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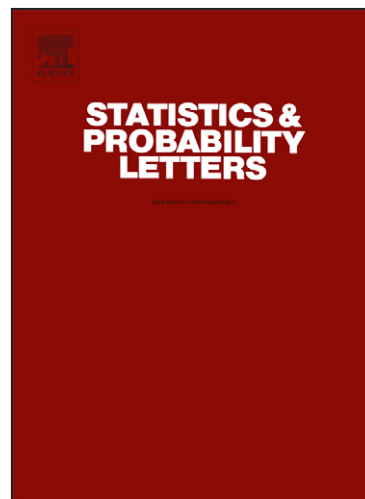
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# The rank of a normally distributed matrix and positive definiteness of a noncentral Wishart distributed matrix

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## Abstract

If  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , then  $S = X'X$  has the noncentral Wishart distribution  $W'_k(n, \Sigma; \Lambda)$ , where  $\Lambda = M'M$ . Here  $\Sigma$  is allowed to be singular. It is well known that if  $\Lambda = 0$ , then  $S$  has a (central) Wishart distribution and  $S$  is positive definite with probability 1 if and only if  $n \geq k$  and  $\Sigma$  is positive definite. We show that if  $S$  has a noncentral Wishart distribution, then  $S$  is positive definite with probability 1 if and only if  $n \geq k$  and  $\Sigma + \Lambda$  is positive definite. This is a consequence of the main result that  $\text{rank } X = \min(n, \text{rank}(\Sigma + \Lambda))$  with probability one.

*Multivariate normal distribution, singular covariance matrix, rank, noncentral Wishart, positive definite.*

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\*This author has deceased between the first version of this paper and the final manuscript.

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## 1 Introduction

It is well known that if  $S \sim W_k(n, \Sigma)$ , then  $S$  is positive definite with probability one if and only if  $n \geq k$  and  $\Sigma$  is positive definite, see, for example, Muirhead (1982), theorem 3.1.4 and Eaton (1983), proposition 8.2. The proof of this result goes back to Dijkstra (1970). In case  $S$  follows a noncentral Wishart distribution,  $S$  is also positive definite if  $n \geq k$ , see Muirhead (1982), page 442 and Eaton (1983) states on page 317 that  $S$  is positive definite with probability one if and only if  $n \geq k$  and  $\Sigma$  is positive definite. Eaton and Perlman (1973) study the non-singularity of (generalized) sample covariance matrices in great detail. To the best of our knowledge, the literature thus gives the following result: if  $\Sigma$  is positive definite, then  $S$  is positive definite with probability one if and only if  $n \geq k$ . We focus on noncentral singular Wishart distributions and drop the condition that  $\Sigma$  is positive definite. For the (central) singular Wishart distribution in particular, reference can be made to Uhlig (1994) and Srivastava (2003).

In section 2 we start with some basic notation. In section 3 we prove a theorem on the rank of  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , which is the basis for our main result on the positive definiteness of the noncentral Wishart matrix  $S$ . We present and prove this result in section 4.

## 2 Notation

If the random  $k \times 1$  vector  $X \sim \mathcal{N}_k(\mu, \Sigma)$ , where  $\mu \in \mathbb{R}^k$  and  $\Sigma$  is a positive semi-definite symmetric  $k \times k$  matrix, then we say that  $X$  follows a multivariate normal distribution with mean  $\mu$  and covariance matrix  $\Sigma$ . We allow  $\Sigma$  to be singular, which includes the possibility that  $X$  lies in an affine subspace with probability one. In the above notation, the distribution of  $X$  is degenerated if  $k < \text{rank } \Sigma$ .

When studying Wishart distributions, notational comfort is provided by using random matrices whose elements are multivariate normally distributed. The random  $n \times k$  matrix  $X$  follows the  $\mathcal{N}_{n \times k}(M, \Omega)$  distribution if

$$\text{vec } X' \sim \mathcal{N}_{nk}(\text{vec } M', \Omega),$$

where  $M = EX$  is an  $n \times k$  matrix and  $\Omega$  is the  $nk \times nk$  covariance matrix of  $\text{vec } X'$ . If the rows  $X_1, \dots, X_n$  of the random  $n \times k$  matrix  $X = (X_1, \dots, X_n)'$  are independently distributed as  $\mathcal{N}_k(\mu_i, \Sigma)$ , then

$$X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma),$$

where  $M = (\mu_1, \dots, \mu_n)'$ .

**Definition 1.** Let  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , then  $S = X'X$  has the noncentral Wishart distribution of dimension  $k$ , degrees of freedom  $n$ , covariance matrix  $\Sigma$  and noncentrality matrix  $\Lambda = M'M$ . This distribution is denoted by  $W'_k(n, \Sigma; \Lambda)$ . If  $M = 0$ , then  $S$  has the Wishart distribution, denoted by  $W_k(n, \Sigma)$ .

### 3 The rank of a normally distributed matrix

Frequently we assume that we have  $n$  independent  $k$ -variate normal observations assembled in a matrix  $X$ , denoted as  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , where  $\Sigma$  is positive semi-definite. Usually, the number of observations  $n$  will be larger than  $k$ , but we will not adopt this assumption, see also Srivastava (2003). If the rank of  $\Sigma$  is known, what can be said about the rank of  $X$ ? The answer is given in the following theorem.

**Theorem 1.** If  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , then

$$\text{rank } X = \min(n, \text{rank}(\Sigma + M'M))$$

with probability one.

Before we prove theorem 1 we want to emphasize the importance of this theorem. In the one-dimensional case, where  $X \sim \mathcal{N}(\mu, \sigma^2)$ , we know that  $X \neq 0$  with probability one either if  $\mu \neq 0$ , or if  $\sigma^2 > 0$ . Or, stated otherwise,  $X \neq 0$  with probability one if and only if  $\sigma^2 + \mu^2 > 0$ , which corresponds with theorem 1. Theorem 1 shows that, even if  $\text{rank } \Sigma < k$ , the rank of  $X$  can still be equal to  $k$ , as long as the matrix of expectations  $M$  is such that  $\text{rank}(\Sigma + M'M) = k$ , provided  $n \geq k$ .

The proof of this theorem is divided into a couple of lemmas. We use the notation of section 2. In lemma 1 we first consider the case  $\text{rank } \Sigma = k$ . The proof of this lemma can be found in Srivastava and Khatri (1979, page 73).

**Lemma 1.** If  $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , where  $\text{rank } \Sigma = k$ , then  $\text{rank } X = \min(n, k)$  with probability one.

**Lemma 2.** Let  $Y \sim \mathcal{N}_{n \times r}(M_1, I_n \otimes \Omega)$  with  $\text{rank } \Omega = r$ , let  $M_2$  be an  $n \times s$  matrix of rank  $s$ , and let  $X = (Y, M_2)$ . Then  $\text{rank } X = \min(n, r + s)$  with probability one.

*Proof.* Note that the case  $s = 0$  is covered by lemma 1. So, we may assume  $s > 0$ . Obviously  $s \leq n$ .

(i) Assume  $n \leq r$ . According to lemma 1 we have

$$1 = \text{P} \{ \text{rank } Y = n \} = \text{P} \{ \text{rank}(Y, M_2) = n \} = \text{P} \{ \text{rank } X = n \}.$$

- (ii) Assume  $r < n \leq r + s$ . Let  $M_2 = (m_{r+1}, \dots, m_{r+s})$ . Define  $Z = XA$ , where the  $(r + s) \times (r + s)$  matrix

$$A = \begin{pmatrix} \Omega^{-\frac{1}{2}} & 0 \\ 0 & I_s \end{pmatrix}$$

is of rank  $r + s$ . We partition  $Z$  as follows:

$$Z = (Y\Omega^{-\frac{1}{2}}, M_2) = (Z_1, M_2),$$

where  $Z_1 = Y\Omega^{-\frac{1}{2}} \sim \mathcal{N}_{n \times r}(M_1\Omega^{-\frac{1}{2}}, I_n \otimes I_r)$ . Let  $Z_1 = (Z_{11}, \dots, Z_{1r})$ , then by construction  $Z_{11}, \dots, Z_{1r}$  are independently distributed with

$$Z_{1j} \sim \mathcal{N}_n(\lambda_j, I_n), \quad j = 1, \dots, r$$

and

$$M_1\Omega^{-\frac{1}{2}} = (\lambda_1, \dots, \lambda_n).$$

Subsequently, consider the  $n \times n$  submatrix  $\tilde{Z}$  of  $Z$  defined by  $\tilde{Z} = (Z_1, m_{r+1}, \dots, m_n)$ , then

$$\begin{aligned} & \mathbb{P}(\text{rank } \tilde{Z} < n) \\ & \leq \sum_{j=1}^r \mathbb{P}(Z_{1j} \in \text{span}\{Z_{11}, \dots, Z_{1,j-1}, Z_{1,j+1}, \dots, Z_{1r}, m_{r+1}, \dots, m_n\}) \\ & = 0. \end{aligned}$$

This implies that

$$\mathbb{P}(\text{rank } X = n) = \mathbb{P}(\text{rank } Z = n) \geq \mathbb{P}(\text{rank } \tilde{Z} = n) = 1.$$

- (iii) Assume that  $n > r + s$ . Let  $M'_2 = (m_1, \dots, m_n)$ . Select  $m_{i_1}, \dots, m_{i_{r+s}}$  such that the  $(r+s) \times s$  matrix  $M_3$  defined by  $M'_3 = (m_{i_1}, \dots, m_{i_{r+s}})$  is of rank  $s$ . Let  $Y' = (Y_1, \dots, Y_n)$  and define  $Z' = (Y_{i_1}, \dots, Y_{i_{r+s}})$ . Now we consider the  $(r + s) \times (r + s)$  random matrix  $(Z, M_3)$ . According to part (ii) we have

$$\mathbb{P}\{\text{rank}(Z, M_3) = r + s\} = 1.$$

Since  $X$  is obtained from  $(Z, M_3)$  by adding rows we now may conclude that  $\text{rank } X = r + s$  with probability one. □

As an immediate extension we have

**Lemma 3.** Let  $Y \sim \mathcal{N}_{n \times r}(M_1, I_n \otimes \Omega)$  with  $\text{rank } \Omega = r$  and let  $M_2$  be an  $n \times (k - r)$  matrix of rank  $s$ . Define  $X = (Y, M_2)$ , then  $\text{rank } X = \min(n, r + s)$  with probability one.

*Proof.* Apply lemma 2 to  $X$  after deleting  $k - r - s$  columns of  $M_2$  in such a way that the remaining part is of the full column rank  $s \leq n$ .  $\square$

By using lemma 3 we are now able to prove theorem 1.

*Proof of theorem 1.* Let  $\Sigma = CDC'$  be a spectral decomposition of  $\Sigma$  with

$$D = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0),$$

such that  $\lambda_1 \geq \dots \geq \lambda_r > 0$ , and let  $C$  be a  $k \times k$  orthogonal matrix partitioned as  $C = (C_1, C_2)$ , where  $C_1$  is of order  $k \times r$ . Now we have  $XC \sim \mathcal{N}_{n \times k}(MC, I_n \otimes D)$ . Let  $Y = XC_1, M_2 = MC_2, \Omega = \text{diag}(\lambda_1, \dots, \lambda_r)$ , then  $XC = (Y, M_2)$  with probability one, and  $Y \sim \mathcal{N}_{n \times r}(MC_1, I_n \otimes \Omega)$ . If  $\text{rank}(MC_2)$  is denoted by  $s$ , then according to lemma 3 we have

$$P \{\text{rank } X = \min(n, r + s)\} = P \{\text{rank}(XC) = \min(n, r + s)\} = 1.$$

Because

$$\text{rank}(MC_2) = \text{rank}(MC_2 C_2' M') = \text{rank}(MC_2 C_2' C_2 C_2' M') = \text{rank}(MC_2 C_2'),$$

it follows that

$$\begin{aligned} s &= \text{rank}(MC_2) \\ &= \text{rank}(MC_2 C_2') \\ &= \text{rank}(M(I_k - C_1 C_1')) \\ &= \text{rank}(M(I_k - \Sigma^+ \Sigma)), \end{aligned}$$

where we used the fact that  $\Sigma^+ = C_1 \Omega^{-1} C_1'$ . For ease of notation define

$$\Xi = (I_k - \Sigma \Sigma^+) M' M (I_k - \Sigma^+ \Sigma),$$

then  $\text{rank}(\Xi) = s$ . Note that  $\Sigma^+ \Sigma = \Sigma \Sigma^+$ . Suppose that  $x \in \mathcal{R}(\Sigma) \cap \mathcal{R}(\Xi)$ , then we can write for some  $y, z \in \mathbb{R}^k$  that

$$x = \Sigma y = \Xi z.$$

Hence

$$\|x\|^2 = y' \Sigma \Xi z = 0,$$

which implies  $x = 0$ . Now it follows that

$$\text{rank}(\Sigma + \Xi) = \text{rank}(\Sigma) + \text{rank}(\Xi) = r + s,$$

see, for example, theorem 3.19 of Magnus and Neudecker (1988), or Marsaglia and Styan (1972). In order to show that  $\text{rank}(\Sigma + M'M) = r + s$ , it suffices to show that the kernel of  $\Sigma + M'M$  and the kernel of  $\Sigma + \Xi$  are equal. Because  $\Sigma + \Xi \geq 0$ , it can be derived that

$$\begin{aligned} (\Sigma + \Xi)x = 0 &\Leftrightarrow x'\Sigma x = 0 \quad \text{and} \quad M(I_k - \Sigma^+\Sigma)x = 0 \\ &\Leftrightarrow \Sigma x = 0 \quad \text{and} \quad Mx = 0 \\ &\Leftrightarrow (\Sigma + M'M)x = 0. \end{aligned}$$

□

#### 4 Positive definiteness of a Wishart matrix

Recall from section 1 that if  $\mathcal{L}(S) = W'_k(n, \Sigma; \Lambda)$  with  $\Sigma$  positive definite, then  $S$  is positive definite with probability one if and only if  $n \geq k$  (Eaton, 1983, page 317). If  $\Lambda = 0$ , so that  $S$  follows a central Wishart distribution, then this result is true. For the case of the noncentral Wishart distribution, however, the rank-condition on  $\Sigma$  is more subtle, as we show in the following theorem.

**Theorem 2.** *Let  $S \sim W'_k(n, \Sigma; \Lambda)$ , then  $S$  is positive definite with probability one if and only if  $n \geq k$  and  $\Sigma + \Lambda$  is positive definite.*

*Proof.* Using definition 1, let  $S = Z'Z$  where  $Z \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$  such that  $\Lambda = M'M$ . On account of theorem 1, it follows that

$$\begin{aligned} 1 &= \text{P} \left\{ x'Sx > 0 \quad \text{for all } 0 \neq x \in \mathbb{R}^k \right\} \\ &= \text{P} \left\{ Zx \neq 0 \quad \text{for all } 0 \neq x \in \mathbb{R}^k \right\} \\ &= \text{P} \{ \text{rank}(Z) = k \} \end{aligned}$$

if and only if  $n \geq k$  and  $\Sigma + M'M = \Sigma + \Lambda$  is positive definite. □

As for the practical relevance of theorem 2, we agree that the theorem is more of mathematical than of statistical interest. If we consider the random matrix  $X\mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$ , then examples can be constructed such that  $\Sigma$  is not positive definite, but  $\Sigma + M'M$  is. Theorem 2 then shows that, even if  $\Sigma$  is not of fullrank,  $S = X'X$  can still be positive definite. Assume, for simplicity, that  $k = 2$ , and consider the random matrix  $X\mathcal{N}_{n \times 2}(M, I_n \otimes \Sigma)$ , where  $M = (0_n, \iota_n)$ ,  $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}$

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