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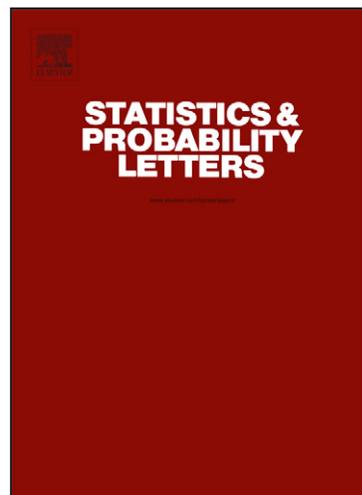
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Characterization of periodically correlated and multivariate stationary discrete time wide Markov processes

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Abstract The aim of this paper is to give an overview of the structure of the class of discrete time wide Markov processes, either periodically correlated or multivariate stationary. We show many properties of their covariance, correlation and reflection coefficients matrices. We characterize these processes chiefly in terms of autoregressive models of order one. Illustrative numerical examples are given.

MCS: 60G12, 60J99, 62M10, 62M20

Key-words: Auto-regressive processes; Multivariate stationary processes; Non stationary processes; Periodically correlated processes; Reflection coefficients; Wide Markov processes.

1 Introduction and basics

The future evolution of a strict Markov process conditional to its past and present evolution depends only on its present. In terms of second order properties, this amounts to considering projection on the linear subspaces spanned by the process, hence the notion of wide sense Markov (WM) processes introduced by Doob (1953). In the literature, WM processes have mainly been considered in the continuous time case: Mehr and McFadden (1965) developed the scalar case with application to first-passage times; Mandrekar (1968) –with application to spectral representation, and Beutler (1963) –with application to differential equations, developed the multivariate case through connection with continuous autoregressive processes. Here, we are interested in the discrete time case; to our knowledge, this has been specifically studied only by Nematollahi and Soltani (2000), in terms of spectral densities.

A scalar one-dimensional second order process is periodically correlated (PC) if its covariance function is periodic. A one-to-one relationship exists between PC processes and multivariate stationary (MS) processes. Hence, results proven for ones can be used in the investigation of the others. Periodically correlated data are often encountered in applications; see Franses (1996), Castro and Girardin (2002) and the references therein.

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We study here mainly PCWM and MSWM processes, using their duality, through their second order structure and connection with autoregressive processes. Many properties of their covariance, correlation and reflection coefficients matrices are presented in Section 2 in order to give a complete overview of their second order structure, especially in terms of number of parameters necessary to characterize them. Some of these properties are specific, others derive simply from properties known in the continuous time case. The WM processes are characterized in terms of wide martingales in Section 3. In the discrete time case, contrary to the continuous time case, all these characterizations are not sufficient to induce the representation of the processes as autoregressive models of order one. This latter definitive characterization is given here in Section 4 using generalized reflection coefficients matrices introduced in Castro and Girardin (2002). The subclass of MSWM processes dual to PCWM processes (called MSD) is especially characterized. Illustrative numerical examples are given.

The following definitions and properties will be necessary. For simplification, we will consider zero-mean real-valued processes indexed by \mathbb{Z} . Note that complex-valued processes indexed by \mathbb{Z} or \mathbb{N} can be studied in a similar way.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $L^2(\Omega)$ denote as usual the space of zero-mean second order random variables X equipped with norm $\|X\|_2 = \sqrt{\mathbb{E}(X^2)}$ and inner product $\langle X, X' \rangle_2 = \mathbb{E}(XX')$. Let further $L_d^2(\Omega)$, for $d \in \mathbb{N}^*$, denote the space of all d -variate random variables $V = (V_1, \dots, V_d)$ such that $V_i \in L^2(\Omega)$, equipped with the Euclidean norm $\|V\| = (\sum_{i=1}^d \|V_i\|_2^2)^{1/2}$ and inner product $\langle V, W \rangle = \text{Trace}(VW^*)$, where the $*$ index denotes transpose. The projection of a d -variate random variable V on a linear subspace \mathcal{S} of $L_d^2(\Omega)$ is the element minimizing the usual sum of squares; see Mandrekar (1968) for details. A real-valued d -variate sequence $\mathbf{Z} = (Z(n))_{n \in \mathbb{Z}}$ is a multivariate second-order stochastic process if $Z(n) \in L_d^2(\Omega)$ for all $n \in \mathbb{Z}$. In the following, we will assume that \mathbf{Z} is a basic process, that is a basis of the linear subspace $\mathcal{H}^{\mathbf{Z}}$ it spans in $L_d^2(\Omega)$; hence, we will not include in the WM processes the independent sequences of variables, which would not be basic. We will also consider the subspaces $\mathcal{H}_{k,l}^{\mathbf{Z}} = \text{Sp}\{Z(n) : k \leq n \leq l\}$ of $\mathcal{H}^{\mathbf{Z}}$ spanned by sub-families, for $k, l \in \overline{\mathbb{Z}}$, and denote by $\widehat{\mathbb{E}}[\cdot | Z(m), k \leq m \leq l]$ the projection on $\mathcal{H}_{k,l}^{\mathbf{Z}}$. A random variable V is orthogonal to $\mathcal{H}_{k,l}^{\mathbf{Z}}$ if and only if some invertible matrix M exists such that $\mathbb{E}[VZ(n)^*M] = 0$ for all $k \leq n \leq l$; see Beutler (1963).

A d -variate (with $d \geq 1$) zero mean second order stochastic process $\mathbf{Z} = (Z(n))_{n \in \mathbb{Z}}$ is a WM process (see Doob 1953) if

$$\widehat{\mathbb{E}}[(Z(n_k)) | Z(m), n_1 \leq m \leq n_{k-1}] = \widehat{\mathbb{E}}[Z(n_k) | Z(n_{k-1})], \quad n_1 < \dots < n_k, \quad (1)$$

with probability one, or equivalently if

$$\widehat{\mathbb{E}}[Z(n) | Z(m), m \leq m_0] = \widehat{\mathbb{E}}[Z(n) | Z(m_0)], \quad m_0 \leq n. \quad (2)$$

Since we are interested in the second order properties of the processes, we will omit the phrase "with probability one".

Finally, a scalar second order process \mathbf{Y} is said to be periodically correlated (PC) if there exists an integer $d \geq 1$ such that $\mathbb{E}Y(n) = \mathbb{E}Y(n+d)$ and $\mathbb{E}[Y(m)Y(n)] = \mathbb{E}[Y(m+d)Y(n+d)]$ for $m, n \in \mathbb{Z}$. Note that \mathbf{Y} is stationary if and only if $d = 1$. A one-to-one relationship is established between the class of scalar non stationary PC processes \mathbf{Y} and the class of MS processes \mathbf{Z} by setting $Z_k(n) = Y(k+dn)$ for the k -th component of the d -variate process \mathbf{Z} ; see Gladyshev (1961).

2 Second order characterizations

Let us investigate the covariance, correlation and reflection coefficients sequences of general WM processes and then of PCWM and MSWM processes. Some precise definitions of these second order notions are necessary before stating the results.

The covariance function R of a d -variate process \mathbf{Z} is well-known to be a positive definite matrix-valued function of two variables; the covariance matrices $R(m, n)$ for $m, n \in \mathbb{Z}$, are defined by $R(m, n)_{kl} = \mathbb{E}[Z_k(m)Z_l(n)]$, for $0 \leq k, l \leq d-1$. For basic processes, these coefficients are never null. The process is stationary (in the weak sense) if $R(m, n) = \mathcal{R}(n-m)$, for $m, n \in \mathbb{Z}$, for a positive definite matrix valued-function \mathcal{R} of one variable. A scalar process is WM if and only if its covariance matrices satisfy the functional equation

$$R(n, n)R(m, u) = R(m, n)R(n, u), \quad m \leq n \leq u \in \mathbb{Z}, \quad (3)$$

as proven by Doob (1953); we will see at the end of this section using reflection coefficients that this triangular characterization does not hold for MSWM processes with $d > 1$.

The correlation function ρ of the process is defined by

$$\rho(m, n) = R(m, n)R(m, m)^{-1}, \quad m, n \in \mathbb{Z}. \quad (4)$$

It is matrix-valued if $d > 1$. If the process is stationary, then $\rho(m, n) = \mathcal{R}(n-m)\mathcal{R}(0)^{-1}$ for $m, n \in \mathbb{Z}$. By properties of projection, we have $\widehat{\mathbb{E}}(Z(n) | Z(m)) = \rho(m, n)Z(m)$ for $m < n$. Therefore, (1) is equivalent to

$$\widehat{\mathbb{E}}[Z(u) | Z(m), Z(n)] = \rho(n, u)Z(n), \quad m \leq n \leq u, \quad (5)$$

a characterization that will be of use several times in the following.

For a scalar stationary process \mathbf{Y} , the reflection coefficients are defined as the partial autocorrelations, that is as the correlations that remain between $Y(k)$ and $Y(l)$ after eliminating the linear effect of intermediate variables (see Brockwell and Davis 1996). For an MS process, this definition involves a square root of a matrix, which is not uniquely defined, hence some generalization of the notion has to be considered. In the probabilistic approach developed in Castro and Girardin (2002), each coefficient is replaced by a triangular array of pairs of coefficients linked to the right and left prediction errors. Specifically, for a scalar non stationary process \mathbf{Y} , the one-step forward residual given $\mathcal{H}_{k+1, l}^Y$ and the one-step backward residual given $\mathcal{H}_{k, l-1}^Y$ are given by

$$P_{k, l} = Y(k) - \widehat{\mathbb{E}}[Y(k) | Y(m), k+1 \leq m \leq l], \quad (6)$$

$$Q_{k, l} = Y(l) - \widehat{\mathbb{E}}[Y(l) | Y(m), k \leq m \leq l-1], \quad (7)$$

for all $k < l$, and $P_{k, k} = Q_{k, k} = Y(k)$ for all $k \in \mathbb{Z}$. Note that both $(P_{i, l})_{k \leq i \leq l}$ and $(Q_{k, i})_{k \leq i \leq l}$ are basis of $\mathcal{H}_{k, l}^Y$. Due to Szegő's theorem, $Q_{k, l}$ tends to the innovation of the process $\widehat{\mathbb{E}}[Y(l) | Y(m), m < l]$ when k tends to infinity (see Helson and Lowdenslager 1958). The (generalized) reflection coefficients $r_{k, l}$ and $r'_{k, l}$ of \mathbf{Y} are thus defined as the ordered pair of complex numbers $(r_{k, l}, r'_{k, l})$ such that

$$P_{k, l} = P_{k, l-1} - r_{k, l}Q_{k+1, l} \quad \text{and} \quad Q_{k, l} = Q_{k+1, l} - r'_{k, l}P_{k, l-1}, \quad 0 \leq k < l; \quad (8)$$

see Castro and Girardin 2002 for the proof of existence. The above definition extends to MS processes through their duality with PC processes. Before specializing to PC and MS processes, let us characterize the second order structure of a general scalar WM process. For any scalar process \mathbf{Y} , this structure is determined by one of the sequence of arrays $(r_{k,l})$ or $(r'_{k,l})$ and the sequence of variances $R(k,k)$, for $k, l \in \mathbb{N}$.

Proposition 1 *The only non zero reflection coefficients of a scalar WM process are equal to its correlations of order one.*

Proof If \mathbf{Y} is a scalar WM process then (1) holds, clearly equivalent to $Q_{k,l} = Q_{l-1,l}$ for $k < l$. Therefore, only the reflection coefficients $r'_{l-1,l}$ are not null. Moreover, (1) and the right-hand formula in (8) jointly imply that

$$Q_{k,l} = Q_{l-1,l} = Y(l) - r'_{l-1,l}Y(l-1). \quad (9)$$

Through (7), we get $\mathbb{E}[Q_{k,l}Y(k)] = 0$; multiplying both sides of (9) by $Y(k)$ and taking expectations yields $R(k,l) = r'_{l-1,l}R(l,l-1)$, for $k < l$, that is $r'_{l-1,l} = \rho(l-1,l)$. \square

According to Proposition 1, the second-order structure of a scalar WM process is completely determined by $(\rho(k,k+1))$ and $(R(k,k))$ for $k \in \mathbb{N}$. A PCWM process \mathbf{Y} with period d also satisfies $r'_{k,k+1} = r'_{k+d,k+d+1}$ and $R(k,k) = R(k+d,k+d)$ for $k \in \mathbb{N}$. Therefore, its second order properties are determined by $\rho(k,k+1)$ and $R(k,k)$ for $0 \leq k \leq d-1$, that is to say by $2d$ parameters.

MSWM processes dual to PCWM processes can be characterized by their covariance functions. Nematollahi and Soltani (2000) have shown that the covariance matrices of any MSWM process dual to a PCWM process are given by

$$R(k+dn, l) = \tilde{g}(d-1)^n \tilde{g}(k-1) [\tilde{g}(l-1)]^{-1} R(l, l), \quad k, l, n \in \mathbb{Z}, \quad (10)$$

where $\tilde{g}(-1) = 1$ and $\tilde{g}(j) = \prod_{k=0}^j \rho(k, k+1)$ for $j \in \mathbb{N}$. This can be completed in the following way.

Theorem 1 *There is a one-to-one correspondence between the class of PCWM processes and the class of MSWM processes such that*

$$\mathcal{R}(n) = c^n A.B', \quad (11)$$

for the constant $c = \tilde{g}(d-1)$ and column vectors $A = (a_i)$ and $B = (b_i)$ defined by $a_i = \tilde{g}(i-1)$ and $b_i = [\tilde{g}(i-1)]^{-1} R(i, i)$ for $0 \leq i \leq d-1$.

Proof Let \mathbf{Y} be a PCWM process with MS dual \mathbf{Z} . If \mathcal{R} denotes the matrix-valued covariance function of \mathbf{Z} and R the covariance function of \mathbf{Y} , then $\mathcal{R}(n)_{kl} = R(k+dn, l)$ for $m, n \in \mathbb{Z}$ and $0 \leq k, l \leq d-1$. Thus (10) clearly yields (11), and hence

$$\mathcal{R}(n+m)\mathcal{R}(0) = \tilde{g}(d-1)^{n+m} A.B'.A.B' = \tilde{g}(d-1)^n A.B'.\tilde{g}(d-1)^m A.B' = \mathcal{R}(n)\mathcal{R}(m).$$

Due to the triangular characterization (3), this proves that \mathbf{Z} is WM.

Conversely let us suppose that \mathbf{Z} is an MS process with covariance matrices satisfying (11). As above, \mathbf{Z} is clearly WM. It remains to prove that its dual PC process \mathbf{Y} is WM too. Due to (11), we have $R(k+dn, l) = c^n a_k b_l$, for $0 \leq k, l \leq d-1$ and $n \in \mathbb{Z}$.

Let $s = k + dn$, $t = j + dm$ and $u = l + dr$. Since \mathbf{Y} is PC, we compute on the one hand

$$R(t, t)R(s, u) = R(j, j)R(k + d(n - r), l) = a_j b_j c^{n-r} a_k b_l,$$

and on the other hand,

$$R(s, t)R(t, u) = R(k + d(n - m), j)R(j + d(m - r), l) = c^{n-m} a_k b_j c^{m-r} a_j b_l.$$

Due to (3), \mathbf{Y} is PCWM. The direct proof gives the expressions of c , A and B in terms of \tilde{g} and R . \square

In the following, we will refer to the MSWM processes dual to PCWM processes (that is satisfying (11)) as to MSD processes. For MSD processes, we get from (11) that \mathcal{R} is fully specified by the matrices $R(k, k)$ and $R(k + d, k)$ for $0 \leq k \leq d - 1$.

The following characterization of WM processes in terms of correlations, due to Doob (1953) for the univariate processes, has been extended to multivariate processes by Beutler (1963) for the continuous time case. As we show here for sake of completeness, the same proof applies to the discrete time case.

Proposition 2 *A second order process \mathbf{Z} with non-singular variance matrices $R(n, n)$ is WM if and only if its correlation matrices satisfy the functional equations*

$$\rho(m, u) = \rho(m, n)\rho(n, u), \quad m \leq n \leq u \in \mathbb{Z}. \quad (12)$$

Proof If \mathbf{Z} is a WM process, then (5) holds. Hence, by definition of projection, $Z(u) - \rho(n, u)Z(n)$ is orthogonal to $Z(m)$, implying $\mathbb{E}([Z(u) - \rho(n, u)Z(n)]Z(m)^*) = 0$, and (12) follows.

Conversely, if (12) holds true, then

$$\begin{aligned} \mathbb{E}([Z(u) - \rho(n, u)Z(n)]\rho(m, m)^{-1}Z(m)) &= \\ &= \rho(m, m)^{-1}\mathbb{E}[Z(u)Z(m)^*] - \rho(n, u)\rho(m, m)^{-1}\mathbb{E}[Z(n)Z(m)^*] = 0. \end{aligned}$$

This relation also holds true for $m = n$. Thus we get (5) that characterizes WM processes. \square

Using in (12) the definition of correlation coefficients (4), we get

$$R(m, u)R(m, m)^{-1} = R(m, n)R(m, m)^{-1}R(n, u)R(n, n)^{-1}. \quad (13)$$

For a scalar process, (13) yields the triangular characterization (3) of the covariances. On the contrary, for an MS process, (13) proves that this triangular characterization does not hold true, but nevertheless yields $\mathcal{R}(n) = [\mathcal{R}(1)\mathcal{R}(0)^{-1}]^n \mathcal{R}(0) = \mathcal{R}(-n)$ for $n \in \mathbb{Z}$. Therefore, the second order structure of an MSWM process is completely determined by $\mathcal{R}(0)$ and $\mathcal{R}(1)$, that is to say by $2d^2$ parameters. Moreover, it implies that its correlation matrices can be written as

$$\rho(n) = \rho(1)^n, \quad n \in \mathbb{Z}, \quad (14)$$

a relation known to be satisfied by multivariate autoregressive processes of order 1, but which does not characterize them.

For a scalar stationary process, $R(m, n) = \mathcal{R}(n - m)$, and hence taking $m = m$, $n = v + m$ and $u = w + n$ in (3) yields the simple characterization $\mathcal{R}(0)\mathcal{R}(v + w) = \mathcal{R}(v)\mathcal{R}(w)$ for $v, w \in \mathbb{Z}$. Therefore, the second order structure of a scalar stationary WM process is completely determined by the values of $\mathcal{R}(0)$ and $\mathcal{R}(1)$, that is to say by two parameters. Moreover, it follows that the correlation coefficients of the process can be written as in (14). This relation is known to be characteristic of scalar autoregressive processes of order one (see Doob 1953 and Priestley 1981 for details). We will come back to this in Section 4.

3 Characterization in terms of wide martingales

Let us characterize the WM processes in terms of wide sense martingales.

A d -variate (with $d \geq 1$) second order process $Z = (Z(n))_{n \in \mathbb{Z}}$ is a wide sense martingale (see Doob 1953) if

$$\widehat{\mathbb{E}}[Z(n_k)|Z(m), n_1 \leq m \leq n_{k-1}] = Z(n_{k-1}), \quad n_1 < \dots < n_k \in \mathbb{Z},$$

(with probability 1) or equivalently if $\widehat{\mathbb{E}}[Z(n)|Z(m), m \leq m_0] = Z(m_0)$ for $m_0 < n \in \mathbb{Z}$.

The following result has been proven in the multivariate continuous time case by Mandrekar (1968). As we show here for the sake of completeness, the same proof applies to the discrete time case.

Theorem 2 *Any WM process \mathbf{Z} with non-singular covariance matrices $R(m, n)$ is the linear transform of a d -variate wide sense martingale \mathbf{M} such that $\mathcal{H}_m^{\mathbf{Z}} = \mathcal{H}_m^{\mathbf{M}}$ for $m \in \mathbb{Z}$.*

Proof If \mathbf{Z} is a WM process, then (5) holds. Let us set

$$\psi(n) = \begin{cases} \rho(m_0, n) & \text{if } m_0 \leq n, \\ \rho(n, m_0)^{-1} & \text{if } n < m_0, \end{cases}$$

for some fixed number m_0 . Clearly, $\rho(m, n) = \psi(n)\psi(m)^{-1}$, and since the matrix $\rho(m, n)$ is non-singular, $\psi(n)$ is a non-singular matrix too. Setting $M(n) = \psi^{-1}(n)Z(n)$ for $n \in \mathbb{Z}$ defines a wide sense martingale, such that $\mathcal{H}_n^{\mathbf{Z}} = \mathcal{H}_n^{\mathbf{M}}$.

Conversely, if \mathbf{Z} is defined by $Z(n) = \psi(n)M(n)$, where $\psi(n)$ is a non-singular $d \times d$ matrix and \mathbf{M} is a d -variate wide sense martingale, then

$$\widehat{\mathbb{E}}[Z(n)|Z(u), u \leq m] = \widehat{\mathbb{E}}[\psi(n)M(n)|\psi(u)M(u), u \leq m] = \psi(n)M(m) = \psi(n)\psi^{-1}(m)Z(m),$$

for $m < n \in \mathbb{Z}$. Therefore, \mathbf{Z} is WM, with correlation function $\rho(m, n) = \psi(n)\psi(m)^{-1}$. \square

Nevertheless, in the discrete case, contrary to the continuous one, it cannot be derived from the above result WM processes are autoregressive processes. We will prove this below in Section 4.

4 Characterization in terms of autoregressive models

An MS process \mathbf{Z} is an autoregressive process (MAR(N)) if it has a representation

$$\sum_{k=0}^N A(k)Z(n-k) = \epsilon(n), \quad n \in \mathbb{Z}, \quad (15)$$

where the coefficients $A(k)$ are $d \times d$ matrices, $A(0)$ is a unit lower triangular matrix and ϵ is a multivariate white noise process with diagonal covariance matrix Σ_ϵ . Note that this representation is equivalent to the more usual one (but less convenient for duality with PC processes) $\sum_{j=0}^N \tilde{A}(j)Z(n-j) = \tilde{\epsilon}(n)$ for $n \in \mathbb{Z}$, where $\tilde{A}(0)$ is the identity matrix, $\tilde{A}(j) = A(0)^{-1}A(j)$, and $\Sigma_{\tilde{\epsilon}} = A(0)^{-1}\Sigma_\epsilon[A(0)^{-1}]^*$ is the modified Cholevski decomposition of the covariance matrix $\Sigma_{\tilde{\epsilon}}$ (see Pagano 1978).

A PC process \mathbf{Y} with period d is a periodic autoregressive process (PAR) if it has a representation

$$Y(n) + \sum_{j=1}^{N_n} \alpha_n(j)Y(n-j) = \varepsilon(n), \quad n \in \mathbb{Z},$$

where $N_n = N_{n+d}$, $\alpha_n(j) = \alpha_{n+d}(j)$ and ε is a white noise process with periodic variance $\sigma_n^2 = \sigma_{n+d}^2$, for $n \in \mathbb{Z}$. Since this can be written

$$Y(k+ld) + \sum_{j=1}^{N_k} a_k(j)Y(k+d(l-j)) = \varepsilon(k+ld), \quad l \in \mathbb{Z}, \quad k = 0, \dots, d-1, \quad (16)$$

obviously related to (15), it follows that \mathbf{Y} is a PAR($d, (N_1, \dots, N_d)$) if and only if its dual \mathbf{Z} is a MAR(N) with $N = \max_k [(N_k - k)/d] + 1$, where $[\cdot]$ denotes the integer part of a real number (see Pagano 1978).

The following two structural characterizations of PCWM and MSWM processes in terms of autoregressive models are essential and constitute the main result of the paper. To our knowledge, only two results exist in the literature in this direction. The first one is the result by Mandrekar (1968) cited in Section 3, and the second is a result by Beutler (1963) stating that any (stationary or not) continuous time process whose covariance function has an exponential form is a WM process, so that any continuous time MAR(1) is a WM process. Neither of their proofs apply to the discrete time.

Theorem 3 1. *The class of MSWM processes is exactly the class of stationary MAR(1) processes. They are dual with the PAR($d, (n_0, \dots, n_{d-1})$) processes such that $1 \leq n_i \leq 2d - i$.*

2. *The class of PCWM processes is exactly the class of PAR($d, (1, \dots, 1)$) processes. The MSD processes are the MAR(1) processes \mathbf{Z} with representation*

$$A(0)Z(n) + A(1)Z(n-1) = \epsilon(n), \quad n \in \mathbb{Z}, \quad (17)$$

where: ϵ is a white noise with diagonal covariance matrix Σ ; $A(0)$ is a unit upper triangular matrix with only $2d - 1$ non zero coefficients, namely $A(0)_{l,l} = 1$ for $0 \leq l \leq d - 1$ and $A(0)_{l,l-1} = -\rho_{\mathbf{Y}}(l-1, l)$ for $1 \leq l \leq d - 1$; $A(1)$ has a unique non zero entry $A(1)_{0,d-1} = -\rho_{\mathbf{Y}}(d-1, d)$.

Proof 1. Let \mathbf{Z} be an MSWM process. From (2) and definition of innovation (see the remarks following (7)), we deduce that $Z(n) - \rho_{\mathbf{Z}}(n-1, n)Z(n-1) = \epsilon(n)$ for $n \in \mathbb{Z}$, where ϵ is the innovation of the process. Since the innovation is known to be a white noise process, this yields (17). Note that since \mathbf{Z} is stationary, $\rho_{\mathbf{Z}}(n-1, n)$ is constant in n .

The one-to-one correspondence is given by the remark following (16).

2. Clearly, a $\text{PAR}(d, (1, \dots, 1))$ process is a WM process. Conversely, let \mathbf{Y} be a PCWM process. By definition, $Q_{k,l+nd}$ tends to the innovation $\varepsilon(l+nd)$ of \mathbf{Y} when k tends to infinity. Moreover, by Proposition 1, $r'_{l+nd-1,l+nd} = \rho_{\mathbf{Y}}(l+nd-1, l+nd)$. Therefore, (9) can be written

$$Y(l+nd) - \rho_{\mathbf{Y}}(l+nd-1, l+nd)Y(l+nd-1) = \varepsilon(l+nd), \quad (18)$$

for $n \in \mathbb{Z}$, and $0 \leq l \leq d-1$. Since $\varepsilon(l+nd)$ is a white noise process, this means that \mathbf{Y} is a $\text{PAR}(d, (1, \dots, 1))$ process.

Finally, let \mathbf{Z} be an MSD process. Then its dual process \mathbf{Y} satisfies (18). Using the remark following (15) yields (17). \square

Using (9), the coefficients of the covariance matrix Σ of the white noise ε in (17) can explicitly be computed as follows.

$$\begin{aligned} \Sigma_{ll} &= R_{\mathbf{Y}}(l, l) - 2\rho_{\mathbf{Y}}(l-1, l)R_{\mathbf{Y}}(l-1, l) + (\rho_{\mathbf{Y}}(l-1, l))^2 R_{\mathbf{Y}}(l-1, l-1) \\ &= R_{\mathbf{Y}}(l, l) - R_{\mathbf{Y}}(l-1, l)^2 / R_{\mathbf{Y}}(l-1, l-1), \quad n \in \mathbb{Z}, 1 \leq l \leq d-1, \\ \Sigma_{00} &= R_{\mathbf{Y}}(d-1, d-1) - R_{\mathbf{Y}}(d-1, d)^2 / R_{\mathbf{Y}}(d-1, d-1). \end{aligned}$$

Therefore, we find again that the second-order structure of MSD processes is fully determined by $2d$ parameters.

Example 1 The $\text{MAR}(1)$ process \mathbf{Z} with representation

$$\begin{pmatrix} 1 & 0 \\ 0.7239 & 1 \end{pmatrix} \begin{pmatrix} Z_0(n) \\ Z_1(n) \end{pmatrix} + \begin{pmatrix} -0.4455 & 0.1464 \\ 0.5279 & -0.6333 \end{pmatrix} \begin{pmatrix} Z_0(n-1) \\ Z_1(n-1) \end{pmatrix} = \begin{pmatrix} \varepsilon_0(n) \\ \varepsilon_1(n) \end{pmatrix}$$

is dual to the $\text{PAR}(2, (2, 3))$ process \mathbf{Y} , with representation

$$\begin{cases} Y(2n) + 0.1464Y(2n-1) - 0.4455Y(2n-2) = \varepsilon_0(n), \\ Y(2n+1) + 0.7239Y(2n) - 0.6333Y(2n-1) + 0.5279Y(2n-2) = \varepsilon_1(n). \end{cases}$$

The MS process \mathbf{Z} is a WM process, but its dual PC process \mathbf{Y} is not a WM process. \triangleleft

Example 2 The $\text{MAR}(1)$ process \mathbf{Z} with representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.1 & 1 & 0 & 0 \\ 0 & -0.2 & 1 & 0 \\ 0 & 0 & -0.3 & 1 \end{pmatrix} Z(n) + \begin{pmatrix} 0 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(n-1) = \begin{pmatrix} \varepsilon(4n) \\ \varepsilon(4n+1) \\ \varepsilon(4n+2) \\ \varepsilon(4n+3) \end{pmatrix}$$

is an MSD process dual to the $\text{PAR}(2, (1, 1, 1, 1))$ process \mathbf{Y} with representation

$$\begin{cases} Y(4n) - 0.4Y(4n-1) = \varepsilon(4n) \\ Y(4n+1) - 0.1Y(4n) = \varepsilon(4n+1) \\ Y(4n+2) - 0.2Y(4n+1) = \varepsilon(4n+2) \\ Y(4n+3) - 0.3Y(4n+2) = \varepsilon(4n+3). \end{cases}$$

The MS process \mathbf{Z} is a WM process, and its dual PC process \mathbf{Y} is a WM process too. \triangleleft

Note that Lai (1978) considered discrete time strict Markov processes with finite state spaces. Using the spectral density, he obtained conditions in terms of the transition matrix for the process to have an autoregressive representation, i.e., to be a WM process. These conditions, involving orthonormal functions and polynomials and marginal distributions, are too complicated to be given here.

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