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## Author's Accepted Manuscript

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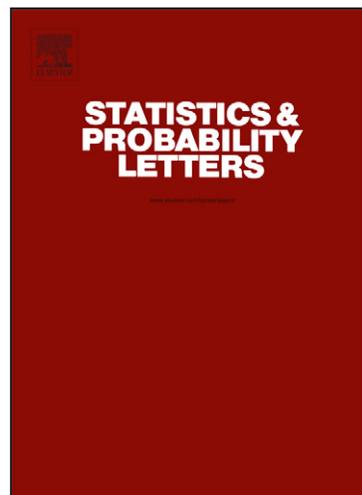
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# Large deviations probabilities for a symmetry test statistic based on delta-sequence density estimation

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## Abstract

The goal of this paper is to provide large deviations limit theorems for statistics, based on the delta-sequence density estimation and designed to symmetry testing of distribution. A general result is stated for any regular delta-sequence and a discussion of hypotheses for the most usual methods is given. The estimation is based upon sequences of independent and identically distributed random variables.

*Keywords:* Symmetry testing; Large deviations; rate functions; delta-sequence; Bahadur exact slope.

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## 1 Introduction

Let  $\mathcal{X}$  be an open symmetric interval of the real line  $\mathbb{R}$ . A sequence  $\{\delta_m(x, u)\}$  of bounded measurable functions on  $\mathcal{X} \times \mathcal{X}$  is a delta-sequence on  $\mathcal{X}$  if, for each  $x \in \mathcal{X}$  and each  $C^\infty$  function  $\varphi$  with support in  $\mathcal{X}$ , we have  $\lim_{m \rightarrow \infty} \int_{\mathcal{X}} \delta_m(x, u) \varphi(u) du = \varphi(x)$ . Let  $X_1, X_2, \dots, X_n$  be a sequence of i.i.d real random variable defined on a probability space  $(\Omega, \mathcal{F}, P)$  and taking values in a set  $\mathcal{X} \subseteq \mathbb{R}$ . Denote by  $F$  the distribution function of  $X$  and by  $f$  its probability density function with respect to the Lebesgue measure over  $\mathcal{X}$ . Define the delta-sequence estimator of  $f$  associated to the sequence  $\{\delta_m(x, u)\}$  by

$$f_n(x) = \frac{1}{n} \sum_{i=1}^n \delta_m(x, X_i),$$

where  $m = m_n$  is a sequence of positive real numbers that tends to infinity with  $n$ .

In this paper, we study the large deviations problem pertaining with the symmetry testing statistics based on the delta-sequence method density estimation. More precisely, we are concerned by the test of the following null hypothesis  $H : "f(x) = f(-x), \forall x \in \mathcal{X} \setminus ]-\xi, \xi["$  against the alternative  $A : "There exists  $x \in \mathcal{X} \setminus ]-\xi, \xi["$  such that  $f(x) \neq f(-x)"$  on the basis of the observation  $X_1, X_2, \dots, X_n$ . For this purpose, we use the statistic$

$$V_{n,\xi} = \sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f_n(x) - f_n(-x)|,$$

which is naturally significant if greater than some positive threshold. Throughout the paper, we examine the following probabilities  $P_{x,n}(\lambda) = P(|f_n(x) - f_n(-x)| > \lambda)$ ,  $P_{n,\xi}(\lambda) =$

$P(V_{n,\xi} > \lambda)$ , that we show to tend to zero at exponential rate for  $\lambda > 0$ . A general result pertaining with any regular delta-sequence is stated and a discussion of hypotheses for the most usual methods is given. The results we state allows to obtain the Bahadur exact slope associated to the statistic  $V_{n,\xi}$  and then to compare the test based on  $V_{n,\xi}$  with any other test provided to have its Bahadur slope. This work follows the result by Osmoukhina (2001) and treating the kernel method. Note that whenever  $x = 0$ , we have  $|f_n(x) - f_n(-x)| = 0$  and therefore,  $P_{x,n}(\lambda) = 0$  for any  $\lambda > 0$ . The symmetry has then to be considered outside a small ball around 0. This problem has not been considered in Osmoukhina (2001) and has lead to a wrong statement in pages 367 – 368 where  $\alpha(t, x, n)$  cannot be equal to  $o(1)$  uniformly over  $\mathbb{R}$ .

## 2 Results

Throughout the paper, assume that  $f$  is a bounded symmetric function and set afterwards some hypotheses upon the density  $f$  and the delta-sequence  $(\delta_{m_n})$ .

(H<sub>1</sub>) For any  $x$  in  $\mathcal{X}$   $\int_{\mathcal{X}} \delta_m(x, u) du = 1$ .

(H<sub>2</sub>) For any  $t > 0$  and for any  $x$  in  $\mathcal{X}$ , the limit

$$L(t, x) := \lim_{n \rightarrow \infty} L_n(t, x) := \lim_{n \rightarrow \infty} m_n \int_{\mathcal{X}} \left[ \exp \left\{ \frac{t}{m_n} \delta_{m_n}(x, u) \right\} - 1 \right] f(u) du, \quad (1)$$

exists.

(H<sub>3</sub>) There exists a function  $I$  such that, for any  $t > 0$  and for any  $x$  in  $\mathcal{X}$ ,  $L(t, x)$  may be split up as follows  $L(t, x) = f(x)I(t)$ . Moreover,  $J(t) := I(t) + I(-t)$  is a twice differentiable function with invertible derivative denoted by  $\psi$ .

(H<sub>4</sub>) For any  $t > 0$  and for any  $x$  in  $\mathcal{X} - \{0\}$ ,

$$\lim_{n \rightarrow \infty} m_n \int_{\mathcal{X}} \left[ \exp \left\{ \frac{t}{m_n} \delta_{m_n}(x, u) \right\} - 1 \right] \left[ \exp \left\{ \frac{-t}{m_n} \delta_{m_n}(-x, u) \right\} - 1 \right] f(u) du = 0. \quad (2)$$

The inverse function is defined over the domain  $D = (0, t_1)$ , where  $t_1 = \sup_t \{\psi(t)\}$ , by

$$\psi^{-1}(t) = \inf \{s : \psi(s) \geq t\},$$

*Remark 1.* For several estimation methods covered by the delta-sequence method, the function  $L(t, x)$  can be factorized under the form  $L(t, x) = f(x)I(t)$ . Examples of such a factorization are given in Berrahou (2003). So the conditions (H<sub>2</sub>)-(H<sub>3</sub>) are satisfied for all the estimation methods discussed in this work.

### Properties of the function $J$

It is easily seen that, when the function  $L_n(t, x)$  is twice differentiable with respect to  $t$ , we have for any  $t \in \mathbb{R}$  and any  $x \in \mathcal{X}$   $\frac{\partial^2}{\partial t^2} L_n(t, x) \geq 0$ . Therefore, the second derivative  $I''$

is nonnegative for any  $t \in \mathbb{R}$  and  $I$  is a convex function. Therefore,  $J$  is strictly convex. The symmetry and convexity of  $J$  imply that  $J' \equiv \psi$  is an odd increasing function and  $\psi(0) = 0$ .

The pointwise large deviations result is given in the following theorem.

**Theorem 2.1** *Assume that the conditions  $(\mathbf{H}_1)$ - $(\mathbf{H}_4)$  are satisfied. If  $m_n \rightarrow \infty$  and  $m_n/n \rightarrow 0$  as  $n \rightarrow \infty$ , then for any  $x$  in  $\mathcal{X} - \{0\}$  and any  $\lambda > 0$ ,*

$$\lim_{n \rightarrow \infty} \frac{m_n}{n} \log P_{x,n}(\lambda) = -\Gamma_x(\lambda),$$

where

$$\Gamma_x(\lambda) = \sup_{t>0} \{t\lambda - f(x)J(t)\} = \begin{cases} \lambda\psi^{-1}\left(\frac{\lambda}{f(x)}\right) - f(x)J\left(\psi^{-1}\left(\frac{\lambda}{f(x)}\right)\right) & \text{if } 0 < \lambda \leq f(x)t_1, \\ \infty & \text{elsewhere.} \end{cases}$$

In the order to state the uniform large deviations result, let us introduce some notations. Define

$$g_\xi(\lambda) = \inf_{x \in \mathcal{X} \setminus ]-\xi, \xi[} \Gamma(\lambda).$$

Assume from now on that  $M_\xi := \sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} f(x) < \infty$ . Set, for  $u \in (0, M_\xi]$  and  $\lambda > 0$ ,  $h_\lambda(u) = \lambda\psi^{-1}(\frac{\lambda}{u}) - uJ(\psi^{-1}(\frac{\lambda}{u}))$ . It is proved in Lemma 3.1 below that for any fixed  $\lambda > 0$ ,  $h_{\lambda,\pm}$  is a nonincreasing function. Therefore, since  $(0, f(x)t_1] \subset (0, M_\xi t_1]$ , it follows that

$$g_\xi(\lambda) = \begin{cases} \lambda\psi^{-1}(\frac{\lambda}{M_\xi}) - M_\xi J(\psi^{-1}(\frac{\lambda}{M_\xi})) & \text{if } 0 < \lambda \leq M_\xi t_1, \\ \infty & \text{elsewhere.} \end{cases}$$

Moreover, as  $\lambda \rightarrow 0$  whenever  $\psi$  is differentiable, we have  $g_\xi(\lambda) = \frac{\lambda^2}{2M_\xi\psi'(0)}(1 + o(1))$ . Note that the rate function  $g_\xi$  is continuous if and only if  $\psi$  is continuous. From now on, we assume that  $\psi$  is a continuous function. The rate function  $g_\xi$  is increasing with respect to  $\xi$ .

The uniform large deviations result given in Theorem 2.2 is stated under the following additional conditions.

- (B<sub>1</sub>) Convergences (1) and (2) are uniform with respect to  $x$  in  $\mathcal{X} \setminus ]-\xi, \xi[$ .
- (B<sub>2</sub>) There exists a sequence  $(H_n)_{n \geq 1}$  of positive real numbers that tends to infinity with  $n$  and there exists a sequence of positive functions  $g_n$ , such that,
  - (i) for any  $\epsilon > 0$ , there exists  $n_0 > 0$  such that for any  $n \geq n_0$ ,  $\sup_{|x| > H_n} |\delta_{m_n}(x, u)| \leq g_n(u) + \epsilon$ ,
  - (ii) for any  $\zeta > 0$ ,  $\lim_{n \rightarrow \infty} \frac{m_n}{n} E(g_n(X_1)) \exp(\zeta n/m_n) = 0$ .
- (B<sub>3</sub>) There exists a partition  $\xi = a_0 < a_1 < \dots < a_{d_n} = H_n$  of  $[\xi, H_n]$ , such that,
  - (i) for any  $\epsilon > 0$ , there exists  $\eta_n > 0$  such that, for any  $1 \leq j \leq d_n$  and for any  $z \in \mathcal{X}$ ,  $\sup_{\{(x,u) \in (a_{j-1}, a_j)^2 : |x-u| \leq \eta_n\}} |\delta_{m_n}(\pm x, z) - \delta_{m_n}(\pm u, z)| \leq \epsilon$ ,
  - (ii)  $\lim_{n \rightarrow \infty} m_n \log(H_n/\eta_n)/n = 0$ .

We give now the uniform version of our results and point out that the rate function depends on the underlying density function via only its supremum over  $\mathcal{X} \setminus [-\xi, \xi]$ .

**Theorem 2.2** *Assume that the conditions  $(\mathbf{H}_1)$ - $(\mathbf{H}_4)$ ,  $(\mathbf{B}_1)$ - $(\mathbf{B}_3)$  are satisfied. If  $m_n \rightarrow \infty$  and  $m_n/n \rightarrow 0$  as  $n \rightarrow \infty$ , then for any  $\lambda > 0$ ,*

$$\lim_{n \rightarrow \infty} \frac{m_n}{n} \log P_{n,\xi}(\lambda) = -g_\xi(\lambda).$$

### Discussion of the conditions

- *Kernel method.* For  $(x, u) \in \mathbb{R}^2$ , set

$$\delta_m(x, u) = mK(m(x - u)),$$

where  $K$  is a positive bounded function such that  $\int_{-\infty}^{\infty} K(x)dx = 1$  and  $\lim_{x \rightarrow \infty} |x| K(x) = 0$ . Parzen (1962) showed that  $\{\delta_m\}$  constitutes a delta-sequence.

In Lemma 2.1, sufficient conditions that allow hypotheses  $(\mathbf{H}_4)$ ,  $(\mathbf{B}_1)$ - $(\mathbf{B}_3)$  to be satisfied are given. In the first place we gather together these conditions.

(A<sub>1</sub>)  $K$  is a lipschitz function.

(A<sub>2</sub>) There exists a sequence  $(H_n)$  of positive real numbers converging to infinity, such that,  $P(|X| > H_n) \leq \epsilon_n$  and for  $\zeta > 0$ ,  $\lim_{n \rightarrow \infty} \frac{\epsilon_n m_n^2 \exp\{\zeta n / m_n\}}{n} = 0$ .

(A<sub>3</sub>)  $\lim_{n \rightarrow \infty} m_n \log(H_n m_n^2)/n = 0$ .

**Lemma 2.1** *Suppose that the conditions  $(\mathbf{A}_1)$ - $(\mathbf{A}_3)$  are satisfied, and that  $f$  is a uniformly continuous function. If  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ , Then the conditions  $(\mathbf{H}_4)$ ,  $(\mathbf{B}_1)$ - $(\mathbf{B}_3)$  are satisfied.*

- *Trigonometric basis method.* Let  $\mathcal{X}$  be the interval  $[-\pi, \pi]$  and  $\{e_k, k \in \mathbb{Z}\}$  be the trigonometric basis defined by  $e_k(x) = (1/\sqrt{2\pi})e^{ikx}$ . Consider the sequence of functions

$$\delta_m(x, u) = \sum_{j=-m}^m e_j(x)e_j(u).$$

Walter (1965) stated that the sequence  $(\delta_m(x, u))$  is a delta-sequence.

**Lemma 2.2** *Assume that  $\sum_{k \in \mathbb{Z}} |\int f(u)e^{-iku}du| < \infty$ . If  $\lim_{n \rightarrow \infty} \frac{m_n}{n} \log(m_n) = 0$  and  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then the conditions  $(\mathbf{H}_4)$ ,  $(\mathbf{B}_1)$ - $(\mathbf{B}_2)$  are satisfied.*

- *Fejér kernel method.* Consider the Fejér kernel defined, for any  $u \in [-\pi, \pi]$ , by

$$F_m(u) = \frac{\sin^2((m+1)u/2)}{2\pi(m+1)\sin^2(u/2)}.$$

Winter (1975) showed that the sequence of function  $\delta_m(x, u) = F_m(x - u)$  constitutes a delta-sequence.

**Lemma 2.3** Assume that the conditions  $(\mathbf{A}_2)$ - $(\mathbf{A}_3)$  are satisfied,  $f$  is a continuous function and  $m_n$  is even. If  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then the conditions  $(\mathbf{H}_4)$ ,  $(\mathbf{B}_1)$ - $(\mathbf{B}_3)$  are satisfied.

### Applications

Large deviations results are useful and efficient tools to study the asymptotic efficiency of tests. This question has been widely investigated; we refer to Bahadur (1971) and the book of Nikitin (1995) for an account of results on this subject. In testing the hypothesis  $H$ , the rejection region associated to the test statistic  $V_{n,\xi}$  is given by  $\{V_{n,\xi} \geq c\}$  where  $c$  is some positive real number. The power function of this test is  $P_f(V_{n,\xi} \geq \lambda)$ , here,  $P_f$  denotes the distribution of observations when  $f$  is the underlying density. For any  $\lambda \in \mathbb{R}$ , define  $G_n(\lambda) = P_f(V_{n,\xi} \leq \lambda)$ . The  $P$ -value relative to the test statistic  $V_{n,\xi}$  is  $L_n = 1 - G_n(V_{n,\xi})$ .

The following corollary gives the asymptotic behavior of the  $P$ -value associated to the statistic  $(V_{n,\xi})$ .

**Corollary 2.3** Assume that  $f_n$  is a uniformly consistent estimator of  $f$ . Under conditions of Theorem 2.2, for any  $f$  that is not symmetric about zero for  $x \in \mathcal{X} \setminus ]-\xi, \xi[$ , we have with  $P_f$ -probability one,

$$\lim_{n \rightarrow \infty} \frac{m_n}{n} \log L_n = -g_\xi \left( \sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| \right).$$

*Remark 2.* From Corollary 2.3 above, we deduce that the Bahadur exact slope relative to the statistic  $(V_{n,\xi})$  is  $2g_\xi(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)|)$ .

## 3 Proofs

**Lemma 3.1** For fixed  $\lambda > 0$ ,  $h_\lambda$  in nonincreasing function on  $(0, \infty)$ .

**Proof** It is easily seen that  $h'_\lambda(u) = -J(\psi^{-1}(\frac{\lambda}{u}))$ , where  $h'_\lambda$  is the first derivative of  $h_\lambda$ . Set  $l(x) = -J(x)$  and observe as stated above (properties of  $J$ ) that  $l'(x) = -\psi(x) \leq 0$  for any  $x > 0$ . Therefore,  $l$  is a nonincreasing function. Moreover, it is easily seen that  $l(0) = 0$ . We achieve the proof by the fact that  $\psi^{-1}(\frac{\lambda}{u}) > 0$  since  $\lambda/u > 0$ .

**Proof of Theorem 2.1** Observe, for any  $x \in \mathcal{X} - \{0\}$  and for any  $\lambda > 0$ , that

$$\begin{aligned} \max\{P(f_n(x) - f_n(-x) > \lambda), P(f_n(-x) - f_n(x) > \lambda)\} &\leq P(|f_n(x) - f_n(-x)| > \lambda) \\ &\leq 2 \max\{P(f_n(x) - f_n(-x) > \lambda), P(f_n(-x) - f_n(x) > \lambda)\}. \end{aligned}$$

As the proofs use the same arguments, we will give here only the details concerning  $P(f_n(x) - f_n(-x) > \lambda)$ . Set  $Y_{n,j}(x) = \frac{1}{m_n} (\delta_{m_n}(x, u) - \delta_{m_n}(-x, u))$ . It is easily seen that

$$P(f_n(x) - f_n(-x) > \lambda) = P\left(\sum_{j=1}^n Y_{n,j}(x) > \frac{n}{m_n} \lambda\right).$$

Define  $\varphi_n^x(t)$  to be the moment generating function of  $\sum_{j=1}^n Y_{n,j}(x)$  and observe that

$\varphi_n^x(t) = [\psi_n^x(t)]^n$ , where

$$\begin{aligned}\psi_n^x(t) &= \mathbb{E}(\exp\{tY_{n,1}(x)\}) = 1 + \int_{\mathcal{X}} \left( \exp\left\{ \frac{t}{m_n} \left( \delta_{m_n}(x, u) - \delta_{m_n}(-x, u) \right) \right\} - 1 \right) f(u) du \\ &:= 1 + \frac{1}{m_n} R_n(t, x).\end{aligned}$$

Observe, further, that

$$\begin{aligned}R_n(t, x) &= m_n \int_{\mathcal{X}} \left[ \exp\left\{ \frac{t}{m_n} \delta_{m_n}(x, u) \right\} - 1 \right] f(u) du + m_n \int_{\mathcal{X}} \left[ \exp\left\{ \frac{-t}{m_n} \delta_{m_n}(-x, u) \right\} - 1 \right] f(u) du \\ &\quad + m_n \int_{\mathcal{X}} \left[ \exp\left\{ \frac{t}{m_n} \delta_{m_n}(x, u) \right\} - 1 \right] \left[ \exp\left\{ \frac{-t}{m_n} \delta_{m_n}(-x, u) \right\} - 1 \right] f(u) du.\end{aligned}$$

Using the conditions  $(\mathbf{H}_2)$ ,  $(\mathbf{H}_4)$ , we obtain that  $R_n(t, x)/m_n \rightarrow 0$  as  $n \rightarrow \infty$ . Taylor series expansion of  $\log(1 + u)$  about  $u = 0$ , yields  $\log \varphi_n^x(t) = \frac{n}{m_n} \left( R_n(t, x) + O\left(\frac{1}{m_n}\right) \right)$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{m_n}{n} \log \varphi_n^x(t) = f(x)(I(t) + I(-t)).$$

The remainder of the proof essentially uses arguments of the proof of the theorem in Plachky and Steinebach (1975). Namely, Chebycheff inequality is applied for deriving the upper bound and an exponential change of measure is used to derive the lower bound. ■

**Proof of Theorem 2.2** For any  $x \in \mathcal{X} \setminus ] - \xi, \xi[$ , we have  $P(V_{n,\xi} > \lambda) \geq P(|f_n(x) - f_n(-x)| > \lambda)$ . Using Theorem 2.1, we obtain for any  $x \in \mathcal{X} \setminus ] - \xi, \xi[$

$$\liminf_{n \rightarrow \infty} \frac{m_n}{n} \log P_{n,\xi}(\lambda) \geq \liminf_{n \rightarrow \infty} \frac{m_n}{n} \log P_n^x(\lambda) \geq -\Gamma_x(\lambda).$$

Thus,

$$\liminf_{n \rightarrow \infty} \frac{m_n}{n} \log P_{n,\xi}(\lambda) \geq -g_\xi(\lambda). \quad (3)$$

Let  $H_n$  be a positive real number tending to infinity with  $n$ . Observe that

$$P_{n,\xi}(\lambda) \leq P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda\right) + P\left(\sup_{x > H_n} |f_n(x) - f_n(-x)| > \lambda\right).$$

For any  $j = 1, \dots, d_n$ , consider the following partition of the interval  $(a_{j-1}, a_j)$ ,  $u_{0,j-1}, u_{1,j-1}, \dots, u_{l,j-1}$ , such that  $a_0 = u_{0,0} = \xi$ ,  $a_{d_n} = H_n$ ,  $u_{l,j-1} = u_{0,j} = a_j$  and  $u_{i,j-1} - u_{i-1,j-1} = \mu((a_{j-1}, a_j))/l$ , where  $\mu$  is the Lebesgue measure. Clearly,

$$\begin{aligned}\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| &= \max\left\{ \max_{1 \leq j \leq d_n} \left\{ \sup_{a_{j-1} < x < a_j} |f_n(x) - f_n(-x)| \right\}; \right. \\ &\quad \left. \max_{0 \leq j \leq d_n} |f_n(a_j) - f_n(-a_j)| \right\}.\end{aligned} \quad (4)$$

For any  $j = 1, \dots, d_n$ , we have

$$\begin{aligned}\sup_{a_{j-1} < x < a_j} |f_n(x) - f_n(-x)| &= \max\left\{ \sup_{u_{0,j-1} < x \leq u_{1,j-1}} |f_n(x) - f_n(u_{1,j-1})|; \right. \\ &\quad \sup_{u_{l-1,j-1} \leq x < u_{l,j-1}} |f_n(x) - f_n(-x)|; \\ &\quad \left. \max_{2 \leq i \leq l-1} \left\{ \sup_{u_{i-1,j-1} < x \leq u_{i,j-1}} |f_n(x) - f_n(-x)| \right\} \right\}.\end{aligned}$$

Observe, for any  $i = 1, \dots, l-1$ , that

$$\begin{aligned} \sup_{u_{i-1,j-1} < x \leq u_{i,j-1}} |f_n(x) - f_n(-x)| &\leq \sup_{u_{i-1,j-1} < x < u_{i,j-1}} |f_n(x) - f_n(u_{i,j-1})| \\ &+ \sup_{u_{i-1,j-1} < x < u_{i,j-1}} |f_n(-u_{i,j-1}) - f_n(-x)| \\ &+ |f_n(u_{i,j-1}) - f_n(-u_{i,j-1})|, \end{aligned}$$

and

$$\begin{aligned} \sup_{u_{l-1,j-1} \leq x < u_{l,j-1}} |f_n(x) - f_n(-x)| &\leq \sup_{u_{l-1,j-1} < x < u_{l,j-1}} |f_n(x) - f_n(u_{l-1,j-1})| \\ &+ \sup_{u_{l-1,j-1} < x < u_{l,j-1}} |f_n(-u_{l-1,j-1}) - f_n(-x)| \\ &+ |f_n(u_{l-1,j-1}) - f_n(-u_{l-1,j-1})|. \end{aligned}$$

Using the condition  $(\mathbf{B}_3)$ , we obtain, for any  $\epsilon \geq 0$ , that there exists  $l_{j-1} \geq \mu((a_{j-1}, a_j))/\eta_n$ , such that, for any  $i = 1, \dots, l_{j-1} - 1$ ,  $\sup_{u_{i-1,j-1} < x < u_{i,j-1}} |f_n(u_{i,j-1}) - f_n(x)| \leq \epsilon$  and  $\sup_{u_{i-1,j-1} < x < u_{i,j-1}} |f_n(-u_{i,j-1}) - f_n(-x)| \leq \epsilon$ . Thus,

$$\sup_{a_{j-1} < x < a_j} |f_n(x) - f_n(-x)| \leq \max_{1 \leq i \leq l_{j-1} - 1} \{|f_n(u_{i,j-1}) - f_n(-u_{i,j-1})| + 2\epsilon\}. \quad (5)$$

By combining (4) and (5), we obtain

$$\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(x)| \leq \max_{1 \leq j \leq d_n, 0 \leq i \leq l_{j-1}} \{|f_n(u_{i,j-1}) - f_n(u_{i,j-1})| + 2\epsilon\}.$$

Therefore, we have

$$P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(x)| > \lambda\right) \leq \sum_{j=1}^{d_n} \sum_{i=0}^{l_{j-1}} P(|f_n(u_{i,j-1}) - f_n(-u_{i,j-1})| > \lambda - 2\epsilon).$$

Thus,

$$\begin{aligned} \log P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda\right) &\leq \\ &\log\left(\sum_{j=1}^{d_n} l_{j-1}\right) + \sup_{x \in \mathcal{X} \setminus [-\xi, \xi]} \{\log P(|f_n(x) - f_n(-x)| > \lambda - 2\epsilon)\}. \end{aligned} \quad (6)$$

On the other hand, we have  $\sup_{x > H_n} |f_n(x) - f_n(-x)| \leq 2 \sup_{|x| > H_n} |f_n(x)|$ . From the condition  $(\mathbf{B}_2)$  and using Markov's inequality, we obtain

$$P\left(2 \sup_{|x| > H_n} |f_n(x)| > \lambda\right) \leq P\left(\sum_{i=1}^n g_n(X_i) > n\left(\frac{\lambda}{2} - \epsilon\right)\right) \leq \frac{E(g_n(X_1))}{\frac{\lambda}{2} - \epsilon}. \quad (7)$$

Notice that

$$P_{n,\xi}(\lambda) \leq P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda\right) + P\left(\sup_{x > H_n} |f_n(x) - f_n(-x)| > \lambda\right).$$

Since,  $\log(1 + u) \leq u$  for  $u \geq 0$ , it follows that

$$\log P_{n,\xi}(\lambda) \leq \log P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda\right) + \frac{P(\sup_{x > H_n} |f_n(x) - f_n(-x)| > \lambda)}{P(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda)}.$$

From (3) and (7), we obtain that

$$\log P_n(\lambda) \leq \log P\left(\sup_{\xi \leq x \leq H_n} |f_n(x) - f_n(-x)| > \lambda\right) + \frac{E(g_n(X_1))}{\frac{\lambda}{2} - \epsilon} \exp\left\{\frac{n}{m_n}(g_\xi(\lambda) + o(1))\right\}. \quad (8)$$

Substituting the expression in the right hand side of (6) to the first term in the right hand side of (8), multiplying afterwards by  $\frac{m_n}{n}$  in both sides of (8) and using then hypotheses **(B<sub>2</sub>)**-**(B<sub>3</sub>)**, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{na_n} \log P_n(\lambda) \leq \sup_{x \in \mathcal{X} \setminus [-\xi, \xi]} (-\Gamma_x(\lambda - 2\epsilon)) \leq -g_\xi(\lambda - 2\epsilon).$$

Since  $g$  is continuous, we achieve the proof by making  $\epsilon$  go to zero. ■

**Proof of Lemma 2.3** Taking the proof of the Corollary 2.2 in Berrahou (2003). If we suppose that  $f$  is bounded and the uniformly continuous instead of only continuous, we obtain that the function  $L_n(t, x)$  converges uniformly to  $f(x)I(t)$  with respect to  $x$ , which implies that the convergence (1) is uniform with respect to  $x$ . We suppose now that the kernel  $K$  is bounded by a positive real number  $A$ . Using Taylor series expansion of the exponential function and the fact that  $K$  is bounded, it follows that, for all  $\alpha > 0$ ,

$$\sup_{|u| \geq \alpha} m_n [\exp\{tK(m_n u)\} - 1] \leq h(t) \sup_{|u| \geq \alpha} m_n K(m_n u),$$

where  $h(t) = (1/A) [\exp\{tA\} - 1]$ . Set

$$I_0 =: m_n \int [\exp\{tK(m_n(x-u))\} - 1] [\exp\{tK(m_n(-x-u))\} - 1] f(u) du \leq I_1 + I_2,$$

where

$$I_1 := m_n \sup_{|x-u| > \alpha} [\exp\{tK(m_n(x-u))\} - 1] \int_{|x-u| > \alpha} [\exp\{tK(m_n(-x-u))\} - 1] f(u) du,$$

and

$$I_2 := m_n \sup_{|x-u| \leq \alpha} [\exp\{tK(m_n(-x-u))\} - 1] \int_{|x-u| \leq \alpha} [\exp\{tK(m_n(x-u))\} - 1] f(u) du.$$

Observe that  $I_1 \leq Ah(t)^2 m_n \sup_{|u| > \alpha} K(m_n u)$  and  $I_2 \leq Ah(t)^2 m_n \sup_{|u| \leq \alpha} K(m_n(-2x-u))$ . The term  $I_1$  is smaller than  $A\epsilon h(t)^2$  and the term  $I_2$  is bounded by  $A\epsilon h(t)^2$ . These facts imply that the condition **(H<sub>4</sub>)** is satisfied. Notice now that, if  $\alpha < 2\xi$  and  $x \in \mathcal{X} \setminus [-\xi, \xi]$ ,

$$I_2 \leq Ah(t)^2 m_n \sup_{|u| \leq \alpha} K(m_n(-2x-u)) \leq Ah(t)^2 m_n \sup_{|u| > 2\xi - \alpha} K(m_n u).$$

Therefore, the term  $I_2$  is bounded by  $A\epsilon h(t)^2$  independently of  $x$ . This implies that the convergence (2) is uniform with respect to  $x$  on  $\mathcal{X} \setminus ]-\xi, \xi[$ . Using the condition  $(\mathbf{A}_1)$ , we obtain, for any  $\epsilon \geq 0$ , that there exists a sequence  $\eta_n = \epsilon/Lm_n^2$ , such that, for any  $z \in \mathcal{X}$ ,

$$\sup_{|x-u| \leq \eta_n} |\delta_{m_n}(x, z) - \delta_{m_n}(u, z)| \leq \epsilon.$$

Here,  $L$  is the lipschitz coefficient. Thus, the condition  $(\mathbf{B}_3)$ (i) is satisfied. Moreover, using the fact that  $\lim_{|x| \rightarrow \infty} |x|K(x) = 0$ , we obtain

$$\sup_{|x| > H_n} |\delta_{m_n}(x, u)| \leq m_n \bar{K} \mathbb{I}_{\{|u| > H_n/2\}} + \sup_{|v| > m_n \frac{H_n}{2}} \frac{2|v|}{H_n} K(v) \mathbb{I}_{\{|u| \leq H_n/2\}} \leq m_n \bar{K} \mathbb{I}_{\{|X_1| > H_n/2\}} + \epsilon,$$

where  $\bar{K} = \sup_{x \in \mathbb{R}} K(x)$ . Consequently, making use of the conditions  $(\mathbf{A}_2)$ - $(\mathbf{A}_3)$ , we conclude that the conditions  $(\mathbf{B}_2)$ ,  $(\mathbf{B}_3)$ (ii) are satisfied.  $\blacksquare$

**Proof of Lemma 2.4** It is shown in Louani (2003) that the convergence (1) is uniform with respect  $x$ . Using Taylor expansion of the exponential function and Fubini's Theorem, it follows that

$$\begin{aligned} T_n &:= (2m_n + 1) \int_{-\pi}^{\pi} \left( \exp \left\{ \frac{t \sum_{k=-m_n}^{m_n} e^{ik(x-u)}}{2\pi(2m_n + 1)} \right\} - 1 \right) \left( \exp \left\{ \frac{-t \sum_{k=-m_n}^{m_n} e^{-ik(x+u)}}{2\pi(2m_n + 1)} \right\} - 1 \right) f(u) du, \\ &= \sum_{p=1}^{\infty} \sum_{l=1}^{\infty} \sum_{-m_n \leq k_1, \dots, k_l \leq m_n} \sum_{-m_n \leq s_1, \dots, s_p \leq m_n} \frac{(-1)^{p+l} t^{p+l} e^{i(k_1 + \dots + k_l)x} e^{-i(s_1 + \dots + s_p)x}}{(2\pi)^{l+p} (2m_n + 1)^{l+p-1} l! p!} \\ &\quad \times \int_{-\pi}^{\pi} e^{-i(k_1 + \dots + k_l)u} e^{-i(s_1 + \dots + s_p)u} f(u) du. \end{aligned}$$

It is easily seen that,

$$\begin{aligned} T_n &= \sum_{p=1}^{\infty} \sum_{l=1}^{lm_n} \sum_{r_1=-lm_n}^{pm_n} \sum_{r_2=-pm_n}^{pm_n} \frac{(-1)^{p+l} t^{p+l} A_{l, m_n, r_1} A_{p, m_n, r_2} e^{i(r_1 - r_2)x}}{(2\pi)^{p+l} (2m_n + 1)^{l+p-1} l! p!} \int_{-\pi}^{\pi} e^{-i(r_1 + r_2)u} f(u) du, \\ &= \sum_{p=1}^{\infty} \sum_{l=1}^{lm_n} \sum_{r_1=-lm_n}^{pm_n} \sum_{r_2=-pm_n}^{pm_n} \frac{(-1)^{p+l} t^{p+l} A_{l, m_n, r_1} A_{p, m_n, r_2} C_{r_1 + r_2}(f) e^{i(r_1 - r_2)x}}{(2\pi)^{p+l-1} (2m_n + 1)^{l+p-1} l! p!}, \end{aligned}$$

where  $C_r(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iru} f(u) du$ , and

$$\begin{aligned} A_{d, m_n, r} &= |\{(k_1, \dots, k_d) \in \{-m_n, \dots, m_n\}^d : k_1 + \dots + k_d = r\}|, \\ &= \sum_{0 \leq j \leq \frac{r+dm_n}{2m_n+1}} (-1)^j C_d^j C_{r+dm_n-j(2m_n+1)+d-1}^{r+dm_n-j(2m_n+1)} \end{aligned}$$

(see, Louani (2003) for details). Set now

$$\zeta_{m_n}(x, t, p, l) := \sum_{r_1 \in \mathbb{Z}} \sum_{r_2 \in \mathbb{Z}} \frac{(-1)^{p+l} t^{p+l} A_{p, m_n, r_2}}{(2\pi)^{p+l-1} (2m_n + 1)^{p+l} l! p!} \mathbb{I}_{\|r_2\|/p, \infty}[(m_n) \theta_{m_n}(x, l, r_1, r_2) e^{-2ir_2 x}],$$

where  $\theta_{m_n}(x, l, r_1, r_2) := \frac{A_{l, m_n, r_1}}{(2m_n+1)^{l-1}} \mathbb{I}_{\|r_1\|/p, \infty}[(m_n) C_{r_1+r_2}(f) e^{i(r_1+r_2)x}]$ . It is easily seen that  $A_{d, m_n, r} \leq \frac{(2d)^d (2m_n+1)^{d-1}}{d!}$ . Thus,

$$|\zeta_{m_n}(x, t, p, l)| \leq \frac{t^{p+l} (2p)^p (2l)^l}{(p!)^2 l! (l-1)! (2\pi)^{p+l-1}} \sum_{r \in \mathbb{Z}} C_r(f).$$

Using the d'Alembert's rule, it follows that  $\sum_{l=1}^{\infty} \frac{(2l)^l}{l!(l-1)!(2\pi)^{l-1}} < \infty$  and  $\sum_{p=1}^{\infty} \frac{(2pt)^p}{(p!)^2(2\pi)^p} < \infty$ . Since  $\sum_{r \in \mathbb{Z}} |c_r(f)| < \infty$ , then we have  $\sum_{p=1}^{\infty} \sum_{l=1}^{\infty} |\zeta_{m_n}| < \infty$ . Furthermore, observe that  $|\theta_{m_n}(x, l, r_1, r_2)| \leq \frac{(2l)^l}{d!} |C_{r_1+r_2}(f)|$ . Since  $\sum_{r \in \mathbb{Z}} |c_r(f)| < \infty$ , then we have  $\sum_{r_1=-lm_n}^{lm_n} \theta_{m_n}(x, l, r_1, r_2) < \infty$ . Observe now that

$$\theta_{m_n}(x, l, r_1, r_2) \xrightarrow{n \rightarrow \infty} \frac{C_{r_1+r_2}(f)e^{i(r_1+r_2)x}}{(l-1)!} \sum_{0 \leq j \leq \frac{l}{2}} (-1)^j C_l^j \left( \frac{l-2j}{2} \right)^{l-1}.$$

By the dominated convergence Theorem, it follows that

$$\sum_{r_1=-lm_n}^{lm_n} \theta_{m_n}(x, l, r_1, r_2) \xrightarrow{n \rightarrow \infty} \frac{f(x)}{(l-1)!} \sum_{0 \leq j \leq \frac{l}{2}} (-1)^j C_l^j \left( \frac{l-2j}{2} \right)^{l-1} := \theta(x, l).$$

Set now  $\zeta'_{m_n}(x, p, l) := \frac{1}{(2m_n+1)^p} \sum_{r_1 \in \mathbb{Z}} \sum_{r_2 \in \mathbb{Z}} A_{p, m_n, r_2} \mathbb{1}_{|r_2|/p, \infty[(m_n)\theta_{m_n}(x, l, r_1, r_2)e^{-2ir_2x}}$ , and  $\zeta''_{m_n}(x, p, l) := \frac{1}{(2m_n+1)^p} \sum_{r_2 \in \mathbb{Z}} A_{p, m_n, r_2} \mathbb{1}_{|r_2|/p, \infty[(m_n)\theta(x, l)e^{-2ir_2x}$ . Obviously,  $|\zeta'_{m_n}(x, p, l) - \zeta''_{m_n}(x, p, l)| \leq \frac{(2p)^p}{(p-1)!} \epsilon$ . It is easily seen that  $\zeta''_{m_n}(x, p, l) = \left( \frac{D_n(-2x)}{2m_n+1} \right)^p \theta(x, l)$ , where  $D_n(x) = \sum_{r=-m_n}^{m_n} e^{irx}$ . It is well-known that if  $0 < |x| \leq \pi$ , then  $|D_n(x)| \leq \frac{\pi}{2|x|}$ , therefore

$$|\zeta''_{m_n}(x, p, l)| \leq \left( \frac{\pi}{4|x|(2m_n+1)} \right)^p |\theta(x, l)|.$$

Thus,  $\zeta''_{m_n}(x, p, l)$  converges to zero for any  $x \in \mathcal{X} - \{0\}$ . Again, by the dominated convergence Theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{p=1, l=1}^{\infty} \zeta_{m_n}(x, t, p, l) &= \sum_{p=1, l=1}^{\infty} \frac{(-1)^{pt^{p+l}}}{(2\pi)^{p+l-1} l! p!} \lim_{n \rightarrow \infty} \zeta'_{m_n}(x, p, l), \\ &= \sum_{p=1, l=1}^{\infty} \frac{(-1)^{pt^{p+l}}}{(2\pi)^{p+l-1} l! p!} \lim_{n \rightarrow \infty} \zeta''_{m_n}(x, p, l) = 0. \end{aligned}$$

Hence, the condition  $(\mathbf{H}_4)$  is satisfied. Suppose now that  $\xi \leq |x| \leq \pi$ , it follows that  $|D_n(x)| \leq \frac{\pi}{2\xi}$ . It is easily seen that

$$|\zeta''_{m_n}(x, p, l)| \leq \left( \frac{\pi}{4\xi(2m_n+1)} \right)^p \frac{M}{(l-1)!} \sum_{0 \leq j \leq \frac{l}{2}} (-1)^j C_l^j \left( \frac{l-2j}{2} \right)^{l-1},$$

where  $M = \sup_{x \in \mathcal{X}} f(x)$ . Therefore,  $\zeta''_{m_n}(x, p, l)$  converges uniformly to zero with respect to  $x$  on  $\mathcal{X} \setminus ]-\xi, \xi[$ . This implies that the convergence (2) is uniform with respect to  $x$  on  $\mathcal{X} \setminus ]-\xi, \xi[$ . Since the elements of the basis  $(e_k)$  are of bounded support, we conclude that the condition  $(\mathbf{B}_2)$  is satisfied. Observe now that

$$|\delta_m(x, z) - \delta_m(u, z)| = \frac{1}{2\pi} \sum_{k=-m_n}^{m_n} |e^{ikx} - e^{iku}| \leq \frac{1}{\pi} \sum_{k=-m_n}^{m_n} |k(x-u)| \leq \frac{6m_n^2}{\pi} |x-u|.$$

The condition  $(\mathbf{B}_3)$  is then satisfied, ■

**Proof of Lemma 2.5** Suppose that  $m_n = 2q_n$ . It is shown in (Berrahou (2003) Corollary 2.4) that

$$\lim_{n \rightarrow \infty} I_n(t) = \sum_{p=1}^{\infty} \frac{t^p}{(2\pi)^{p-1} p! (2p-1)!} \sum_{0 \leq j \leq p} (-1)^j C_{2p}^j (p-j)^{2p-1},$$

where  $I_n(t) := 2q_n \int_{-\pi}^{\pi} \left( \exp \left\{ \frac{t}{2q_n} F_{2q_n}(u) \right\} - 1 \right) du$ . Using the continuity of  $f$  which is actually a uniform continuity on the compact set  $[-\pi, \pi]$ , we obtain, for any  $\epsilon > 0$ , that there exists  $\delta > 0$  such that for any  $x \in [-\pi, \pi]$  and for any  $|u| \leq \delta$ ,  $|f(x+u) - f(x)| \leq \epsilon$ . It is well-known that, if  $u \in [-\pi, \pi]$ ,  $F_{2q_n}(u) \leq (2q_n + 1)/4$ . Then, we obtain

$$2q_n \int_{|u| \leq \delta} \left[ \exp \left\{ \frac{t}{2q_n} F_{2q_n}(u) \right\} - 1 \right] du \leq H(t) \int_{|u| \leq \delta} F_{2q_n}(u) du,$$

where  $H(t) = (4/5) [\exp\{5t/4\} - 1]$ . Observe now that for any  $\delta > 0$ ,

$$\begin{aligned} |L_n(t, x) - f(x)I_n(t)| &\leq 2q_n \int_{-\pi}^{\pi} \left[ \exp \left\{ \frac{t}{2q_n} F_{2q_n}(u) \right\} - 1 \right] |f(x+u) - f(x)| du, \\ &\leq 2q_n \int_{|u| \leq \delta} \left[ \exp \left\{ \frac{t}{2q_n} F_{2q_n}(u) \right\} - 1 \right] |f(x+u) - f(x)| du, \\ &\quad + 2M \left( 2q_n \int_{|u| \geq \delta} \left[ \exp \left\{ \frac{t}{2q_n} F_{2q_n}(u) \right\} - 1 \right] du \right), \end{aligned}$$

where  $M := \sup_x f(x)$ . The first term in the last inequality may be made smaller than  $(4/5)\{\exp(5t/4) - 1\}\epsilon$  and the second term is bounded by  $(8/5)\{\exp(5t/4) - 1\}M\epsilon$ , which implies that the convergence (1) is uniform with respect to  $x$ . Set

$$I_0 := m_n \int \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(x-u) \right\} - 1 \right] \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(-x-u) \right\} - 1 \right] f(u) du \leq I_1 + I_2,$$

where

$$I_1 := m_n \sup_{|x-u| > \alpha} \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(x-u) \right\} - 1 \right] \int_{|x-u| > \alpha} \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(-x-u) \right\} - 1 \right] f(u) du,$$

and

$$I_2 := m_n \sup_{|x-u| \leq \alpha} \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(-x-u) \right\} - 1 \right] \int_{|x-u| \leq \alpha} \left[ \exp \left\{ \frac{t}{m_n} F_{2q_n}(x-u) \right\} - 1 \right] f(u) du.$$

Observe that  $I_1 \leq (5/4)H(t)^2 \sup_{|u| > \alpha} F_{2q_n}(u)$  and  $I_2 \leq (5/4)H(t)^2 \sup_{|u| \leq \alpha} F_{2q_n}(-2x-u)$ . The term  $I_1$  is smaller than  $(5/4)H(t)^2\epsilon$  and the term  $I_2$  is bounded by  $(5/4)H(t)^2\epsilon$  for any  $x \neq 0$ . This implies that the  $(\mathbf{H}_4)$  is satisfied. Observe now that, if  $\alpha < 2\xi$  and  $x \in \mathcal{X} \setminus ]-\xi, \xi[$ , we have  $I_2 \leq (5/4)H(t)^2 \sup_{|u| \geq 2\xi - \alpha} F_{2q_n}(u)$ . Then the term  $I_2$  is bounded by  $(5/4)H(t)^2\epsilon$  and this implies that the convergence (2) is uniform with respect to  $x$  on  $\mathcal{X} \setminus ]\xi, \xi[$ . It easily seen that

$$\delta_{2q_n}(x, z) = \frac{1}{2\pi(2q_n + 1)} \left( \sum_{k=-q_n}^{q_n} e^{-ik(x-z)} \right)^2 = \frac{1}{2\pi(2q_n + 1)} \sum_{-2q_n \leq k \leq 2q_n} A_k e^{-ik(x-z)},$$

where  $A_k = |\{(k_1, k_2) \in \{-q_n \leq k_1, k_2 \leq q_n\}^2 : k_1 + k_2 = k\}| \leq 4(2q_n + 1)$ , (see Louani (2003) for details). Thus,  $|\delta_{2q_n}(x, z) - \delta_{2q_n}(u, z)| \leq \frac{32q_n^2}{\pi}|x - u|$ . The condition  $(\mathbf{B}_3)(i)$  is then satisfied. Observe now that

$$\begin{aligned} \sup_{|x| > H_n} |\delta_{m_n}(x, u)| &\leq \frac{2q_n + 1}{4} \mathbb{I}_{\{|u| > H_n/2\}} + \sup_{|x-u| > H_n/2} F_{2q_n}(x - u) \mathbb{I}_{\{|u| \leq H_n/2\}}, \\ &\leq \frac{2q_n + 1}{4} \mathbb{I}_{\{|u| > H_n/2\}} + \epsilon. \end{aligned}$$

Consequently, making use of the conditions  $(\mathbf{A}_2)$ - $(\mathbf{A}_3)$ , we conclude that the conditions  $(\mathbf{B}_2)$ ,  $(\mathbf{B}_3)(ii)$  are satisfied. ■

**Proof of Corollary 2.3** Making use of the uniform consistency of the estimator  $f_n$ , it follows that  $V_{n,\xi} \rightarrow \sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)|$  in  $P_f$ -probability as  $n \rightarrow \infty$ . Therefore, for an arbitrarily fixed  $f$  such that  $f(x) \neq f(-x)$  for an  $x \in \mathcal{X} \setminus ]-\xi, \xi[$ , and  $\epsilon > 0$ , we have, for  $n$  large enough

$$P_f\left(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| - \epsilon \leq V_{n,\xi} \leq \sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| + \epsilon\right) > 1 - \delta,$$

with  $\delta > 0$  arbitrarily small. Since  $G_n$  is a monotone function, then we have

$$1 - G_n\left(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| + \epsilon\right) \leq L_n \leq 1 - G_n\left(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| - \epsilon\right).$$

Making use of Theorem 2.2, it follows that

$$\begin{aligned} -g_\xi\left(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| + \epsilon\right) &\leq \liminf_{n \rightarrow \infty} \frac{m_n}{n} L_n, \\ &\leq \limsup_{n \rightarrow \infty} \frac{m_n}{n} L_n \leq -g_\xi\left(\sup_{x \in \mathcal{X} \setminus ]-\xi, \xi[} |f(x) - f(-x)| - \epsilon\right). \end{aligned}$$

Since  $g_\xi$  is continuous, we achieve the proof by making  $\epsilon$  tend to zero. ■

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