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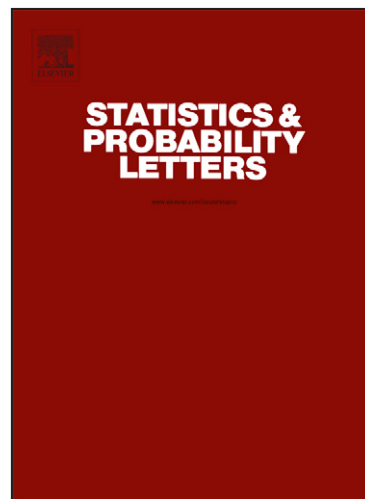
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# On the existence of moments of ladder heights

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## Abstract

Let  $S_0, S_n = \sum_{k=1}^n X_k$ , where  $X_k, k = \overline{1, n}$  are iid random variables. Denote  $N^- = \min \{n \geq 1 : S_n \leq 0\}$ ,  $N^+ = \min \{n \geq 1 : S_n > 0\}$ . Necessary and sufficient conditions are found for the existence of the moments of ladder heights  $S_{N^-}$  and  $S_{N^+}$ . These conditions are taken into consideration when calculating the moments of ladder heights in the cases  $EX_k > 0$  and  $EX_k < 0$ .

Keywords: random walk, ladder heights, moments of ladder heights.

Let  $X, X_1, X_2, \dots$  be independent identically distributed random variables with the distribution function  $F(x)$ . Let  $S_n = X_1 + \dots + X_n, n \geq 1, S_0 = 0$ .

Denote

$$N^- = \min \{n \geq 1 : S_n \leq 0\}, \quad N^+ = \min \{n \geq 1 : S_n > 0\}.$$

Then  $Z_- := S_{N^-}$  is the first nonpositive sum in the random walk  $\{S_n, n \geq 1\}$  or, in other words, the first descending ladder height, and  $Z_+ := S_{N^+}$  is the first positive sum or the first ascending ladder height.

Let  $EX = 0, X^+ = \max(0, X), X^- = \min(0, X)$ . In 1960, Spitzer [8] showed that, in the case  $EX = 0$ , if  $E|X|^{k+1} < \infty$ , then  $E|Z_-|^k < \infty, k = 1, 2$ .

For proving Spitzer used an analytic method based on generating functions connected with fluctuations of the random walk  $\{S_n, n \geq 0\}$  and Tauberian arguments. Lai [6] generalized Spitzer's results showing that, if  $E|X|^{k+1} < \infty$ , then  $E|Z_-|^k < \infty$ , for  $k = 1, 2, \dots$ . Chow and Lai [2] showed by probabilistic methods that, if  $E|X^-|^{p+1} < \infty$ , then  $E|Z_-|^p < \infty$  for any  $p > 0$ . Doney [4] generalized this result. He investigated the moments of functions of ladder heights and proved that, if  $\varphi$  was a nonnegative monotonously increasing differentiable function on  $[0, \infty)$  such that  $\lim_{x \rightarrow \infty} \varphi(x) = \infty$  and  $\sup_{x \rightarrow \infty} \varphi(x+K)/\varphi(x) < \infty$  for any  $K > 0$ , and  $\int_0^x \varphi(x) dx = \Phi(x)$ , then

$$\begin{aligned} 1) E\{\Phi(X^+)\} < \infty &\Rightarrow E\{\varphi(Z_+)\}, \\ 2) E(|Z_-|) < \infty, \quad E\{\varphi(Z_+)\} < \infty &\Rightarrow E\{\Phi(X^+)\} \leq E(|Z_-|) \cdot E\{\varphi(Z_+)\}. \end{aligned} \tag{1}$$

Hence it follows that for  $\beta > 1$   $E(X^+)^\beta < \infty \Rightarrow E(Z_+^{\beta-1}) < \infty$ , and if  $E|Z_-| < \infty$ , then  $EZ_+^{\beta-1} \Rightarrow E(X^+)^\beta \leq \beta E|Z_-| \cdot EZ_+^{\beta-1}$ . Moreover, Doney pointed in his work [4] (see also [5]) to the unsolved problem: find the necessary and sufficient conditions in terms of distribution  $F$  for  $EZ_+ < \infty$  in the case  $E(X^+)^2 = E(X^-)^2 = \infty$ . In our opinion, this problem is solved in Theorem 1 proved by us.

Further, when investigating the necessary and sufficient conditions, Chow [3] proved that, if  $EX = 0$ ,  $E|X| > 0$  and  $p \geq 1$ , then  $E|Z_-|^p < \infty$  if only if

$$\int_0^\infty x^{p+1} dP\{|X^-| \leq x\} / \int_0^\infty y(y \wedge x) dF(y) < \infty.$$

In the works of Lai [2], A.V. Nagaev [7] and of the author [1], analytic representations for the moments of the variables  $Z_+$  and  $Z_-$  are given in terms of some characteristics of distribution  $F$ . In the papers [2] and [7], only the case

$EX = 0$  was investigated. In the author's work [1], all the qualitatively different cases  $EX = 0$ ,  $EX > 0$ , and  $EX < 0$  were considered.

Let us introduce our results.

**Theorem 1** *The following relations hold:*

$$\begin{aligned} E(X^+)^p < \infty &\Leftrightarrow E(Z_+)^p < \infty, \quad p > 0, \\ E(|X^-|)^p < \infty &\Leftrightarrow E(|Z_-|)^p < \infty. \end{aligned} \quad (2)$$

**Proof.** First note that  $E(Z_+)^p < \infty$  follows from  $E(X^+)^p < \infty$ . Thus, let  $E(X^+)^p < \infty$ . We have

$$\begin{aligned} EZ_+^p &= E(Z_+^p | Z_+ = S_1) \cdot P\{Z_+ = S_1\} + E(Z_+^p | Z_+ = S_2) \cdot P\{Z_+ = S_2\} + \\ &+ \dots + E(Z_+^p | Z_+ = S_n) \cdot P\{Z_+ = S_n\} + \dots \end{aligned} \quad (3)$$

Hence, since

$$E(Z_+^p | Z_+ = S_n) \leq E(X_n^+)^p, \quad n \geq 1, \quad P\{Z_+ = S_n\} = P\{N^+ = n\},$$

we get

$$\begin{aligned} EZ_+^p &\leq E(X_1^+)^p \cdot P\{N^+ = 1\} + E(X_2^+)^p \cdot P\{N^+ = 2\} + \dots + E(X_n^+)^p \cdot \\ &\cdot P\{N^+ = n\} + \dots = E(X^+)^p \cdot P\{N^+ \geq 1\} = E(X^+)^p. \end{aligned}$$

Consequently,  $E(X^+)^p < \infty \Rightarrow EZ_+^p < \infty$ .

Let now  $EZ_+^p < \infty$ . We see that  $EZ_+^p < \infty \Rightarrow E(X^+)^p < \infty$ . We obtain

$$EZ_+^p = E(Z_+^p | Z_+ = S_1) \cdot P\{Z_+ = S_1\} + E(Z_+^p | Z_+ \neq S_1) \cdot P\{Z_+ \neq S_1\}. \quad (4)$$

Since  $EZ_+^p < \infty$  and both summands on the right-hand side of relation (4) are positive, these summands are also finite. Consequently,  $E(Z_+^p | Z_+ = S_1) < \infty$ . But  $E(Z_+^p | Z_+ = S_1) = E(X_1^+)^p$ . Hence we obtain that  $E(X_1^+)^p < \infty$ .

In the proof of the second relation in (2) one should write the sign "-" instead of "+".

The result of Doney (1) now can be generalized as follows.

**Theorem 2** *Let  $\varphi$  be a non-negative monotonically increasing function on  $(0, \infty)$ . Then*

$$\begin{aligned} E(\varphi(X^+)) < \infty &\Leftrightarrow E(\varphi(Z_+)) < \infty, \\ E(\varphi(|X^-|)) < \infty &\Leftrightarrow E(\varphi(|Z_-|)) < \infty. \end{aligned}$$

For  $\varphi(x) = x^p$ ,  $p > 0$ , Theorem 2 yields the statements of Theorem 1.

**Proof.** The proof of Theorem 2 is analogous to that of Theorem 1, because we can draw conclusions from the equalities

$$\begin{aligned} E(\varphi(Z_+)) &= E(\varphi(Z_+) | \varphi(Z_+) = \varphi(S_1)) \cdot P\{Z_+ = S_1\} + \\ &+ E(\varphi(Z_+) | \varphi(Z_+) = \varphi(S_2)) \cdot P\{Z_+ = S_2\} + \dots \end{aligned}$$

$$\begin{aligned} E(\varphi(Z_+)) &= E(\varphi(Z_+) | \varphi(Z_+) = \varphi(S_1)) \cdot P\{Z_+ = S_1\} + \\ &+ E(\varphi(Z_+) | \varphi(Z_+) \neq \varphi(S_1)) \cdot P\{Z_+ \neq S_1\} \end{aligned}$$

analogous as that from equalities (3) and (4), respectively.

We have mentioned above that in [1], analytical expressions of moments of the variables  $Z_+$  and  $Z_-$  are given through the characteristics of distribution  $F$  in the cases  $EX > 0$  and  $EX < 0$ . The first three moments of these variables

are calculated there in recurrent way. (Using the same algorithm, it is possible to calculate moments of a higher order in a recurrent way).

However, in [1] the requirements as to the existence of the moments of distribution  $F$  are overstated. In view of the statement of Theorem 1, the conditions of Theorems 1-3 in [1] could be weakened. Let us reformulate these theorems under the new weakened conditions.

**Theorem 3** *Let  $EX > 0$  and  $E|X^-| < \infty$ . Then*

$$EZ_- = e^{-A} \sum_{n=1}^{\infty} \int_{-\infty}^{0+} x dF_n(x), \quad A = \sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leq 0\}.$$

*In addition, if  $E|X^-|^2 < \infty$ , then*

$$EZ_-^2 = e^{-A} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x^2 dF_n(x) - e^{-A} \left( \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x dF_n(x) \right)^2,$$

*and if  $E|X^-|^3 < \infty$ , then*

$$EZ_-^3 = e^{-A} \left[ \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x^3 dF_n(x) - 3 \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x^2 dF_n(x) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^0 x dF_n(x) + \left( \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x dF_n(x) \right)^3 \right].$$

*The rows on the right-hand sides of the given equalities converge.*

**Theorem 4** *Let  $E|X| < \infty$  and  $EX > 0$ . Then*

$$EZ_+ = EXe^A.$$

Moreover, if  $EX^2 < \infty$ , then

$$EZ_+^2 = EX^2e^A + 2EXe^A \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x dF_n(x),$$

and if  $E|X|^3 < \infty$ , then

$$\begin{aligned} EZ_+^3 &= EX^3e^A + 3EX^2e^A \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x dF_n(x) + 3EXe^A \left( \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x dF_n(x) \right)^2 + \\ &+ 3EXe^A \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0+} x^2 dF_n(x). \end{aligned}$$

The rows appearing in the given relations converge.

Analogous theorems can be presented for the case  $EX < 0$  (analogue of Theorem 3 in [1]).

The proof of these theorems remains the same as in [1] except those sites regarding the existence of moments of the variables  $Z_+$  and  $Z_-$ . In these sites one ought to make use of Theorem 1.

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