat{X}_i, \hat{X}_j)$  are standard normal and (12) holds, we can apply Theorem 4.2.1 in Leadbetter et al. (1983). This and (15) yield  $D_1 \ll$

$$m \exp\left(-\frac{u_m^2 + u_n^2}{2(1+\delta)}\right) \sum_{t=1}^{n-1} r(t) \ll \frac{m(\log n)^{\frac{1}{1+\delta}}}{m^{\frac{1}{1+\delta}} n^{\frac{1}{1+\delta}}} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \ll \frac{m(\log n)^{\frac{1}{1+\delta}} L(n) n^{1-\alpha}}{m^{\frac{1}{1+\delta}} n^{\frac{1}{1+\delta}}}$$



$$= \frac{m^{1-\frac{1}{1+\delta}} (\log n)^{\frac{1}{1+\delta}} L(n)}{n^{1-\frac{1}{1+\delta}} n^{\frac{2}{1+\delta}+\alpha-2}} \ll \frac{(\log n)^{\frac{1}{1+\delta}} L(n)}{n^{\frac{2}{1+\delta}+\alpha-2}} \ll \frac{n^\epsilon}{n^{\frac{2}{1+\delta}+\alpha-2}} \text{ for any } \epsilon > 0.$$

Since, by (12),  $\delta < \alpha/(2-\alpha)$ , then  $\frac{2}{1+\delta} + \alpha - 2 > 0$  and

$$D_1 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (27)$$

In order to estimate the component  $D_2$  in (26), let us denote by  $Y_n$  a standard normal random variable, which is independent of  $S_n/\sigma_n$ . We can write that

$D_2 = \sum_{i=1}^m (P\{X_i \leq u_m, S_n/\sigma_n \leq y\} - P\{Y_n \leq u_m, S_n/\sigma_n \leq y\})$ . Let  $\lambda$  satisfy (11). As the vectors  $(X_i, S_n/\sigma_n)$ ,  $(Y_n, S_n/\sigma_n)$  are standard normal and (11) holds, then, due to Theorem 4.2.1 in Leadbetter et al. (1983), and (10), (15),

$$D_2 \ll \sum_{i=1}^m \text{Cov}(X_i, S_n/\sigma_n) \exp\left(-\frac{u_m^2}{2(1+\lambda)}\right) \ll \frac{m(\log m)^{\frac{1}{2(1+\lambda)}}}{n^\beta m^{\frac{1}{1+\lambda}}} \ll 1/n^\gamma \quad (28)$$

for some  $\gamma > 0$ , where the last relation follows from the fact that  $\beta > 1 - \frac{1}{1+\lambda}$ . We also wish to estimate  $D_3$  in (26). Let  $\mu$  satisfy (25). By using similar reasoning to that, which led us to the first derivation in (28), we obtain

$$D_3 \ll \exp\left(-\frac{u_n^2}{2(1+\mu)}\right) \sum_{i=m+1}^n \text{Cov}(X_i, S_m/\sigma_m). \quad (29)$$

Notice that  $\sum_{i=m+1}^n \text{Cov}(X_i, S_m/\sigma_m) = \frac{1}{\sigma_m} \left( \sum_{t=m}^{n-1} r(t) + \dots + \sum_{t=1}^{n-m} r(t) \right) < \frac{m}{\sigma_m} \sum_{t=1}^{n-1} r(t)$ . Thus, by (29), (15), assumption (2) and relation (9),

$$\begin{aligned} D_3 &\ll \frac{(\log n)^{\frac{1}{2(1+\mu)}}}{n^{\frac{1}{1+\mu}}} \frac{m}{L(m)^{1/2} m^{1-\alpha/2}} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \ll \frac{(\log n)^{\frac{1}{2(1+\mu)}} m^{\alpha/2} L(n) n^{1-\alpha}}{n^{\frac{1}{1+\mu}} L(m)^{1/2}} \\ &\ll \frac{(\log n)^{\frac{1}{2(1+\mu)}} n^{\alpha/2} L(n)}{n^{\frac{1}{1+\mu}+\alpha-1}} = \frac{(\log n)^{\frac{1}{2(1+\mu)}} L(n)}{n^{\frac{1}{1+\mu}+\alpha/2-1}} \ll \frac{n^\epsilon}{n^{\frac{1}{1+\mu}+\alpha/2-1}} \text{ for any } \epsilon > 0. \end{aligned}$$

Since, by (25),  $\mu < \alpha/(2-\alpha)$ , then  $\frac{1}{1+\mu} + \alpha/2 - 1 > 0$  and

$$D_3 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (30)$$

It remains to estimate  $D_4$  in (26). Notice that, if  $\text{Cov}(S_m/\sigma_m, S_n/\sigma_n) \leq 1/2$ , then, by Theorem 4.2.1 in Leadbetter et al. (1983),

$$D_4 \leq \frac{\text{Cov}(S_m/\sigma_m, S_n/\sigma_n)}{2\pi\sqrt{1 - \{\text{Cov}(S_m/\sigma_m, S_n/\sigma_n)\}^2}} \leq \frac{\text{Cov}(S_m/\sigma_m, S_n/\sigma_n)}{\pi\sqrt{3}}. \quad (31)$$

In turn, provided  $\text{Cov}(S_m/\sigma_m, S_n/\sigma_n) > 1/2$ , we can estimate  $D_4$  in (26) as follows. Observe that, since  $S_m/\sigma_m, S_n/\sigma_n$  are associated and have bounded,

continuous density, then, by Lemma 2.2 in Bagai and Prakasa Rao (1991), there exists a positive constant  $C_1$ , neither depending on  $m$ ,  $n$  nor on  $y$ , such that  $D_4 \leq C_1 \{Cov(S_m/\sigma_m, S_n/\sigma_n)\}^{1/3}$ . Thus, if  $Cov(S_m/\sigma_m, S_n/\sigma_n) > 1/2$ , then

$$D_4 \leq C_1 \frac{Cov(S_m/\sigma_m, S_n/\sigma_n)}{\{Cov(S_m/\sigma_m, S_n/\sigma_n)\}^{2/3}} < C_1 2^{2/3} Cov(S_m/\sigma_m, S_n/\sigma_n). \quad (32)$$

Put  $C := \max(1/\pi\sqrt{3}, C_1 2^{2/3})$ . By (31), (32),  $D_4 \leq C \cdot Cov(S_m/\sigma_m, S_n/\sigma_n)$ . Consequently

$$D_4 \ll Cov(S_m/\sigma_m, S_n/\sigma_n). \quad (33)$$

Let us estimate the covariance on the r.h.s. of (33). We have

$$\begin{aligned} Cov\left(\frac{S_m}{\sigma_m}, \frac{S_n}{\sigma_n}\right) &= Cov\left(\frac{S_m}{\sigma_m}, \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) + E\left(\frac{S_m}{\sigma_m}\right) \left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) \\ &\quad - E\left(\frac{S_m}{\sigma_m}\right) E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) =: F_1 + F_2 + F_3. \end{aligned} \quad (34)$$

We now estimate  $F_1$ - $F_3$  in (34). By (2), (9) and Karamata's theorem, we obtain

$$\begin{aligned} F_1 &\ll \frac{1}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} m \sum_{t=m+1}^{n+2m-1} \frac{L(t)}{t^\alpha} \\ &\ll \frac{1}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} m (n+2m)^{1-\alpha} L(n+2m). \end{aligned} \quad (35)$$

In order to bound  $F_2$  in (34), notice that  $F_2 \leq E\left|\frac{S_m}{\sigma_m}\right| \left|\frac{(S_{n+2m} - S_n) - S_{2m}}{\sigma_n}\right|$ . By the Cauchy-Schwarz inequality, the stationarity of the sequence  $\{X_i\}$  and the fact that  $EX_i X_j > 0$  for all  $i, j$ , we deduce that

$$F_2 \leq \sqrt{E\left(\frac{S_m}{\sigma_m}\right)^2 E\left(\frac{(S_{n+2m} - S_n) - S_{2m}}{\sigma_n}\right)^2} \leq \sqrt{2 \frac{\sigma_{2m}^2}{\sigma_n^2}} = \sqrt{2} \frac{\sigma_{2m}}{\sigma_n}. \quad (36)$$

This, relation (9) and definition of the slowly varying function at infinity imply

$$F_2 \ll \frac{\sigma_{2m}}{\sigma_n} \sim \frac{L(2m)^{1/2} (2m)^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}} \ll \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}}. \quad (37)$$

Since, in view of the stationarity of the sequence  $\{X_i\}$ , the last component  $F_3$  in (34) is equal to zero, then, by (33)-(35) and (37),

$$D_4 \ll \frac{m(n+2m)^{1-\alpha} L(n+2m)}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} + \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}}. \quad (38)$$

Finally, due to (26)-(28), (30) and (38), relation (22) is fulfilled in the case, when assumption (2) is satisfied with some  $0 < \alpha < 1$ . ■

**Proof of Lemma 2(b).** Suppose that (2) holds for some  $\alpha > 1$ . We have

$$\begin{aligned} & |Cov(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})| \\ & =: \mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3 + \mathbb{D}_4, \end{aligned} \quad (39)$$

where  $\mathbb{D}_1$ - $\mathbb{D}_4$  are defined as  $D_1$ - $D_4$  in (26), but for the case (2) holds for some  $\alpha > 1$ . By applying Theorem 4.2.1 in Leadbetter et al. (1983), it is easy to check that

$$\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (40)$$

Thus, it remains to estimate the component  $\mathbb{D}_4$ . By (33)

$$\mathbb{D}_4 \ll Cov(S_m/\sigma_m, S_n/\sigma_n) =: \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3, \quad (41)$$

with  $\mathbb{F}_1$ - $\mathbb{F}_3$ , defined as  $F_1$ - $F_3$  in (34), but for the case (2) holds with some  $\alpha > 1$ .

$$\text{As } \mathbb{F}_1 = Cov\left(\frac{S_m}{\sigma_m}, \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) \leq \frac{1}{\sigma_m \sigma_n} m \left( \sum_{t=m+1}^{\infty} \frac{L(t)}{t^\alpha} - \sum_{t=n+2m}^{\infty} \frac{L(t)}{t^\alpha} \right),$$

it follows from the facts that:  $\alpha > 1$ ,  $\sigma_i \geq i^{1/2}$ , and from Karamata's theorem that

$$\mathbb{F}_1 \ll (m^{1/2}/n^{1/2})(L(m)/m^{\alpha-1}) \ll m^{1/2}/n^{1/2}. \quad (42)$$

In order to estimate  $\mathbb{F}_2 = E\left(\frac{S_m}{\sigma_m}\right)\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right)$ , notice that, as (2) holds for some  $\alpha > 1$ , then  $\sigma_{2m} \ll m^{1/2}$ . Since in addition,  $\sigma_n \geq n^{1/2}$ , we have, by (36),

$$\mathbb{F}_2 \ll \sigma_{2m}/\sigma_n \ll m^{1/2}/n^{1/2}. \quad (43)$$

As moreover,  $\mathbb{F}_3 = -E\left(\frac{S_m}{\sigma_m}\right)E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) = 0$ , then, due to (41)-(43),

$$\mathbb{D}_4 \ll m^{1/2}/n^{1/2}. \quad (44)$$

Relations (39), (40), (44) imply (23) in the case (2) holds for some  $\alpha > 1$ . ■

**Proof of Lemma 2(c).** Assume that (2) is satisfied with  $\alpha = 1$ . We have

$$\begin{aligned} & |Cov(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})| \\ & =: \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4, \end{aligned} \quad (45)$$

where  $\mathcal{D}_1$ - $\mathcal{D}_4$  are defined as  $D_1$ - $D_4$  in (26), but for the case (2) holds with  $\alpha = 1$ . By using Theorem 4.2.1 in Leadbetter et al. (1983), it is easy to verify that

$$\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 \ll 1/n^\gamma \text{ for some } \gamma > 0. \quad (46)$$

Thus, it remains to estimate the component  $\mathcal{D}_4$ . By (33)

$$\mathcal{D}_4 \ll Cov(S_m/\sigma_m, S_n/\sigma_n) =: \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3, \quad (47)$$

with  $\mathcal{F}_1$ - $\mathcal{F}_3$  defined as  $F_1$ - $F_3$  in (34), but for the case, when (2) holds with  $\alpha = 1$ . We have

$$\mathcal{F}_1 = \frac{1}{\sigma_m \sigma_n} Cov(S_m, S_{n+2m} - S_{2m}) \leq \frac{1}{\sigma_m \sigma_n} m \sum_{t=m+1}^{n+2m-1} r(t). \quad (48)$$

By the same reasoning as in the excerpt of the proof of Theorem 1 in Mielniczuk (2002) (see p. 396, lines 7-15), we obtain

$$\sigma_n^2 = n + 2 \sum_{t=1}^{n-1} (n-t) r(t) \sim n \left( 1 + 2 \sum_{t=1}^{n-1} r(t) \right) = n \tilde{L}(n). \quad (49)$$

By definition of  $\tilde{L}(n)$ , we have  $m \sum_{t=m+1}^{n+2m-1} r(t) = m \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right) / 2$ .

This and relations (48), (49) yield

$$\mathcal{F}_1 \ll \frac{1}{m^{1/2} \tilde{L}(m)^{1/2} n^{1/2} \tilde{L}(n)^{1/2}} m \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right). \quad (50)$$

In turn, by (36),  $\mathcal{F}_2 = E \left( \frac{S_m}{\sigma_m} \right) \left( \frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n} \right) \leq E \left| \frac{S_m}{\sigma_m} \right| \left| \frac{(S_{n+2m} - S_n) - S_{2m}}{\sigma_n} \right| \ll \sigma_{2m} / \sigma_n$ . Hence, from (49) and the fact that  $\tilde{L}(\cdot)$  is slowly varying, we get

$$\mathcal{F}_2 \ll \left( m^{1/2} \tilde{L}(m)^{1/2} \right) / \left( n^{1/2} \tilde{L}(n)^{1/2} \right). \quad (51)$$

As moreover,  $F_3 = 0$ , then, by (47), (50) and (51),

$$\mathcal{D}_4 \ll \frac{m^{1/2} \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right)}{n^{1/2} \tilde{L}(m)^{1/2} \tilde{L}(n)^{1/2}} + \frac{m^{1/2} \tilde{L}(m)^{1/2}}{n^{1/2} \tilde{L}(n)^{1/2}}. \quad (52)$$

Relations (45), (46), (52) imply (24) in the case, when (2) is satisfied with  $\alpha = 1$ . This completes the whole proof of Lemma 2. ■

The following lemma will be also used in our further considerations.

**Lemma 3 (for comparison - see also Ho and Hsing (1996)).** Under the assumptions of Theorem 1 on  $X_1, X_2, \dots$ , and  $r(t)$ ,  $\{u_n\}$ , we have

$$\lim_{n \rightarrow \infty} P \{M_n \leq u_n, S_n / \sigma_n \leq y\} = e^{-\tau} \Phi(y) \text{ for all } y \in \mathbb{R} \text{ and } \tau \text{ defined in (3)}.$$

**Proof.** Notice that, as  $X_1, X_2, \dots$  is a standard normal sequence,  $r(t)$ ,  $\{u_n\}$  satisfy (2), (3), respectively, then, by Theorem 4.3.3 in Leadbetter et al. (1983),

$$\lim_{n \rightarrow \infty} P \{M_n \leq u_n\} = e^{-\tau} \text{ for } \tau \text{ defined in (3)}. \quad (53)$$

Let  $y$  be an arbitrary real number and let, for each  $n \geq 1$ ,  $Y_n$  denote a random variable, which has the same distribution as  $S_n / \sigma_n$ , but is independent of  $(X_1, \dots, X_n)$ . From the estimations of  $A_1, \mathbb{A}_1$  in the proof of Lemma 1,  $|P \{M_n \leq u_n, S_n / \sigma_n \leq y\} - P \{M_n \leq u_n\} P \{Y_n \leq y\}| \ll 1/n^\gamma$  for some  $\gamma > 0$ . As: (53) holds,  $1/n^\gamma \rightarrow 0$ , as  $n \rightarrow \infty$ , and, for each  $n$ ,  $Y_n$  is standard normal, we get  $\lim_{n \rightarrow \infty} P \{M_n \leq u_n, S_n / \sigma_n \leq y\} = \lim_{n \rightarrow \infty} P \{M_n \leq u_n\} P \{Y_n \leq y\} = e^{-\tau} \Phi(y)$ , which yields the desired result. ■

#### 4. Proof of main result

In this section, we present the proof of Theorem 1. In our proof, we make an extensive use of the earlier proved Lemmas 1-3.

**Proof of Theorem 1(i).** First, we will show that for any real  $y$

$$\frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} (I\{M_n \leq u_n, S_n/\sigma_n \leq y\} - P\{M_n \leq u_n, S_n/\sigma_n \leq y\}) \xrightarrow{\text{a.s.}} 0, \quad (54)$$

as  $N \rightarrow \infty$ . By Lemma 3.1 in Csaki and Gonchigdanzan (2002), in order to prove (54), it suffices to show that the following property occurs for some  $\varepsilon > 0$

$$\text{Var} \left( \sum_{n=1}^N \frac{1}{n} I\{M_n \leq u_n, S_n/\sigma_n \leq y\} \right) \ll (\log N)^2 (\log \log N)^{-(1+\varepsilon)}. \quad (55)$$

Put  $\xi_n := I\{M_n \leq u_n, S_n/\sigma_n \leq y\}$ . We have

$$\begin{aligned} \text{Var} \left( \sum_{n=1}^N \frac{1}{n} I\{M_n \leq u_n, S_n/\sigma_n \leq y\} \right) &= \text{Var} \left( \sum_{n=1}^N \frac{1}{n} \xi_n \right) \\ &\leq \sum_{n=1}^N \frac{1}{n^2} \text{Var}(\xi_n) + 2 \sum_{1 \leq m < n \leq N} \frac{1}{mn} |\text{Cov}(\xi_m, \xi_n)| =: \sum_1 + \sum_2. \end{aligned} \quad (56)$$

Since  $\xi_n^2 \leq 1$ , we obtain that

$$\sum_1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \quad (57)$$

Thus, we only need to estimate  $\sum_2$  in (56). Let  $m < n$ . Observe that

$$\begin{aligned} |\text{Cov}(\xi_m, \xi_n)| &\leq 2E |I\{M_n \leq u_n, S_n/\sigma_n \leq y\} - I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\}| \\ &\quad + |\text{Cov}(I\{M_m \leq u_m, S_m/\sigma_m \leq y\}, I\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\})|. \end{aligned}$$

This and Lemmas 1, 2 imply that:

(a) if condition (2) is satisfied with some  $0 < \alpha < 1$ , then for some  $\gamma > 0$

$$|\text{Cov}(\xi_m, \xi_n)| \ll \frac{1}{n^\gamma} + \frac{m}{n} + \frac{m(n+2m)^{1-\alpha} L(n+2m)}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} + \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}}, \quad (58)$$

(b) if condition (2) is satisfied with some  $\alpha > 1$ , then for some  $\gamma > 0$

$$|\text{Cov}(\xi_m, \xi_n)| \ll 1/n^\gamma + m^{1/2}/n^{1/2}, \quad (59)$$

(c) if condition (2) is satisfied with  $\alpha = 1$ , then for some  $\gamma > 0$

$$|\text{Cov}(\xi_m, \xi_n)| \ll \frac{1}{n^\gamma} + \frac{m}{n} + \frac{m^{1/2} \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right)}{n^{1/2} \tilde{L}(m)^{1/2} \tilde{L}(n)^{1/2}} + \frac{m^{1/2} \tilde{L}(m)^{1/2}}{n^{1/2} \tilde{L}(n)^{1/2}}. \quad (60)$$

Let us consider the case (a) first. By (56) and (58), we get

$$\begin{aligned} \sum_2 &\ll \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{1}{n^\gamma} + \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m}{n} \\ &+ \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m(n+2m)^{1-\alpha} L(n+2m)}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} + \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}} \\ &=: \sum_3 + \sum_4 + \sum_5 + \sum_6 \ll \log N + \sum_5 + \sum_6, \end{aligned} \quad (61)$$

where the last relation follows from the obvious fact that  $\sum_3 + \sum_4 \ll \log N$ . Thus, it remains to estimate  $\sum_5, \sum_6$  in (61). It is easy to check that

$$\sum_5 \ll \sum_{m=1}^{N-1} \frac{1}{L(m)^{1/2} m^{1-\alpha/2}} \sum_{n=m+1}^N \left(1 + \frac{2m}{n}\right)^{1-\alpha} \frac{L(n+2m)}{L(n)^{1/2}} \frac{1}{n^{1+\alpha/2}}. \quad (62)$$

For a fixed  $m$ , put  $\bar{L}(n) := L(n+2m)/L(n)^{1/2}$ . By using (62) and the same reasoning as in the excerpt of the proof of Theorem 1 in Mielniczuk (2002) (see p. 395, lines 7-14), we obtain

$$\sum_5 \ll \sum_{m=1}^{N-1} \frac{\bar{L}(m) m^{-\alpha/2}}{L(m)^{1/2} m^{1-\alpha/2}} = \sum_{m=1}^{N-1} \frac{\bar{L}(m)}{L(m)^{1/2} m} = \sum_{m=1}^{N-1} \frac{L(3m)}{L(m)} \frac{1}{m} \ll \log N. \quad (63)$$

In addition, we can estimate  $\sum_6$  in (61) as follows. Observe that, by the facts that  $L(\cdot)$  is slowly varying and  $0 < \alpha < 1$ , we have

$$\sum_6 \ll \sum_{m=1}^{N-1} \frac{L(m)^{1/2}}{m^{\alpha/2}} \sum_{n=m+1}^{\infty} \frac{1}{L(n)^{1/2} n^{2-\alpha/2}} \ll \sum_{m=1}^{N-1} \frac{L(m)^{1/2}}{m^{\alpha/2}} \frac{1}{L(m)^{1/2} m^{1-\alpha/2}}.$$

Hence  $\sum_6 \ll \log N$ . This and (61), (63) yield

$$\sum_2 \ll \log N. \quad (64)$$

Due to (56), (57) and (64), we conclude that (55) holds for arbitrary real  $y$  and any positive  $\varepsilon$ . Consequently, by the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan (2002), condition (54) is satisfied as well. This, Lemma 3 and the regularity property of logarithmic means imply the statement (i) of Theorem 1 for the case, when (2) is satisfied with some  $0 < \alpha < 1$ .

We now consider the case (b), i.e. the case, when (2) holds with some  $\alpha > 1$ .

By (56) and (59), we get  $\sum_2 \ll \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{1}{n^\gamma} + \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m^{1/2}}{n^{1/2}} \ll \log N$ .

This, (56) and (57) imply that condition (55) holds for arbitrary real  $y$  and any positive  $\varepsilon$ . Therefore (54) occurs. Hence, also in the case, when (2) is satisfied with some  $\alpha > 1$ , the statement (i) of Theorem 1 is true.

Finally, let us consider the case (c), assuming that (2) holds with  $\alpha = 1$ . It follows from (56) and (60) that

$$\begin{aligned}
\sum_2 &\ll \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{1}{n^\gamma} + \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m}{n} \\
&+ \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m^{1/2} \left( \tilde{L}(n+2m) - \tilde{L}(m+1) \right)}{n^{1/2} \tilde{L}(m)^{1/2} \tilde{L}(n)^{1/2}} + \sum_{1 \leq m < n \leq N} \frac{1}{mn} \frac{m^{1/2} \tilde{L}(m)^{1/2}}{n^{1/2} \tilde{L}(n)^{1/2}} \\
&=: \sum_7 + \sum_8 + \sum_9 + \sum_{10} \ll \log N + \sum_9 + \sum_{10}. \quad (65)
\end{aligned}$$

Let us estimate the term  $\sum_9$  in (65). By reasoning identically as in Mielniczuk

$$\begin{aligned}
(2002), \text{ p. 397, we get } \sum_9 &= \sum_{m=1}^{N-1} \frac{1}{m^{1/2} \tilde{L}(m)^{1/2}} \sum_{n=m+1}^N \frac{\tilde{L}(n+2m) - \tilde{L}(m+1)}{n^{3/2} \tilde{L}(n)^{1/2}} \\
&\ll \sum_{m=1}^{N-1} \frac{1}{m^{1/2} \tilde{L}(m)^{1/2}} \frac{\tilde{L}(3m) - \tilde{L}(m+1)}{m^{1/2} \tilde{L}(m)^{1/2}} = \sum_{m=1}^{N-1} \frac{\tilde{L}(3m) - \tilde{L}(m+1)}{m \tilde{L}(m)} \ll \log N.
\end{aligned}$$

In order to estimate  $\sum_{10}$  in (65), observe that, as  $\tilde{L}(\cdot)$  is slowly varying,

$$\sum_{10} \ll \sum_{m=1}^{N-1} \frac{\tilde{L}(m)^{1/2}}{m^{1/2}} \sum_{n=m+1}^N \frac{1}{n^{3/2} \tilde{L}(n)^{1/2}} \ll \sum_{m=1}^{N-1} \frac{\tilde{L}(m)^{1/2}}{m^{1/2}} \frac{1}{m^{1/2} \tilde{L}(m)^{1/2}} \ll \log N.$$

The estimates for  $\sum_9$ ,  $\sum_{10}$ , together with (65), yield  $\sum_2 \ll \log N$ , which, by (56), (57), implies that (55) holds for arbitrary real  $y$  and any  $\varepsilon > 0$ . Therefore (54) occurs. Thus, (4) holds also in the case (2) is satisfied with  $\alpha = 1$ . ■

**Proof of Theorem 1(ii).** It is easily seen that Theorem 1(ii) is a special case of the earlier proved Theorem 1(i), with:  $u_n := x/a_n + b_n$ ,  $\tau := e^{-x}$ . ■

**Acknowledgement.** I thank an anonymous Referee for valuable remarks.

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