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# On estimation of the shape parameter of the gamma distribution 

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#### Abstract

The problem of estimation of an unknown shape parameter under the sample drawn from the gamma distribution, where the scale parameter is also unknown, is considered. A new estimator, called the maximum likelihood scale invariant estimator, is proposed. It is established that both the bias and the variance of this estimator are less than that of the usual maximum likelihood estimator. A property of the psi function is also obtained.


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[^0]
## 1 Introduction

Let a sample $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be drawn from the gamma distribution $\Gamma(\alpha, \sigma)$ with an unknown shape parameter $\alpha>0$ and an unknown scale parameter $\sigma>0$, whose density function has the form

$$
p(u ; \alpha, \sigma)=\frac{u^{\alpha-1} e^{-u / \sigma}}{\sigma^{\alpha} \Gamma(\alpha)}, \quad u>0 .
$$

Consider the problem of estimation of $\alpha$. One of the most popular estimators is the well-known maximum likelihood estimator (ML-estimator) (e.g. [3, Sections 9.3, 9.4], [4], [5], [6]). Let

$$
\mathbf{p}(x ; \alpha, \sigma)=\sigma^{-n \alpha}(\Gamma(\alpha))^{-n}\left(\prod_{j=1}^{n} x_{j}\right)^{\alpha-1} \exp \left(-\frac{1}{\sigma} \sum_{k=1}^{n} x_{k}\right)
$$

be the corresponding likelihood function. The ML-estimators of $\alpha$ and $\sigma$ are determined by the equations:

$$
\left\{\begin{array}{l}
\ln \sigma+\Psi(\alpha)=\sum_{j=1}^{n} \ln x_{j} / n \\
\alpha-\sum_{k=1}^{n} x_{k} /(n \sigma)=0
\end{array}\right.
$$

where $\Psi(\alpha):=\frac{d}{d \alpha} \ln \Gamma(\alpha):=(\ln \Gamma(\alpha))^{\prime}$ is the so-called Euler psi (digamma) function.
From those equations one can obtain the ML-estimators $\alpha^{*}$ and $\sigma^{*}$ of $\alpha$ and $\sigma$, respectively. Namely, $\alpha^{*}$ is the root of the equation

$$
g(\alpha):=\ln \alpha-\Psi(\alpha)=\ln \bar{x}-\frac{1}{n} \sum_{j=1}^{n} \ln x_{j}
$$

while

$$
\sigma^{*}=\frac{\bar{x}}{\alpha^{*}} .
$$

Here, $\bar{x}$ is the sample mean, i.e.

$$
\bar{x}=\frac{1}{n} \sum_{k=1}^{n} x_{k} .
$$

Observe that the function $g$ is strictly decreasing and takes values in $(0, \infty)$ (e.g. Theorem 1 of [2]). Therefore, the estimator $\alpha^{*}$ is well-defined and unique.

Observe also that the estimator $\alpha^{*}$ is scale invariant. Furthermore, one can easily see that

$$
\mathrm{E}_{(\alpha, \sigma)}\left(\ln \bar{x}-\frac{1}{n} \sum_{j=1}^{n} \ln x_{j}\right)=\Psi(n \alpha)-\Psi(\alpha)-\ln n=g(\alpha)-g(n \alpha):=g_{n}(\alpha)
$$

The question arises: why wouldn't one take the root of the equation

$$
\begin{equation*}
g_{n}(\alpha)=\ln \bar{x}-\frac{1}{n} \sum_{j=1}^{n} \ln x_{j} \tag{1}
\end{equation*}
$$

as an estimator of an unknown shape parameter $\alpha$ ? Such an estimator would coincide with that based on the method of moments.

It turns out that such a choice has quite a deep reasoning. Since in our scheme $\sigma$ becomes a nuisance parameter, it is natural to apply the maximum likelihood principle to the measure defined on the $\sigma$-algebra of the scale invariant sets generated by the underlying gamma distribution. As it is known (e.g. [7, Subsection 3.2.2], [8, Section $8.3]$ ), the density corresponding to this measure, with respect to that generated by the standard normal distribution, is given as follows:

$$
\mathbf{q}(x ; \alpha)=\frac{\int_{0}^{\infty} t^{n-1} \mathbf{p}(t x ; \alpha, \sigma) d t}{\int_{0}^{\infty} t^{n-1} \mathbf{s}(t x) d t}=\frac{2 \pi^{n / 2} \Gamma(n \alpha)\left(\sum_{i=1}^{n} x_{i}^{2}\right)^{n / 2}\left(\prod_{i=1}^{n} x_{i}\right)^{\alpha-1}}{\Gamma(n / 2)(\Gamma(\alpha))^{n}\left(\sum_{i=1}^{n} x_{i}\right)^{n \alpha}},
$$

where

$$
\mathbf{s}(x)=(2 \pi)^{-n / 2} \exp \left(-\frac{1}{2} \sum_{k=1}^{n} x_{k}^{2}\right)
$$

Then by direct calculations one can obtain that the maximum likelihood scale invariant estimator (IML-estimator) $\alpha^{* *} \in \arg \max _{\alpha>0} \mathbf{q}(x ; \alpha)$ is the root of the equation (1).


Figure 1: Graphs of the functions $g, g_{n}, g^{\prime}, g_{n}^{\prime}$ for $n=2$.

The estimator $\alpha^{* *}$ is also scale invariant, well-defined and unique since the function $g_{n}$ is strictly decreasing and takes values in $(0, \infty)$ (see Lemma 1 in Section 3).

It is worth noting that the scale invariance of the maximum likelihood estimator of a shape parameter, is a quite common property in the case when the scale is also unknown. Indeed, consider the likelihood function

$$
\mathbf{p}(x ; \alpha, \sigma)=\sigma^{-n} \prod_{j=1}^{n} p\left(\sigma^{-1} x_{j} ; \alpha, 1\right)
$$

where $\alpha$ is a shape parameter taking values in $\left(\alpha_{-}, \alpha_{+}\right)$. Assume that

$$
\max _{\alpha \in\left(\alpha_{-}, \alpha_{+}\right), \sigma>0} \mathbf{p}(x ; \alpha, \sigma)=\max _{\alpha \in\left(\alpha_{-}, \alpha_{+}\right)} \max _{\sigma>0} \mathbf{p}(x ; \alpha, \sigma) .
$$

Let

$$
\hat{\sigma}(x ; \alpha) \in \arg \max _{\sigma>0} \mathbf{p}(x ; \alpha, \sigma)
$$

Observe that for any $\lambda>0$

$$
\mathbf{p}(\lambda x ; \alpha, \sigma)=\lambda^{-n} \mathbf{p}(x ; \alpha, \sigma / \lambda),
$$

whence

$$
\hat{\sigma}(\lambda x ; \alpha)=\lambda \hat{\sigma}(x ; \alpha)
$$

Thus,

$$
\begin{aligned}
\left(\alpha^{*}, \sigma^{*}\right) & \in \arg \max _{\alpha \in\left(\alpha_{-}, \alpha_{+}\right),}{ }_{\sigma>0} \mathbf{p}(x ; \alpha, \sigma)=\arg \max _{\alpha \in\left(\alpha_{-}, \alpha_{+}\right)} \mathbf{p}(x ; \alpha, \hat{\sigma}(x ; \alpha)) \\
& =\arg \max _{\alpha \in\left(\alpha_{-}, \alpha_{+}\right)}\left((\hat{\sigma}(x ; \alpha))^{-n} \prod_{j=1}^{n} p\left((\hat{\sigma}(x ; \alpha))^{-1} x_{j} ; \alpha, 1\right)\right) .
\end{aligned}
$$

It is evident that $\alpha^{*}(\lambda x)=\alpha^{*}(x)$, i.e. the estimator $\alpha^{*}$ is scale invariant. Therefore, it is reasonable to apply the method presented here also for other distributions.

The goal of this paper is to compare two estimators of $\alpha$ : the ML-estimator and the IML-estimator. We show that both the bias and the variance of the IML-estimator $\alpha^{* *}$ are less than that of the ML-estimator $\alpha^{*}$.

The paper is organized as follows. In Section 2 we establish the main result while all the auxiliary results are proved in Section 3.

## 2 Main result

Theorem. If a sample $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is drawn from $\Gamma(\alpha, \sigma)$ distribution, then

$$
\begin{equation*}
\mathrm{E} \alpha^{*}>\mathrm{E} \alpha^{* *}>\alpha, \quad \operatorname{Var} \alpha^{*}>\operatorname{Var} \alpha^{* *} \tag{2}
\end{equation*}
$$

Proof. Since the functions $g$ and $g_{n}$ are strictly decreasing, the inverse functions $g^{-1}$ and $g_{n}^{-1}$ are well-defined. According to the definitions,

$$
\alpha^{*}=g^{-1}(T(x)), \quad \alpha^{* *}=g_{n}^{-1}(T(x))
$$

where

$$
T(x):=\ln \bar{x}-\frac{1}{n} \sum_{j=1}^{n} \ln x_{j} .
$$

Denote

$$
f(u):=g^{-1}(u)-g_{n}^{-1}(u), \quad u>0 .
$$

The function $f$ is positive. Indeed, since

$$
g_{n}\left(g^{-1}(u)\right)=g\left(g^{-1}(u)\right)-g\left(n g^{-1}(u)\right)=u-g\left(n g^{-1}(u)\right)<u
$$

and the function $g_{n}$ is strictly decreasing, we obtain

$$
g^{-1}(u)>g_{n}^{-1}(u), \quad u>0
$$

Therefore, $\mathrm{E} \alpha^{*}>\mathrm{E} \alpha^{* *}$.
Now let us show that for any $\alpha>0$

$$
\mathrm{E} \alpha^{* *}=\mathrm{E} g_{n}^{-1}(T(x)) \gg .
$$

Observe that the function $g_{n}^{-1}$ is strictly convex since from Lemma 1 in Section 3 it follows that

$$
\frac{\partial^{2} g_{n}^{-1}(u)}{\partial u^{2}}=-\frac{g_{n}^{\prime \prime}\left(g_{n}^{-1}(u)\right)}{\left(g_{n}^{\prime}\left(g_{n}^{-1}(u)\right)\right)^{3}}>0, \quad u>0
$$

Since $\mathrm{E} T(x)=g_{n}(\alpha)$, by the Jensen inequality we get

$$
\mathrm{E} g_{n}^{-1}(T(x))>g_{n}^{-1}(\mathrm{E} T(x))=g_{n}^{-1}\left(g_{n}(\alpha)\right)=\alpha
$$

It remains to prove the inequality for variances. From the evident equalities

$$
\alpha^{*}-\mathrm{E} \alpha^{* *}=\alpha^{*}-\alpha^{* *}+\alpha^{* *}-\mathrm{E} \alpha^{* *}, \quad \alpha^{*}-\mathrm{E} \alpha^{* *}=\alpha^{*}-\mathrm{E} \alpha^{*}+\mathrm{E} \alpha^{*}-\mathrm{E} \alpha^{* *}
$$

one can obtain

$$
\operatorname{Var} \alpha^{*}+\left(\mathrm{E} \alpha^{*}-\mathrm{E} \alpha^{* *}\right)^{2}=\operatorname{Var} \alpha^{* *}+\mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)^{2}+2 \mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right)
$$

Therefore,

$$
\operatorname{Var} \alpha^{*}-\operatorname{Var} \alpha^{* *}=\operatorname{Var}\left(\alpha^{*}-\alpha^{* *}\right)+2 \mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right)
$$

In order to show the second part of (2), it is enough to establish that

$$
\mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right)>0 .
$$

Denote $c:=\mathrm{E} \alpha^{* *}$ and observe that

$$
\begin{aligned}
\mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right) & =\mathrm{E}\left(g^{-1}(T(x))-g_{n}^{-1}(T(x))\right)\left(g_{n}^{-1}(T(x))-c\right) \\
& =\int_{0}^{\infty}\left(g^{-1}(u)-g_{n}^{-1}(u)\right)\left(g_{n}^{-1}(u)-c\right) p_{T(x)}(u) d u
\end{aligned}
$$

where $p_{T(x)}$ is the density of $T(x)$.
Let

$$
\Delta(u):=g_{n}^{-1}(u)-c, \quad u>0
$$

Observe that

$$
\begin{aligned}
& \Delta(u)>0, \quad 0<u<g_{n}(c), \\
& \Delta(u)=0, \quad u=g_{n}(c) \\
& \Delta(u)<0, \quad u>g_{n}(c) .
\end{aligned}
$$

Since, as it is proved in Lemma 3 of Section 3, the function $f$ is strictly decreasing, we obtain

$$
\int_{0}^{g_{n}(c)} f(u) \Delta(u) p_{T(x)}(u) d u>f\left(g_{n}(c)\right) \int_{0}^{g_{n}(c)} \Delta(u) p_{T(x)}(u) d u
$$

and

$$
\int_{g_{n}(c)}^{\infty} f(u) \Delta(u) p_{T(x)}(u) d u>f\left(g_{n}(c)\right) \int_{g_{n}(c)}^{\infty} \Delta(u) p_{T(x)}(u) d u
$$

Therefore,

$$
\begin{aligned}
\mathrm{E}\left(\alpha^{*}-\alpha^{* *}\right)\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right) & =\int_{0}^{\infty} f(u) \Delta(u) p_{T(x)}(u) d u>f\left(g_{n}(c)\right) \int_{0}^{\infty} \Delta(u) p_{T(x)}(u) d u \\
& =f\left(g_{n}(c)\right) \mathrm{E}\left(\alpha^{* *}-\mathrm{E} \alpha^{* *}\right)=0
\end{aligned}
$$

Remark 1. Since

$$
\mathrm{E}\left(\alpha^{*}-\alpha\right)^{2}=\operatorname{Var} \alpha^{*}+\left(E \alpha^{*}-\alpha\right)^{2}, \quad \mathrm{E}\left(\alpha^{* *}-\alpha\right)^{2}=\operatorname{Var} \alpha^{* *}+\left(E \alpha^{* *}-\alpha\right)^{2}
$$

from the theorem it follows that

$$
\mathrm{E}\left(\alpha^{*}-\alpha\right)^{2}>\mathrm{E}\left(\alpha^{* *}-\alpha\right)^{2} .
$$

Let us apply the Monte-Carlo simulation to confirm the results of the theorem. Given $\alpha$ and $\sigma$, we generate 10000 samples drawn from $\Gamma(\alpha, \sigma)$ distribution for $n=$ 10, 20, 30, 50, 100. Next, we solve numerically two equations: $g(\alpha)=T(x)$ and $g_{n}(\alpha)=T(x)$ with respect to $\alpha$, and obtain 10000 values of $\alpha^{*}$ and $\alpha^{* *}$. Taking their means, we estimate $\mathrm{E} \alpha^{*}$ and $\mathrm{E} \alpha^{* *}$ and then calculate $\left(\mathrm{E} \alpha^{*}-\alpha\right)^{2}$, $\left(\mathrm{E} \alpha^{* *}\right.$ $\alpha)^{2}$, Var $\alpha^{*}$, Var $\alpha^{* *}$. The corresponding programs are written in C++. The results are presented in Table 1.

| $\alpha=0.125$ |  |  |  |  | $\alpha=0.5$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ |
| 10 | 0.00159 | 0.00126 | 0.00119 | 0.00118 | 0.00410 | 0.00140 | 0.02770 | 0.02760 |
| 20 | 0.00120 | 0.00097 | 0.00100 | 0.00100 | 0.00258 | 0.00114 | 0.02008 | 0.01941 |
| 30 | 0.00100 | 0.00084 | 0.00080 | 0.00079 | 0.00120 | 0.00052 | 0.01392 | 0.01347 |
| 50 | 0.00082 | 0.00072 | 0.00048 | 0.00047 | 0.00056 | 0.00027 | 0.00849 | 0.00821 |
| 100 | 0.00068 | 0.00064 | 0.000228 | 0.000225 | 0.00011 | 0.00005 | 0.00367 | 0.00361 |
| $\alpha=1$ |  |  |  |  | $\alpha=2$ |  |  |  |
| $n$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ |
| 10 | 0.0145 | 0.0042 | 0.1171 | 0.1165 | 0.0536 | 0.0136 | 0.4885 | 0.4876 |
| 20 | 0.0112 | 0.0048 | 0.0865 | 0.0841 | 0.0463 | 0.0197 | 0.3744 | 0.3705 |
| 30 | 0.0078 | 0.0038 | 0.0647 | 0.0625 | 0.0320 | 0.0148 | 0.2785 | 0.2706 |
| 50 | 0.0026 | 0.0012 | 0.0371 | 0.0358 | 0.0128 | 0.0059 | 0.1672 | 0.1615 |
| 100 | 0.0006 | 0.0003 | 0.0174 | 0.0171 | 0.0029 | 0.0013 | 0.0771 | 0.0756 |
| $\alpha=4$ |  |  |  |  | $\alpha=16$ |  |  |  |
| $n$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ |
| 10 | 0.216 | 0.051 | 2.027 | 2.026 | 42.83 | 12.07 | 198.44 | 127.58 |
| 20 | 0.186 | 0.077 | 1.541 | 1.535 | 3.09 | 1.21 | 25.69 | 25.44 |
| 30 | 0.142 | 0.062 | 1.190 | 1.137 | 2.04 | 0.86 | 19.02 | 18.62 |
| 50 | 0.062 | 0.029 | 0.71 | 0.69 | 1.03 | 0.47 | 12.50 | 12.05 |
| 100 | 0.014 | 0.007 | 0.335 | 0.328 | 0.24 | 0.11 | 5.556 | 5.446 |
| $\alpha=32$ |  |  |  |  | $\alpha=128$ |  |  |  |
| $n$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ | $v^{*}$ | $v^{* *}$ | Var $\alpha^{*}$ | Var $\alpha^{* *}$ |
| 10 | 174.38 | 2.43 | 774.92 | 560.11 | 191.55 | 32.38 | 2102.56 | 2099.77 |
| 20 | 11.82 | 4.38 | 103.39 | 102.07 | 173.42 | 62.90 | 1619.45 | 1615.24 |
| 30 | 8.62 | 3.59 | 78.95 | 76.43 | 143.47 | 60.76 | 1288.96 | 1247.04 |
| 50 | 3.81 | 1.69 | 47.84 | 46.48 | 55.77 | 23.48 | 781.47 | 759.32 |
| 100 | 0.89 | 0.38 | 21.96 | 21.53 | 14.14 | 5.982 | 357.58 | 350.46 |

Table 1: Numerical calculations of $v^{*}=\left(\mathrm{E} \alpha^{*}-\alpha\right)^{2}, v^{* *}=\left(\mathrm{E} \alpha^{* *}-\alpha\right)^{2}, \operatorname{Var} \alpha^{*}, \operatorname{Var} \alpha^{* *}$.

## 3 Proofs of auxiliary results

Lemma 1. For any $n>1$ the function $g_{n}(u)=\Psi(n u)-\Psi(u)-\ln n, u>0$, is strictly decreasing and strictly convex.

Proof. The integral representation

$$
\begin{equation*}
\Psi^{\prime}(u)=\int_{0}^{\infty} \frac{t e^{-u t}}{1-e^{-t}} d t, \quad u>0 \tag{3}
\end{equation*}
$$

(e.g. [1, formula 6.4.1]) yields

$$
\begin{equation*}
g_{n}^{\prime}(u)=n \Psi^{\prime}(n u)-\Psi^{\prime}(u)=\int_{0}^{\infty} \frac{t}{1-e^{-t}}\left(n e^{-n u t}-e^{-u t}\right) d t, \quad u>0 \tag{4}
\end{equation*}
$$

Denote

$$
h(t):=\frac{t}{1-e^{-t}}, \quad \Delta_{n, u}(t):=n e^{-n u t}-e^{-u t}, \quad t>0 .
$$

The function $h$ is strictly increasing since

$$
h^{\prime}(t)=\frac{1-(1+t) e^{-t}}{\left(1-e^{-t}\right)^{2}}>0, \quad t>0
$$

due to the inequality $e^{t}>1+t, t>0$. The function $\Delta_{n, u}$ is such that

$$
\begin{aligned}
& \Delta_{n, u}(t)>0, \quad 0<t<t_{n}(u)=\frac{\ln n}{(n-1) u}, \\
& \Delta_{n, u}(t)=0, \quad t=t_{n}(u), \\
& \Delta_{n, u}(t)<0, \quad t>t_{n}(u) .
\end{aligned}
$$

This implies that

$$
\int_{0}^{t_{n}(u)} h(t) \Delta_{n, u}(t) d t<h\left(t_{n}(u)\right) \int_{0}^{t_{n}(u)} \Delta_{n, u}(t) d t
$$

and

$$
\int_{t_{n}(u)}^{\infty} h(t) \Delta_{n, u}(t) d t<h\left(t_{n}(u)\right) \int_{t_{n}(u)}^{\infty} \Delta_{n, u}(t) d t .
$$

Therefore,

$$
g_{n}^{\prime}(u)<h\left(t_{n}(u)\right) \int_{0}^{\infty} \Delta_{n, u}(t) d t=0
$$

since

$$
\int_{0}^{\infty} \Delta_{n, u}(t) d t=0
$$

Further, by differentiation of both sides of (4) we obtain

$$
g_{n}^{\prime \prime}(u)=\int_{0}^{\infty} \frac{t}{1-e^{-t}}\left(t e^{-u t}-n^{2} t e^{-n u t}\right) d t, \quad u>0
$$

Denote

$$
\bar{\Delta}_{n, u}(t):=t e^{-u t}-n^{2} t e^{-n u t}, \quad t>0
$$

and observe that

$$
\begin{aligned}
& \bar{\Delta}_{n, u}(t)>0, \quad t>\bar{t}_{n}(u)=\frac{2 \ln n}{(n-1) u} \\
& \bar{\Delta}_{n, u}(t)=0, \quad t=\bar{t}_{n}(u) \\
& \bar{\Delta}_{n, u}(t)<0, \quad 0<t<\bar{t}_{n}(u)
\end{aligned}
$$

This implies that

$$
\int_{0}^{\bar{t}_{n}(u)} h(t) \bar{\Delta}_{n, u}(t) d t>h\left(\bar{t}_{n}(u)\right) \int_{0}^{\bar{t}_{n}(u)} \bar{\Delta}_{n, u}(t) d t
$$

and

$$
\int_{\bar{\epsilon}_{n}(u)}^{\infty} h(t) \bar{\Delta}_{n, u}(t) d t>h\left(\bar{t}_{n}(u)\right) \int_{\bar{t}_{n}(u)}^{\infty} \bar{\Delta}_{n, u}(t) d t .
$$

Therefore,

$$
g_{n}^{\prime \prime}(u)>h\left(\bar{t}_{n}(u)\right) \int_{0}^{\infty} \bar{\Delta}_{n, u}(t) d t=0
$$

since

$$
\int_{0}^{\infty} \bar{\Delta}_{n, u}(t) d t=0
$$

In the next lemma a property of the psi function is established. This result will be helpful in proving the last lemma.

Lemma 2. Let the function $\theta$ be defined by the equality

$$
g^{\prime}(\theta(u))=g_{n}^{\prime}(u), \quad u>0
$$

Then

$$
g(\theta(u))>g_{n}(u), \quad u>0
$$

Proof. First of all, observe that the function $\theta$ is well-defined since both the functions $g^{\prime}$ and $g_{n}^{\prime}$ take values in $(-\infty, 0)$, are negative and strictly increasing (see Theorem 1 of [2] and Lemma 1 above).

Now we prove that

$$
\begin{equation*}
g^{\prime}(\sqrt{n /(n-1)} u)>g_{n}^{\prime}(u), \quad u>0 \tag{5}
\end{equation*}
$$

Consider the function

$$
\rho(u):=g_{n}^{\prime}(u)-g^{\prime}(\sqrt{n /(n-1)} u)=g^{\prime}(u)-n g^{\prime}(n u)-g^{\prime}(\sqrt{n /(n-1)} u), \quad u>0
$$

Making use of (3) one can write the representation (Binet's formula):

$$
g(u)=\int_{0}^{\infty} \varphi(t) e^{-u t} d t, \quad u>0
$$

where

$$
\varphi(t):=\frac{1}{1-e^{-t}}-\frac{1}{t}, \quad t>0
$$

Therefore,

$$
\rho(u)=\int_{0}^{\infty} \varphi(t)\left(n t e^{-n u t}+t e^{-\sqrt{n /(n-1)} u t}-t e^{-u t}\right) d t, \quad u>0
$$

The function $\varphi$ is strictly increasing and

$$
\lim _{t \rightarrow 0} \varphi(t)=\frac{1}{2}, \quad \lim _{t \rightarrow \infty} \varphi(t)=1
$$

(see the proof of Theorem 1 in [2]). Denote

$$
\begin{aligned}
\Delta_{n}(z) & :=n z e^{-n z}+z e^{-\sqrt{n /(n-1)} z}-z e^{-z} \\
& =z e^{-z}\left(n e^{-(n-1) z}+e^{-(\sqrt{n /(n-1)}-1) z}-1\right), \quad z>0
\end{aligned}
$$

Then

$$
\rho(u)=\frac{1}{u^{2}} \int_{0}^{\infty} \varphi\left(\frac{z}{u}\right) \Delta_{n}(z) d z, \quad u>0
$$

Observe that there exists $z_{0}>0$ such that

$$
\begin{aligned}
& \Delta_{n}(z)>0, \quad 0<z<z_{0}, \\
& \Delta_{n}(z)=0, \quad z=z_{0}, \\
& \Delta_{n}(z)<0, \quad z>z_{0} .
\end{aligned}
$$

The standard reasoning leads to the inequalities

$$
\begin{aligned}
& \int_{0}^{z_{0}} \varphi\left(\frac{z}{u}\right) \Delta_{n}(z) d z<\varphi\left(\frac{z_{0}}{u}\right) \int_{0}^{z_{0}} \Delta_{n}(z) d z, \\
& \int_{z_{0}}^{\infty} \varphi\left(\frac{z}{u}\right) \Delta_{n}(z) d z<\varphi\left(\frac{z_{0}}{u}\right) \int_{z_{0}}^{\infty} \Delta_{n}(z) d z .
\end{aligned}
$$

Therefore,

$$
u^{2} \rho(u)=\int_{0}^{\infty} \varphi\left(\frac{z}{u}\right) \Delta_{n}(z) d z<\varphi\left(\frac{z_{0}}{u}\right) \int_{0}^{\infty} \Delta_{n}(z) d z=0
$$

since

$$
\int_{0}^{\infty} \Delta(z) d z=0
$$

and (5) holds.
Since the function $g^{\prime}$ is strictly increasing, from (5) it follows that

$$
\theta(u)<\sqrt{n /(n-1)} u, \quad u>0
$$

On the other hand,

$$
\theta(u)>u, \quad u>0
$$

since

$$
g_{n}^{\prime}(u)-g^{\prime}(u)=-n g^{\prime}(n u)>0, \quad u>0 .
$$

Denote

$$
\lambda(u):=g(\theta(u))-g_{n}(u), \quad u>0 .
$$

Since $\theta^{\prime}(u)>1$ and $g^{\prime}(u)<0$, we obtain

$$
\lambda^{\prime}(u)=g^{\prime}(\theta(u)) \theta^{\prime}(u)-g_{n}^{\prime}(u)=g^{\prime}(\theta(u))\left(\theta^{\prime}(u)-1\right)<0, \quad u>0
$$

Hence, the function $\lambda$ is strictly decreasing. Now we establish that $\lambda(u)>0$ for all sufficiently large $u$. Due to the representations

$$
g(u)=\frac{1}{2 u}(1+o(1)), \quad g_{n}(u)=\frac{1}{2 u}\left(1-\frac{1}{n}\right)(1+o(1)), \quad u \rightarrow \infty
$$

(e.g. [1, formula 6.3.18]), we get

$$
\lambda(u)=\frac{1}{2}\left(\frac{1}{\theta(u)}-\frac{1}{u}\left(1-\frac{1}{n}\right)\right)(1+o(1)), \quad u \rightarrow \infty
$$

Simple calculations leads to the conclusion that $\lambda(u)>0$, as $u \rightarrow \infty$, for $\theta(u)<$ $\sqrt{n /(n-1)} u$. Thus, $\lambda(u)>0$ for all $u>0$.

In the proof of Lemma 2 it is established that $u<\theta(u)<\sqrt{n /(n-1)} u$. The numerical calculations, made with the help of Maple for $n=2,10$, 50 , and presented in Table 2, confirm those inequalities.

|  | $n=2$ |  | $n=10$ |  | $n=50$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $u$ | $\theta(u)$ | $\sqrt{n /(n-1)} u$ | $\theta(u)$ | $\sqrt{n /(n-1)} u$ | $\theta(u)$ | $\sqrt{n /(n-1)} u$ |
| 0.1 | 0.134913918 | 0.141421356 | 0.103577225 | 0.105409255 | 0.100565420 | 0.101015254 |
| 0.5 | 0.651452797 | 0.707106781 | 0.517553390 | 0.527046277 | 0.503191582 | 0.505076272 |
| 1 | 1.316748502 | 1.414213562 | 1.039115016 | 1.054092553 | 1.007252736 | 1.010152544 |
| 3 | 4.090593081 | 4.242640686 | 3.140261683 | 3.162277660 | 3.026250755 | 3.030457633 |
| 5 | 6.905201515 | 7.071067810 | 5.246643827 | 5.270462768 | 5.046219652 | 5.050762721 |
| 10 | 13.96570362 | 14.14213562 | 10.51571989 | 10.54092553 | 10.09672350 | 10.10152544 |
| 50 | 70.52586082 | 70.71067810 | 52.67831719 | 52.70462768 | 50.50261897 | 50.50762721 |
| 100 | 141.2355025 | 141.4213562 | 105.3828081 | 105.4092553 | 101.0102207 | 101.0152544 |

Table 2: Numerical calculations of bounds for $\theta(u)$.

Lemma 3. For any $n>1$ the function $f(u)=g^{-1}(u)-g_{n}^{-1}(u), u>0$, is strictly decreasing.

Proof. Clearly,

$$
f^{\prime}(u)=\frac{1}{g^{\prime}\left(g^{-1}(u)\right)}-\frac{1}{g_{n}^{\prime}\left(g_{n}^{-1}(u)\right)}, \quad u>0
$$

For any $u>0$ we have

$$
\begin{equation*}
g\left(g^{-1}(u)\right)=g_{n}\left(g_{n}^{-1}(u)\right)<g\left(\theta\left(g_{n}^{-1}(u)\right)\right), \tag{6}
\end{equation*}
$$

where the function $\theta$ is defined as in Lemma 2. Since the function $g$ is strictly decreasing, from (6) it follows that

$$
g^{-1}(u)>\theta\left(g_{n}^{-1}(u)\right), \quad u>0
$$

while since the function $g^{\prime}$ is strictly increasing, we get

$$
g_{n}^{\prime}\left(g_{n}^{-1}(u)\right)=g^{\prime}\left(\theta\left(g_{n}^{-1}(u)\right)\right)<g^{\prime}\left(g^{-1}(u)\right), \quad u>0 .
$$

Thus, $f^{\prime}(u)<0$ for all $u>0$.

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