HAL open science

Sharp ill-posedness results for the KdV and mKdV equations on the torus

Luc Molinet

▶ To cite this version:

Luc Molinet. Sharp ill-posedness results for the KdV and mKdV equations on the torus. 2011. hal-00593976v2

HAL Id: hal-00593976

https://hal.science/hal-00593976v2

Preprint submitted on 3 Jun 2011 (v2), last revised 20 Jul 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sharp ill-posedness results for the KdV and mKdV equations on the torus

Luc Molinet

L.M.P.T., Université François Rabelais Tours, Fédération Denis Poisson-CNRS, Parc Grandmont, 37200 Tours, France

June 3, 2011

AMS Subject Classification : 35Q53, 35A01, 35D30.

Abstract

We establish a new a priori bound for L^2-bounded sequences of solutions to the mKdV equations on the torus. This first enables us to construct weak solutions in L^2 for this equation and to check that the "solutions" constructed by Kappeler and Topalov in the defocusing case satisfy the equation in some weak sense. In a second time, we prove that the solution-map associated with the mKdV and the KdV equation are discontinuous for the $H^s(\mathbb{T})$ topology for respectively $s < 0$ and $s < -1$. These last results are sharp.

1 Introduction

In this paper we study different properties of the Cauchy problems posed on the flat torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ associated with the Korteweg-de Vries (KdV) equation

$$w_t + w_{xxx} - 6ww_x = 0 \tag{1.1}$$

and the modified Korteweg-de Vries (mKdV) equation

$$v_t + v_{xxx} \mp 6v^2 v_x = 0 \tag{1.2}$$

Here, w and v are real-valued functions on $\mathbb{T}$. For some results we will have to distinguish between two mKdV equations depending on the sign in front of the nonlinear term. (1.2) is called the defocussing mKdV equation when there is a minus sign in front of the nonlinear term and the focussing mKdV

equation when it is a plus sign. The Cauchy problem associated with these equations in space of rough functions on the torus has been extensively studied these last two decades. In a seminal paper [1], Bourgain proved that the Cauchy problem associated with the KdV equation is globally well-posed in $H^s(\mathbb{T})$, $s \geq 0$, whereas the one associated with the mKdV equation is globally well-posed in $H^s(\mathbb{T})$, $s \geq 1$, and locally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. The local well-posedness of the KdV equation was pushed down to $H^s(\mathbb{T})$, $s \geq -1/2$ by Kenig, Ponce and Vega [13] (see [6] for the global-wellposedness of the KdV and the mKdV equations in $H^s(\mathbb{T})$ for respectively $s \geq -1/2$ and $s \geq 1/2$.) The local well-posednesss results proved in these papers mean the following : for any initial data $u_0 \in H^s(\mathbb{T})$ there exists a time $T = T(\|u_0\|_{H^s}) > 0$ only depending on $\|u_0\|_{H^s}$ and a solution u that satisfies the equation at least in some weak sense and is unique in some function space $X \hookrightarrow C([0,T]; H^s(\mathbb{T}))$. Moreover, for any $R > 0$, the flow-map $u_0 \mapsto u$ is continuous from the ball centered at the origin with radius R of $H^s(\mathbb{T})$ into $C([0,T(R)]; H^s(\mathbb{T}))$. Note that in all these works, by a change of variables, the study of the KdV equation is actually restricted to initial data with mean value zero and the mKdV equation is substituting by the following "renormalized" equation :

$$u_t + u_{xxx} \mp 6(u^2 - \fint u^2)u_x = 0 \tag{1.3}$$

where $\fint u^2$ denotes the mean value of u^2. The best results quoted above are known to be sharp if one requires moreover the smoothness of the flow-map (cf. [3]) or the uniform continuity on bounded sets of the solution-map (cf. [4]) associated respectively with the KdV equation on space of functions with mean value zero and with (1.3). On the other hand, they have been improved if one only requires the continuity of the flow-map. In this direction, in [12]-[11], Kappeler and Topalov introduced the following notion of solutions which a priori does not always corresponds to the solution in the sense of distribution : *A continuous curve $\gamma : (a,b) \to H^\beta(\mathbb{T})$ with $0 \in (a,b)$ and $\gamma(0) = u_0$ is called a solution of KdV equation (resp. mKdV equation) in $H^\beta(\mathbb{T})$ with initial data u_0 iff for any C^∞-sequence of initial data $\{u_{0,n}\}$ converging to u_0 in $H^\beta(\mathbb{T})$ and for any $t \in]a,b[$, the sequence of emanating solutions $\{u_n\}$ of the KdV equation (resp. mKdV equation) satisfies : $u_n(t) \to \gamma(t)$ in $H^\beta(\mathbb{T})$*

Note that a solution in the sense of this definition is necessarily unique. With this notion of solution they proved the global well-posedness of the KdV and the defocusing mKV equations in $H^s(\mathbb{T})$ for respectively $s \geq -1$ and $s \geq 0$, with a solution-map which is continuous from $H^{-1}(\mathbb{T})$ (resp.

$L^2(\mathbb{T})$) into $C(\mathbb{R}; H^{-1}(\mathbb{T}))$ (resp. $C(\mathbb{R}; L^2(\mathbb{T}))$). Their proof is based on the inverse scattering method and thus depends in a crucial way of the complete integrability of these equations. It is worth noticing that, by Sobolev embedding theorem, their solutions of the defocussing mKdV equation satisfy the equation in the distributional sense as soon as $s \geq 1/6$. Independently, Takaoka and Tsutsumi ([21]) extended the local well-posedness of the mKdV equation (with the classical notion of solutions) to $H^s(\mathbb{T})$ for $s > 3/8$ by modifying in a suitable way the Bourgain's space used as resolution space. This approach has been very recently improved by Nakanashi, Takaoka and Tsutsumi [19] and local well-posedness has been pushed to $H^s(\mathbb{T})$ for $s > 1/3$ (local existence of solutions is shown in $H^s(\mathbb{T})$ for $s > 1/4$)

In this paper we first establish an a priori estimate for L^2-bounded sequences of solutions to the mKdV equation. To this aim we slightly modify the spaces introduced by Ionescu-Kenig and Tataru in [9]. Recall that these spaces are constructed by localizing in time the Bourgain spaces with a localization in time that depends inversely on the space frequencies of the functions (see [14] and [5] for previous works in this direction). Note that, to some extent, this approach is a version for the Bourgain's spaces of the approach developped by Koch and Tzvetkov [15] in Strichartz spaces. Once our a priori estimate is established we translate it in the Bourgain's type spaces introduced in [9] . This enables us to pass to the limit on the nonlinear term by separating resonant and non resonant parts. Following some ideas developped in [16], we then derive a non continuity result for the mKdV equation in $H^s(\mathbb{T})$ for $s < 0$. On the other hand, we obtain the existence of weak L^2-solutions of (1.3) and prove that the L^2-solutions constructed in [12] of the defocusing mKdV equation satisfy the equation in some weak sense. Finally, we follow some ideas of [17] and use properties of the Riccati map proved in [10] to derive a non continuity result for the KdV equation in $H^s(\mathbb{T})$ for $s < -1$.

1.1 Statement of the results

Our results can be summarized in the two following theorems. The first one deals with the discontinuity of the solution-map associated with the KdV and mKdV equations.

Theorem 1.1. *The Cauchy problems associated with the KdV equation and the mKdV equation are ill-posed in $H^s(\mathbb{T})$ for respectively $s < -1$ and $s < 0$. More precisely,*

i) for any $T > 0$ and any $s < -1$, the solution-map $u_0 \mapsto u$ associated with the KdV equation is discontinous at any $u_0 \in H_0^\infty(\mathbb{T})$ from $H_0^\infty(\mathbb{T})$, endowed with the topology inducted by $H^s(\mathbb{T})$, into $\mathcal{D}'(]0,T[\times\mathbb{T})$.

ii) for any $T > 0$ and any $s < 0$, the solution-map $u_0 \mapsto u$ associated with the mKdV equation is discontinous at any non constant function $u_0 \in H^\infty(\mathbb{T})$ from $H^\infty(\mathbb{T})$, endowed with the topology inducted by $H^s(\mathbb{T})$, into $\mathcal{D}'(]0,T[\times\mathbb{T})$.

Remark 1.1. *Actually we prove the following assertions :*

i') For any $T > 0$, the solution-map $u_0 \mapsto u$ associated with the KdV equation is discontinous, at any $u_0 \in H_0^\infty(\mathbb{T})$, from $H_0^\infty(\mathbb{T})$ endowed with the weak topology of $H^{-1}(\mathbb{T})$ into $\mathcal{D}'(]0,T[\times\mathbb{T})$.

ii') For any $T > 0$, the solution-map $u_0 \mapsto u$ associated with the mKdV equation is discontinous, at any non constant function $u_0 \in H^\infty(\mathbb{T})$, from $H^\infty(\mathbb{T})$ endowed with the weak topology of $L^2(\mathbb{T})$ into $\mathcal{D}'(]0,T[\times\mathbb{T})$.

ii") Let $u_0 \in H^\infty(\mathbb{T})$ be a non constant function. There exists no $T > 0$ such that for all $t \in]0,T[$ the flow-map $u_0 \mapsto u(t)$ associated with the mKdV equation is continuous, at u_0 , from $H^\infty(\mathbb{T})$ endowed with the weak topology of $L^2(\mathbb{T})$ into $\mathcal{D}'(\mathbb{T})$.

Remark 1.2. *Our proof of the ill-posedness of the KdV equation below $H^{-1}(\mathbb{T})$ is heavily related to the algebraic structure of the equation via the Miura map. However, it is interesting to notice that, in [18], a similar result is proved for the KdV-Burgers equation with a completely different method.*

The second one deals with the existence of weak L^2-solutions to the mKdV equation.

Theorem 1.2.

i) For any $u_0 \in L^2(\mathbb{T})$ there exists a weak solution $u \in C_w(\mathbb{R}; L^2(\mathbb{T})) \cap (\cup_{s<0}\tilde{F}^{s,1/2})$ of mKdV such that $u(0) = u_0$. Moreover, $u(t) \to u_0$ in $L^2(\mathbb{T})$ as $t \to 0$.

ii) The $C(\mathbb{R}; L^2(\mathbb{T}))$-functions determined by the unique continuous extension to $C(\mathbb{R}; L^2(\mathbb{T}))$ of the $C(\mathbb{R}; H^\infty(\mathbb{T}))$ solution-map of the defocusing mKdV equation, constructed in [11], are weak solutions of the mKdV equation and belong to $\cup_{s<0}\tilde{F}^{s,1/2}$.

Remark 1.3. *The function spaces $\tilde{F}^{s,1/2}$ are defined by* (2.4) *when substituting the linear KdV group $U(\cdot)$ by the linear group $V(\cdot)$ defined in* (5.11)*. Our notion of weak solution to mKdV is described in Definition 5.2 (see also* (5.12) *for assertion ii)).*

Remark 1.4. *Once, the second assertion of Theorem 1.2 is established, the first assertion seems to have no more interest for the defocusing mKdV equation (note that the first assertion is, up to our knowledge, the only available existence result of global weak $L^2(\mathbb{T})$-solutions for the focusing mKdV equation). However, the proof of assertion 2 uses the complete integrability of the equation which is not a priori conserved by perturbations. On the other hand, the proof of assertion 1 seems to be widely more tractable and for instance certainly works for a wide class of perturbations of the defocusing mKdV equation. For instance,*

$$\textit{Damped mKdV}: \quad u_t + u_{xxx} + \nu u \mp u^2 u_x = 0, \quad \nu > 0$$

$$\textit{KdV-mKdV}: \quad u_t + u_{xxx} \mp u^2 u_x \mp u u_x = 0,$$

Remark 1.5. *We also construct in Proposition 5.1 global weak solutions for the "renormalized" mKdV equation* (1.3)*.*

This paper is organized as follows: In the next section we introduce the notations and the functions spaces we will work with. We also give some useful estimates for time-localized functions. In Section 3 we recall general linear estimates in such functions spaces and some linear and bilinear estimates relating to the KdV group. Section 4 is devoted to the proof of the uniform bound for $L^2(\mathbb{T})$-bounded sequence of solutions to mKdV. We prove Theorem 1.2 in Section 5 and Theorem 1.1 in Section 6. Finally, in the appendix we first give a simplified proof of the continuous embedding in $L^4(\mathbb{R}\times\mathbb{T})$ of some Bourgain's space related to the KdV group. Then, for sake of completeness, we prove some needed bilinear estimates and sketch the proof of some properties of the Riccati map $u \mapsto u' - u^2 - \fint u^2$.

2 Notations and functional spaces

2.1 Notations

For $x, y \in \mathbb{R}_+^*$, $x \sim y$ means that there exist C_1, $C_2 > 0$ such that $C_1 x \leq y \leq C_2 x$. $x \lesssim y$ and $x \gtrsim y$ mean that there exists $C_2 > 0$ such that respectively $x \leq C_2 y$ and $x \geq C_2 y$. For a Banach space X, we denote by $\|\cdot\|_X$ the norm in X.

We will use the same notations as in [6] and [7] to deal with Fourier transform of space periodic functions with a large period λ. $(d\xi)_\lambda$ will be the renormalized counting measure on $\lambda^{-1}\mathbb{Z}$:

$$\int a(\xi)\,(d\xi)_\lambda = \frac{1}{\lambda}\sum_{\xi\in\lambda^{-1}\mathbb{Z}} a(\xi) \quad .$$

As written in [7], $(d\xi)_\lambda$ is the counting measure on the integers when $\lambda = 1$ and converges weakly to the Lebesgue measure when $\lambda \to \infty$. In all the text, all the Lebesgue norms in ξ will be with respect to the measure $(d\xi)_\lambda$. For a $(2\pi\lambda)$-periodic function φ, we define its space Fourier transform on $\lambda^{-1}\mathbb{Z}$ by

$$\hat{\varphi}(\xi) = \int_{\lambda\mathbb{T}} e^{-i\xi x}\, f(x)\, dx, \quad \forall \xi \in \lambda^{-1}\mathbb{Z} \quad .$$

We denote by $U(\cdot)$ the free group associated with the linearized Korteweg-de Vries equation,

$$\widehat{U(t)\varphi}(\xi) = e^{ip(\xi)t}\,\hat{\varphi}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z} \quad , \; p(\xi) = \xi^3 \; .$$

The Lebesgue spaces L^q_λ, $1 \le q \le \infty$, for $(2\pi\lambda)$-periodic functions, will be defined as usually by

$$\|\varphi\|_{L^q_\lambda} = \Big(\int_{\lambda\mathbb{T}} |\varphi(x)|^q\, dx\Big)^{1/q}$$

with the obvious modification for $q = \infty$.
We define the Sobolev spaces H^s_λ for $(2\pi\lambda)$-periodic functions by

$$\|\varphi\|_{H^s_\lambda} = \|\langle\xi\rangle^s \varphi(\xi)\|_{L^2_\xi} = \|J^s_x \varphi\|_{L^2_\lambda} \quad ,$$

where $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$ and $\widehat{J^s_x\varphi}(\xi) = \langle\xi\rangle^s \widehat{\varphi}(\xi)$.
Note that the closed subspace of zero mean value functions of H^s_λ will be denoted by $H^s_{0,\lambda}$ (it is equipped with the H^s_λ-norm).
In the same way, for a function $u(t,x)$ on $\mathbb{R}\times\lambda\mathbb{T}$, we define its space-time Fourier transform by

$$\hat{u}(\tau,\xi) = \mathcal{F}_{t,x}(u)(\tau,\xi) = \int_{\mathbb{R}}\int_{\lambda\mathbb{T}} e^{-i(\tau t+\xi x)}\, u(t,x)\, dxdt, \quad \forall(\tau,\xi) \in \mathbb{R}\times\lambda^{-1}\mathbb{Z} \quad .$$

$L^p_t L^q_\lambda$ and $L^p_T L^q_\lambda$ will denote respectively the Lebesgue spaces

$$\|u\|_{L^p_t L^q_\lambda} = \Big(\int_{\mathbb{R}} \|u(t,\cdot)\|^p_{L^q_\lambda}\, dt\Big)^{1/p} \quad \text{and} \quad \|u\|_{L^p_T L^q_\lambda} = \Big(\int_0^T \|u(t,\cdot)\|^p_{L^q_\lambda}\, dt\Big)^{1/p}$$

with the obvious modification for $p=\infty$.
For any $(s,b)\in\mathbb{R}^2$, we define the Bourgain space $X_\lambda^{s,b}$, of $(2\pi\lambda)$-periodic (in x) functions as the completion of $\mathcal{S}(\lambda\mathbb{T}\times\mathbb{R})$ for the norm

$$\|u\|_{X_\lambda^{s,b}}=\|\langle\tau-p(\xi)\rangle^b\langle\xi\rangle^s\hat{u}\|_{L^2_{\tau,\xi}}=\|\langle\tau\rangle^b\langle\xi\rangle^s\mathcal{F}_{t,x}(U(-t)u)\|_{L^2_{\tau,\xi}}\ , \tag{2.1}$$

For $T>0$ and a function space B_λ, we denote by $B_{T,\lambda}$ the corresponding restriction in time space endowed with the norm

$$\|u\|_{B_{T,\lambda}}=\inf_{w\in B_\lambda}\{\|w\|_{B_\lambda}\,,\,w(\cdot)\equiv u(\cdot)\text{ on }]-T,T[\,\}\ .$$

Finally, for all function spaces of $(2\pi\lambda)$-periodic functions, we will drop the index λ when $\lambda=1$.

2.2 Bourgain's spaces on frequency dependent time intervals

We will need a Littlewood-Paley analysis. Let $\psi\in C_0^\infty(\mathbb{R})$ be an even function such that $\psi\geq 0$, $\operatorname{supp}\psi\subset[-2,2]$, $\psi\equiv 1$ on $[-1,1]$. We set $\eta_0:=\psi$ and for all $k\in\mathbb{N}^*$, $\eta_k(\xi):=\psi(2^{-k}\xi)-\psi(2^{-k+1}\xi)$ and $\eta_{\leq k}:=\psi(2^{-k}\cdot)=\sum_{j=0}^k\eta_k$. We also set $\tilde{\eta}_0:=\psi(2^{-1}\cdot)$ and for all $k\in\mathbb{N}^*$, $\tilde{\eta}_k(\xi):=\psi(2^{-k-1}\xi)-\psi(2^{-k+2}\xi)$ The Fourier multiplicator operators by η_j, $\tilde{\eta}_j$ and $\eta_{\leq j}$ will be denoted respectively by Δ_j, $\tilde{\Delta}_j$ and S_j, and the projection on the constant Fourier mode will be denoted by P_0 , i.e. for any $u\in L^2(\lambda\mathbb{T})$

$$\widehat{\Delta_j u}:=\eta_j\widehat{u},\quad \widehat{\tilde{\Delta}_j u}:=\tilde{\eta}_j\widehat{u},\quad \widehat{S_j u}:=\eta_{\leq j}\hat{u}\quad\text{and}\quad P_0(u)=\frac{1}{2\pi\lambda}\int_{\lambda\mathbb{T}}u(x)\,dx.$$

By a slight abuse of notations, we will also define the operator Δ_j, $\tilde{\Delta}_j$ and P_0 on $L^2(\lambda\mathbb{T}\times\mathbb{R})$-functions by the same formula. Finally, for any $l\in\mathbb{N}$ we define the functions ν_l on $\lambda^{-1}\mathbb{Z}\times\mathbb{R}$ by

$$\nu_l(\xi,\tau):=\eta(\tau-p(\xi))\ . \tag{2.2}$$

Let $0\leq b\leq 1/2$, $k\in\mathbb{N}$, $t_0\in\mathbb{R}$ and $f\in C^\infty(\lambda\mathbb{T}\times]t_0-2^{-k},t_0+2^{-k}[)$, we define

$$\|f\|_{F^b_{\lambda,k,t_0}}:=\inf_{\tilde{f}\in X_\lambda^{0,b,1}}\left\{\|\tilde{f}\|_{X_\lambda^{0,b,1}},\ \tilde{f}=f\text{ on }]t_0-2^k,t_0+2^k[\right\}$$

where $X_\lambda^{0,b,1}$ is the Bourgain's type space defined by

$$X_\lambda^{0,b,1}=\left\{\begin{array}{l} f\in\mathcal{S}'(\lambda\mathbb{T}\times\mathbb{R}),\\ \|f\|_{X_\lambda^{0,b,1}}:=\displaystyle\sum_{j=0}^\infty 2^{jb}\|\nu_j(\xi,\tau)\widehat{f}\|_{L^2_{\xi,\tau}}<\infty\end{array}\right\}$$

Our a priori estimate will take place in the normed space G_λ defined as the completion of $C^\infty(\lambda\mathbb{T}\times\mathbb{R})$ for the norm

$$\|f\|^2_{G_\lambda} := \sup_{t\in\mathbb{R}} \sum_{k\geq 0} \|\Delta_k f\|^2_{F^{1/2}_{\lambda,k,t}} \ . \tag{2.3}$$

Once our a priori estimate will be established we will make use of the spaces $F^{s,b}_\lambda$, introduced in [9], that are endowed with the norm

$$\|f\|^2_{F^{s,b}_\lambda} := \sum_{k\geq 0} \Big(\sup_{t\in\mathbb{R}} 2^{ks}\|\Delta_k f\|_{F^b_{\lambda,k,t}}\Big)^2 \ . \tag{2.4}$$

To handle the nonlinear term, for $k\in\mathbb{N}$, we will also need to introduce the function space $Z_{\lambda,k}$ defined as the completion of $L^2(\lambda\mathbb{T}\times\mathbb{R})$ for the following norm :

$$\|f\|_{Z_{\lambda,k}} = \sum_{j\geq 0} 2^{j/2}\|\langle\tau - p(\xi) + i2^k\rangle^{-1}\widehat{f}\|_{L^2} \ . \tag{2.5}$$

Finally, for $t_0\in\mathbb{R}$ and $f\in L^2(\mathbb{T}\times]t_0-2^k, t_0+2^k[)$, we define

$$\|f\|_{Z_{\lambda,k,t_0}} := \inf_{\tilde{f}\in Z_{\lambda,k}} \Big\{\|\tilde{f}\|_{Z_{\lambda,k}},\ \tilde{f} = f \text{ on }]t_0-2^k, t_0+2^k[\Big\} \ .$$

2.3 Some useful estimates for localized in time functions

The following lemma, established in [9] will be useful in the linear estimates and also in the nonlinear estimates when we will localize the functions on time interval of length of order 2^{-l}.

Lemma 2.1. *Let be given $\lambda\geq 1$ and $f\in X^{0,1/2,1}_\lambda$. Then for all $l\in\mathbb{N}$ it holds*

$$2^{l/2}\Big\|\eta_{\leq l}(\tau-p(\xi))\int_{\mathbb{R}} |\widehat{f}(\xi,\tau')|\, 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}\, d\tau'\Big\|_{L^2_{\xi,\tau}} \lesssim \|f\|_{X^{0,1/2,1}_\lambda} \tag{2.6}$$

and

$$\sum_{j\geq l+1} 2^{j/2}\Big\|\eta_j(\tau-p(\xi))\int_{\mathbb{R}} |\widehat{f}(\xi,\tau')|\, 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}\, d\tau'\Big\|_{L^2_{\xi,\tau}} \lesssim \|f\|_{X^{0,1/2,1}_\lambda} \tag{2.7}$$

Proof. From the definition of the space $X^{0,1/2,1}_\lambda$, it is easy to check that for all $f\in X^{0,1/2,1}_\lambda$,

$$\Big\|\int_{\mathbb{R}} |\widehat{f}(\xi,\tau')|\, d\tau'\Big\|_{L^2_\xi} \lesssim \|f\|_{X^{0,1/2,1}_\lambda} \ . \tag{2.8}$$

With this estimate in hand, the left-hands side member of (2.6) can be clearly bounded from above by

$$2^{l/2}2^{-l}\sup_{\xi\in\mathbb{R}}\|\eta_{\leq l}(\tau-p(\xi))\|_{L^2_\tau}\|f\|_{X^{0,1/2,1}_\lambda}\lesssim\|f\|_{X^{0,1/2,1}_\lambda}\ .$$

Now to prove (2.7), we first notice that by the mean-value theorem,

$$|\eta_j(\tau-p(\xi))-\eta_j(\tau'-p(\xi))|\lesssim 2^{-j}|\tau-\tau'|$$

and it thus would be sufficient to estimate

$$\sum_{j\geq l+1}2^{j/2}\Big\|\int_{\mathbb{R}}\eta_j(\tau'-p(\xi))|\widehat{f}(\xi,\tau')|\,2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}\,d\tau'\Big\|_{L^2_{\xi,\tau}}$$

$$+\sum_{j\geq l+1}2^{-j/2}\Big\|\int_{\mathbb{R}}|\widehat{f}(\xi,\tau')|\,2^{-l}\frac{|\tau-\tau'|}{(1+2^{-l}|\tau-\tau'|)^4}\,d\tau'\Big\|_{L^2_{\xi,\tau}}\ .$$

By identifying a convolution term and applying generalized Young's inequality, we can bound the first term by

$$\sum_{j\geq l+1}2^{j/2}\|\nu_j\widehat{f}\|_{L^2}\int_{\mathbb{R}}2^{-l}(1+2^{-l}|\tau'|)^{-4}\,d\tau'\lesssim\sum_{j\geq l+1}2^{j/2}\|\nu_j\widehat{f}\|_{L^2}$$

To treat the second term, we use Minskowsky inequality and (2.8) to get

$$I_2\lesssim\sum_{j\geq l+1}2^{-j/2}2^{-l}\|f\|_{X^{0,1/2,1}_\lambda}\Big\|\frac{|\tau|}{(1+2^{-l}|\tau|)^4}\Big\|_{L^2_\tau}\lesssim\sum_{j\geq l+1}2^{-j/2}2^{l/2}\|f\|_{X^{0,1/2,1}_\lambda}$$

which is acceptable. □

Remark 2.1. *A very useful corollary of the preceding lemma is the following: Let $f\in X^{0,1/2,1}_\lambda$ and $\gamma\in C^\infty_c(\mathbb{R})$ with support in $]-2,2[$. Then for all $k\in\mathbb{N}$, it holds*

$$\|\gamma(2^kt)f\|_{X^{0,1/2,1}_\lambda}\lesssim\|f\|_{X^{0,1/2,1}_\lambda}\tag{2.9}$$

and

$$\|\eta_{\leq k}(\tau-p(\xi))\mathcal{F}_{xt}(\gamma(2^kt)f)\|_{L^2_{\xi,\tau}}\lesssim 2^{-k/2}\|f\|_{X^{0,1/2,1}_\lambda}\tag{2.10}$$

Remark 2.2. *It is easy to check that G_λ is continuously embedded in $L^\infty(\mathbb{R};L^2_\lambda(\mathbb{T}))$. Indeed, for any $u\in G_\lambda$, $t_0\in\mathbb{R}$ and $k\in\mathbb{N}$, taking a function $\tilde{u}\in X^{0,1/2,1}_\lambda$ such that $\tilde{u}\equiv u$ on $]t_0-2^{-k},t_0+2^{-k}[$ and*

$$\|\tilde{u}\|_{X^{0,1/2,1}_\lambda}\leq 2\|u\|_{F^{1/2}_{\lambda,k,t_0}}\ ,$$

it holds

$$\mathcal{F}_x(\Delta_k u(t_0))(\xi) = \int_{\mathbb{R}} \mathcal{F}(\Delta_k \tilde{u}(\xi,\tau)) e^{it_0\tau}\, d\tau$$

According to (2.8)*, this leads to*

$$\|\Delta_k u(t_0)\|_{L^2_\lambda} \lesssim \|\Delta_k \tilde{u}\|_{X^{0,1/2,1}_\lambda} \lesssim \|\Delta_k u\|_{F^{1/2}_{\lambda,k,t_0}} \ .$$

Squaring and summing in k one obtains that

$$\|u\|_{L^\infty_t L^2(\mathbb{T}))} \lesssim \|u\|_{G_\lambda} \ . \tag{2.11}$$

On the other hand, it seems pretty clear that G_λ is not included in $C(\mathbb{R}; L^2_\lambda(\mathbb{T}))$.

3 Some linear and bilinear estimates

3.1 General linear estimates

We first derive linear estimates that do not depend on the dispersive linear group associated with our functional space. We mainly follow [14]-[9].

Lemma 3.1. *$\forall \varphi \in L^2_\lambda$ and all $k \in \mathbb{N}$, it holds*

$$\|U(t)\varphi\|_{F^{1/2}_{\lambda,k,0}} \lesssim \|\varphi\|_{L^2_\lambda} \ . \tag{3.1}$$

Proof. Clearly, it suffices to prove that for any $k \in \mathbb{N}$,

$$\|\eta_0(2^k t) U(\cdot)\Delta_k \varphi\|_{X^{0,1/2,1}_\lambda} \lesssim \|\Delta_k \varphi\|_{L^2_\lambda} \ .$$

Notice that the left-hand side member of the above inequality is bounded by

$$\Big(\sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau) 2^{-k} \hat{\eta}_0(2^{-k}\tau)\|_{L^2_\tau}\Big) \|\Delta_k \varphi\|_{L^2_\lambda} \ .$$

Since $\eta_0 \in C^\infty_c(R)$, $\widehat{\eta_0}$ decays at least as $(1+|y|)^{-4}$ and thus

$$\|\eta_j(\tau) 2^{-k} \hat{\eta}_0(2^{-k}\tau)\|_{L^2_\tau} \lesssim \|\eta_j(\tau) 2^{-k} (1+2^{-k}|\tau|)^{-4})\|_{L^2_\tau} \lesssim 2^{-k} 2^{j/2} \min(1, 2^{4(k-j)}) \ . \tag{3.2}$$

Hence,

$$\sum_{j=0}^{k+2} 2^{j/2} \|\eta_j(\tau) 2^{-k} \hat{\eta}_0(2^{-k}\tau)\|_{L^2_\tau} \lesssim 1$$

and

$$\sum_{j=k+2}^{\infty} 2^{j/2} \|\eta_j(\tau) 2^{-k} \hat{\eta}_0(2^{-k}\tau)\|_{L^2} \lesssim \sum_{j=k+2}^{\infty} 2^{-3j} 2^{3k} \lesssim 1 \ . \quad \square$$

Lemma 3.2. *For any $k \in \mathbb{N}$ and any $f \in Z_{\lambda,k,0}$ it holds*

$$\Big\| \int_0^t U(t-t')f(t')\,dt' \Big\|_{F^{1/2}_{\lambda,k,0}} \lesssim \|f\|_{Z_{\lambda,k,0}} \ . \tag{3.3}$$

Proof. Let $\tilde{f} \in Z_{\lambda,k}$ be and extension of f such that $\|\tilde{f}\|_{Z_{\lambda,k}} \leq 2\|f\|_{Z_{\lambda,k,0}}$. We set

$$v := \eta_0(2^k t) \int_0^t U(t-t')\tilde{f}(t')\,dt' \ .$$

Then

$$\begin{aligned}
\mathcal{F}_{tx}(v)(\xi,\tau) &= \mathcal{F}_t\Big[\eta_0(2^k t)\int_{\mathbb{R}} \frac{e^{it\tau}-e^{itp(\xi)}}{i(\tau-p(\xi))}\mathcal{F}_{t,x}(\tilde{f})\,d\tau\Big](\xi,\tau) \\
&= \mathcal{F}_t(\eta_0(2^k t)) * \Big[\frac{\mathcal{F}_{t,x}(\tilde{f})}{i(\tau-p(\xi))}\Big] - \mathcal{F}_t(e^{itp(\xi)}\eta_0(2^k t))\int_{\mathbb{R}} \frac{\mathcal{F}_{t,x}(\tilde{f})}{i(\tau'-p(\xi))}\,d\tau' \\
&= \int_{\mathbb{R}} \mathcal{F}_{t,x}(\tilde{f})(\tau',\xi)\Big[\frac{2^{-k}\hat{\eta}_0(2^{-k}(\tau-\tau'))-2^{-k}\hat{\eta}_0(2^{-k}(\tau-p(\xi)))}{i(\tau'-p(\xi))}\Big]\,d\tau'
\end{aligned}$$

Now we claim that

$$\begin{aligned}
I &= \frac{\Big|2^{-k}\hat{\eta}_0(2^{-k}(\tau-\tau'))-2^{-k}\hat{\eta}_0(2^{-k}(\tau-p(\xi)))\Big|}{|\tau'-p(\xi)|}|\tau'-p(\xi)+i2^k| \\
&\lesssim 2^{-k}(1+2^{-k}|\tau-\tau'|)^{-4}+2^{-k}(1+2^{-k}|\tau-p(\xi)|)^{-4} \qquad (3.4)
\end{aligned}$$

Assuming (3.4) for a while, it follows that

$$\begin{aligned}
\|v\|_{X^{0,1/2,1}} \lesssim\ & \sum_{j=0}^{\infty} 2^{j/2}\Big\|\eta_j(\tau-p(\xi))\int_{\mathbb{R}} \frac{\widehat{\tilde{f}}(\xi,\tau')}{\langle\tau'-p(\xi)+i2^k\rangle}2^{-k}(1+2^{-k}|\tau-\tau'|)^{-4}\,d\tau'\Big\|_{L^2_{\xi,\tau}} \\
&+\sum_{j=0}^{\infty} 2^{j/2}\Big\|\eta_j(\tau-p(\xi))2^{-k}(1+2^{-k}|\tau-p(\xi)|)^{-4}\int_{\mathbb{R}} \frac{\widehat{\tilde{f}}(\xi,\tau')}{\langle\tau'-p(\xi)+i2^k\rangle}\,d\tau'\Big\|_{L^2_{\xi,\tau}}
\end{aligned}$$

The desired bound on the first term of the above right-hand side member follows directly from (2.6)-(2.7). To obtain the desired bound on the second term, we combine (2.8) and (3.2).
It thus remains to prove (3.4). For this we first notice that, since $\eta_0(y) = \eta_0(|y|)$, by the mean-value theorem there exists $\theta \in]|\tau-\tau'|, |\tau-p(\xi)|[$ such that

$$\Big|2^{-k}\hat{\eta}_0(2^{-k}(\tau-\tau'))-2^{-k}\hat{\eta}_0(2^{-k}(\tau-p(\xi)))\Big| \lesssim 2^{-2k}\hat{\eta}_0'(2^{-k}\theta)|\tau'-p(\xi)| \ . \tag{3.5}$$

Furthermore, since $\eta_0 \in \mathcal{S}(\mathbb{R})$,

$$|\hat{\eta}_0(y)| + |\hat{\eta}_0'(y)| \lesssim (1+|y|)^{-10} .$$

Let us now separate three cases :

- $|\tau - p(\xi)| \leq 2^k$. Then (3.4) follows directly from (3.5).
- $|\tau - p(\xi)| \geq 2^k$ and $|\tau - \tau'| \sim |\tau - p(\xi)|$. Then we must have $|\theta| \sim |\tau - p(\xi)|$ and $|\tau' - p(\xi) + i2^k| \lesssim |\tau - p(\xi)|$.Therefore (3.5) leads to

$$I \lesssim 2^{-2k}(1+2^{-k}|\tau - p(\xi)|)^{-5}|\tau - p(\xi)| \lesssim 2^{-k}(1+2^{-k}|\tau - p(\xi)|)^{-4}$$

- $|\tau - p(\xi)| \geq 2^k$ and $|\tau - \tau'| \not\sim |\tau - p(\xi)|$. Then $|\tau' - p(\xi)| \sim (|\tau - p(\xi)| \vee |\tau' - \tau|) \gtrsim 2^k$ and (3.4) follows directly from the decay of $\hat{\eta}_0$. □

3.2 Specific linear and bilinear estimates

We will also need estimates that are specific for Bourgain's spaces associated with the KdV linear group. We first recall the following Strichartz's type estimate proved in [1] (we give a simplified proof of this estimate in the appendix) :

Lemma 3.3. *For any $\lambda \geq 1$ and any $u \in X_\lambda^{0,1/3}$, it holds*

$$\|u\|_{L^4_{t,\lambda}} \lesssim \|u\|_{X_\lambda^{0,1/3}} \tag{3.6}$$

Finally we will make a frequent use of the following bilinear estimates that can be deduced for instance from [22] (we give a proof of these estimates in the appendix since we need to quantify the dependence of these estimates with respect to the period λ):

Lemma 3.4. *Let $\lambda \geq 1$ and let u_1 and u_2 be two real valued L^2 functions defined on $\mathbb{R} \times (\lambda^{-1}\mathbb{Z})$ with the following support properties*

$$(\tau, \xi) \in \operatorname{supp} u_i \Rightarrow \langle \tau - \xi^3 \rangle \lesssim L_i,\ i = 1, 2.$$

Then for any $N > 0$ the following estimates holds:

$$\|u_1 \star u_2\|_{L^2_\tau L^2(|\xi| \geq N)} \lesssim (L_1 \wedge L_2)^{1/2} \Big(\frac{(L_1 \vee L_2)^{1/4}}{N^{1/4}} + \lambda^{-1/2} \Big) \|u_1\|_{L^2} \|u_2\|_{L^2} , \tag{3.7}$$

and

$$\|\Lambda[N](u_1,u_2)\|_{L^2_{\tau,\xi}} \lesssim (L_1\wedge L_2)^{1/2}\Big(\frac{(L_1\vee L_2)^{1/2}}{N}+\lambda^{-1/2}\Big)\|u_1\|_{L^2}\|u_2\|_{L^2}. \tag{3.8}$$

where $\Lambda[N]:(L^2(\mathbb{R}\times\lambda^{-1}\mathbb{Z}))^2\to L^\infty(\mathbb{R}\times\lambda^{-1}\mathbb{Z})$ *is defined by*

$$\Lambda[N](u_1,u_2)(\xi,\tau)=\int_{\mathbb{R}}\int_{\Big||\xi_1|-|\xi-\xi_1|\Big|\geq\frac{N}{100}} u_1(\xi_1,\tau_1)u_2(\xi-\xi_1,\tau-\tau_1)\,(d\xi_1)_\lambda\,d\tau_1$$

4 A priori estimate for smooth solutions to (1.3)

As in previous works on mKdV on the torus (cf. [1],[6]), we actually work with the "renormalized " mKdV equation (1.3) instead of the mKdV equation itself. This permits to cancel some resonant part in the nonlinear term. Recall that for $v\in C(\mathbb{R};H^\infty_\lambda)$, a smooth solution to mKdV with initial data v_0, the L^2-norm of v is a constant of the motion and thus $u(t,x):=v(t,x\mp\frac{6t}{2\pi\lambda}\|v_0\|^2_{L^2_\lambda})$ satisfies (1.3).

Denoting by $N(u)$ the nonlinear term of (1.3), it holds for any $\xi\in\lambda^{-1}\mathbb{Z}$,

$$\begin{aligned}
\mathcal{F}_x[N(u)](\xi) &= -\frac{6i}{\lambda^2}\sum_{\substack{\xi_1+\xi_2+\xi_3=\xi\\ \xi_1+\xi_2\neq 0}}\hat{u}(\xi_1)\hat{u}(\xi_2)\xi_3\hat{u}(\xi_3)\\
&= -\frac{2i\xi}{\lambda^2}\Big[\sum_{\substack{\xi_1+\xi_2+\xi_3=\xi\\ (\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)\neq 0}}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\Big]\\
&\quad+\frac{6i\xi}{\lambda^2}\hat{v}(\xi)\hat{v}(\xi)\hat{v}(-\xi)\\
&:= -i\xi\Big[\Big(\mathcal{F}_x\Big[A(u,u,u)\Big](\xi)+\mathcal{F}_x\Big[B(u,u,u)\Big](\xi)\Big)\Big]
\end{aligned}$$

i.e.

$$6(u^2-P_0(u^2))u_x=\partial_x\Big(A(u,u,u)+B(u,u,u)\Big)\,. \tag{4.1}$$

According to the resonance relation (4.13), A is non resonant whereas B is a resonant term. As pointing out in [21], A is a "good term" as far as one wants to solve the equation in $H^s(\mathbb{T})$ for $s\geq 1/4$. On the other hand, B is a bad term as soon as one wants to solve the equation below $H^{1/2}(\mathbb{T})$, giving rise to rapid oscillations that breaks the uniform continuity on bounded set of the flow-map.

Proposition 4.1. *Let $\lambda \geq 1$ and $u \in C(\mathbb{R}; H^\infty(\lambda\mathbb{T}))$ be a solution to* (1.3)*. Then,*

$$\|u\|_{G_\lambda} \lesssim \|u\|_{L^\infty(\mathbb{R};L^2_\lambda)} + \|u\|^3_{G_\lambda} \tag{4.2}$$

Proof. From the definition of the norm G_λ, we have to bound $\sup_{t_0\in\mathbb{R}} \sum_{k\geq 0} \|\Delta_k u\|^2_{F^{1/2}_{\lambda,k,t_0}}$.

We use that for any $(t_0, t) \in \mathbb{R}^2$, it holds

$$u(t) = U(t-t_0)u(t_0) + \int_{t_0}^t U(t-t')\partial_x\Big(A(u(t')) + B(u(t')\Big)dt' .$$

By translation in time we can always assume that $t_0 = 0$ and according to Lemmas 3.1-3.2,

$$\|U(t)\Delta_k u(0)\|_{F^{1/2}_{\lambda,k,0}} \lesssim \|\Delta_k u(0)\|_{L^2_\lambda}$$

and

$$\Big\|\int_0^t U(t-t')\partial_x\Delta_k(A(u(t'))+B(u(t'))dt'\Big\|_{F^{1/2}_{\lambda,k,0}} \lesssim \|\Delta_k\partial_x(A(u))+B(u))\|_{Z_{\lambda,k,0}} .$$

Since $\sum_{k\geq 0}\|\Delta_k u(0)\|^2_{L^2_\lambda} \sim \|u\|^2_{L^\infty(\mathbb{R};L^2_\lambda)}$, it remains to prove that

$$\sum_{k=0}^\infty 2^k\Big(\|\Delta_k A(u)\|^2_{Z_{\lambda,k,0}} + \|\Delta_k B(u)\|^2_{Z_{\lambda,k,0}}\Big) \lesssim \|u\|^6_{G_\lambda} .$$

This is the aim of the two following lemmas.

Lemma 4.1. *Let be given $t_0 \in \mathbb{R}$, $\lambda \geq 1$ and $u_i \in G_\lambda$ for $i = 1, 2, 3$. Then it holds*

$$\sum_{k=0}^\infty 2^k\|\Delta_k\Big(B(u_1, u_2, u_3)\Big)\|^2_{Z_{\lambda,k,t_0}} \lesssim \prod_{i=1}^3 \|u_i\|^2_{G_\lambda} .$$

Proof. By translation in time, we can take $t_0 = 0$. For any fixed $k \in \mathbb{N}$, we take a time extension $\tilde{u}_1$ of u_1 such that $\|\Delta_k\tilde{u}_1\|_{X^{0,1/2,1}_\lambda} \leq 2\|\Delta_k u_1\|_{F^{1/2}_{\lambda,k,0}}$. Then, in view of the structure of B, it holds

$$2^k\|\Delta_k\Big(B(u_1, u_2, u_3)\Big)\|_{Z_{\lambda,k,0}} \lesssim \sum_{l\geq 0} 2^{l/2}2^k\Big\|\eta_l(\tau-\xi^3)\langle\tau-\xi^3+i2^k\rangle^{-1}\mathcal{F}_{xt}\Big[\Delta_k B(v_1, v_2, v_3)\Big]\Big\|_{L^2}$$

$$\lesssim \sum_{l\geq 0} 2^{l/2}2^k \Big\| \eta_l(\tau-\xi^3)\langle \tau-\xi^3+i2^k\rangle^{-1}\mathcal{F}_{xt}\Big[B(\Delta_k v_1,\tilde{\Delta}_k v_2,\tilde{\Delta}_k v_3)\Big]\Big\|_{L^2}$$

where $v_1=\eta_0(2^kt)\tilde{u}_1$ and $v_i=\eta_0(2^kt)u_i$, $i=2,3$. By duality it suffices to prove that

$$\begin{aligned} I_k &:= 2^k\Big|\Big(\mathcal{F}_{xt}\Big[B(\Delta_k v_1,\tilde{\Delta}_k v_2,\tilde{\Delta}_k v_3)\Big],\langle \tau-\xi^3+i2^k\rangle^{-1}\widehat{w}\Big)_{L^2}\Big| \\ &\lesssim \sup_l(2^{-l/2}\|\nu_l\widehat{w}\|_{L^2})\|\Delta_k v_1\|_{X^{0,1/2,1}_\lambda}\prod_{i=2}^3\|S_{k+1}v_i\|_{X^{0,1/2,1}_\lambda} \qquad (4.3)\end{aligned}$$

Indeed, first, according to (2.9) for any $k\in\mathbb{N}^*$ and any $u\in C(\mathbb{R}\times\lambda\mathbb{T})$,

$$\|\eta_0(2^kt)S_{k-1}u\|^2_{X^{0,1/2,1}_\lambda}\sim\sum_{j=0}^{k-1}\|\eta_0(2^kt)\Delta_j u\|^2_{X^{0,1/2,1}_\lambda}\leq\sum_{j=0}^{k-1}\|\Delta u\|^2_{X_{\lambda,j,0}}\leq\|u\|^2_{G_\lambda}\ . \qquad (4.4)$$

Second, taking a C^∞-function $\gamma:\mathbb{R}\mapsto[0,1]$ with compact support in $[-1,1]$ satisfying $\gamma\equiv 1$ on $[-1/2,1/2]$ and $\sum_{m\in\mathbb{Z}}\gamma(t-m)\equiv 1$ on $\mathbb{R}$, we get for any $j\geq k\in\mathbb{N}$,

$$\begin{aligned}\|\eta_0(2^kt)\Delta_j u\|_{X^{0,1/2,1}_\lambda} &\leq \sum_{|m|\leq 2^{j-k}}\|\gamma(2^jt-m)\eta_0(2^kt)\Delta_j u\|_{X^{0,1/2,1}_\lambda} \\ &\leq \sum_{|m|\leq 2^{j-k}}\|\Delta_j u\|_{F^{1/2}_{\lambda,j,2^{-j}m}}\lesssim 2^{j-k}\|u\|_{G_\lambda}\ . \qquad (4.5)\end{aligned}$$

Therefore, (4.3) will lead to

$$2^k\|\Delta_k\Big(B(u_1,u_2,u_3)\Big)\|_{Z_{\lambda,k,0}}\lesssim\|\Delta_k u_1\|_{X_{\lambda,k,0}}\|\prod_{i=2}^3\|u_i\|_{G_\lambda}$$

which will gives the result by squaring and summing in k.

Since the norms in the right-hand side of (4.3) only see the size of the modulus of the Fourier transform of the functions we can assume that all the functions have non negative Fourier transforms. In view of the structure of B, using Cauchy-Schwarz, we get

$$I_k\lesssim 2^k\|\eta_k\widehat{v}_1*\tilde{\eta}_k\widehat{v}_2\|_{L^2(\langle\xi\rangle\sim 2^k)}\|\Big(\langle\tau-\xi^3+i2^k\rangle^{-1}\tilde{\eta}\widehat{w}\Big)*\tilde{\eta}_k\check{\widehat{v}}_3\|_{L^2(\langle\xi\rangle\sim 2^k)}$$

where $\check{\widehat{v}}(\tau,\xi)=\widehat{v}(-\tau,-\xi)$ for all $(\tau,\xi)\in\mathbb{R}\times\lambda^{-1}\mathbb{Z}$. For $k=0,1,2$ this yields directly the result by using the Strichartz inequality (3.6). For $k\geq 3$

we introduce the following notation : we set

$$\tilde{\nu}_l(\tau,\xi) := \eta_l(\tau-\xi^3) \text{ for } l>k \quad \text{and} \quad \tilde{\nu}_l(\tau,\xi) := \sum_{j=0}^{k} \eta_j(\tau-\xi^3) \text{ for } l=k\ . \tag{4.6}$$

According to Lemma 3.4 we obtain

$$\begin{aligned} I_k &\lesssim \sum_{\min(l_1,l_2,l_3)\geq k} \sum_{l\geq 0} 2^k 2^{(l_1\wedge l_2)/2}\Big(2^{(l_1\vee l_2)/4}2^{-k/4}+1\Big)\|\tilde{\nu}_{l_1}\eta_k\widehat{v}_1\|_{L^2}\|\tilde{\nu}_{l_2}\tilde{\eta}_k\widehat{v}_2\|_{L^2} \\ &\quad 2^{(l\wedge l_3)/2}\Big(2^{(l\vee l_3)/4}2^{-k/4}+1\Big)2^{-(l\vee k)}\|\tilde{\nu}_{l_3}\tilde{\eta}_k\widehat{v}_3\|_{L^2}\|\nu_l\tilde{\eta}_k\widehat{w}\|_{L^2} \\ &\lesssim \sup_l(2^{-l/2}\|\nu_l\tilde{\eta}_k\widehat{w}\|_{L^2})\sup_{l\geq k}(2^{l/2}\|\tilde{\nu}_l\eta_k\widehat{v}_1\|_{L^2})\prod_{i=2}^{3}\sup_{l\geq k}\Big(2^{l/2}\|\tilde{\nu}_l\tilde{\eta}_k\widehat{v}_i\|_{L^2}\Big) \\ &\lesssim \sup_l(2^{-l/2}\|\nu_l\tilde{\eta}_k\widehat{w}\|_{L^2})\|\Delta_k v_1\|_{X^{0,1/2,1}_\lambda}\prod_{i=2}^{3}\|S_{k+1}v_i\|_{X^{0,1/2,1}_\lambda} \end{aligned} \tag{4.7}$$

where we used (2.10) in the last step. □

Lemma 4.2. *Let be given $t_0\in\mathbb{R}$, $\lambda\geq 1$ and $u_i\in G_\lambda$ for $i=1,2,3$. Then it holds*

$$J := \sum_{k=0}^{\infty} 2^k\|\Delta_k\Big(A(u_1,u_2,u_3)\Big)\|^2_{Z_{\lambda,k,t_0}} \lesssim \prod_{i=1}^{3}\|u_i\|^2_{G_\lambda}\ . \tag{4.8}$$

Proof. Again by translation in time, we can take $t_0=0$. Denoting by ξ_i the Fourier modes of u_i, we can always assume by symmetry that $|\xi_1|\geq|\xi_2|\geq|\xi_3|$. We divide A in different terms corresponding to regions of $(\lambda\mathbb{Z})^3$.
1.$|\xi_1|\leq 2^4$.
Then it holds $|\xi_1+\xi_2+\xi_3|\leq 2^6$. By Sobolev inequalities and (3.6),

$$J \lesssim \sum_{k=0}^{7}\Big[\prod_{i=1}^{3}\|\eta_0(2^k t)S_6 u_i\|_{X^{0,1/2,1}_\lambda}\Big]^2\ ,$$

which is acceptable thanks to (4.4)-(4.5).

2.$|\xi_1|\geq 2^4$ and $|\xi_1|\leq 4|\xi|$.
In this region, it holds $|\xi|\sim|\xi_1|$. Rewriting $\eta_k(\xi)$ as $\eta_k(\xi_1)+\eta_k(\xi)-\eta_k(\xi_1)$ and noticing that by the mean-value theorem,

$$|\eta_k(\xi)-\eta_k(\xi_1)| \lesssim \min\Big(1, 2^{-k}\Big||\xi|-|\xi_1|\Big|\Big)\ ,$$

we infer that $J \lesssim \sum_{k=0}^{\infty}(J_{1,k}^2 + J_{2,k}^2)$ with

$$J_{1,k} := 2^k \Big\| \tilde{\eta}_k(\xi)\langle \tau - \xi^3 + i2^k\rangle^{-1}\mathcal{F}_{tx}\Big(A_2(\Delta_k u_1, u_2, u_3)\Big)\Big\|_{F^{1/2}_{\lambda,k,0}}$$

and

$$J_{2,k} := 2^k \Big\| \tilde{\eta}_k(\xi)\Big[1\wedge(2^{-k}\Big||\xi|-|\xi_1|\Big|\Big]\langle \tau-\xi^3+i2^k\rangle^{-1}\mathcal{F}_{tx}\Big(A_2(u_1, u_2, u_3)\Big)\Big\|_{F^{1/2}_{\lambda,k,0}} ,$$

where

$$\mathcal{F}_x\Big(A_2(v_1, v_2, v_3)\Big)(\xi) := \frac{1}{\lambda^2}\sum_{\substack{(\xi_1,\xi_2,\xi_3)\in\Theta(\xi)\\ 2^4\le|\xi_1|\le 4|\xi|}} \hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3)$$

and

$$\Theta(\xi) := \Big\{(\xi_1,\xi_2,\xi_3)\in\lambda^{-1}\mathbb{Z},\ \sum_{i=1}^{3}\xi_i = \xi,\ \prod_{\substack{i,j=1\\ i\neq j}}^{3}(\xi_i+\xi_j)\neq 0 \text{ and } |\xi_3|\le|\xi_2|\le|\xi_1|\Big\} .$$

• **Estimate on $J_{1,k}$**
For any fixed $k\in\mathbb{N}$, we take a time extension $\tilde{u}_1$ of u_1 such that $\|\Delta_k\tilde{u}_1\|_{X^{0,1/2,1}_\lambda} \le 2\|\Delta_k u_1\|_{F^{1/2}_{\lambda,k,0}}$. We set $v_1 = \eta_0(2^k t)\tilde{u}_1$ and $v_i = \eta_0(2^k t)u_i$, $i = 2, 3$. By duality it suffices to prove that

$$\begin{aligned} H_{1,k} &:= 2^k\Big|\Big(\mathcal{F}_{xt}\Big[A_2(\Delta_k v_1, v_2, v_3)\Big], \langle\tau-\xi^3+i2^k\rangle^{-1}\widehat{w}\Big)_{L^2}\Big| \\ &\lesssim \sup_l(2^{-l/2}\|\nu_l\widehat{w}\|_{L^2})\|\Delta_k v_1\|_{X^{0,1/2,1}_\lambda}\prod_{i=2}^{3}\|S_{k+3}v_i\|_{X^{0,1/2,1}_\lambda} \qquad (4.9)\end{aligned}$$

which is acceptable thanks to (4.4)-(4.5). Again, since the norms in the right-hand side of (4.9) only see the size of the modulus of the Fourier transform of the functions we can assume that all the functions have non negative Fourier transforms. We separate four cases :
A. $\xi\xi_1 \le 0$. Then $|\xi_2+\xi_3| = |\xi-\xi_1| \sim |\xi|$. Proceeding as in (4.7), we get

$$\begin{aligned} H_{1,k} &\lesssim 2^k\|\Big(\langle\tau-\xi^3+i2^k\rangle^{-1}\tilde{\eta}_k\widehat{w}\Big) * \eta_k\check{\hat{v}_1}\|_{L^2(|\xi|\sim 2^k)}\|\eta_{\le k+3}\widehat{v_2} * \eta_{\le k+3}\widehat{v_3}\|_{L^2(|\xi|\sim 2^k)} \\ &\lesssim \sup_l(2^{-l/2}\|\nu_l\widehat{\tilde{\Delta}_k w}\|_{L^2})\|\Delta_k v_1\|_{X^{0,1/2,1}_\lambda}\prod_{i=2}^{3}\|S_{k+3}v_i\|_{X^{0,1/2,1}_\lambda} . \qquad (4.10)\end{aligned}$$

B. $\xi\xi_2 \leq 0$. Then $|\xi_1+\xi_3| = |\xi-\xi_2| \sim |\xi|$. Therefore, exchanging the role of v_1 and v_2, we can proceed exactly as in the previous case.
C. $\xi\xi_3 \leq 0$. Then $|\xi_1+\xi_2| = |\xi-\xi_3| \sim |\xi|$. Therefore, exchanging the role of v_1 and v_3, we can proceed exactly as in the case **A**.
D. ξ_1, ξ_2 and ξ_3 are of the same sign. Then $|\xi-\xi_3| = |\xi_1+\xi_2| \sim |\xi|$. Therefore we can proceed exactly as in the previous case.

• **Estimate on** $J_{2,k}$
For any fixed $k \in \mathbb{N}$, we set $v_i := \eta_0(2^k t)u_i$, $i = 1,2,3$. By duality and (4.4)-(4.5), it suffices to prove that

$$\begin{aligned} H_{2,k} &:= 2^k\Big|\Big(\Big[1\wedge(2^{-k}\Big||\xi|-|\xi_1|\Big|\Big]\mathcal{F}_{xt}\Big[A_2(v_1,v_2,v_3)\Big], \langle\tau-\xi^3+i2^k\rangle^{-1}\widehat{w}\Big)_{L^2}\Big| \\ &\lesssim 2^{-k/8}\sup_l(2^{-l/2}\|\nu_l\widehat{w}\|_{L^2})\prod_{i=1}^{3}\|S_{k+3}v_i\|_{X^{0,1/2,1}_\lambda} \end{aligned} \tag{4.11}$$

First to estimate the contribution of the region $\Big||\xi|-|\xi_1|\Big| \leq |\xi|^{3/4}$, we proceed exactly as for $J_{1,k}$ by separating the four cases **A**, **B**, **C** and **D** to obtain

$$H_{2,k} \lesssim 2^{-k/4}\sup_l(2^{-l/2}\|\nu_l\widehat{\tilde{\Delta}_k w}\|_{L^2})\prod_{i=1}^{3}\|S_{k+3}v_i\|_{X^{0,1/2,1}_\lambda}$$

which is acceptable.
Now in the region $\Big||\xi|-|\xi_1|\Big| \geq |\xi|^{3/4}$, we notice that $|\xi_2+\xi_3| = |\xi-\xi_1| \geq |\xi|^{3/4}$. Therefore, setting

$$\sigma := \sigma(\tau,\xi) = \tau-\xi^3 \text{ and } \sigma_i := \sigma(\tau_i,\xi_i),\ i=1,2,3, \tag{4.12}$$

we claim that the well-known resonance relation

$$\sigma-\sigma_1-\sigma_2-\sigma_3 = 3(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3) \tag{4.13}$$

leads to (recall that $|\xi| \sim |\xi_1|$)

$$\max(|\sigma|,|\sigma_i|) \gtrsim \lambda^{-1}|\xi|^{7/4} \tag{4.14}$$

Indeed, either we are in the cases **B** ,**C** or **D** described above and then there exists $i \in 2,3$ such that $|\xi_1-\xi_i| \sim |\xi|$ so that we are done or we are in the case **A**. In this last case, we first notice that $|\xi_2+\xi_3| \sim |\xi|$. Second, since $\xi\xi_1 \leq 0$ and $\Big||\xi|-|\xi_1|\Big| \geq |\xi|^{3/4}$ we must have $|\xi| \leq |\xi_1|-|\xi|^{3/4}$ and

$|\xi| = |\xi_2| + |\xi_3| - |\xi_1|$. It follows that $|\xi_3| \leq |\xi_1| - \frac{1}{2}|\xi|^{3/4}$ and ensures that (4.14) holds also in this region.
Using (3.7)-(3.8) and the notations (4.6), we get

$$\begin{aligned}
H_{2,k} \lesssim\ & 2^k \|\eta_{\leq k+2}\widehat{v_2} * \eta_{\leq k+3}\widehat{v_3}\|_{L^2(|\xi|\gtrsim 2^{3k/4})} \\
& \Big\|\Lambda[2^{3k/4}]\Big(\langle\tau-\xi^3+i2^k\rangle^{-1}\tilde{\eta}_k\widehat{w}\,,\,\eta_{\leq k+3}\check{\widehat{v_1}}\Big)\Big\|_{L^2} \\
\lesssim & \sum_{\substack{\min(l_1,l_2,l_3)\geq k, l\geq 0\\ \max(2^l,2^{l_i})\geq\lambda^{-1}2^{7k/4}}} 2^k 2^{(l_2\wedge l_3)/2}\Big(2^{(l_2\vee l_3)/4}2^{-\frac{3}{16}k}+\lambda^{-1/2}\Big)\|\tilde{\nu}_{l_2}\eta_{\leq k+3}\widehat{v_2}\|_{L^2}\|\tilde{\nu}_{l_3}\eta_{\leq k+3}\widehat{v_3}\|_{L^2} \\
& 2^{(l_1\wedge l)/2}\Big(2^{(l_1\vee l)/4}2^{-\frac{3}{16}k}+\lambda^{-1/2}\Big)^{1/4}\Big(2^{(l_1\vee l)/2}2^{-3k/4}+\lambda^{-1/2}\Big)^{3/4}2^{-(l\vee k)} \\
& \qquad \|\tilde{\nu}_{l_1}\eta_{\leq k+3}\widehat{v_1}\|_{L^2}\|\nu_l\tilde{\eta}_k\widehat{w}\|_{L^2} \\
\lesssim\ & 2^{-\frac{k}{32}}\sup_l(2^{-l/2}\|\nu_l\tilde{\eta}_k\widehat{w}\|_{L^2})\prod_{i=1}^3\sup_{l\geq k}\Big(2^{l/2}\|\tilde{\nu}_l S_{k+3}v_i\|_{L^2_\lambda}\Big) \\
\lesssim\ & 2^{-\frac{k}{32}}\sup_l(2^{-l/2}\|\nu_l\widehat{\tilde{\Delta}_k w}\|_{L^2})\prod_{i=1}^3\|S_{k+3}v_i\|_{X^{0,1/2,1}_\lambda}
\end{aligned}$$

where in the last step we used (2.9)-(2.10).

3.$|\xi_1| \geq 2^4$ and $|\xi_1| \geq 4|\xi|$.
In this region it holds

$$|\xi_2+\xi_3| \geq \frac{3}{4}|\xi_1|,\ |\xi_2| \geq \frac{3}{8}|\xi_1| \text{ and } |\xi_3| \leq \frac{5}{8}|\xi_1| \tag{4.15}$$

The two first above inequalities are clear. To prove the third one, we first notice that in this region $\xi_1\xi_2 \leq 0$ and we proceed by contradiction by assuming that $|\xi_3| > \frac{5}{8}|\xi_1|$. Then we first notice that if $\xi_1\xi_3 \geq 0$ then we must have $|\xi| \geq |\xi_3| > \frac{5}{8}|\xi_1|$ which contradicts $|\xi_1| \geq 4|\xi|$. Second, if $\xi_1\xi_3 \leq 0$ then we have $|\xi| = \Big||\xi_2|+|\xi_3|-|\xi_1|\Big| > \frac{5}{4}|\xi_1| - |\xi_1| = \frac{1}{4}|\xi_1|$ which again contradicts $|\xi_1| \geq 4|\xi|$.
Therefore the resonance relation yields

$$\max(|\sigma|,|\sigma_i|) \gtrsim \lambda^{-1}|\xi_1|^2\ . \tag{4.16}$$

In this region $J \lesssim \sum_{k\in\mathbb{N}} J_k^2$ where, for any fixed $k\in\mathbb{N}$, J_k is defined by

$$J_k := \sum_{k_1\geq k}\sum_{l\geq 0} 2^{l/2}2^k\Big\|\eta_l(\tau-\xi^3)\langle\tau-\xi^3+i2^k\rangle^{-1}\tilde{\eta}_k(\xi)\mathcal{F}_{tx}\Big(A_3(\Delta_{k_1}v_1,v_2,v_3)\Big)\Big\|_{L^2}$$

with

$$\mathcal{F}_x\Big(A_3(v_1,v_2,v_3)\Big)(\xi) := \frac{1}{\lambda^2} \sum_{\substack{(\xi_1,\xi_2,\xi_3)\in\Lambda(\xi)\\ |\xi_1|\geq 2^4,\, |\xi_1|\geq 4|\xi|}} \hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3)$$

and $v_i := \eta_0(2^k t)u_i,\ i=1,2,3.$
Let γ be a C^∞-function $\gamma : \mathbb{R} \mapsto [0,1]$ with compact support in $[-1,1]$ satisfying

$$\gamma \equiv 1 \text{ on } [-1/2,1/2] \text{ and } \sum_{m\in\mathbb{Z}} \gamma(t-m)^3 \equiv 1 \text{ on } \mathbb{R}. \tag{4.17}$$

We set

$$v_i^{k_1,m} := \eta_0(2^k t)\gamma(2^{k_1}t-m)u_i = \gamma(2^{k_1}t-m)v_i,\ i=1,2,3,\ m\in\mathbb{Z}$$

Clearly, $A(v_1,v_2,v_3) \equiv \sum_{|m|\lesssim 2^{k_1-k}} A(v_1^{k_1,m}, v_2^{k_1,m}, v_3^{k_1,m})$. Therefore, by duality it suffices to prove that

$$\begin{aligned} I_k \ :=\ & \sum_{k_1\geq k}\sum_{|m|\lesssim 2^{k_1-k}}\sum_{l\geq 0} 2^{l/2}2^k \Big|\Big(\mathcal{F}_{xt}\Big[A_3(\Delta_{k_1}v_1^{k_1,m}, v_2^{k_1,m}, v_3^{k_1,m})\Big], \\ & \qquad \langle\tau-\xi^3+i2^k\rangle^{-1}\eta_l(\tau-\xi^3)\tilde{\eta}_k\widehat{w}\Big)_{L^2}\Big| \\ \lesssim\ & 2^{-k/4}\sup_l(2^{-l/2}\|\nu_j\tilde{\eta}_k\widehat{w}\|_{L^2})\| \sup_{m\in\mathbb{Z}}\prod_{i=1}^3 \|S_{k_1+1}v_i^{k_1,m}\|_{X_\lambda^{0,1/2,1}} \ . \end{aligned} \tag{4.18}$$

Indeed, this will be acceptable, since by proceeding as in (4.4)-(4.5), it is easily checked that

$$\begin{aligned} \|S_{k_1+1}v_i^{k_1,m}\|^2_{X_\lambda^{0,1/2,1}} \ \leq\ & \sum_{j=0}^{k_1-1}\|\Delta_j u_i\|^2_{F^{1/2}_{\lambda,j,m2^{-k_1}}} + \sum_{q=-1}^{1}\|\Delta_{k_1}u_i\|^2_{F^{1/2}_{\lambda,k_1,m2^{-k_1}+q2^{-(k_1+1)}}} \\ & + \sum_{q=-2}^{2}\|\Delta_{k_1+1}u_i\|^2_{F^{1/2}_{\lambda,k_1+1,m2^{-k_1}+q2^{-(k_1+2)}}} \lesssim \|u_i\|^2_{G_\lambda}\ . \end{aligned}$$

Since the norms in the right-hand side of (4.18) only see the size of the modulus of the Fourier transform of the functions we can assume that all the functions have non negative Fourier transforms. We will use the following notations :

$$\tilde{\nu}_l(\tau,\xi) := \eta_l(\tau-\xi^3) \text{ for } l>k_1 \ \text{ and } \quad \tilde{\nu}_l(\tau,\xi) := \eta_{\leq k_1}(\tau-\xi^3) \text{ for } l=k_1\ . \tag{4.19}$$

In view of (4.15), in this region it holds $\Big||\xi_1|-|\xi_3|\Big| \gtrsim |\xi_1|$ and $|\xi-\xi_2| \gtrsim |\xi_1|$. Lemma 3.4, (4.16) and (2.9)-(2.10) thus lead to

$$
\begin{aligned}
I_k \lesssim & \sum_{k_1\geq k}\sum_{|m|\lesssim 2^{k_1-k}}\sum_{l\geq 0} 2^{l/2}2^k \Big\|\Lambda[2^{k_1}]\Big(\eta_{k_1}\widehat{v_1^{k_1,m}}\,,\,\eta_{\leq k_1+1}\widehat{v_3^{k_1,m}}\Big)\Big\|_{L^2} \\
& \qquad \Big\|\Big(\langle\tau-\xi^3+i2^k\rangle^{-1}\eta_k\widehat{w}\Big) * \eta_{\leq k_1+1}\check{\widehat{v_2^{k_1,m}}}\Big\|_{L^2(|\xi|\gtrsim 2^{k_1})} \\
\lesssim & \sum_{k_1\geq k}\sum_{|m|\lesssim 2^{k_1-k}}\sum_{\substack{l_1,l_2,l_3\geq k_1,l\geq 0\\ \max(2^l,2^{l_i})\geq\lambda^{-1}2^{2k_1}}} 2^{l/2}2^k 2^{(l_1\wedge l_3)/2}(2^{(l_1\vee l_3)/2}2^{-k_1}+\lambda^{-1/2})^{3/4} \\
& (2^{(l_1\vee l_3)/4}2^{-k_1/4}+\lambda^{-1/2})^{1/4}2^{(l\wedge l_2)/2}(2^{(l\vee l_2)/4}2^{-k_1/4}+\lambda^{-1/2})2^{-(l\vee k)} \\
& \|\tilde{\nu}_{l_2}\eta_{\leq k+1}\widehat{v_2^{k_1,m}}\|_{L^2}\|\tilde{\nu}_{l_3}\eta_{\leq k_1+1}\widehat{v_3^{k_1,m}}\|_{L^2}\|\tilde{\nu}_{l_1}\eta_{k_1}\widehat{v_1^{k_1,m}}\|_{L^2}\|\nu_l\eta_k\widehat{w}\|_{L^2} \\
\lesssim & \sum_{k_1\geq k}\sum_{|m|\lesssim 2^{k_1-k}} 2^k 2^{-\frac{17}{16}k_1}\sup_l\Big(2^{l/2}\|\nu_l\eta_k\widehat{w}\|_{L^2}\Big)\prod_{i=1}^3\sup_{l\geq k_1}\Big(2^{\frac{15}{32}l}\|\tilde{\nu}_l\eta_{\leq k_1+1}\widehat{v_i^{k_1,m}}\|_{L^2}\Big) \\
\lesssim & \sum_{k_1\geq k} 2^{-\frac{k_1}{16}}\sup_l\Big(2^{l/2}\|\nu_l\eta_k\widehat{w}\|_{L^2}\Big)\sup_{|m|\lesssim 2^{k_1-k}}\Big[\prod_{i=1}^3\|S_{k_1+1}v_i^{k_1,m}\|_{X_\lambda^{0,15/32,1}}\Big] \qquad (4.20)
\end{aligned}
$$

which yields (4.18) by summing in k_1 and concludes the proof of the lemma. □

Corollary 4.1. *There exists $\varepsilon_0>0$ such that any $\lambda\geq 1$ and any solution $u\in C(\mathbb{R};H^\infty(\lambda\mathbb{T}))$ to (1.3) satisfying $\|u_0\|_{L^2_\lambda}\leq\varepsilon_0$, it holds*

$$\|u\|_{G_\lambda}\lesssim\|u_0\|_{L^2_\lambda}\,. \qquad (4.21)$$

Proof. We are going to implement a continuity argument on the space period. Recall that if $u(t,x)$ is a smooth global $2\lambda\pi$-periodic solution of (1.3) with initial data u_0 then $u_\beta(t,x)=\beta^{-1}u(\beta^{-3}t,\beta^{-1}x)$ is a $(2\pi\lambda\beta)$-periodic solution of (1.3) emanating from $u_{0,\beta}=\beta^{-1}u_0(\beta^{-1}x)$. Moreover,

$$\|u_{0,\beta}\|_{L^2_{\lambda\beta}}=\beta^{-1/2}\|u_0\|_{L^2_\lambda}\text{ and }\|u_{0,\beta}\|_{H^1_{\lambda\beta}}\leq\beta^{-1/2}\|u_0\|_{H^1_\lambda}\,.$$

From the conservation of the L^2-norm and of the Energy,

$$E(u):=\frac{1}{2}\int_{\mathbb{T}}u_x^2\mp\frac{1}{6}\int_{\mathbb{T}}u^4\,,$$

and Sobolev inequalities (in the focusing case) , we get

$$\|u_\beta\|_{L^\infty(\mathbb{R};L^2_{\lambda\beta})}=\|u_{0,\beta}\|_{L^2_{\lambda\beta}}\text{ and }\|u_\beta\|_{L^\infty(\mathbb{R};H^1_{\lambda\beta})}\lesssim\|u_{0,\beta}\|_{H^1_{\lambda\beta}}(1+\|u_{0,\beta}\|^2_{L^2_{\lambda\beta}})$$

In particular, it follows from (4.2) that for $\beta \geq 1$,

$$\|u_\beta\|_{G_{\lambda\beta}} \leq C \left(\|u_0\|_{L^2_\lambda} + \|u_\beta\|^3_{G_{\lambda\beta}} \right) , \tag{4.22}$$

for some constant $C > 0$. Now, from classical linear estimates in Bourgain's spaces and the Duhamel formula, it holds

$$\begin{aligned} \|u_\beta\|_{G_{\lambda\beta}} &\lesssim \sup_{t_0\in\mathbb{R}} \|\eta_0(t-t_0)u_\beta\|_{X^{0,1/2,1}_{\lambda\beta}} \lesssim \|u_\beta(t_0)\|_{L^2_{\lambda\beta}} + \|(u_\beta^2 - P_0(u_\beta^2))\partial_x u_\beta\|_{L^2(]t_0-2,t_0+2[;L^2_{\lambda\beta})} \\ &\lesssim \|u_\beta\|_{L^\infty(\mathbb{R};L^2_{\lambda\beta})} + \|u_\beta\|^3_{L^\infty(\mathbb{R};H^1_{\lambda\beta})} \lesssim \beta^{-1/2}\|u_0\|_{L^2_\lambda} + \beta^{-3/2}\|u_0\|^3_{H^1_\lambda} . \end{aligned}$$

Therefore for $\beta \geq 1$ large enough (depending on $\|u_0\|_{H^1}$), $\|u_\beta\|_{G_{\lambda\beta}} \leq C\|u_0\|_{L^2_\lambda}$. Noticing that $\beta \mapsto \|u_\beta\|_{G_{\lambda\beta}}$ is continuous on $\mathbb{R}^*_+$, the result follows by (4.22) and a classical continuity argument.

5 Proof of Theorem 1.2

In this section we follow the process proposed in [16] to identify the limit of a L^2-bounded sequence of solutions to mKdV. To pass to the limit on the nonlinear term in (1.3), we will make use of the space $F^{s,b}_{\lambda,T}$ introduced in [9]. We will need the following lemma which states that for any $s < 0$, $b < 1/2$ and $T > 0$, G_λ is compactly embedded in $F^{s,b}_{\lambda,T}$ and that any bounded sequence in G_λ is uniformly equi-continuous with values in $H^s(\mathbb{T})$ for $s < 0$.

Lemma 5.1. *Let $\lambda \geq 1$ and $\{u_n\}_{n\in\mathbb{Z}}$ be a bounded sequence of G_λ. Then*

1. *For any $T > 0$, $s < 0$ and $b < 1/2$, $\{u_n\}_{n\in\mathbb{Z}}$ is relatively compact in $F^{s,b}_{\lambda,T}$.*
2. *For any $\varepsilon > 0$ and $s < 0$, there exists $\delta_{\varepsilon,s} > 0$ such that $\forall (t_1,t_2) \in \mathbb{R}^2$,*

$$|t_1 - t_2| < \delta_{\varepsilon,s} \Rightarrow \|u_n(t_1) - u_n(t_2)\|_{H^s_\lambda} < \varepsilon, \ \forall n \in \mathbb{N} .$$

Proof. First we observe that for any $s < 0$ and any $u \in G_\lambda$,

$$\begin{aligned} \|u\|^2_{F^{s,1/2}_\lambda} &= \sum_{k\in\mathbb{N}} 2^{2ks} \sup_{t\in\mathbb{R}} \|\Delta_k u\|^2_{X^{1/2}_{\lambda,k,t}} \\ &\lesssim (\sum_{k\in\mathbb{N}} 2^{2ks}) \sup_{k\in\mathbb{N}} \sup_{t\in\mathbb{R}} \|\Delta_k u\|^2_{X^{1/2}_{\lambda,k,t}} \\ &\leq \left(C(s)\|u\|_{G_\lambda} \right)^2 \end{aligned} \tag{5.1}$$

Hence $G_\lambda \hookrightarrow F_\lambda^{s,1/2}$. Second, proceeding as in Remark 2.2, it is easy to check that there exists $C_0 > 0$ such that for all $s < 0$ and $v \in F_\lambda^{s,1/2}$,

$$\|v\|_{L^\infty(\mathbb{R};H_\lambda^s)} \leq C_0 \|v\|_{F_\lambda^{s,1/2}} . \tag{5.2}$$

Let us now prove assertion 2. We set $M := \sup_{n\in\mathbb{N}} \|u_n\|_{G_\lambda}$. For any given $\varepsilon > 0$ and $s < 0$, we denote by $k_{\varepsilon,s}$ the smaller integer such that

$$\Big(\sum_{k=k_{\varepsilon,s}}^{\infty} 2^{2ks}\Big)^{1/2} < \frac{\varepsilon}{4C_0\, C(s)M} . \tag{5.3}$$

From (5.1) -(5.3) we infer that

$$\Big\|\sum_{k=k_{\varepsilon,s}+1}^{\infty} \Delta_k u_n(t_1) - \sum_{k=k_{\varepsilon,s}+1}^{\infty} \Delta_k u_n(t_2)\Big\|_{H_\lambda^s} \leq 2C_0 \Big\|\sum_{k=k_{\varepsilon,s}+1}^{\infty} \Delta_k u_n\Big\|_{F_\lambda^s} \leq \varepsilon/2 .$$

Now, by (2.8), for $|t_1 - t_2| < 2^{-k_{\varepsilon,s}}/4$,

$$\begin{aligned}
\Big\|\sum_{k=0}^{k_{\varepsilon,s}}\Big(\Delta_k u_n(t_1) \;-\; \Delta_k u_n(t_2)\Big)\Big\|_{H_\lambda^s}^2 &= \Big\|\sum_{k=0}^{k_{\varepsilon,s}}\Big(\gamma(2^k t_1 - t_1)\Delta_k u_n(t_1) - \gamma(2^k t_2 - t_1)\Delta_k u_n(t_2)\Big)\Big\|_{H_\lambda^s}^2 \\
&\sim \sum_{k=0}^{k_{\varepsilon,s}} 2^{2ks}\Big\|\int_{\mathbb{R}} \mathcal{F}_{t,x}\Big(\Delta_k\gamma(2^k t - t_1)u_n\Big)(\xi,\tau)(e^{it_1\tau} - e^{it_2\tau})\, d\tau\Big\|_{L^2_\xi}^2 \\
&\lesssim |t_1 - t_2|^2 \sum_{k=0}^{k_{\varepsilon,s}}\Big\|\int_{\mathbb{R}} \Big|\mathcal{F}_{t,x}\Big(\Delta_k\gamma(2^k t - t_1)u_n\Big)(\xi,\tau)\Big|\, d\tau\Big\|_{L^2_\xi}^2 \\
&\lesssim |t_1 - t_2|^2 \sum_{k=0}^{k_{\varepsilon,s}}\Big\|\Delta_k\gamma(2^k t - t_1)u_n\Big\|_{X_\lambda^{0,1/2,1}}^2 \\
&\lesssim |t_1 - t_2|^2 M^2 .
\end{aligned}$$

This gives the desired result for $|t_1 - t_2| \leq \delta_{\varepsilon,s} := \min(\frac{\varepsilon}{2M}, 2^{-k_{\varepsilon,s}}/4)$.

Let us now prove the first assertion. We will use a diagonal extraction argument. Let us fix $s' < 0$, $b' < 1/2$ and $T > 0$. We notice that for any fixed $k_0 \in \mathbb{N}$, there exists $C_{k_0} > 0$ such that

$$\|S_{k_0} v\|_{F_{\lambda,T}^{s',b'}} \lesssim \|S_{k_0} v\|_{X_{\lambda,T}^{s',b'}} \leq C_{k_0} \|S_{k_0} v\|_{F_{\lambda,T}^{s',b'}}$$

Indeed, the first inequality is obvious and the second one follows from the following chain of inequalities for any $v \in F^{s',b'}_{\lambda,T}$,

$$\begin{aligned}
\|S_{k_0}v\|^2_{X^{s',b'}_{\lambda,T}} &\leq \sum_{k=0}^{k_0+1} 2^{2ks'} \|\eta_0(\frac{t}{2T})\Delta_k S_{k_0}\tilde{v}\|^2_{X^{0,b'}_{\lambda}} \\
&\lesssim \sum_{k=0}^{k_0+1} 2^k T \sup_{t\in\mathbb{R}} 2^{2ks'} \|\Delta_k S_{k_0}\tilde{v}\|^2_{F^{b'}_{\lambda,k,t}} \\
&\lesssim \Big(C_{k_0}\|S_{k_0}v\|_{F^{s',b'}_{\lambda,T}}\Big)^2
\end{aligned}$$

where $\tilde{v}$ is an extension of v such that $\|S_{k_0}v\|_{F^{s',b'}_{\lambda}} \leq 2\|S_{k_0}v\|_{F^{s',b'}_{\lambda,T}}$. Since, according to (5.1), $\{u_n\}_{n\geq 0}$ is bounded in $F^{s',1/2}_{\lambda}$, it follows that $\{S_{k_0}u_n\}_{n\geq 0}$ is bounded in $X^{s',b'}_{\lambda}$ for any $k_0 \geq 0$. Now, using that for $s < s'$ and $b < b'$, $X^{s',b'}_{\lambda,T+1}$ is compactly embedded in $X^{s,b}_{\lambda,T+1}$, we deduce that there exists a subsequence $\{u_{n_q}\}$ of $\{u_n\}$ and a sequence $\{w_k\} \subset F^{s,b}_{\lambda,T+1}$ such that for any $k \in \mathbb{N}$,

$$S_k u_{n_q} \to w_k \text{ in } F^{s,b}_{\lambda,T+1} \ . \tag{5.4}$$

We define $w \in \mathcal{S}'(]-T-1, T+1[\times\mathbb{T})$ by $\Delta_k w = \Delta_k w_{k+1}$ for all $k \in \mathbb{N}$. Clearly, for all $k_0 \geq 0$,

$$\sum_{k=0}^{k_0} 2^{2ks} \sup_{t\in -]T,T[} \|\Delta_k w\|^2_{F^{s,b}_{\lambda,t}} \lesssim \sum_{k=0}^{\infty} 2^{2ks} \sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \|\Delta_k u_n\|^2_{F^{s,b}_{\lambda,t}} \lesssim \sup_{n\in\mathbb{N}} \|u_n\|^2_{G_\lambda} \ .$$

which ensures that $w \in F^{s,b}_{\lambda,T}$. Moreover, combining (5.1) and (5.4), it follows that $\|u_{n_q} - w\|_{F^{s,b}_{\lambda,T}} \to 0$ as $q \to \infty$. □

As noticed in [16], B has got a nice structure for passing to the limit in the sense of distributions. More precisely, we have the following lemma:

Lemma 5.2. *For any $\lambda \geq 1$ and $T > 0$, the operator $\partial_x B$ is continuous from $(F^{-1/3,1/3}_{\lambda,T})^3$ into $X^{-4,-1/3}_{\lambda,T}$.*

Proof. Taking $w \in X^{4,1/3}_{\lambda}$ supported in time in $]-2T, 2T[$ and extensions $\tilde{u}_i \in F^{-1/3,1/3}_{\lambda}$ of $u_i \in F^{-1/3,1/3}_{\lambda,T}$ such that $\|\tilde{u}_i\|_{F^{-1/3,1/3}_{\lambda}} \leq 2\|u_i\|_{F^{-1/3,1/3}_{\lambda,T}}$, it

holds

$$
\begin{aligned}
I &:= \Big|\Big(w, \partial_x B(u_1,u_2,u_3)\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big| \\
&\lesssim \sum_{k=0}^{\infty}\Big|\Big(\Delta_k w_x, B(\tilde\Delta_k\tilde u_1, \tilde\Delta_k\tilde u_2, \tilde\Delta_k\tilde u_3)\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big| \\
&\lesssim \sum_{k=0}^{\infty}\sum_{\substack{|m|\lesssim T2^k\\ i=1,2,3}}\Big|\Big(\Delta_k w_x, B(\tilde\Delta_k v_1^{k,m_1}, \tilde\Delta_k v_2^{k,m_2}, \tilde\Delta_k v_3^{k,m_3})\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big|
\end{aligned}
$$

where for any $k\in\mathbb{N}$, $i\in\{1,2,3\}$ and $m\in\mathbb{Z}$, we set $v_i^{k,m} := \gamma(2^k t - m)\tilde u_i$ with γ defined as in (4.17). (3.6) and Bernstein inequality then ensure that

$$
\begin{aligned}
I &\lesssim \sum_{k=0}^{\infty}\sum_{|m|\lesssim T2^k} 2^{2k}\|\mathcal{F}^{-1}(|\widehat{\Delta_k w}|)\|_{L^4_\lambda}\prod_{i=1}^{3}2^{-k/3}\|\mathcal{F}^{-1}(|\widehat{\tilde\Delta_k v_i^{k,m}}|)\|_{L^4_\lambda} \\
&\lesssim T\sum_{k=0}^{\infty}2^{3k}\|\Delta_k w\|_{X^{0,1/3}_\lambda}\prod_{i=1}^{3}\sup_{m\in\mathbb{Z}}\|\tilde\Delta_k v_i^{k,m}\|_{X^{-1/3,1/3}_\lambda} \qquad (5.5)
\end{aligned}
$$

But on account of (2.9) (with obvious modification for $k=0$),

$$
\begin{aligned}
\|\tilde\Delta_k v_i^{k,m}\|_{X^{-1/3,1/3}_\lambda} &\lesssim \sum_{j=-1}^{1}\|\Delta_{k+j}\gamma(2^k t - m)\tilde u_i\|_{X^{-1/3,1/3}_\lambda} \\
&\lesssim \sum_{j=-1}^{1}2^{-(k+j)/3}\sup_{t\in\mathbb{R}}\|\Delta_{k+j}\tilde u_i\|_{F^{1/3}_{\lambda,k+j,t}} \\
&\lesssim \|\tilde u_i\|_{F^{-1/3,1/3}_\lambda} \lesssim \|u_i\|_{F^{-1/3,1/3}_{\lambda,T}}\ .
\end{aligned}
$$

Therefore, (5.5) leads to

$$
I \lesssim T\|w\|_{X^{4,1/3}_\lambda}\prod_{i=1}^{3}\|u_i\|_{F^{-1/3,1/3}_{\lambda,T}} \qquad (5.6)
$$

which concludes the proof of the lemma. □

Let us now prove a continuity result for the non resonant part A.

Lemma 5.3. *For any $\lambda\geq 1$ and $T>0$, the operator $\partial_x A$ is continuous from $(F^{-2^{-6},15/32}_{\lambda,T})^3$ into $X^{-4,-1/2}_{\lambda,T}$.*

Proof. Taking $w \in X_\lambda^{4,1/2}$ supported in time in $]-2T,2T[$ and extensions $\tilde{u}_i \in F_\lambda^{-2^{-6},15/32}$ of $u_i \in F_{\lambda,T}^{-2^{-6},15/32}$ such that $\|\tilde{u}_i\|_{F_\lambda^{-2^{-6},15/32}} \leq 2\|u_i\|_{F_{\lambda,T}^{-2^{-6},15/32}}$, it holds

$$\begin{aligned} J &:= \Big|\Big(w, \partial_x A(u_1,u_2,u_3)\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big| \\ &\lesssim \sum_{(k,k_1,k_2,k_3)\in\mathbb{N}^4} \Big|\Big(\Delta_k w_x, A(\Delta_{k_1}\tilde{u}_1, \Delta_{k_2}\tilde{u}_2, \Delta_{k_3}\tilde{u}_3)\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big| \end{aligned}$$

By symmetry we may assume that $k_1 \geq k_2 \geq k_3$. Now, the sum in the reion $k_1 \leq 6k+10$ can clearly be treated exactly in the same way as we treat B in the preceding lemma. It thus remains to consider the sum over the region $k_1 > 6k+10$. We follow the proof of Lemma 4.2. First we notice that (4.15) -(4.16) hold in this region. Then setting,

$$v_i^{k_1,m} := \gamma(2^{k_1}t - m)\tilde{u}_i,\ i=1,2,3,\ m\in\mathbb{Z},$$

with γ defined as in (4.17), we obtain in this region

$$J \lesssim \sum_{k\geq 0} \sum_{\substack{k_1\geq k_2\geq k_3\\ k_1\geq 6k+10}} \sum_{|m|\lesssim T2^{k_1}} \Big|\Big(\Delta_k w_x, A(\Delta_{k_1}v_1^{k_1,m}, \Delta_{k_2}v_2^{k_1,m}, \Delta_{k_3}v_3^{k_1,m})\Big)_{L^2(\mathbb{R}\times\lambda\mathbb{T})}\Big|$$

Proceeding exactly as in (4.20) we get

$$\begin{aligned} J &\lesssim \sum_{k\geq 0} \sum_{\substack{k_1\geq k_2\geq k_3\\ k_1\geq 6k+10}} 2^{-\frac{k_1}{16}} \|\Delta_k w\|_{X_\lambda^{2,1/2}} \prod_{i=1}^3 \sup_{m\in\mathbb{Z}} \|\Delta_{k_i} v_i^{k_1,m}\|_{X_\lambda^{0,15/32}} \\ &\lesssim T\sum_{k\geq 0}\sum_{k_1\geq 0} 2^{-\frac{k_1}{16}} k_1^2 \|\Delta_k w\|_{X_\lambda^{2,1/2}} \sup_{m\in\mathbb{Z}}(\|\Delta_{k_1} v_1^{k_1,m}\|_{X_\lambda^{0,15/32}}) \prod_{i=2}^3 \sup_{\substack{0\leq k_3\leq k_2\leq k_1\\ m\in\mathbb{Z}}} \|\Delta_{k_i} v_i^{k_1,m}\|_{X_\lambda^{0,15/32}} \\ &\lesssim T\|w\|_{X_\lambda^{3,1/2}} \prod_{i=1}^3 \sup_{t\in\mathbb{R},k\in\mathbb{N}} 2^{-2^{-6}k} \|\Delta_k \tilde{u}_i\|_{F_{\lambda,k,t}^{15/32}} \\ &\lesssim T\|w\|_{X_\lambda^{3,1/2}} \prod_{i=1}^3 \|u_i\|_{F_{\lambda,T}^{-2^{-6},15/32}} \end{aligned} \tag{5.7}$$

which completes the proof of the lemma . □

For any $T>0$, let us define the operator Υ_T which to u associates

$$\Upsilon_T(u) := \mathcal{F}_{tx}^{-1}\Big(\Gamma_T(u)\Big)$$

where

$$\Gamma_T(u)(\xi,\tau) := \frac{6i\xi}{\lambda^2}\int_{\tau_1+\tau_2+\tau_3=\tau}\Big[\widehat{\psi_T u}(\xi,\tau_1)\widehat{\psi_T u}(\xi,\tau_2)\widehat{\psi_T u}(-\xi,\tau_3)$$
$$-\frac{1}{3}\sum_{\substack{\xi_1+\xi_2+\xi_3=\xi\\(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)\neq 0}}\widehat{\psi_T u}(\xi_1,\tau_1)\widehat{\psi_T u}(\xi_2,\tau_2)\widehat{\psi_T u}(\xi_3,\tau_3)\Big]d\tau_1\,d\tau_2\,d\tau_3\,, \quad (5.8)$$

with $\psi_T(\cdot) := \psi(\cdot/T)$ for ψ defined as in Section 2.2.
In view of (4.1), for any smooth function $u \in \mathcal{S}(\lambda\mathbb{T}\times\mathbb{R})$, $\Upsilon_T(u) \equiv 6(u^2 - P_0(u^2))u_x$ on $]-T,T[$. From the two above lemmas we infer that Υ_T can be continuously extended in $F_\lambda^{-2^{-6},15/32}$ with values in $X_\lambda^{-4,-1/2}$. In particular, for any $u \in F_\lambda^{-2^{-6},15/32}$ and any $T > 0$, this operator defined an element of $X_\lambda^{-4,-1/2}$ with a $X_\lambda^{-4,-1/2}$-norm which is, according to (5.6) and (5.7), of order at most $O(T)$. This ensures that we can pass to the limit in $\mathcal{S}'$ on $\Gamma_T(u)$ as $T\to\infty$. We can thus define the operator Γ from $F_\lambda^{-2^{-6},15/32}$ into $\mathcal{S}'(\lambda\mathbb{T}\times\mathbb{R})$ by setting

$$\langle\Gamma(u),\phi\rangle_{\mathcal{S}',\mathcal{S}}, := \lim_{T\to\infty}\langle\Gamma_T(u),\phi\rangle_{\mathcal{S}',\mathcal{S}}, \quad \forall\phi\in\mathcal{S}(\lambda\mathbb{T}\times\mathbb{R})$$

Obviously, $\mathcal{F}_{tx}^{-1}(\Gamma(u)) \equiv 6(u^2 - P_0(u^2))u_x$ on $\lambda\mathbb{T}\times\mathbb{R}$ for any $u\in\mathcal{S}(\lambda\mathbb{T}\times\mathbb{R})$.

Definition 5.1. *We will say that a function u is a weak solution of* (1.3) *if it satisfies* (1.3) *in the sense of distributions, when $(u^2 - P_0(u^2))u_x$ is interpreted as the inverse Fourier transform of $\Gamma(u)$.*

Proposition 5.1. *Let $\{u_{0,n}\}\subset H^\infty(\mathbb{T})$ be such that $u_{0,n}\rightharpoonup u_0$ in $L^2(\mathbb{T})$. Then there exist a weak solution $u\in C_w(\mathbb{R};L^2(\mathbb{T}))\cap\Big(\bigcup_{s<0}F^{s,1/2}\Big)$ to* (1.3)*, with $u(0)=u_0$, and a subsequence of emanating solutions $\{u_{n_k}\}$ to* (1.3) *such that for all $T>0$ and $\phi\in L^2(\mathbb{T})$,*

$$(u_{n_k}(t),\phi)_{L^2(\mathbb{T})}\to(u(t),\phi)_{L^2(\mathbb{T})} \text{ in } C([-T,T])\,. \quad (5.9)$$

Moreover, $P_0(u(t)) = P_0(u_0)$ for all $t\in\mathbb{R}$.

Proof. We proceed as in [16]. By Banach-Steinhaus theorem, the sequence $\{u_{0,n}\}$ is bounded in $L^2(\mathbb{T})$. We start by assuming that $\sup_{n\in\mathbb{N}}\|u_{0,n}\|_{L^2}\le\varepsilon_0$ so that the conclusions of Corollary 4.1 hold. From the conservation of the L^2-norm, the sequence of emanating solutions $\{u_n\}$ to (1.3) is bounded in $L^\infty(\mathbb{R};L^2(\mathbb{T}))$ and thus, up to a subsequence, $\{u_n\}$ converges weakly star in $L^\infty(\mathbb{R};L^2(\mathbb{T}))$ to some $u\in L^\infty(\mathbb{R};L^2(\mathbb{T}))$. In particular, $\{\partial_t u_n\}$ and $\{\partial_x^3 u_n\}$

converge in the distributional sense to respectively u_t and u_{xxx}. It remains to pass to the limit on the nonlinear term. By Corollary 4.1, $\{u_n\}$ is bounded in G. From Lemmas 5.1 and 5.3-5.2 it follows that $\partial_x(A(u_n)+B(u_n))$ converges to $\mathcal{F}_{tx}^{-1}(\Gamma(u))$ in the distributional sense on $]-T,T[\times\mathbb{T}$. Since this holds for all $T>0$, the convergence holds actually in the distributional sense on $\mathbb{R}\times\mathbb{T}$. Therefore, u is a weak solution to (1.3) in the sense of Definition 5.1. Note also that, in view of (5.1), $u\in\cup_{s<0}F^{s,1/2}$. Moreover, according to assertion 2 of Lemma 5.1, for any time-independent 2π-periodic smooth function ϕ, the family $\{t\mapsto(u_n(t),\phi)_{L^2_x}\}$ is bounded and uniformly equi-continuous on $[-T,T]$. Ascoli's theorem then ensures that $\{t\mapsto(u_{n_k},\phi)\}$ converges to $t\mapsto(u,\phi)$ in $C([-T,T])$. Since $\{u_n\}$ is bounded in $L^\infty(\mathbb{R};L^2(\mathbb{T}))$, this convergence also holds for any $\phi\in L^2(\mathbb{T})$. This proves (5.9) and that $u\in C_w(\mathbb{R};L^2(\mathbb{T}))$. In particular, $u(0)=u_0$ and for all $t\in\mathbb{R}$,

$$\int_{\mathbb{T}}u(t)\,dx=\lim_{k\to\infty}\int_{\mathbb{T}}u_{n_k}(t)\,dx=\lim_{k\to\infty}\int_{\mathbb{T}}u_{n_k}(0)\,dx=\int_{\mathbb{T}}u_0\,dx\ .$$

This concludes the proof of the proposition whenever $\sup_{n\in\mathbb{N}}\|u_{0,n}\|_{L^2}\le\varepsilon_0$. Now, if $\sup_{n\in\mathbb{N}}\|u_{0,n}\|_{L^2}=M>\varepsilon_0$ we use the dilation symmetry of (1.3). Recall that if $u(t,x)$ is a smooth global 2π-periodic solution of (1.3) with initial data u_0 then $u^\lambda(t,x)=\lambda^{-1}u(\lambda^{-3}t,\lambda^{-1}x)$ is a $2\pi\lambda$-periodic solution of (1.3) emanating from $u_0^\lambda=\lambda^{-1}u_0(\lambda^{-1}x)$. Setting $\lambda_M=M^2/\varepsilon^2$ so that

$$\sup_{n\in\mathbb{N}}\|u_{0,n}^{\lambda_M}\|_{L^2_{\lambda_M}}\le\varepsilon_0\ ,$$

it follows from above that the conclusions of the proposition hold if one replaces $\{u_{0,n}\}$, $\{u_n\}$ and u by respectively $\{u_{0,n}^{\lambda_M}\}$, $\{u_n^{\lambda_M}\}$ and $u^{\lambda_M}\in C_w(\mathbb{R};L^2(\lambda_0\mathbb{T}))\cap\left(\cup_{s<0}F^{s,1/2}_{\lambda_M}\right)$. This ensures[1] that these conclusions also hold for $\{u_{0,n}\}$, $\{u_n\}$ and u with $u(t,x):=\lambda_M u^{\lambda_M}(\lambda_M^3t,\lambda_M x)$ and completes the proof of the proposition. □

Proof of Theorem 1.2 : Again we have to introduce a notion of weak solution for mKdV:

Definition 5.2. *We will say that a function $v\in C_w(\mathbb{R};L^2(\mathbb{T}))$ is a weak solution of* (1.2) *with initial data v_0 if it satisfies the following equation in the sense of distributions,*

$$v_t+v_{xxx}\mp6(v^2-P_0(v^2))v_x\pm6P_0(v_0^2)v_x=0\ , \tag{5.10}$$

when $(v^2-P_0(v^2))v_x$ is interpreted as the inverse Fourier transform of $\Gamma(v)$.

[1] It can be easily checked that the dilation symmetry $u\mapsto\lambda u(\lambda^3t,\lambda x)$ is an isomorphism from $F^{s,1/2}$ into $F^{s,1/2}_\lambda$ for any $\lambda\ge1$ and $s\in\mathbb{R}$.

Of course, any smooth solution of the mKdV equation is a weak solution since the L^2-norm is a constant of the motion for smooth solutions.

Let $v_0 \in L^2(\mathbb{T})$. It is easy to construct a sequence $\{v_{0,n}\} \subset H^\infty(\mathbb{T})$ such that $\|v_{0,n}\|_{L^2} = \|v_0\|_{L^2}$ for any $n \geq 0$ and $v_{0,n} \to v_0$ in $L^2(\mathbb{T})$. The emanating solutions v_n satisfies

$$v_{n,t} + v_{n,xxx} \pm 6P_0(v_0^2)v_x \mp 6(v_n^2 - P_0(v_n^2))v_x = 0 \ .$$

We will consider $6P_0(v_0^2)v_x$ as a part of the linear group of the equation. The u_n satisfy the same Duhamel formula as the solution of (1.3) where we substitute the linear group of the KdV equation $U(t)$ by the linear group $V(t)$ defined by

$$\widehat{V(t)\varphi}(\xi) = e^{iq(\xi)t}\,\hat{\varphi}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z} \quad , \ q(\xi) = \xi^3 \mp 6P_0(v_0^2)\xi \ . \tag{5.11}$$

It is direct to check that all linear estimates remain true when changing the functional spaces in consequence. Also the bilinear estimates in Lemma 3.4 remain true since the resonance relation remain unchanged (see the proof of this lemma in the appendix). Therefore, the results of Section 4 and the conclusions of Proposition 5.1 remain true when substituting the function spaces associated with $U(\cdot)$ by those associated with $V(\cdot)$ and (1.3) by (5.10). We thus obtain a weak solution $v \in C_w(\mathbb{R}; L^2(\mathbb{T})) \cap \left(\cup_{s<0}\tilde{F}^{s,1/2}\right)$ to (5.10) with initial data v_0, where $\tilde{F}^{s,1/2}$ is defined as $F^{s,1/2}$ but for the group $V(\cdot)$. Moreover, by the weak convergence result (5.9), $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2}$ for all $t \in \mathbb{R}$ and thus the weak continuity of v ensures that $v(t) \to v(0)$ in $L^2(\mathbb{T})$ as $t \to 0$. This completes the proof of assertion *i*).

Now in the defocusing case, according to [11], the sequence $\{v_n\}$ of solutions to mKdV emanating from $\{v_{0,n}\}$ converges in $C(\mathbb{R}; L^2(\mathbb{T}))$ to some function w such that $\|w(t)\|_{L^2} = \|v_0\|_{L^2}$ for all $t \in \mathbb{R}$. By the uniqueness of the limit in $\mathcal{D}'(\mathbb{R} \times \mathbb{T})$, $w \equiv v$ on $\mathbb{R}$ and thus w is a weak solution of the defocussing mKdV equation in the sense of Definition 5.2. Actually, using the conservation of the L^2-norm for w, we also obtain that w satisfy the following equation in the sense of distributions,

$$w_t + w_{xxx} - 6(w^2 - P_0(w^2))w_x + 6P_0(w^2)w_x = 0 \ , \tag{5.12}$$

when $(w^2 - P_0(w^2))w_x$ is interpreted as the inverse Fourier transform of $\Gamma(w)$. This concludes the proof of assertion *ii*).

6 Proof of Theorem 1.1

6.1 The mKdV equation

We will prove that the solution-map $u_0 \mapsto u$ is not continuous at any $u_0 \in H^\infty(\mathbb{T})$ from $L^2(\mathbb{T})$ equipped with its weak topology into $\mathcal{D}'(]0,T[\times\mathbb{T})$. This obviously leads to the desired result since $L^2(\mathbb{T})$ is compactly embedded in $H^s(\mathbb{T})$ for any $s<0$. Since the sign in front of the nonlinear term will not play any role in the proof, we choose to take the plus sign to simplify the notations.

Let $u_0 \in H^\infty(\mathbb{T})$ be a non constant function and $\kappa \neq 0$ be a real number. We set

$$u_{0,n} = u_0 + \kappa \cos(nx)$$

so that $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$ and $\|u_{0,n}\|_{L^2}^2 \to \|u_0\|_{L^2}^2 + \kappa^2\pi$. According to Proposition 5.1 there exists a subsequence $\{u_{n_k}\}$ of the emanating solutions $\{u_n\}$ to (1.3) and $u \in C_w(\mathbb{R}; L^2(\mathbb{T}))$ a weak solution of (1.3) , with $u(0) = u_0$, satisfying $u_{n_k}(t) \rightharpoonup u(t)$ in $L^2(\mathbb{T})$ for all $t \in \mathbb{R}$. Let now $\{v_{n_k} := u_{n_k}(\cdot, \cdot - \frac{6t}{2\pi}\|u_{0,n_k}\|_{L^2}^2)\}$ be the associated subsequence of solutions to mKdV emanating from $\{u_{0,n_k}\}$. We proceed by contradiction. Assuming that the solution-map is continuous at u_0 from $L^2(\mathbb{T})$ equipped with its weak topology into $\mathcal{D}'(]0,T[\times\mathbb{T})$, we obtain that $\{v_{n_k}\}$ converges in the sense of distributions in $]0,T[\times\mathbb{T}$ to the solution $v \in C^\infty(\mathbb{R}; H^\infty(\mathbb{T}))$ of mKdV emanating from u_0. It follows that $\{u_{n_k}\}$ converges in the same sense to $v\Big(\cdot, \cdot + \frac{6t}{2\pi}(\|u_0\|_{L^2}^2 + \kappa^2\pi)\Big)$ and thus

$$u \equiv v\Big(\cdot, \cdot + \frac{6t}{2\pi}(\|u_0\|_{L^2}^2 + \kappa^2\pi)\Big) \text{ on }]0,T[\ . \tag{6.1}$$

This ensures that u is actually a strong solution of (1.3) and satisfies this equation everywhere on $]0,T[\times\mathbb{T}$. On the other hand, according to (6.1), u is also solution of

$$u_t + u_{xxx} + 6(u^2 - P_0(u^2) - \kappa^2/2)u_x = 0\ .$$

This forces u_x to be identically vanishing on $]0,T[$ which contradicts that $u(0) = u_0$ is not a constant and $u \in C_w(\mathbb{R}; L^2(\mathbb{T}))$.

Note that in the contradiction process, we can replace the assumption on the continuity of the solution-map by the assumption that the flow-map $u_0 \mapsto u(t)$ is continuous from $L^2(\mathbb{T})$, equipped with its weak topology, into $\mathcal{D}'(\mathbb{T})$ for all $t \in]0,T[$. Since for each $t \in \mathbb{R}$, $u_{n_k}(t) \rightharpoonup u(t)$ in $L^2(\mathbb{T})$ we also get a contradiction. This proves assertion *ii")* of Remark 1.1.

6.2 The KdV equation

First, recall that Miura discovered that the Miura map $M(u) := u' + u^2$ maps smooth solutions to the defocusing mKdV equation into smooth solutions to the KdV equation. Actually, it was observed in [8] that the Riccati map

$$R(u) := u' + u^2 - P_0(u^2)$$

maps smooth solutions to the defocusing version of equation (1.3) (i.e. with the + sign in front of the nonlinear term) into smooth solutions to the KdV equation (1.1), i.e. if u is a smooth solution of the defocusing (1.3) then $R(u)$ is a smooth solution of (1.1). Moreover, according to [10], this map enjoys the following property :

Theorem 6.1 ([10])**.** *The Riccati map R is an isomorphism from $L^2_0(\mathbb{T})$ into $H^{-1}_0(\mathbb{T})$.*

Actually, we will only need the Riccati map to be a bijection from $H^\infty_0(\mathbb{T})$ into itself. For sake of completeness, we give in the appendix the outline of the proof of this property.

Now, let $w_0 \in H^\infty_0(\mathbb{T})$ and $\kappa \in \mathbb{R}^*$ be given. We set $\theta_0 := w_0 - \kappa^2 \cos x$, $u_0 := R^{-1}(\theta_0) \in H^\infty_0(\mathbb{T})$[2] and

$$u_{0,n} := u_0 + \kappa[\cos(nx) + \cos((n+1)x)] \ .$$

Clearly, $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$, $\|u_{0,n}\|^2_{L^2} \to \|u_0\|^2_{L^2} + 2\kappa^2\pi$ and $R(u_{0,n}) \rightharpoonup \theta_0 + \kappa^2 \cos(x) = w_0$ in $H^{-1}(\mathbb{T})$. According to Proposition 5.1 there exists a subsequence $\{u_{n_k}\}$ of the emanating solutions $\{u_n\}$ to (1.3) and $u \in C_w(\mathbb{R}; L^2_0(\mathbb{T}))$ such that $u_{n_k}(t) \rightharpoonup u(t)$ in $L^2(\mathbb{T})$ for all $t \in \mathbb{R}$ and u is a weak solution to (1.3). To identify $R(u)$ we will need the following lemma

Lemma 6.1. *The operator $u \mapsto u^2 - P_0(u^2)$ is continuous from $F_T^{-1/16,7/16}$ into $\mathcal{D}'(]-T,T[\times\mathbb{T})$.*

Proof. We set

$$C(u,u) := u^2 - P_0(u^2) = \sum_{\xi\in\mathbb{Z}^*} \Big[\sum_{\substack{(\xi_1,\xi_2)\in\mathbb{Z}^2 \\ \xi_1+\xi_2=\xi}} \hat{u}(\xi_1)\hat{u}(\xi_2) \Big] e^{i\xi x}$$

[2]It is easy to check that $u_0 \in L^2(\mathbb{T})$ and $R(u_0) \in H^\infty_0(\mathbb{T})$ ensure that $u_0 \in H^\infty_0(\mathbb{T})$.

Taking $w \in \mathcal{D}(]-T,T[\times\mathbb{T})$ and extensions $\tilde{u}_i \in F^{-1/16,7/16}$ of $u_i \in F_T^{-1/16,7/16}$ such that $\|\tilde{u}_i\|_{F^{-1/16,7/16}} \leq 2\|u_i\|_{F_T^{-1/16,7/16}}$, it holds

$$\begin{aligned} J &:= \left|\Big(w, C(u_1,u_2)\Big)_{L^2(\mathbb{R}\times\mathbb{T})}\right| \\ &\lesssim \sum_{(k,k_1,k_2)\in\mathbb{N}^3} \left|\Big(\Delta_k w, C(\Delta_{k_1}\tilde{u}_1, \Delta_{k_2}\tilde{u}_2)\Big)_{L^2(\mathbb{R}\times\mathbb{T})}\right| \end{aligned}$$

By symmetry we may assume that $k_1 \geq k_2$. We set

$$v_i^{k,m} := \gamma(2^{k_1}t - m)\tilde{u}_i,\ i=1,2,\ m\in\mathbb{Z},$$

where $\gamma : \mathbb{R} \mapsto [0,1]$ is a C^∞-function with compact support in $[-1,1]$ satisfying $\gamma \equiv 1$ on $[-1/2,1/2]$ and $\sum_{m\in\mathbb{Z}}\gamma(t-m)^2 \equiv 1$ on $\mathbb{R}$. We then obtain

$$J \lesssim \sum_{\substack{(k,k_2,k_3)\in\mathbb{Z}\\ k_1\geq k_2}} \sum_{|m|\lesssim T2^{k_1}} \left|\Big(\Delta_k w, C(\Delta_{k_1}v_1^{k_1,m}, \Delta_{k_2}v_2^{k_2,m})\Big)_{L^2(\mathbb{R}\times\mathbb{T})}\right|.$$

We separate $\mathbb{Z}^3$ into two regions.
1. The region : $k_1 \leq 4k+4$. Then by (3.6) and (2.9), we can write

$$\begin{aligned} J &\lesssim \sum_{\substack{(k,k_2,k_3)\in\mathbb{Z}\\ k_1\geq k_2}} \sum_{|m|\lesssim T2^{k_1}} \|\Delta_k w\|_{L^2}\|\Delta_{k_1}v_1^{k_1,m}\|_{L^4}\|\Delta_{k_2}v_2^{k_2,m}\|_{L^4} \\ &\lesssim T \sum_{\substack{(k,k_2,k_3)\in\mathbb{Z}\\ k_1\geq k_2}} 2^{k_1}2^{2k_1/3}\|\Delta_k w\|_{L^2}\prod_{i=1}^{2}\sup_{m\in\mathbb{Z}} 2^{-k_1/3}\|\Delta_{k_i}v_i^{k_i,m}\|_{X^{0,1/3}} \\ &\lesssim T(\sup_{k\in\mathbb{N}} 2^{5k/3}\|\Delta_k w\|_{L^2})\prod_{i=1}^{2}\sup_{k_i\in\mathbb{N}}\sup_{m\in\mathbb{Z}}\|\Delta_{k_i}v_i^{k_i,m}\|_{X^{-1/3,1/3}} \\ &\lesssim T\|w\|_{L^2(\mathbb{R};H^2(\lambda\mathbb{T}))}\|u_1\|_{F_T^{-1/3,1/3}}\|u_2\|_{F_T^{-1/3,1/3}} \end{aligned} \tag{6.2}$$

which is acceptable.
2. The region : $k_1 > 4k+4$. Note that in this region, $k_2 \geq k_1 - 2 \geq 4k+2$. In this region we will need the well-known resonance relation

$$|\sigma - \sigma_1 - \sigma_2| = |3\xi\xi_1\xi_2| \gtrsim 2^{2k_1}\ , \tag{6.3}$$

where $\xi = \xi_1 + \xi_2$ and $\sigma, \sigma_1, \sigma_2$ are defined by (4.12). We subdivide this region into subregions with respect to the maximum of $(|\sigma|, |\sigma_i|)$.

• $|\sigma| = \max(|\sigma|, |\sigma_1|, \sigma_2|)$. In this subregion, making use of (6.3), (3.6) and (2.9), we get

$$\begin{aligned} J &\lesssim \sum_{\substack{(k,k_2,k_3)\in\mathbb{Z}^3\\ k_1\geq k_2,\, k_1>4k+4}} \sum_{|m|\lesssim T2^{k_1}} \|\Delta_k w\|_{L^2} \|\Delta_{k_1} v_1^{k_1,m}\|_{L^4} \|\Delta_{k_2} v_2^{k_2,m}\|_{L^4} \\ &\lesssim T \sum_{\substack{(k,k_2,k_3)\in\mathbb{Z}\\ k_1\geq k_2,\, k_1>4k+4}} 2^{k_1} 2^{2k_1/3} 2^{-2k_1} \|\Delta_k w\|_{X^{0,1}} \prod_{i=1}^{2} \sup_{m\in\mathbb{Z}} 2^{-k_1/3} \|\Delta_{k_i} v_i^{k_i,m}\|_{X^{0,1/3}} \\ &\lesssim T\lambda \|w\|_{X^{0,1}} \|u_1\|_{F_T^{-1/3,1/3}} \|u_2\|_{F_T^{-1/3,1/3}} \ . \end{aligned} \tag{6.4}$$

• $|\sigma_1| = \max(|\sigma|, |\sigma_1|, \sigma_2|)$. In this subregion, defining $\tilde{\nu}_k$, $k \geq k_1$ as in (4.19) and making use of (6.3), (3.7) and (2.9)-(2.10), we get

$$\begin{aligned} J &\lesssim \sum_{\substack{(k,k_1,k_2)\in\mathbb{Z}^3\\ k_1\geq k_2,\, k_1>4k+4}} \sum_{|m|\lesssim T2^{k_1}} \sum_{l\geq 0, l_2\geq k_1} 2^{l/2} 2^k \|\eta_{k_1} \widehat{\Delta w} * \tilde{\nu}_{l_2} \check{\widehat{v_2^{k_2,m}}}\|_{L^2(|\xi|\gtrsim 2^{k_1})} \|\Delta_{k_1} v_1^{k_1,m}\|_{L^2} \\ &\lesssim T \sum_{\substack{(k,k_1,k_2)\in\mathbb{Z}^3\\ k_1\geq k_2,\, k_1>4k+4}} \sum_{l\geq 0, l_2\geq k_1} 2^{l/2} 2^{k_1} (2^{l_2/4} 2^{-k_1/4} + 1) \|\Delta_k w\|_{L^2} \\ &\qquad \sup_{m\in\mathbb{Z}} \|\tilde{\nu}_{l_2} \widehat{\Delta_{k_2} v_2^{k_2,m}}\|_{L^2}\, 2^{-7k_1/8} \sup_{m\in\mathbb{Z}} \|\Delta_{k_1} v_1^{k_1,m}\|_{X^{0,7/16}} \\ &\lesssim T \sum_{\substack{(k,k_1,k_2)\in\mathbb{Z}^3\\ k_1\geq k_2,\, k_1>4k+4}} \sum_{l\geq 0} 2^{l/2} 2^{k_1} 2^{-21k_1/16} \|\Delta_k w\|_{L^2} \| \sup_{\substack{m\in\mathbb{Z}\\ l_2\geq k_1}} \|\tilde{\nu}_{l_2} \widehat{\Delta_{k_2} v_2^{k_2,m}}\|_{L^2} \\ &\qquad \sup_{m\in\mathbb{Z}} \|\Delta_{k_1} v_1^{k_1,m}\|_{X^{0,7/16}} \\ &\lesssim T \|w\|_{X^{0,1}} \prod_{i=1}^{2} \|u_i\|_{F_T^{-1/16,7/16}} \ . \end{aligned} \tag{6.5}$$

• $|\sigma_2| = \max(|\sigma|, |\sigma_1|, \sigma_2|)$. Then we can proceed exactly as in the preceding case by exchanging the role of $v_1^{k_1,m}$ and $v_2^{k_2,m}$. Gathering (6.2), (6.4) and (6.5), we obtain the desired continuity result. □

By Corollary 4.1, Lemma 5.1 and possibly dilation arguments as in the proof of Proposition 5.1, we know that the sequence $\{u_n\}$ is relatively compact in $F_T^{-1/16,7/16}$ for any $T > 0$. From the above lemma we thus infer that

$$R(u_n) \rightharpoonup R(u) \text{ weak star in } L^\infty(\mathbb{R}; H^{-1}(\mathbb{T})) \ .$$

We proceed now by contradiction. Let us assume that the solution-map is continuous at $w_0 = \theta_0 + \kappa^2 \cos x$ from $H^{-1}(\mathbb{T})$ equipped with its weak

topology into $\mathcal{D}'(]0,T[\times\mathbb{T})$. Since $w_0 \in H_0^\infty(\mathbb{T})$, we deduce from Lemma 6.1 that $w_0 = R(\Theta_0)$ for some $\Theta_0 \in H_0^\infty(\mathbb{T})$ with $\Theta_0 \neq u_0$. By the continuity of the solution-map at w_0, $R(u)$ must be equal on $]0,T[$ to the solution of KdV emanating from w_0 which is nothing else but $R(\Theta)$ where Θ is the smooth solution to (1.3) emanating from Θ_0. This ensures that $u(t) \in H_0^\infty(\mathbb{T})$ for all $t \in]0,T[$ and the injectivity of R on $H_0^\infty(\mathbb{T})$ then forces $u = \Theta$ on $]0,T[$. This contradicts that $u(0) = u_0 \neq \Theta_0 = \Theta(0)$ and both functions are weakly continous with values in $L^2(\mathbb{T})$.

Acknowledgements: L.M. was partially supported by the ANR project "Equa-Disp".

7 Appendix

7.1 A simplified proof of (3.6)

We give below a very simple proof of (3.6). Recall that this estimate was first established in [1]. To simplify the notations we take $\lambda = 1$ (this corresponds to 2π-periodic in space functions) but it easy to check that the proof is exactly the same for any $\lambda \geq 1$ (see for instance Section 7.2 for the slight modifications in the case $\lambda \geq 1$). By the triangle inequality, we write

$$\begin{aligned} \|v\|_{L^4_{tx}}^2 = \|v^2\|_{L^2} = \|\widehat{v} \star \widehat{v}\|_{L^2} &\lesssim \sum_{l_1 \geq 0, l_2 \geq 0} \Big\| (\beta_{l_1}|\hat{v}|) \star (\beta_{l_2}|\hat{v})| \Big\|_{L^2} \\ &\lesssim \sum_{l_1 \geq l_2 \geq 0} \Big\| (\beta_{l_1}|\hat{v}|) \star (\beta_{l_2}|\hat{v}|) \Big\|_{L^2} \quad . \end{aligned}$$

where the β_l, $l \in \mathbb{N}$, are defined in (2.2). The proof of (3.6) will then follow from the following lemma.

Lemma 7.1. *Let u_1 and u_2 be $L^2(\mathbb{Z}\times\mathbb{R})$-real valued functions then for any $(l_1, l_2) \in \mathbb{N}^2$,*

$$\Big\| (\nu_{l_1} u_1) \star (\nu_{l_2} u_2) \Big\|_{L^2} \lesssim \left(2^{l_1} \wedge 2^{l_2}\right)^{1/2} \left(2^{l_1} \vee 2^{l_2}\right)^{1/6} \|\nu_{l_1} u_1\|_{L^2} \, \|\nu_{l_2} u_2\|_{L^2} \quad . \tag{7.1}$$

Indeed, with this lemma in hand, rewritting l_1 as $l_1 = l_2 + l$ with $l \in \mathbb{N}$,

we get the following chain of inequalities

$$\begin{aligned}
\sum_{l_1\geq l_2\geq 0}\Big\|(\nu_{l_1}|\hat{v}|)\star(\nu_{l_2}|\hat{v}|)\Big\|_{L^2} &\lesssim \sum_{l\geq 0}\sum_{l_2\geq 0}2^{l_2/2}2^{(l_2+l)/6}\|\nu_{l_2+l}\widehat{v}\|_{L^2}\|\nu_{l_2}\widehat{v}\|_{L^2}\\
&\lesssim \sum_{l\geq 0}\sum_{l_2\geq 0}2^{l_2/3}\|\nu_{l_2}\hat{v}\|_{L^2}2^{-l/6}2^{(l_2+l)/3}\|\nu_{l_2+l}\widehat{v}\|_{L^2}\\
&\lesssim \sum_{l\geq 0}2^{-l/6}\Big(\sum_{l_2}2^{l_2/3}\|\nu_{l_2}\hat{v}\|^2_{L^2}\Big)^{1/2}\Big(\sum_{l_2\geq 0}2^{(l_2+l)/3}\|\nu_{l_2+l}\widehat{v}\|^2_{L^2}\Big)^{1/2}\\
&\lesssim \|v\|^2_{X^{0,1/3}} \quad .
\end{aligned}$$

It thus remains to prove Lemma 7.1. Following the arguments given in [2] (see also [[20], page 460]for more details) we can assume that u and w are supported in $\{(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}_+\}$. Let us recall that these arguments are based on the fact that the operator $\mathbf{j}:L^2(\mathbb{R}\times\mathbb{Z})\to L^2(\mathbb{R}\times\mathbb{Z})$, defined by $\mathbf{j}(u)(\tau,\xi)=u(-\tau,-\xi)$, is an isometry of $L^2(\mathbb{R}\times\mathbb{Z})$ satisfying, for any real-valued $L^1\cap L^2$-functions u_1 and u_2,

$$\|u_1\star u_2\|_{L^2(\mathbb{R}\times\mathbb{Z})}=\|u_1\star\mathbf{j}(u_2)\|_{L^2(\mathbb{R}\times\mathbb{Z})}\ .$$

By Cauchy-Schwarz in (τ_1,ξ_1) we infer that

$$\begin{aligned}
\|(\beta_{l_1}u_1)\star(\beta_{l_2}u_2)\|^2_{L^2} &= \int_\tau\sum_{\xi\in\mathbb{Z}}\Big|\int_{\tau_1}\sum_{\xi_1\in\mathbb{Z}}(\beta_{l_1}u_1)(\tau_1,\xi_1)\,(\beta_{l_2}u_2)(\tau-\tau_1,\xi-\xi_1)\Big|^2\\
&\lesssim \int_\tau\sum_{\xi\in\mathbb{Z}}\alpha(\tau,\xi)\int_{\tau_1}\sum_{\xi_1\in\mathbb{Z}}\Big|(\beta_{l_1}u_1)(\tau_1,\xi_1)\,(\beta_{l_2}u_2)(\tau-\tau_1,\xi-\xi_1)\Big|^2\\
&\lesssim \sup_{\tau\in\mathbb{R},\xi\in\mathbb{Z}_+}\alpha(\tau,\xi)\,\|\beta_{l_1}u_1\|^2_{L^2}\|\beta_{l_2}u_2\|^2_{L^2} \quad ,
\end{aligned}$$

where

$$\begin{aligned}
\alpha(\tau,\xi) &\lesssim mes\Big\{(\tau_1,\xi_1)\in\mathbb{R}\times\mathbb{Z}_+\ /\ \xi-\xi_1\in\mathbb{Z}_+,\\
&\qquad\qquad \langle\tau_1-\xi_1^3\rangle\sim 2^{l_1}\text{ and }\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle\sim 2^{l_2}\Big\}\\
&\lesssim (2^{l_1}\wedge 2^{l_2})\,\#A(\tau,\xi)\ ,
\end{aligned}$$

with

$$A(\tau,\xi):=\{\xi_1\geq 0/\ \xi-\xi_1\geq 0\text{ and }\langle\tau-\xi_1^3-(\xi-\xi_1)^3\rangle\lesssim 2^{l_1}\vee 2^{l_2}\}\ .$$

We separate two regions. In the region $\xi^3 \geq 2^{l_1} \vee 2^{l_2}$, we notice that $\partial_y^2[\tau - y^3 - (\xi - y)^3] = -6\xi$ which leads to

$$\#A(\tau,\xi) \lesssim \Big(\frac{2^{l_1} \vee 2^{l_2}}{\xi}\Big)^{1/2} + 1 \lesssim \Big(2^{l_1} \vee 2^{l_2}\Big)^{1/3} .$$

In the region $0 \leq \xi^3 \leq 2^{l_1} \vee 2^{l_2}$, we use that $0 \leq \xi_1 \leq \xi$ to obtain that

$$\#A(\tau,\xi) \leq \#\{\xi_1,\ 0 \leq \xi_1^3 \leq 2^{l_1} \vee 2^{l_2}\} \lesssim \Big(2^{l_1} \vee 2^{l_2}\Big)^{1/3} .$$

This completes the proof of (7.1). □

7.2 Proof of (3.7)-(3.8)

We take $\lambda \geq 1$. Let $A \subset \lambda^{-1}\mathbb{Z}$ and let, for any $\xi \in \lambda^{-1}\mathbb{Z}$, $B(\xi) \subset \lambda^{-1}\mathbb{Z}$. To prove (3.7)-(3.8) we first notice that, by Cauchy-Schwarz,

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \frac{1}{\lambda} \sum_{\xi_1 \in B(\xi)} u_1(\tau_1,\xi_1)\, u_2(\tau-\tau_1,\xi-\xi_1)\, d\tau_1 \Big\|^2_{L^2(A\times\mathbb{R})} \\
&\lesssim \int_\tau \frac{1}{\lambda} \sum_{\xi\in A} \alpha(\tau,\xi) \int_{\tau_1} \frac{1}{\lambda} \sum_{\xi_1\in B(\xi)} \Big| u_1(\tau_1,\xi_1)\, u_2(\tau-\tau_1,\xi-\xi_1)\Big|^2 d\tau_1\, d\tau \\
&\lesssim \sup_{\tau\in\mathbb{R},\xi\in A} \alpha(\tau,\xi)\, \|u_1\|^2_{L^2} \|u_2\|^2_{L^2} \quad ,
\end{aligned}$$

where

$$\alpha(\tau,\xi) \lesssim mes\Big\{(\tau_1,\xi_1) \in \mathbb{R}\times B(\xi) \,/\, (\tau_1,\xi_1) \in \mathrm{supp}(u_1) \text{ and } (\tau-\tau_1,\xi-\xi_1) \in \mathrm{supp}(u_2)\Big\} .$$

Therefore, assuming that $\mathrm{supp}(u_i) \subset \{(\tau,\xi)/\langle \tau - \xi^3\rangle \lesssim L_i\}$, we get

$$\alpha(\tau,\xi) \lesssim (L_1 \wedge L_2)\, mes[C(\tau,\xi)]$$

with

$$C(\tau,\xi) := \{\xi_1 \in B(\xi)/\langle \tau - \xi_1^3 - (\xi-\xi_1)^3\rangle \lesssim L_1 \vee L_2\} .$$

(3.7) then follows by noticing that $\partial_y^2[\tau - y^3 - (\xi - y)^3] = -6\xi$ and thus, for any $|\xi| \geq N > 0$,

$$mes[C(\tau,\xi)] = \frac{1}{\lambda}\#C(\tau,\xi) \lesssim \frac{1}{\lambda}\Big[\Big(\lambda\, \frac{(L_1 \vee L_2)^{1/2}}{N^{1/2}}\Big) \vee 1\Big] .$$

Finally, (3.8) follows by noticing that

$$\partial_y[\tau - y^3 - (\xi-y)^3] = -3\Big(y + (\xi - y)\Big)\Big(y - (\xi - y)\Big)$$

and thus, on $B(\xi) = \{\xi_1 \in \lambda^{-1}\mathbb{Z} \,/\, \Big||\xi_1| - |\xi - \xi_1|\Big| \geq N > 0\}$, it holds

$$\Big|\Big(\xi_1 + (\xi - \xi_1)\Big)\Big(\xi_1 - (\xi - \xi_1)\Big)\Big| \gtrsim N^2$$

which leads to

$$mes[C(\tau, \xi)] \lesssim \frac{1}{\lambda}\Big[\Big(\lambda\, \frac{L_1 \vee L_2}{N^2}\Big) \vee 1\Big] \,.$$

7.3 Outline of the proof of the bijectivity of the Riccati map from $H_0^\infty(\mathbb{T})$ into itself.

We follow the arguments in [8]. For $u \in H_0^\infty(\mathbb{T})$ we denote by L_u the Schrödinger operator with potential u, i.e.

$$L_u := -\frac{d}{dx^2} + u$$

with domain $H^2(\mathbb{T})$. One can associate to L_u the energy $E_u(\cdot)$ defined on $H^1(\mathbb{T})$ by

$$E_u(\phi) := \langle L_u\phi, \phi\rangle = \int_{\mathbb{T}} \phi_x^2 + u\phi^2 \,.$$

Since L_u is a self adjoint operator with compact resolvent, it has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq ...$ with $\lambda_n \to +\infty$. By the definition of E_u one must have $E_u(\phi) \geq \lambda_1 \int_{\mathbb{T}} \phi^2$ for any $\phi \in H^1(\mathbb{T})$ with equality if and only if ϕ is a λ_1-eigenfunction. For $u \not\equiv 0$, noticing that $E(1) = 0$ and that 1 is not an eigenfunction, it follows that the first eigenvalue λ_1 is negative. Then, by standard arguments, one can check that λ_1 is a simple eigenvalue with an eigenfunction that is a non vanishing $H^\infty(\mathbb{T})$-function. On the other hand, if $u \equiv 0$ it is well-known that the first eigenvalue of L_0 is 0 and that the associated eigenspace is spanned by 1. In both cases, we normalized this eigenfunction by requiring it to be positive and L^2-normalized and we call it ϕ_1. Introducing the logarithmic derivative v of ϕ_1, defined by

$$v := \frac{d}{dx}\ln(\phi_1) = \frac{\phi_{1,x}}{\phi_1} \in H_0^\infty(\mathbb{T})$$

we observe that

$$v_x = \frac{\phi_{1,xx}}{\phi_1} - \Big(\frac{\phi_{1,x}}{\phi_1}\Big)^2 = u - \lambda_1 - v^2$$

and thus $u = v_x + v^2 + \lambda_1$. Taking the means of both sides of this equality it leads to $\lambda_1 = -P_0(v^2)$ which ensures that $u = R(v)$. This proves that

R is surjective from $H_0^\infty(\mathbb{T})$ into itself. Now, let $w \in H_0^\infty(\mathbb{T})$ be such that $R(w) = u$. Setting

$$\rho := \exp(\int_0^x w(s)\, ds)$$

it is easy to check that $w = \rho'/\rho$. Observing that

$$(\frac{d}{dx} + w)(-\frac{d}{dx} + w) = -\frac{d^2}{dx^2} + R(w) + P_0(w^2) = -\frac{d^2}{dx^2} + u + P_0(w^2)\,,$$

easy calculations then lead to

$$E_u(\phi) = \int_{\mathbb{T}} (\phi' - w\phi)^2 - P_0(w^2) \int_{\mathbb{T}} \phi^2, \qquad \forall \phi \in H^1(\mathbb{T})\ .$$

It follows that $E(\rho) = -P_0(w^2) \int_{\mathbb{T}} \rho^2$ and $E(\phi_1) \geq -P_0(w^2) \int_{\mathbb{T}} \phi_1^2$. This ensures that $-P_0(w^2) = \lambda_1$. Therefore, $\rho/\sqrt{\int \rho^2} = \phi_1$ and thus $w = \phi_1'/\phi_1 = v$. This proves the injectivity of R in $H_0^\infty(\mathbb{T})$.

References

[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations, I. Schrödinger equations II. The KdV equation,* GAFA, **3** (1993), 209–262.

[2] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation,* GAFA, **3** (1993), 315–341.

[3] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data,* Sel. Math. New. Ser. **3** (1993), pp. 115–159.

[4] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation and low regularity ill-posedness for canonical defocusing equations,* Amer. J. Math., **125** (2003), no. 6, 1235–1293.

[5] M. Christ, J. Colliander, and T. Tao, *A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order,* J. Funct. Anal., **2**54 (2008), no. 2, 368395.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness results for periodic and non-periodic KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$*,* J. Amer. Math. Soc. **1**6 (2003), 705–749.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Multilinear estimates for periodic KdV equations, and applications,* J. Funct. Analysis **2**11 (2004), 173–218.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Symplectic nonsqueezing of the Korteweg-de Vries flow,* Acta Math. **1**95 (2005), 197252.

[9] A. D. Ionescu, C. Kenig, D. Tataru, *Global well-posedness of the initial value problem for the KP-I equation in the energy space*, Invent. Math. **1**73 2 (2008), 265–304.

[10] T. Kappeler and P. Topalov, *Riccati map of $L^2_0(\mathbb{T})$ and its applications*, J. Math. Anal. Appl. **309** (2005), 544–566.

[11] T. Kappeler and P. Topalov, *Global well-posedness of mKdV in $L^2(\mathbb{T},\mathbb{R})$,* Comm. Partial Differential Equations **30** (2005), no. 1-3, 435–449.

[12] T. Kappeler and P. Topalov, *Global wellposedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$,* Duke Math. J. **1**35 (2006), no. 2, 327–360.

[13] C. E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation,* J. Amer. Math. Soc., **9**, no. 2 (1996), 573–603.

[14] H. Koch and D.Tataru, *A priori bounds for the 1D cubic NLS in negative Sobolev spaces* Int. Math. Res. Not. (2007), no. 16.

[15] H. Koch and N. Tzvetkov *Local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$*, Int. Math. Res. Not. 26, (2003) 1449-1464.

[16] L. Molinet, *On ill-posedness for the periodic cubic Schrödinger equation,* Mathematical Res. Lett. **16** (2009), 111–120.

[17] L. Molinet, *A note on ill-posedness for the KdV equation,* Differential Integral Equations **24** (2011), 759–765.

[18] L. Molinet and S. Vento, *Sharp ill-posedness and well-posedness results for the KdV-Burgers equation: the periodic case,* To appear in Trans. A.M.S.

[19] K. Nakanishi, H. Takaoka and Y. Tsutsumi, *Local well-posedness in low regularity of the mKdV equation with periodic boundary condition,* Disc. Cont. Dyn. Systems **28** (2010), no. 4, 1635–1654.

[20] J.- C. Saut and N. Tzvetkov, *On the periodic KP-I type equations*, Comm. Math. Phys. **221** (2001), 451-476.

[21] H. Takaoka and Y. Tsutsumi, *Well-posedness of the Cauchy problem for the modied KdV equation with periodic boundary condition,* Int. Math. Res. Not. (2004), 3009–3040.

[22] T. Tao, *Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations,* Amer. J. Math. **123** (2001), no.5, 839–908.