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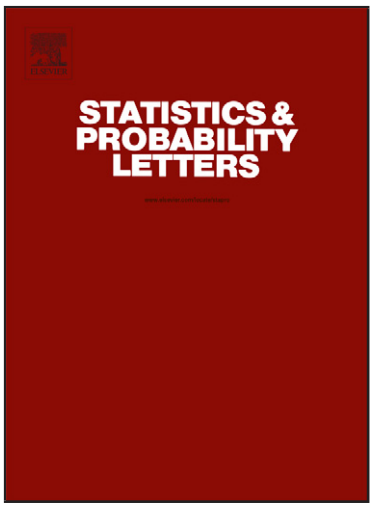

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The almost sure central limit theorems in the joint version for the maxima and sums of certain stationary Gaussian sequences

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Abstract

Suppose that $X_1, X_2, ...$ is a standardized stationary Gaussian sequence. Let: $M_n := \max(X_1, ..., X_n)$, $S_n := \sum_{i=1}^{n} X_i$, $\sigma_n := \sqrt{Var\,(S_n)}$, and $a_n > 0$, b_n denote suitable normalizing constants. Our goal is to prove the almost sure central limit theorem for the sequence $\{a_n\,(M_n - b_n)\,, S_n/\sigma_n\}$, under certain additional assumptions on the covariance function $r\,(t) := Cov\,(X_1, X_{1+t})$.

MSC: primary 60F15, secondary 60F05.

Key words and phrases: Almost sure central limit theorem; Extreme values; Partial sums; Stationary Gaussian sequences; Associated random variables; Slowly varying functions at infinity; Karamata's theorem.

1. Introduction

The almost sure central limit theorem (ASCLT) has become an intensively studied subject in recent research. In our paper, we are concerned with the ASCLT in its two-dimensional version. A general outline of the investigations concerning the ASCLT in such a form may be described as follows. Suppose that $X_1, X_2, ...$ is a sequence of r.v.'s, $f_1, f_2, ..., f_k, ...$ and $g_1, g_2, ..., g_k, ...$ are some real-valued measurable functions, defined on $\mathbb{R}, \mathbb{R}^2, ..., \mathbb{R}^k, ...$, respectively. We seek conditions under which, for some nondegenerate d.f. H, defined on $\mathbb{R}^2$,

$$\frac{1}{D_N}\sum_{n=1}^{N} d_n I\left\{f_n\left(X_1, X_2, ..., X_n\right) \le x, g_n\left(X_1, X_2, ..., X_n\right) \le y\right\} \overset{\text{a.s.}}{\to} H\left(x, y\right) \quad (1)$$

for all $(x, y) \in C_H$, where: $\{d_n\}$ is some sequence of weights, $D_N = \sum_{n=1}^{N} d_n$, I stands for the indicator function, C_H denotes the set of continuity points of H.

In our investigations, we will restrict ourselves to the situation, when (1) is satisfied with: $d_n = 1/n$, $D_N \sim \log N$, $f_n\left(X_1, ..., X_n\right) := a_n\left(M_n - b_n\right)$, $g_n\left(X_1, ..., X_n\right) := S_n/\sigma_n$, where: $M_n := \max(X_1, ..., X_n)$, $S_n := X_1 + ... + X_n$, $\sigma_n := \sqrt{Var\,(S_n)}$, and $a_n > 0$, b_n are certain normalizing constants. The purpose of our paper is to prove that if: $X_1, X_2, ...$ is a standardized stationary Gaussian sequence, the covariance function $r\,(t) := Cov\,(X_1, X_{1+t})$ satisfies

$$r(t) = L\,(t)\,/t^{\alpha},\ t = 1, 2, ..., \quad (2)$$

where $\alpha > 0$ and $L\,(\cdot)$ is a positive, slowly varying function at infinity, and if moreover, the numerical sequence $\{u_n\}$ fulfills the relation

$$\lim_{n\to\infty} n\,(1 - \Phi\,(u_n)) = \tau \text{ for some } \tau,\ 0 \le \tau < \infty, \quad (3)$$

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where Φ is the standard normal d.f., then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{M_n\leq u_n, S_n/\sigma_n\leq y\right\}=e^{-\tau}\Phi\left(y\right)\ \text{a.s. for any } y\in\mathbb{R}. \quad (4)$$

As a direct conclusion, we will also show that if, for $n\geq 3$:

$$a_n:=\left(2\log n\right)^{1/2},\ b_n:=\left(2\log n\right)^{1/2}-\frac{1}{2}\left(2\log n\right)^{-1/2}\left(\log\log n+\log 4\pi\right), \quad (5)$$

then for all $x,y\in\mathbb{R}$

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{a_n\left(M_n-b_n\right)\leq x, S_n/\sigma_n\leq y\right\}=\exp\left(-e^{-x}\right)\Phi\left(y\right)\ \text{a.s.}. \quad (6)$$

It should be mentioned that, although $r\left(t\right)=L\left(t\right)/t^{\alpha}$ in (2) is defined only for positive integers, the slowly varying functions $L\left(\cdot\right)$ are in general defined on the real line or on the $\mathbb{R}_+$ semi-line.

The earlier given notations - M_n, S_n, σ_n, $r\left(t\right)$, Φ - will be intensively used throughout the paper. For our convenience, we will also introduce the following one $M_{m,n}:=\max(X_{m+1},...,X_n)$. Furthermore, $f\left(n\right)\ll g\left(n\right)$ and $f\left(n\right)\sim g\left(n\right)$ will stand for $f\left(n\right)=\mathcal{O}\left(g\left(n\right)\right)$ and $f\left(n\right)/g\left(n\right)\to 1$, as $n\to\infty$, respectively.

2. Main result

Our main result is the following almost sure central limit theorem in the joint version for the maxima and sums of some standardized stationary Gaussian sequences.

Theorem 1 . *Let $X_1,X_2,...$ be a standardized stationary Gaussian sequence. Assume that the covariance function $r(t):=Cov(X_1,X_{1+t})$ satisfies (2) for some $\alpha>0$. Then:*
(i) If the numerical sequence $\{u_n\}$ fulfills (3), then (4) holds,
(ii) If $\{a_n\}$, $\{b_n\}$ are such as in (5), then (6) holds.

3. Auxiliary results

In this section, we state and prove some lemmas, which will be needed for the proof of our main result. Here, we present the first one.

Lemma 1 . *Let $X_1,X_2,...$ be a standardized stationary Gaussian sequence. Assume that the covariance function $r(t):=Cov\left(X_1,X_{1+t}\right)$ satisfies assumption (2) with some $\alpha>0$. Suppose moreover that the sequence $\{u_n\}$ fulfills (3), and that $m<n$. Then for any $y\in\mathbb{R}$ and some $\gamma>0$*

$$E\left|I\left\{M_n\leq u_n, S_n/\sigma_n\leq y\right\}-I\left\{M_{m,n}\leq u_n, S_n/\sigma_n\leq y\right\}\right|\ll 1/n^{\gamma}+m/n. \quad (7)$$

Proof. Assume first that (2) holds with some $0<\alpha<1$. Let $1\leq i\leq n$. We have

$$0<Cov\left(X_i,S_n/\sigma_n\right)<\frac{2}{\sigma_n}\sum_{t=0}^{n-1}r(t)=\frac{2}{\sigma_n}+\frac{2}{\sigma_n}\sum_{t=1}^{n-1}\frac{L\left(t\right)}{t^{\alpha}}\ \text{for some } 0<\alpha<1. \quad (8)$$

As $0 < \alpha < 1$, then, by the two times application of Karamata's theorem, we get (see also Mielniczuk (2002), p. 394)

$$\sigma_n \sim C(\alpha) L(n)^{1/2} n^{1-\alpha/2}, \text{ where } C(\alpha) = \{2/(1-\alpha)(2-\alpha)\}^{1/2}. \quad (9)$$

Due to (8), (9), we have for some $0 < \alpha < 1$ that

$$Cov(X_i, S_n/\sigma_n) \ll \frac{1}{L(n)^{1/2} n^{1-\alpha/2}} \sum_{t=1}^{n-1} \frac{L(t)}{t^\alpha} \ll \frac{L(n) n^{1-\alpha}}{L(n)^{1/2} n^{1-\alpha/2}} = \frac{L(n)^{1/2}}{n^{\alpha/2}}.$$

Since $L(n)$ is a slowly varying function at infinity, then $L(n)^{1/2} \ll n^\epsilon$ for arbitrary $\epsilon > 0$. Consequently

$$Cov(X_i, S_n/\sigma_n) \ll \frac{n^\epsilon}{n^{\alpha/2}} \text{ for any } \epsilon > 0 \text{ and some } 0 < \alpha < 1.$$

Therefore

$$0 < \sup_{1 \le i \le n} Cov(X_i, S_n/\sigma_n) \ll 1/n^\beta \text{ for some } 0 < \beta < 1/2. \quad (10)$$

Since $\lim_{n\to\infty} 1/n^\beta = 0$, there exist numbers λ, n_0, such that

$$0 < \sup_{1 \le i \le n} Cov(X_i, S_n/\sigma_n) < \lambda < \beta/(1-\beta) < 1 \text{ for any } n > n_0. \quad (11)$$

In turn, by assumption (2), we have $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} L(t)/t^\alpha = 0$. Thus, there exist numbers δ, n_1, such that

$$0 < \sup_{t > n_1} r(t) = \delta < \alpha/(2-\alpha) < 1. \quad (12)$$

Relations (10)-(12) will be intensively used in the next stages of our proof.

Let y be an arbitrary real number and $m < n$. We have

$$\begin{aligned} & E\left|I\{M_n \le u_n, S_n/\sigma_n \le y\} - I\{M_{m,n} \le u_n, S_n/\sigma_n \le y\}\right| \\ = \; & P\{M_{m,n} \le u_n, S_n/\sigma_n \le y\} - P\{M_n \le u_n, S_n/\sigma_n \le y\}. \end{aligned}$$

Suppose moreover that Y_n is a random variable, which has the same distribution as S_n/σ_n, but is independent of $(X_1, ..., X_n)$. We can write that

$$\begin{aligned} & E\left|I\{M_n \le u_n, S_n/\sigma_n \le y\} - I\{M_{m,n} \le u_n, S_n/\sigma_n \le y\}\right| \\ \le \; & \left|P\{M_n \le u_n, S_n/\sigma_n \le y\} - P\{M_n \le u_n\} P\{Y_n \le y\}\right| \\ + \; & \left|P\{M_{m,n} \le u_n, S_n/\sigma_n \le y\} - P\{M_{m,n} \le u_n\} P\{Y_n \le y\}\right| \\ + \; & \left(P\{M_{m,n} \le u_n\} - P\{M_n \le u_n\}\right) =: A_1 + A_2 + A_3. \end{aligned} \quad (13)$$

Since $(X_1, ..., X_n, S_n/\sigma_n)$, $(X_1, ..., X_n, Y_n)$ are standard normal random vectors and (11), (12) hold, then, by Theorem 4.2.1 in Leadbetter et al. (1983), the fact that Y_n is independent of $(X_1, ..., X_n)$ and relation (10), we obtain

$$A_1 + A_2 \ll \sum_{i=1}^{n} Cov(X_i, S_n/\sigma_n) \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \ll n \frac{1}{n^\beta} \exp\left(-\frac{u_n^2}{2(1+\lambda)}\right) \quad (14)$$

for some $0 < \beta < 1/2$, where λ is such as in (11).
As $\{u_n\}$ satisfies (3), then, by (4.3.4(i)), (4.3.4(ii)) in Leadbetter et al. (1983),

$$\exp\left(-\frac{u_n^2}{2\left(1+\lambda\right)}\right) \sim L\frac{\left(\log n\right)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}} \text{ for some } L, \text{ not depending on } n. \quad (15)$$

From (14) and (15), we have

$$A_1 + A_2 \ll n\frac{1}{n^{\beta}}\frac{\left(\log n\right)^{\frac{1}{2(1+\lambda)}}}{n^{\frac{1}{1+\lambda}}} = \frac{\left(\log n\right)^{\frac{1}{2(1+\lambda)}}}{n^{\beta+\frac{1}{1+\lambda}-1}}. \quad (16)$$

As, by (11), $\lambda < \beta/\left(1-\beta\right)$, we have $\beta + \frac{1}{1+\lambda} - 1 > 0$. Thus, by (16),

$$A_1 + A_2 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \quad (17)$$

It remains only to estimate the term A_3 in (13). By going through the first lines in the proof of Lemma 2.4 from Csaki and Gonchigdanzan (2002), we see that

$$\begin{aligned} A_3 \quad \leq \quad & \left|P\left\{M_n \leq u_n\right\} - \Phi^n\left(u_n\right)\right| + \left|P\left\{M_{m,n} \leq u_n\right\} - \Phi^{n-m}\left(u_n\right)\right| \\ & + \left\{\Phi^{n-m}\left(u_n\right) - \Phi^n\left(u_n\right)\right\} \quad =: \; B_1 + B_2 + B_3. \end{aligned} \quad (18)$$

Our goal now is to give the bounds for B_1-B_3 in (18). Let δ be such as in (12). By Theorem 4.2.1 in Leadbetter et al. (1983), relation (15) and assumption (2), $B_1 + B_2 \ll n\exp\left(-\frac{u_n^2}{1+\delta}\right)\sum_{t=1}^{n-1} r(t) \ll \frac{\left(\log n\right)^{\frac{1}{1+\delta}}}{n^{\frac{2}{1+\delta}-1}}\sum_{t=1}^{n-1}\frac{L\left(t\right)}{t^{\alpha}} \ll \frac{\left(\log n\right)^{\frac{1}{1+\delta}} L\left(n\right) n^{1-\alpha}}{n^{\frac{2}{1+\delta}-1}} = \frac{\left(\log n\right)^{\frac{1}{1+\delta}} L\left(n\right)}{n^{\frac{2}{1+\delta}+\alpha-2}}$. As, by (12), $\delta < \alpha/\left(2-\alpha\right)$, we have $\frac{2}{1+\delta} + \alpha - 2 > 0$. Since $\left(\log n\right)^{\frac{1}{1+\delta}} L\left(n\right) \ll n^{\epsilon}$ for arbitrary $\epsilon > 0$, we obtain

$$B_1 + B_2 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \quad (19)$$

Moreover, the bound

$$B_3 \leq m/n \quad (20)$$

follows from the fact that $x^{n-m} - x^n \leq m/n$, if $0 \leq x \leq 1$. By (18)-(20)

$$A_3 \ll 1/n^{\gamma} + m/n \text{ for some } \gamma > 0. \quad (21)$$

Relations (13), (17), (21) yield (7), provided (2) holds with some $0 < \alpha < 1$.

Assume now that (2) holds with some $\alpha \geq 1$. Since $r(\cdot)$ is positive, we get $\sigma_n = \sqrt{n + 2\sum_{t=1}^{n-1}(n-t)r\left(t\right)} \geq n^{1/2}$. This and the fact that $\alpha \geq 1$ imply $0 < Cov\left(X_i, S_n/\sigma_n\right) < \frac{2}{n^{1/2}} + \frac{2}{n^{1/2}}\sum_{t=1}^{n-1}\frac{L\left(t\right)}{t} \ll \frac{L\left(n\right)\log n}{n^{1/2}}$. As $L\left(n\right)\log n \ll n^{\epsilon}$

for any $\epsilon > 0$, we obtain that (10), (11) hold also in the case, when (2) is satisfied for some $\alpha \geq 1$. Let $\mathbb{A}_1$-$\mathbb{A}_3$ be defined as A_1-A_3 in (13), but for the case, when (2) holds with some $\alpha \geq 1$. By using Theorem 4.2.1 in Leadbetter et al. (1983) and reasoning as in (14)-(17), we get $\mathbb{A}_1 + \mathbb{A}_2 \ll 1/n^{\gamma}$ for some $\gamma > 0$. Let moreover, $\mathbb{A}_3$, $\mathbb{B}_1$-$\mathbb{B}_3$ be defined as A_3, B_1-B_3 in (13), (18), but for the case, when (2) holds with some $\alpha \geq 1$. Since $\lim_{t\to\infty} r(t) = 0$, then (see the remark before Lemma 4.3.2 in Leadbetter et al. (1983)) $0 < \sup_{t\geq 1} r(t) = \delta < 1$. By Theorem 4.2.1 in Leadbetter et al. (1983), we have $\mathbb{B}_1 + \mathbb{B}_2 \ll n \exp\left(-\frac{u_n^2}{1+\delta}\right) \sum_{t=1}^{n-1} r(t) \ll \frac{(\log n)^{\frac{1}{1+\delta}}}{n^{\frac{2}{1+\delta}-1}} \sum_{t=1}^{n-1} \frac{L(t)}{t} \ll \frac{(\log n)^{\frac{1}{1+\delta}+1} L(n)}{n^{\frac{2}{1+\delta}-1}}$. As $(\log n)^{\frac{1}{1+\delta}+1} L(n) \ll n^{\epsilon}$ for any $\epsilon > 0$, we obtain that $\mathbb{B}_1 + \mathbb{B}_2 \ll 1/n^{\gamma}$ for some $\gamma > 0$. As moreover, $\mathbb{B}_3 \leq m/n$, we get $\mathbb{A}_3 \leq \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3 \ll 1/n^{\gamma} + m/n$ for some $\gamma > 0$. Consequently $\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 \ll 1/n^{\gamma} + m/n$ for some $\gamma > 0$, which yields (7) in the case, when (2) holds with some $\alpha \geq 1$. This completes the whole proof of Lemma 1. ■

We also need to prove the following statement.

Lemma 2 . *Suppose that $X_1, X_2, ...$ is a standardized stationary Gaussian sequence, the sequence $\{u_n\}$ fulfills condition (3), and $m < n$. We have:*
(a) If the covariance function $r(t) := Cov(X_1, X_{1+t})$ satisfies assumption (2) with some $0 < \alpha < 1$, then for any $y \in \mathbb{R}$ and some $\gamma > 0$

$$\begin{aligned} & |Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right)| \\ \ll\ & \frac{1}{n^{\gamma}} + \frac{m\,(n+2m)^{1-\alpha} L(n+2m)}{L(m)^{1/2} m^{1-\alpha/2} L(n)^{1/2} n^{1-\alpha/2}} + \frac{L(m)^{1/2} m^{1-\alpha/2}}{L(n)^{1/2} n^{1-\alpha/2}}, \end{aligned} \tag{22}$$

(b) If the covariance function $r(t) := Cov(X_1, X_{1+t})$ satisfies assumption (2) with some $\alpha > 1$, then for any $y \in \mathbb{R}$ and some $\gamma > 0$

$$|Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right)| \ll \frac{1}{n^{\gamma}} + \frac{m^{1/2}}{n^{1/2}}, \tag{23}$$

(c) If the covariance function $r(t) := Cov(X_1, X_{1+t})$ satisfies assumption (2) with $\alpha = 1$, then for any $y \in \mathbb{R}$ and some $\gamma > 0$

$$\begin{aligned} & |Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right)| \\ \ll\ & \frac{1}{n^{\gamma}} + \frac{m^{1/2}\left(\widetilde{L}(n+2m) - \widetilde{L}(m+1)\right)}{n^{1/2}\widetilde{L}(m)^{1/2}\widetilde{L}(n)^{1/2}} + \frac{m^{1/2}\widetilde{L}(m)^{1/2}}{n^{1/2}\widetilde{L}(n)^{1/2}}, \end{aligned} \tag{24}$$

where $\widetilde{L}(n) := 1 + 2\sum_{t=1}^{n-1} r(t)$.

Proof of Lemma 2(a). We assume that (2) holds with some $0 < \alpha < 1$. Let $i \geq m + 1$. By assumption (2) and relation (9), we have

$$0 < Cov\left(X_i, S_m/\sigma_m\right) \leq \frac{1}{\sigma_m}\sum_{t=i-m}^{i-1}\frac{L\left(t\right)}{t^{\alpha}} \sim \frac{1}{C\left(\alpha\right)L\left(m\right)^{1/2}m^{1-\alpha/2}}\sum_{t=i-m}^{i-1}\frac{L\left(t\right)}{t^{\alpha}}.$$

Since $L\left(t\right) \ll t^{\epsilon}$ for arbitrary $\epsilon > 0$, we can write that

$$\begin{aligned} Cov\left(X_i, S_m/\sigma_m\right) &\ll \frac{1}{L\left(m\right)^{1/2}m^{1-\alpha/2}}\sum_{t=i-m}^{i-1}\frac{t^{\alpha/4}}{t^{\alpha}} \leq \frac{1}{L\left(m\right)^{1/2}m^{1-\alpha/2}}\sum_{t=1}^{m}\frac{1}{t^{3\alpha/4}} \\ &= \frac{1}{L\left(m\right)^{1/2}m^{\alpha/4}m^{1-3\alpha/4}}\sum_{t=1}^{m}\frac{1}{t^{3\alpha/4}} \leq \frac{1}{L\left(m\right)^{1/2}m^{\alpha/4}}\sum_{t=1}^{m}\frac{1}{t^{1-3\alpha/4}t^{3\alpha/4}} \\ &\ll \frac{\log m}{L\left(m\right)^{1/2}m^{\alpha/4}} \ll \frac{m^{\epsilon}}{m^{\alpha/4}} \text{ for any } \epsilon > 0. \end{aligned}$$

Hence, we get that $0 < \sup_{i\geq m+1} Cov\left(X_i, S_m/\sigma_m\right) \ll 1/m^{\eta}$ for some $0 < \eta < 1/4$. Consequently, there exist numbers μ, m_0, such that

$$0 < \sup_{i\geq m+1} Cov\left(X_i, S_m/\sigma_m\right) < \mu < \alpha/(2-\alpha) < 1 \text{ for all } m > m_0. \tag{25}$$

We will use (25) later in our proof. Let: y be an arbitrary real number, $m < n$. We know that: $(-X_1, ..., -X_m, -S_m/\sigma_m, -X_{m+1}, ..., -X_n, -S_n/\sigma_n)$ is a standard Gaussian vector, and (since $r\left(\cdot\right)$ is positive) that: $Cov\left(-X_i, -X_j\right) > 0$, $Cov\left(-X_i, -S_j/\sigma_j\right) > 0$, $Cov\left(-S_i/\sigma_i, -S_j/\sigma_j\right) > 0$ for all i, j. Therefore, by Theorem in Pitt (1991), $-X_1, ..., -X_m, -S_m/\sigma_m, -X_{m+1}, ..., -X_n, -S_n/\sigma_n$ are the associated r.v.'s. Hence, due to Theorem 2 in Newman (1984),

$$\begin{aligned} 0 \;&\leq\; Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right) \\ &\leq \sum_{i=1}^{m}\sum_{j=m+1}^{n}\left(P\left\{-X_i > -u_m, -X_j > -u_n\right\} - P\left\{-X_i > -u_m\right\}P\left\{-X_j > -u_n\right\}\right) \\ &\quad + \sum_{i=1}^{m}\left(P\left\{-X_i > -u_m, -S_n/\sigma_n > -y\right\} - P\left\{-X_i > -u_m\right\}P\left\{-S_n/\sigma_n > -y\right\}\right) \\ &\quad + \sum_{i=m+1}^{n}\left(P\left\{-X_i > -u_n, -S_m/\sigma_m > -y\right\} - P\left\{-X_i > -u_n\right\}P\left\{-S_m/\sigma_m > -y\right\}\right) \\ &\quad + \left(P\left\{-S_m/\sigma_m > -y, -S_n/\sigma_n > -y\right\} - P\left\{-S_m/\sigma_m > -y\right\}P\left\{-S_n/\sigma_n > -y\right\}\right) \\ &=: D_1 + D_2 + D_3 + D_4. \end{aligned} \tag{26}$$

Our task now is to estimate all the components D_1-D_4 in (26).

Let $\left\{\hat{X}_n\right\}$ denote an i.i.d. standard normal sequence. Obviously, we have $D_1 = \sum_{i=1}^{m}\sum_{j=m+1}^{n}\left(P\left\{X_i \leq u_m, X_j \leq u_n\right\} - P\left\{\hat{X}_i \leq u_m, \hat{X}_j \leq u_n\right\}\right)$. As the random vectors (X_i, X_j), $\left(\hat{X}_i, \hat{X}_j\right)$ are standard normal and (12) holds, we can apply Theorem 4.2.1 in Leadbetter et al. (1983). This and (15) yield $D_1 \ll m\exp\left(-\frac{u_m^2+u_n^2}{2\left(1+\delta\right)}\right)\sum_{t=1}^{n-1}r\left(t\right) \ll \frac{m\left(\log n\right)^{\frac{1}{1+\delta}}}{m^{\frac{1}{1+\delta}}n^{\frac{1}{1+\delta}}}\sum_{t=1}^{n-1}\frac{L\left(t\right)}{t^{\alpha}} \ll \frac{m\left(\log n\right)^{\frac{1}{1+\delta}}L\left(n\right)n^{1-\alpha}}{m^{\frac{1}{1+\delta}}n^{\frac{1}{1+\delta}}}$

$$= \frac{m^{1-\frac{1}{1+\delta}} (\log n)^{\frac{1}{1+\delta}} L(n)}{n^{1-\frac{1}{1+\delta}} n^{\frac{2}{1+\delta}+\alpha-2}} \ll \frac{(\log n)^{\frac{1}{1+\delta}} L(n)}{n^{\frac{2}{1+\delta}+\alpha-2}} \ll \frac{n^{\epsilon}}{n^{\frac{2}{1+\delta}+\alpha-2}} \text{ for any } \epsilon > 0.$$

Since, by (12), $\delta < \alpha/(2-\alpha)$, then $\frac{2}{1+\delta} + \alpha - 2 > 0$ and

$$D_1 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \tag{27}$$

In order to estimate the component D_2 in (26), let us denote by Y_n a standard normal random variable, which is independent of S_n/σ_n. We can write that $D_2 = \sum_{i=1}^{m} \left(P\{X_i \leq u_m, S_n/\sigma_n \leq y\} - P\{Y_n \leq u_m, S_n/\sigma_n \leq y\}\right)$. Let λ satisfy (11). As the vectors $(X_i, S_n/\sigma_n)$, $(Y_n, S_n/\sigma_n)$ are standard normal and (11) holds, then, due to Theorem 4.2.1 in Leadbetter et al. (1983), and (10), (15),

$$D_2 \ll \sum_{i=1}^{m} Cov(X_i, S_n/\sigma_n) \exp\left(-\frac{u_m^2}{2(1+\lambda)}\right) \ll \frac{m (\log m)^{\frac{1}{2(1+\lambda)}}}{n^{\beta} m^{\frac{1}{1+\lambda}}} \ll 1/n^{\gamma} \tag{28}$$

for some $\gamma > 0$, where the last relation follows from the fact that $\beta > 1 - \frac{1}{1+\lambda}$. We also wish to estimate D_3 in (26). Let μ satisfy (25). By using similar reasoning to that, which led us to the first derivation in (28), we obtain

$$D_3 \ll \exp\left(-\frac{u_n^2}{2(1+\mu)}\right) \sum_{i=m+1}^{n} Cov(X_i, S_m/\sigma_m). \tag{29}$$

Notice that $\sum_{i=m+1}^{n} Cov(X_i, S_m/\sigma_m) = \frac{1}{\sigma_m}\left(\sum_{t=m}^{n-1} r(t) + ... + \sum_{t=1}^{n-m} r(t)\right) < \frac{m}{\sigma_m} \sum_{t=1}^{n-1} r(t)$. Thus, by (29), (15), assumption (2) and relation (9),

$$D_3 \ll \frac{(\log n)^{\frac{1}{2(1+\mu)}}}{n^{\frac{1}{1+\mu}}} \frac{m}{L(m)^{1/2} m^{1-\alpha/2}} \sum_{t=1}^{n-1} \frac{L(t)}{t^{\alpha}} \ll \frac{(\log n)^{\frac{1}{2(1+\mu)}} m^{\alpha/2} L(n) n^{1-\alpha}}{n^{\frac{1}{1+\mu}} L(m)^{1/2}}$$

$$\ll \frac{(\log n)^{\frac{1}{2(1+\mu)}} n^{\alpha/2} L(n)}{n^{\frac{1}{1+\mu}+\alpha-1}} = \frac{(\log n)^{\frac{1}{2(1+\mu)}} L(n)}{n^{\frac{1}{1+\mu}+\alpha/2-1}} \ll \frac{n^{\epsilon}}{n^{\frac{1}{1+\mu}+\alpha/2-1}} \text{ for any } \epsilon > 0.$$

Since, by (25), $\mu < \alpha/(2-\alpha)$, then $\frac{1}{1+\mu} + \alpha/2 - 1 > 0$ and

$$D_3 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \tag{30}$$

It remains to estimate D_4 in (26). Notice that, if $Cov(S_m/\sigma_m, S_n/\sigma_n) \leq 1/2$, then, by Theorem 4.2.1 in Leadbetter et al. (1983),

$$D_4 \leq \frac{Cov(S_m/\sigma_m, S_n/\sigma_n)}{2\pi\sqrt{1 - \{Cov(S_m/\sigma_m, S_n/\sigma_n)\}^2}} \leq \frac{Cov(S_m/\sigma_m, S_n/\sigma_n)}{\pi\sqrt{3}}. \tag{31}$$

In turn, provided $Cov(S_m/\sigma_m, S_n/\sigma_n) > 1/2$, we can estimate D_4 in (26) as follows. Observe that, since S_m/σ_m, S_n/σ_n are associated and have bounded,

continous density, then, by Lemma 2.2 in Bagai and Prakasa Rao (1991), there exists a positive constant C_1, neither depending on m, n nor on y, such that $D_4 \leq C_1 \{Cov\,(S_m/\sigma_m, S_n/\sigma_n)\}^{1/3}$. Thus, if $Cov\,(S_m/\sigma_m, S_n/\sigma_n) > 1/2$, then

$$D_4 \leq C_1 \frac{Cov\,(S_m/\sigma_m, S_n/\sigma_n)}{\{Cov\,(S_m/\sigma_m, S_n/\sigma_n)\}^{2/3}} < C_1 2^{2/3} Cov\,(S_m/\sigma_m, S_n/\sigma_n)\,. \tag{32}$$

Put $C := \max\left(1/\pi\sqrt{3}, C_1 2^{2/3}\right)$. By (31), (32), $D_4 \leq C \cdot Cov\,(S_m/\sigma_m, S_n/\sigma_n)$. Consequently

$$D_4 \ll Cov\,(S_m/\sigma_m, S_n/\sigma_n)\,. \tag{33}$$

Let us estimate the covariance on the r.h.s. of (33). We have

$$Cov\left(\frac{S_m}{\sigma_m}, \frac{S_n}{\sigma_n}\right) = Cov\left(\frac{S_m}{\sigma_m}, \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) + E\left(\frac{S_m}{\sigma_m}\right)\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right)$$

$$-E\left(\frac{S_m}{\sigma_m}\right) E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) =: F_1 + F_2 + F_3. \tag{34}$$

We now estimate F_1-F_3 in (34). By (2), (9) and Karamata's theorem, we obtain

$$\begin{aligned} F_1 &\ll \frac{1}{L\,(m)^{1/2}\, m^{1-\alpha/2} L\,(n)^{1/2}\, n^{1-\alpha/2}} m \sum_{t=m+1}^{n+2m-1} \frac{L\,(t)}{t^{\alpha}} \\ &\ll \frac{1}{L\,(m)^{1/2}\, m^{1-\alpha/2} L\,(n)^{1/2}\, n^{1-\alpha/2}} m\,(n+2m)^{1-\alpha}\, L\,(n+2m)\,. \end{aligned} \tag{35}$$

In order to bound F_2 in (34), notice that $F_2 \leq E\left|\frac{S_m}{\sigma_m}\right| \left|\frac{(S_{n+2m} - S_n) - S_{2m}}{\sigma_n}\right|$. By the Cauchy-Schwarz inequality, the stationarity of the sequence $\{X_i\}$ and the fact that $EX_iX_j > 0$ for all i, j, we deduce that

$$F_2 \leq \sqrt{E\left(\frac{S_m}{\sigma_m}\right)^2 E\left(\frac{(S_{n+2m} - S_n) - S_{2m}}{\sigma_n}\right)^2} \leq \sqrt{2\frac{\sigma_{2m}^2}{\sigma_n^2}} = \sqrt{2}\frac{\sigma_{2m}}{\sigma_n}. \tag{36}$$

This, relation (9) and definition of the slowly varying function at infinity imply

$$F_2 \ll \frac{\sigma_{2m}}{\sigma_n} \sim \frac{L\,(2m)^{1/2}\,(2m)^{1-\alpha/2}}{L\,(n)^{1/2}\, n^{1-\alpha/2}} \ll \frac{L\,(m)^{1/2}\, m^{1-\alpha/2}}{L\,(n)^{1/2}\, n^{1-\alpha/2}}. \tag{37}$$

Since, in view of the stationarity of the sequence $\{X_i\}$, the last component F_3 in (34) is equal to zero, then, by (33)-(35) and (37),

$$D_4 \ll \frac{m\,(n+2m)^{1-\alpha}\, L\,(n+2m)}{L\,(m)^{1/2}\, m^{1-\alpha/2} L\,(n)^{1/2}\, n^{1-\alpha/2}} + \frac{L\,(m)^{1/2}\, m^{1-\alpha/2}}{L\,(n)^{1/2}\, n^{1-\alpha/2}}. \tag{38}$$

Finally, due to (26)-(28), (30) and (38), relation (22) is fulfilled in the case, when assumption (2) is satisfied with some $0 < \alpha < 1$. ■

Proof of Lemma 2(b). Suppose that (2) holds for some $\alpha > 1$. We have

$$\begin{aligned} &|Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right)| \\ &=: \mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3 + \mathbb{D}_4, \end{aligned} \tag{39}$$

where $\mathbb{D}_1$-$\mathbb{D}_4$ are defined as D_1-D_4 in (26), but for the case (2) holds for some $\alpha > 1$. By applying Theorem 4.2.1 in Leadbetter et al. (1983), it is easy to check that

$$\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \tag{40}$$

Thus, it remains to estimate the component $\mathbb{D}_4$. By (33)

$$\mathbb{D}_4 \ll Cov\left(S_m/\sigma_m, S_n/\sigma_n\right) =: \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3, \tag{41}$$

with $\mathbb{F}_1$-$\mathbb{F}_3$, defined as F_1-F_3 in (34), but for the case (2) holds with some $\alpha > 1$. As $\mathbb{F}_1 = Cov\left(\frac{S_m}{\sigma_m}, \frac{S_{n+2m} - S_{2m}}{\sigma_n}\right) \leq \frac{1}{\sigma_m \sigma_n} m \left(\sum_{t=m+1}^{\infty} \frac{L(t)}{t^{\alpha}} - \sum_{t=n+2m}^{\infty} \frac{L(t)}{t^{\alpha}}\right)$, it follows from the facts that: $\alpha > 1$, $\sigma_i \geq i^{1/2}$, and from Karamata's theorem that

$$\mathbb{F}_1 \ll (m^{1/2}/n^{1/2})(L(m)/m^{\alpha-1}) \ll m^{1/2}/n^{1/2}. \tag{42}$$

In order to estimate $\mathbb{F}_2 = E\left(\frac{S_m}{\sigma_m}\right)\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right)$, notice that, as (2) holds for some $\alpha > 1$, then $\sigma_{2m} \ll m^{1/2}$. Since in addition, $\sigma_n \geq n^{1/2}$, we have, by (36),

$$\mathbb{F}_2 \ll \sigma_{2m}/\sigma_n \ll m^{1/2}/n^{1/2}. \tag{43}$$

As moreover, $\mathbb{F}_3 = -E\left(\frac{S_m}{\sigma_m}\right) E\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right) = 0$, then, due to (41)-(43),

$$\mathbb{D}_4 \ll m^{1/2}/n^{1/2}. \tag{44}$$

Relations (39), (40), (44) imply (23) in the case (2) holds for some $\alpha > 1$. ■

Proof of Lemma 2(c). Assume that (2) is satisfied with $\alpha = 1$. We have

$$\begin{aligned} &|Cov\left(I\left\{M_m \leq u_m, S_m/\sigma_m \leq y\right\}, I\left\{M_{m,n} \leq u_n, S_n/\sigma_n \leq y\right\}\right)| \\ &=: \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4, \end{aligned} \tag{45}$$

where $\mathcal{D}_1$-$\mathcal{D}_4$ are defined as D_1-D_4 in (26), but for the case (2) holds with $\alpha = 1$. By using Theorem 4.2.1 in Leadbetter et al. (1983), it is easy to verify that

$$\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 \ll 1/n^{\gamma} \text{ for some } \gamma > 0. \tag{46}$$

Thus, it remains to estimate the component $\mathcal{D}_4$. By (33)

$$\mathcal{D}_4 \ll Cov\left(S_m/\sigma_m, S_n/\sigma_n\right) =: \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3, \tag{47}$$

with $\mathcal{F}_1$-$\mathcal{F}_3$ defined as F_1-F_3 in (34), but for the case, when (2) holds with $\alpha = 1$. We have

$$\mathcal{F}_1 = \frac{1}{\sigma_m \sigma_n} Cov\left(S_m, S_{n+2m} - S_{2m}\right) \leq \frac{1}{\sigma_m \sigma_n} m \sum_{t=m+1}^{n+2m-1} r(t). \tag{48}$$

By the same reasoning as in the excerpt of the proof of Theorem 1 in Mielniczuk (2002) (see p. 396, lines 7-15), we obtain

$$\sigma_n^2 = n + 2\sum_{t=1}^{n-1}(n-t)\,r(t) \sim n\left(1 + 2\sum_{t=1}^{n-1} r(t)\right) = n\widetilde{L}(n). \tag{49}$$

By definition of $\widetilde{L}(n)$, we have $m\sum_{t=m+1}^{n+2m-1} r(t) = m\left(\widetilde{L}(n+2m) - \widetilde{L}(m+1)\right)/2$. This and relations (48), (49) yield

$$\mathcal{F}_1 \ll \frac{1}{m^{1/2}\widetilde{L}(m)^{1/2}\, n^{1/2}\widetilde{L}(n)^{1/2}} m\left(\widetilde{L}(n+2m) - \widetilde{L}(m+1)\right). \tag{50}$$

In turn, by (36), $\mathcal{F}_2 = E\left(\frac{S_m}{\sigma_m}\right)\left(\frac{S_n}{\sigma_n} - \frac{S_{n+2m}-S_{2m}}{\sigma_n}\right) \le E\left|\frac{S_m}{\sigma_m}\right|\left|\frac{(S_{n+2m}-S_n)-S_{2m}}{\sigma_n}\right|$ $\ll \sigma_{2m}/\sigma_n$. Hence, from (49) and the fact that $\widetilde{L}(\cdot)$ is slowly varying, we get

$$\mathcal{F}_2 \ll \left(m^{1/2}\widetilde{L}(m)^{1/2}\right) / \left(n^{1/2}\widetilde{L}(n)^{1/2}\right). \tag{51}$$

As moreover, $F_3 = 0$, then, by (47), (50) and (51),

$$\mathcal{D}_4 \ll \frac{m^{1/2}\left(\widetilde{L}(n+2m) - \widetilde{L}(m+1)\right)}{n^{1/2}\widetilde{L}(m)^{1/2}\,\widetilde{L}(n)^{1/2}} + \frac{m^{1/2}\widetilde{L}(m)^{1/2}}{n^{1/2}\widetilde{L}(n)^{1/2}}. \tag{52}$$

Relations (45), (46), (52) imply (24) in the case, when (2) is satisfied with $\alpha = 1$. This completes the whole proof of Lemma 2. ■

The following lemma will be also used in our further considerations.

Lemma 3 ***(for comparison - see also Ho and Hsing (1996)).*** *Under the assumptions of Theorem 1 on $X_1, X_2, ...,$ and $r(t)$, $\{u_n\}$, we have*

$$\lim_{n\to\infty} P\{M_n \le u_n, S_n/\sigma_n \le y\} = e^{-\tau}\Phi(y) \text{ for all } y \in \mathbb{R} \text{ and } \tau \text{ defined in (3).}$$

Proof. Notice that, as $X_1, X_2, ...$ is a standard normal sequence, $r(t)$, $\{u_n\}$ satisfy (2), (3), respectively, then, by Theorem 4.3.3 in Leadbetter et al. (1983),

$$\lim_{n\to\infty} P\{M_n \le u_n\} = e^{-\tau} \text{ for } \tau \text{ defined in (3).} \tag{53}$$

Let y be an arbitrary real number and let, for each $n \ge 1$, Y_n denote a random variable, which has the same distribution as S_n/σ_n, but is independent of $(X_1, ..., X_n)$. From the estimations of A_1, $\mathbb{A}_1$ in the proof of Lemma 1, $|P\{M_n \le u_n, S_n/\sigma_n \le y\} - P\{M_n \le u_n\}P\{Y_n \le y\}| \ll 1/n^\gamma$ for some $\gamma > 0$. As: (53) holds, $1/n^\gamma \to 0$, as $n \to \infty$, and, for each n, Y_n is standard normal, we get $\lim_{n\to\infty} P\{M_n \le u_n, S_n/\sigma_n \le y\} = \lim_{n\to\infty} P\{M_n \le u_n\}P\{Y_n \le y\} = e^{-\tau}\Phi(y)$, which yields the desired result. ■

4. Proof of main result

In this section, we present the proof of Theorem 1. In our proof, we make an extensive use of the earlier proved Lemmas 1-3.

Proof of Theorem 1(i). First, we will show that for any real y

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(I\left\{M_n \le u_n, S_n/\sigma_n \le y\right\} - P\left\{M_n \le u_n, S_n/\sigma_n \le y\right\}\right) \overset{\text{a.s.}}{\to} 0, \quad (54)$$

as $N \to \infty$. By Lemma 3.1 in Csaki and Gonchigdanzan (2002), in order to prove (54), it suffices to show that the following property occurs for some $\varepsilon > 0$

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left\{M_n \le u_n, S_n/\sigma_n \le y\right\}\right) \ll (\log N)^2 (\log\log N)^{-(1+\varepsilon)}. \quad (55)$$

Put $\xi_n := I\left\{M_n \le u_n, S_n/\sigma_n \le y\right\}$. We have

$$\begin{aligned} Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left\{M_n \le u_n, S_n/\sigma_n \le y\right\}\right) &= Var\left(\sum_{n=1}^{N}\frac{1}{n}\xi_n\right) \\ \le \sum_{n=1}^{N}\frac{1}{n^2}Var\left(\xi_n\right) + 2\sum_{1\le m<n\le N}\frac{1}{mn}\left|Cov\left(\xi_m,\xi_n\right)\right| &=: \sum\nolimits_1 + \sum\nolimits_2. \quad (56) \end{aligned}$$

Since $\xi_n^2 \le 1$, we obtain that

$$\sum\nolimits_1 \le \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty. \quad (57)$$

Thus, we only need to estimate $\sum_2$ in (56). Let $m < n$. Observe that

$$\begin{aligned} \left|Cov\left(\xi_m,\xi_n\right)\right| \le{}& 2E\left|I\left\{M_n \le u_n, S_n/\sigma_n \le y\right\} - I\left\{M_{m,n} \le u_n, S_n/\sigma_n \le y\right\}\right| \\ &+ \left|Cov\left(I\left\{M_m \le u_m, S_m/\sigma_m \le y\right\}, I\left\{M_{m,n} \le u_n, S_n/\sigma_n \le y\right\}\right)\right|. \end{aligned}$$

This and Lemmas 1, 2 imply that:

(a) if condition (2) is satisfied with some $0 < \alpha < 1$, then for some $\gamma > 0$

$$\left|Cov\left(\xi_m,\xi_n\right)\right| \ll \frac{1}{n^\gamma} + \frac{m}{n} + \frac{m\left(n+2m\right)^{1-\alpha}L\left(n+2m\right)}{L\left(m\right)^{1/2}m^{1-\alpha/2}L\left(n\right)^{1/2}n^{1-\alpha/2}} + \frac{L\left(m\right)^{1/2}m^{1-\alpha/2}}{L\left(n\right)^{1/2}n^{1-\alpha/2}}, \quad (58)$$

(b) if condition (2) is satisfied with some $\alpha > 1$, then for some $\gamma > 0$

$$\left|Cov\left(\xi_m,\xi_n\right)\right| \ll 1/n^\gamma + m^{1/2}/n^{1/2}, \quad (59)$$

(c) if condition (2) is satisfied with $\alpha = 1$, then for some $\gamma > 0$

$$\left|Cov\left(\xi_m,\xi_n\right)\right| \ll \frac{1}{n^\gamma} + \frac{m}{n} + \frac{m^{1/2}\left(\widetilde{L}\left(n+2m\right) - \widetilde{L}\left(m+1\right)\right)}{n^{1/2}\widetilde{L}\left(m\right)^{1/2}\widetilde{L}\left(n\right)^{1/2}} + \frac{m^{1/2}\widetilde{L}\left(m\right)^{1/2}}{n^{1/2}\widetilde{L}\left(n\right)^{1/2}}. \quad (60)$$

Let us consider the case (a) first. By (56) and (58), we get

$$\sum\nolimits_2 \ll \sum_{1\le m<n\le N} \frac{1}{mn}\frac{1}{n^{\gamma}} + \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m}{n}$$
$$+ \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m\left(n+2m\right)^{1-\alpha} L\left(n+2m\right)}{L\left(m\right)^{1/2} m^{1-\alpha/2} L\left(n\right)^{1/2} n^{1-\alpha/2}} + \sum_{1\le m<n\le N} \frac{1}{mn}\frac{L\left(m\right)^{1/2} m^{1-\alpha/2}}{L\left(n\right)^{1/2} n^{1-\alpha/2}}$$
$$=: \sum\nolimits_3 + \sum\nolimits_4 + \sum\nolimits_5 + \sum\nolimits_6 \ll \log N + \sum\nolimits_5 + \sum\nolimits_6, \tag{61}$$

where the last relation follows from the obvious fact that $\sum_3 + \sum_4 \ll \log N$. Thus, it remains to estimate $\sum_5$, $\sum_6$ in (61). It is easy to check that

$$\sum\nolimits_5 \ll \sum_{m=1}^{N-1} \frac{1}{L\left(m\right)^{1/2} m^{1-\alpha/2}} \sum_{n=m+1}^{N} \left(1+\frac{2m}{n}\right)^{1-\alpha} \frac{L\left(n+2m\right)}{L\left(n\right)^{1/2}} \frac{1}{n^{1+\alpha/2}}. \tag{62}$$

For a fixed m, put $\bar{L}\left(n\right) := L\left(n+2m\right)/L\left(n\right)^{1/2}$. By using (62) and the same reasoning as in the excerpt of the proof of Theorem 1 in Mielniczuk (2002) (see p. 395, lines 7-14), we obtain

$$\sum\nolimits_5 \ll \sum_{m=1}^{N-1} \frac{\bar{L}\left(m\right) m^{-\alpha/2}}{L\left(m\right)^{1/2} m^{1-\alpha/2}} = \sum_{m=1}^{N-1} \frac{\bar{L}\left(m\right)}{L\left(m\right)^{1/2}} \frac{1}{m} = \sum_{m=1}^{N-1} \frac{L\left(3m\right)}{L\left(m\right)} \frac{1}{m} \ll \log N. \tag{63}$$

In addition, we can estimate $\sum_6$ in (61) as follows. Observe that, by the facts that $L\left(\cdot\right)$ is slowly varying and $0<\alpha<1$, we have

$$\sum\nolimits_6 \ll \sum_{m=1}^{N-1} \frac{L\left(m\right)^{1/2}}{m^{\alpha/2}} \sum_{n=m+1}^{\infty} \frac{1}{L\left(n\right)^{1/2} n^{2-\alpha/2}} \ll \sum_{m=1}^{N-1} \frac{L\left(m\right)^{1/2}}{m^{\alpha/2}} \frac{1}{L\left(m\right)^{1/2} m^{1-\alpha/2}}.$$

Hence $\sum_6 \ll \log N$. This and (61), (63) yield

$$\sum\nolimits_2 \ll \log N. \tag{64}$$

Due to (56), (57) and (64), we conclude that (55) holds for arbitrary real y and any positive ε. Consequently, by the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan (2002), condition (54) is satisfied as well. This, Lemma 3 and the regularity property of logarithmic means imply the statement (i) of Theorem 1 for the case, when (2) is satisfied with some $0<\alpha<1$.

We now consider the case (b), i.e. the case, when (2) holds with some $\alpha>1$. By (56) and (59), we get $\sum_2 \ll \sum_{1\le m<n\le N} \frac{1}{mn}\frac{1}{n^{\gamma}} + \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m^{1/2}}{n^{1/2}} \ll \log N$. This, (56) and (57) imply that condition (55) holds for arbitrary real y and any positive ε. Therefore (54) occurs. Hence, also in the case, when (2) is satisfied with some $\alpha>1$, the statement (i) of Theorem 1 is true.

Finally, let us consider the case (c), assuming that (2) holds with $\alpha=1$. It follows from (56) and (60) that

$$\sum\nolimits_2 \ll \sum_{1\le m<n\le N} \frac{1}{mn}\frac{1}{n^{\gamma}} + \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m}{n}$$
$$+ \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m^{1/2}\left(\widetilde{L}(n+2m)-\widetilde{L}(m+1)\right)}{n^{1/2}\widetilde{L}(m)^{1/2}\,\widetilde{L}(n)^{1/2}} + \sum_{1\le m<n\le N} \frac{1}{mn}\frac{m^{1/2}\widetilde{L}(m)^{1/2}}{n^{1/2}\widetilde{L}(n)^{1/2}}$$
$$=: \sum\nolimits_7 + \sum\nolimits_8 + \sum\nolimits_9 + \sum\nolimits_{10} \ll \log N + \sum\nolimits_9 + \sum\nolimits_{10}. \qquad (65)$$

Let us estimate the term $\sum_9$ in (65). By reasoning identically as in Mielniczuk (2002), p. 397, we get $\sum_9 = \sum_{m=1}^{N-1} \frac{1}{m^{1/2}\widetilde{L}(m)^{1/2}} \sum_{n=m+1}^{N} \frac{\widetilde{L}(n+2m)-\widetilde{L}(m+1)}{n^{3/2}\widetilde{L}(n)^{1/2}}$
$\ll \sum_{m=1}^{N-1} \frac{1}{m^{1/2}\widetilde{L}(m)^{1/2}} \frac{\widetilde{L}(3m)-\widetilde{L}(m+1)}{m^{1/2}\widetilde{L}(m)^{1/2}} = \sum_{m=1}^{N-1} \frac{\widetilde{L}(3m)-\widetilde{L}(m+1)}{m\widetilde{L}(m)} \ll \log N.$

In order to estimate $\sum_{10}$ in (65), observe that, as $\widetilde{L}(\cdot)$ is slowly varying,
$\sum_{10} \ll \sum_{m=1}^{N-1} \frac{\widetilde{L}(m)^{1/2}}{m^{1/2}} \sum_{n=m+1}^{N} \frac{1}{n^{3/2}\widetilde{L}(n)^{1/2}} \ll \sum_{m=1}^{N-1} \frac{\widetilde{L}(m)^{1/2}}{m^{1/2}} \frac{1}{m^{1/2}\widetilde{L}(m)^{1/2}} \ll \log N.$
The estimates for $\sum_9$, $\sum_{10}$, together with (65), yield $\sum_2 \ll \log N$, which, by (56), (57), implies that (55) holds for arbitrary real y and any $\varepsilon > 0$. Therefore (54) occurs. Thus, (4) holds also in the case (2) is satisfied with $\alpha = 1$. ■

Proof of Theorem 1(ii). It is easily seen that Theorem 1(ii) is a special case of the earlier proved Theorem 1(i), with: $u_n := x/a_n + b_n$, $\tau := e^{-x}$. ■

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