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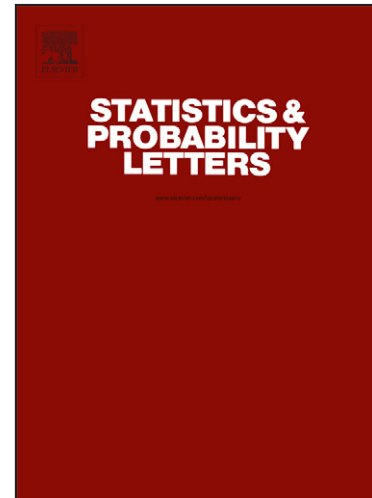
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On the statistical properties of a stationary process sampled by a stationary point process

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Abstract

Let $X = \{X_t\}_{t \in \mathcal{T}}$, where $\mathcal{T} = \mathbb{R}$ or \mathbb{Z} , be a strictly stationary process, which is assumed to be strongly mixing. In this paper, we are concerned with the stationarity and the mixing properties of the process obtained from X by a random sampling, that is, $\{X_{t_n}\}_{n \in \mathbb{Z}}$, where $\{t_n\}_{n \in \mathbb{Z}}$ is a real point process. This study is done for φ , β , ρ and α -mixing processes.

Key words: Mixing properties; Palm's probability; Point processes; Stationarity

AMS 1991 Subject Classifications: Primary 60G10, 60G55, Secondary 62M15, 62G20

1 Introduction

The efficiency of continuous time processes in modeling several real phenomena was established a long time ago. These processes are used in communication problems (Cf Davenport and Root (1958) for linear filtering and prediction theory), in seismology (Cf Carpenter (1967) for determining the nature of a seismic event), in oceanography (Cf Hasselmann et al. (1963) for oceanic waves study), in physics (Cf MacDonald and Ness (1961) concerning the fundamental earth vibration periods) and in several fields of medicine.

For continuous time phenomena, data are often collected by using a sampling

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scheme. In order to make this idea clear and to show the crucial helpfulness of the results of this work in many purposes, we look more closely at the interesting problem of functional estimation for continuous time processes (Priestley (1981), Bosq (1998)). Let $X = \{X_t\}_{t \in \mathcal{T}}$ be a real continuous-time stochastic process ($\mathcal{T} \subset \mathbb{R}$), which is strictly stationary and strongly mixing (*see* section 2). It is known that, in order to estimate consistently the spectral density function of X from the observations $\{X_t, t \in (0, T)\}$ with $T > 0$, one has to “smooth” the associated periodogram. This requires to compute numerically the finite integral $\int_0^T X_t \exp(-i\lambda t) dt$. However, in practical applications the observations of X are not obtained in an analytical form. Thus, one could not get a useful estimate of the spectral density function of X (Cf Brillinger (1972 and 1981) and Priestley (1981)). To overcome this difficulty, we observe the process X at instants $\{t_n\}_{n \in \mathbb{Z}}$, which are to be determined. In spectral estimation theory, in order to avoid the aliasing phenomenon to occur, one has to use a random sampling (Cf Shapiro and Silverman (1960)). Therefore, we introduce discrete time process $\tilde{X} = \{X_{t_n}\}_{n \in \mathbb{Z}}$, where $\{t_n\}_{n \in \mathbb{Z}}$ is a sequence of real numbers which is independent of X . Moreover, in order to do some statistical studies on X , we have to know how the statistical properties may be transmitted from X to \tilde{X} .

Some nice investigations, among others, were done on stationary processes, point processes and on the mixing properties along with some examples by Neveu (1976), Yoshihara (1992), Doukhan (1994), Politis and Romano (1999) and the references therein. Concerning the association between stationary processes and point processes, the nearest work to our is the investigation done by Rolski (1984) to establish the ergodicity of the sampled process, and by Resnick and Samorodnitsky (2004) for α -stable processes, however, these works do not answer the questions posed here.

The estimation of the spectral density function for continuous time process from sampling design was studied by Masry(1978), Lii and Masry (1994) and Rachdi (1999, 2003 and 2004).

The main goal of this paper is to study how to transmit the stationarity and the mixing properties from X to the sampled process \tilde{X} . Afterward, one of the immediate applications of these results, is to obtain the almost complete convergence of the spectral density estimates (Cf Rachdi (2004)).

This paper is organized as follows. In section 2, some preliminaries on the mixing properties and some notations are presented. Then, in section 3, the stationary point processes are introduced and the transmission of the stationarity property from X to \tilde{X} is established. Section 4 is devoted to the establishment of the mixing properties of \tilde{X} . The proofs of all the results are presented in section 5.

2 Preliminaries

Let \mathcal{T} be one of the following sets: the set of real numbers \mathbb{R} or \mathbb{R}_+ or the set of relative integers \mathbb{Z} . Let $X = \{X_t\}_{t \in \mathcal{T}}$ be a strictly stationary process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in some measurable space (E, \mathcal{E}) . Let $\mathcal{X} = \sigma\{X_s, s \in \mathcal{T}\}$ be the σ -algebra generated by X , and for all $t \in \mathcal{T}$,

$$\mathcal{X}_t = \sigma\{X_s, s \leq t\} \text{ and } \mathcal{Y}_t = \sigma\{X_s, s > t\}.$$

In order to estimate the correlation between \mathcal{X}_t and \mathcal{Y}_t various coefficients are used (Cf Bosq (1998)):

$$\begin{aligned} \alpha &= \alpha(\mathcal{X}_t, \mathcal{Y}_t) = \sup_{\substack{X \in \mathcal{X}_t \\ Y \in \mathcal{Y}_t}} |\mathbf{P}(X \cap Y) - \mathbf{P}(X)\mathbf{P}(Y)|, \\ \beta &= \beta(\mathcal{X}_t, \mathcal{Y}_t) = \mathbb{E} \sup_{Y \in \mathcal{Y}_t} |\mathbf{P}(Y) - \mathbf{P}(Y/\mathcal{X}_t)|, \\ \varphi &= \varphi(\mathcal{X}_t, \mathcal{Y}_t) = \sup_{\substack{X \in \mathcal{X}_t, \mathbf{P}(X) > 0 \\ Y \in \mathcal{Y}_t}} |\mathbf{P}(Y) - \mathbf{P}(Y/X)|, \\ \rho &= \rho(\mathcal{X}_t, \mathcal{Y}_t) = \sup_{\substack{X \in L^2(\mathcal{X}_t) \\ Y \in L^2(\mathcal{Y}_t)}} |\text{cor}(X, Y)| \end{aligned}$$

where $\text{cor}(X, Y)$ denotes the correlation coefficient between X and Y . These coefficients satisfy: $2\alpha \leq \beta \leq \varphi$ and $4\alpha \leq \rho \leq 2\sqrt{\varphi}$. Now, the process X is said to be α -mixing (or strongly mixing) if

$$\alpha(s) = \sup_{t \in \mathcal{T}} \alpha(\mathcal{X}_t, \mathcal{Y}_{t+s}) \rightarrow 0, \text{ when } s \rightarrow +\infty$$

where the *sup* may be omitted if X is stationary. Similarly one defines the β -mixing (or absolute regularity), the φ -mixing and the ρ -mixing. For other formulations of the definitions of α , β , φ and ρ -mixing processes with some specific examples readers can see the books by Doukhan (1994) and Yoshihara (1992).

Let $\Xi = E^{\mathbb{R}}$. For $\xi \in \Xi$ and $s \in \mathbb{R}$, set $X_s(\xi) = \xi(s)$. We denote by ψ_t the shift operator defined by: $X_s \circ \psi_t = X_{s+t}$ for $(s, t) \in \mathcal{T}^2$.

Let \mathbf{P}_1 be a probability measure on (Ξ, \mathcal{X}) such that: $\forall t \in \mathcal{T}, \mathbf{P}_1 \circ \psi_t = \mathbf{P}_1$. The process $X = \{X_t\}_{t \in \mathcal{T}}$ is strictly stationary under \mathbf{P}_1 , that is,

$$\mathbb{E}_1(F(X_{t_1+h}, \dots, X_{t_n+h})) = \mathbb{E}_1(F(X_{t_1}, \dots, X_{t_n})), \forall (t_1, \dots, t_n, h) \in \mathcal{T}^{n+1} \quad (1)$$

for any $\mathcal{E}^{\otimes n}$ -measurable function $F : E^n \longrightarrow \mathbb{R}_+$.

3 Stationarity property of the sampled process

3.1 Stationary point processes

Let M be the set of \mathbb{N} -valued Radon measures m such that,

$$\forall t \in \mathcal{T}, m(\{t\}) \leq 1 \text{ and } m(-\infty, 0) = m(0, +\infty) = +\infty$$

and \mathcal{M} denotes the σ -algebra on M generated by the functions $m \mapsto m(B)$ where $B \in \mathcal{B}(\mathcal{T})$ the borelean σ -algebra.

For all $m \in M$, we write $m = \sum_{n \in \mathbb{Z}} \delta_{t_n(m)}$ where δ_x being the mass of a Dirac function at the point x , and with $\{t_n(m)\}_{n \in \mathbb{Z}}$ is a strictly increasing sequence of real numbers, such that

$$t_0(m) \leq 0 < t_1(m) \text{ and } \lim_{n \rightarrow \pm\infty} t_n(m) = \pm\infty$$

with $\forall n \in \mathbb{N}$, the maps $t_n : M \longrightarrow \mathcal{T}$ are measurable (Cf Neveu (1977)).

Consider the point process $N = \sum_{n=-\infty}^{+\infty} \delta_{t_n}$. Then N is a random Radon measure defined on \mathcal{T} , in the sense that, for all $B \in \mathcal{B}(\mathbb{R})$, $N(B)$ is an integer random variable, which counts the number of instants t_n belonging to B . The process N is then more specifically defined by the following formulation:

$$\forall m \in M, \forall B \in \mathcal{B}, N(m, B) = \sum_{n=-\infty}^{+\infty} \delta_{t_n(m)}(B)$$

For $\tau_t(m) = \sum_{n \in \mathbb{Z}} \delta_{t_n - t(m)}$, if $t_k(m) \leq t < t_{k+1}(m)$, if $t_k(m) \leq t < t_{k+1}(m)$, we have

$$t_n(\tau_t(m)) = t_{n+k}(m) - t,$$

with especially

$$t_n \circ \tau_{t_k} = t_{n+k} - t_k.$$

Let \mathbf{P}_2 be a probability measure on (M, \mathcal{M}) such that

$$\forall t \in \mathcal{T}, \mathbf{P}_2 \circ \tau_t = \mathbf{P}_2.$$

then, with the probability measure \mathbf{P}_2 , the process N is then a stationary point process. Moreover, under the probability \mathbf{P}_2 , there is no reason for the increment sequence $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ to be stationary. Therefore, we construct the Palm space associated to this stationary point process for which $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is stationary.

For this purpose, let

$$\widehat{M} = \{m \in M / m(\{0\}) = 0\} = \{m \in M / t_0(m) = 0\}.$$

For any integer n , the image of M by τ_{t_n} is \widehat{M} . But τ_{t_0} is the identity function on \widehat{M} and $\tau_{t_n} = (\tau_{t_1})^n$ for $n \in \mathbb{Z}$.

Hence,

$$\text{for } n \in \mathbb{Z}, t_{n+1} - t_n = t_1 \circ (\tau_{t_1})^n \text{ on } \widehat{M}.$$

Thus,

$$t_n = \begin{cases} \sum_{k=0}^{n-1} t_1 \circ (\tau_{t_1})^k, & \text{for } n \geq 0, \\ -\sum_{k=-1}^n t_1 \circ (\tau_{t_1})^k & \text{for } n \leq 0. \end{cases}$$

it follows that, there exists a probability measure $\widehat{\mathbf{P}}_2$ on \widehat{M} defined by the formula

$$\int_M \mathbf{P}_2(dm) \int_{\mathcal{T}} N(m, ds) f(\tau_s(m), s, m) = c \int_{\widehat{M}} \widehat{\mathbf{P}}_2(dm) \int_{\mathbb{R}} \lambda(ds) f(m, s, \tau_{-s}(m)).$$

where f is a positive and measurable function on $M \times \mathbb{R} \times M$ and λ is the Lebesgue measure on \mathbb{R} , with $c = 1/\widehat{\mathbb{E}}_2(t_1) = \mathbb{E}_2(N(0, 1)) < +\infty$.

The probability measure $\widehat{\mathbf{P}}_2$ is invariant by τ_{t_1} , that is:

$$\widehat{\mathbf{P}}_2 \circ \tau_{t_1} = \widehat{\mathbf{P}}_2$$

Thus the sequence $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is stationary under $\widehat{\mathbf{P}}_2$.

Conversely, if $Y = \{Y_n\}_{n \in \mathbb{Z}}$ is a strictly stationary sequence such that $\mathbf{P}(Y > 0) = 1$ and $\mathbb{E}(Y) < +\infty$, then, there exists a probability measure \mathbf{P}_2 on (M, \mathcal{M}) for which N is a stationary point process and such that, under $\widehat{\mathbf{P}}_2$, the processes $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ and $\{Y_n\}_{n \in \mathbb{Z}}$ are identically distributed.

The probability measure $\widehat{\mathbf{P}}_2$ is called the *Palm's probability* and the space $(\widehat{M}, \mathcal{M} \cap \widehat{M}, \widehat{\mathbf{P}}_2)$ is the *Palm's space* of the point process N (Cf Neveu (1977) and Yoshihara (1992)).

3.2 The fundamental space

In the following, we will denote

$$\Omega = \Xi \times M, \mathcal{F} = \mathcal{X} \otimes \mathcal{M}, \mathbf{P} = \mathbf{P}_1 \times \widehat{\mathbf{P}}_2, \theta_t(\omega) = \theta_t(\xi, m) = (\psi_t(\xi), \tau_t(m)).$$

The processes X and N defined on Ω by

$$\forall \omega = (\xi, m) \in \Omega, X(\omega) = X(\xi) = \{X_t(\xi)\}_{t \in \mathcal{T}} \text{ and } N(\omega, \cdot) = N(m, \cdot)$$

are obviously independent, where the abuse notation $X(\omega) = X(\xi)$ and $N(\omega, \cdot) = N(m, \cdot)$ means that X depends only on the first coordinate of ω whereas N depends only on the second coordinate of ω . Their distributions are respectively

\mathbf{P}_1 and $\widehat{\mathbf{P}}_2$. For all $t \in \mathcal{T}$ and for all Borelean set B , we have

$$X_t = X_0 \circ \theta_t \text{ and } N(\theta_t(\omega), B) = N(\omega, B + t).$$

For all $n \in \mathbb{N}$, set $t_n(\omega) = t_n(m)$. Then, we have,

$$\theta_{t_0} = Id_{\Xi \times \widehat{M}} \text{ on } \Xi \times \widehat{M}, \text{ and } \theta_{t_n} = (\theta_{t_1})^n, \text{ for all } n \in \mathbb{Z}.$$

Proposition 1 *The probability measure \mathbf{P} is invariant by θ_{t_1} and the process $\{X_{t_n}\}_{n \in \mathbb{Z}}$ is strictly stationary.*

4 Mixing property of the sampled process

To answer questions asked in the introduction, we establish and prove the following Theorem.

Theorem 2 *Let $X = \{X_t\}_{t \in \mathcal{T}}$ be a strictly stationary process, let $\{t_n\}_{n \in \mathbb{Z}}$ be a sequence of random variables independent of X and such that $t_0 = 0$, $\mathbf{P}(t_1 > 0) = 1$ and $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is a strictly stationary sequence.*

- (1) *If X is α -mixing and if $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is α' -mixing, then the process $\{X_{t_n}\}_{n \in \mathbb{Z}}$ is an α -mixing stationary process, with mixing coefficients $\{\mathbb{E}(\alpha(t_n)) + \alpha'_n\}_{n \in \mathbb{N}}$.*
- (2) *Let $\kappa \in \{\varphi, \rho, \beta\}$. If X is κ -mixing and if $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is a sequence of independent variables, then the process $\{X_{t_n}\}_{n \in \mathbb{Z}}$ is a κ -mixing stationary process, with mixing coefficients $\{\mathbb{E}(\kappa(t_n))\}_{n \in \mathbb{N}}$.*

To prove this Theorem, we will need the following Lemma.

Lemma 3

- (1) *Let $\kappa \in \{\alpha, \varphi\}$. The process X is κ -mixing if and only if there exists a non-increasing function $\kappa : \mathcal{T}_+ \rightarrow \mathbb{R}_+$ converging to 0, and such that, for any $s \in \mathcal{T}_+$, any \mathcal{X}_0 -measurable random variable U , $0 \leq U \leq 1$, and any \mathcal{Y}_s -measurable random variable V , $0 \leq V \leq 1$,*

$$|\mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)| \leq \kappa(s)\mathbb{E}(U^\epsilon),$$

where $\epsilon = 0$ if $\kappa = \alpha$ and $\epsilon = 1$ if $\kappa = \varphi$.

- (2) *The process X is β -mixing if and only if there exists a non-increasing function $\beta : \mathcal{T}_+ \rightarrow \mathbb{R}_+$ converging to 0, and such that, for any $s \in \mathcal{T}_+$, and a bounded positive \mathcal{X}_0 -measurable random function $\{\gamma_t\}_{t \in \mathbb{R}_+}$ such that $\mathbb{E}(\gamma_t) \leq \beta(t)$:*

$$|\mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)| \leq \mathbb{E}(\gamma_s U) \text{ for } s \in \mathcal{T}_+$$

and for any \mathcal{X}_0 -measurable random variable U , $0 \leq U \leq 1$, and any \mathcal{Y}_s -measurable random variable V , $0 \leq V \leq 1$.

5 Proofs

Proof of Proposition 1. As $X_{t_n} = X_0 \circ \theta_{t_n}$, to prove the strict stationarity of the stochastic process $\{X_{t_n}\}_{n \in \mathbb{Z}}$, it is enough to prove that the measure \mathbf{P} is invariant by θ_{t_1} , which is equivalent to show that, for any positive random variable defined on Ω , say Z , $\mathbb{E}(Z \circ \theta_{t_1}) = \mathbb{E}(Z)$. \square

Proof of Lemma 3. The proof of this Lemma becomes simpler, when one observes that

$$\mathbb{E}(U) = \int_0^{+\infty} \mathbf{P}(U > t) dt \text{ and that } \mathbb{E}(UV) = \int_0^{+\infty} \int_0^{+\infty} \mathbf{P}(U > t, V > t') dt dt'.$$

That is, if X is α -mixing, we can write

$$\begin{aligned} \left| \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V) \right| &= \left| \int_0^1 \int_0^1 \mathbf{P}(U > t, V > t') dt dt' - \int_0^1 \mathbf{P}(U > t) dt \int_0^1 \mathbf{P}(V > t') dt' \right| \\ &\leq \left| \int_0^1 \int_0^1 (\mathbf{P}(U > t, V > t') - \mathbf{P}(U > t)\mathbf{P}(V > t')) dt dt' \right| \\ &\leq \int_0^1 \int_0^1 \left| \mathbf{P}(U > t, V > t') - \mathbf{P}(U > t)\mathbf{P}(V > t') \right| dt dt' \\ &\leq \alpha(s) \end{aligned}$$

The proof for the φ -mixing case is similar to that for the α -mixing case. For the β -mixing case, one may consider $\gamma_t = \sup\{|\mathbf{P}(B/\mathcal{X}_0) - \mathbf{P}(B)| : B \in \mathcal{Y}_t\}$. \square

Proof of Theorem 2. Recall that

$$\mathcal{X}_0 = \sigma\{X_t, t \leq 0\} \text{ and } \mathcal{Y}_0 = \sigma\{X_t, t > 0\}.$$

Set

$$\mathcal{M}_0 = \sigma\{N(u, v), u < v \leq 0\} = \sigma\{t_k, k \leq 0\},$$

$$\mathcal{N}_0 = \sigma\{N(u, v), 0 < u < v\} = \sigma\{t_k, k \geq 1\}, \mathcal{F}_0 = \mathcal{X}_0 \vee \mathcal{M}_0 \text{ and } \mathcal{G}_0 = \mathcal{Y}_0 \vee \mathcal{N}_0.$$

Note that $\mathcal{M} = \mathcal{M}_0 \vee \mathcal{N}_0$ and $\mathcal{X} = \mathcal{X}_0 \vee \mathcal{Y}_0$ are independent. Denote,

$$\mathcal{A}_0 = \sigma\{X_{t_n}, n \leq 0\} \subseteq \mathcal{F}_0 \text{ and } \mathcal{B}_0 = \sigma\{X_{t_n}, n \geq 1\} \subseteq \mathcal{G}_0$$

and for $k \geq 0$, $\mathcal{B}_k = \sigma\{X_{t_{n+k}}, n \geq 1\}$.

If a random variable Z_k is \mathcal{B}_k -measurable, then there exists a \mathcal{G}_0 -measurable random variable V , such that $Z_k = V \circ \theta_{t_k}$. We denote

$$\mathcal{N}_k = (\tau_{t_k})^{-1}(\mathcal{N}_0) = \sigma\{t_{k+n} - t_k, n \geq 1\}$$

If $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is a sequence of independent random variables, then the σ -algebra \mathcal{M}_0 and \mathcal{N}_k are independent, for $k \geq 0$.

- (1) Assume that X is α -mixing and $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is α' -mixing. Let U be a \mathcal{F}_0 -measurable random variable such that $0 \leq U \leq 1$, and V is a \mathcal{G}_0 -measurable random variable with $0 \leq V \leq 1$. We have that

$$\begin{aligned} \mathbb{E}(U(V \circ \theta_{t_k})) &= \int_M \hat{\mathbf{P}}_2(dm) \int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) V(\psi_{t_k(m)}(\xi), \tau_{t_k(m)}(m)) \\ &\leq \int_M \hat{\mathbf{P}}_2(dm) \alpha(t_k(m)) \\ &\quad + \int_M \hat{\mathbf{P}}_2(dm) \left(\int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \int_{\Xi} \mathbf{P}_1(d\xi) V(\xi, \tau_{t_k(m)}(m)) \right) \\ &\leq \mathbb{E}(\alpha(t_k)) + \alpha'_k + \mathbb{E}(U)\mathbb{E}(V) \end{aligned}$$

Since the converse inequality may be proved analogously, we have thus the result for the α -mixing case by Lemma 3.

- (2) Here we show the result just for $\kappa = \varphi$ and ρ . The β -mixing case may be demonstrated by a similar formulation as for the φ -mixing.
- Assume that X is φ -mixing and $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is a sequence of independent random variables.

$$\begin{aligned} \mathbb{E}(U(V \circ \theta_{t_k})) &= \int_M \hat{\mathbf{P}}_2(dm) \int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) V(\psi_{t_k(m)}(\xi), \tau_{t_k(m)}(m)) \\ &\leq \int_M \hat{\mathbf{P}}_2(dm) \left(\varphi(t_k(m)) \int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \right) \\ &\quad + \int_M \hat{\mathbf{P}}_2(dm) \left(\int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \int_{\Xi} \mathbf{P}_1(d\xi) V(\xi, \tau_{t_k(m)}(m)) \right) \\ &= \mathbb{E}(\varphi(t_k))\mathbb{E}(U) + \mathbb{E}(U)\mathbb{E}(V), \end{aligned}$$

the last equality is justified in the first place by the independence of \mathcal{M}_0 and \mathcal{N}_0 and by the independence of \mathcal{M}_0 and \mathcal{N}_k in the second place. Thus, the result is obtained as for the α -mixing case.

- Assume that X is ρ -mixing and that $\{t_{n+1} - t_n\}_{n \in \mathbb{Z}}$ is a sequence of independent random variables. Let $U \in L^2(\mathcal{X}_0)$ and $V \in L^2(\mathcal{Y}_t)$. Since \mathcal{M}_0 and the σ -algebra generated by t_k are independent, from the independence of the σ -algebra generated by t_k and \mathcal{N}_k and the Schwartz inequality, we obtain that

$$\begin{aligned}
 & \mathbb{E}(U(V \circ \theta_{t_k})) \\
 &= \int_M \hat{\mathbf{P}}_2(dm) \int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) V(\psi_{t_k(m)}(\xi), \tau_{t_k(m)}(m)) \\
 &\leq \int_M \hat{\mathbf{P}}_2(dm) \left(\int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \int_{\Xi} \mathbf{P}_1(d\xi) V(\xi, \tau_{t_k(m)}(m)) \right. \\
 &\quad \left. + \rho(t_k(m)) \left(\int_{\Xi} \mathbf{P}_1(d\xi) (U(\xi, m))^2 - \left(\int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \right)^2 \right)^{1/2} \right. \\
 &\quad \left. \times \left(\int_{\Xi} \mathbf{P}_1(d\xi) (V(\xi, \tau_{t_k(m)}))^2 - \left(\int_{\Xi} \mathbf{P}_1(d\xi) V(\xi, \tau_{t_k(m)}) \right)^2 \right)^{1/2} \right) \\
 &\leq \mathbb{E}(U) \mathbb{E}(V) + \mathbb{E}(\rho(t_k)) \left(\left(\int_M \hat{\mathbf{P}}_2(dm) \left(\int_{\Xi} \mathbf{P}_1(d\xi) (U(\xi, m))^2 - \left(\int_{\Xi} \mathbf{P}_1(d\xi) U(\xi, m) \right)^2 \right) \right)^{1/2} \right. \\
 &\quad \left. \times \left(\int_M \hat{\mathbf{P}}_2(dm) \left(\int_{\Xi} \mathbf{P}_1(d\xi) (V(\xi, \tau_{t_k(m)}))^2 - \left(\int_{\Xi} \mathbf{P}_1(d\xi) V(\xi, \tau_{t_k(m)}) \right)^2 \right) \right)^{1/2} \right) \\
 &\leq \mathbb{E}(U) \mathbb{E}(V) + \mathbb{E}(\rho(t_k)) \sigma_U \sigma_V.
 \end{aligned}$$

Thus the result of ρ -mixing is obtained, which completes the proof of the theorem. \square

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