tau-recurrent sequences and modular forms

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Abstract. In this paper we deal with *Drinfeld modular forms*, defined and taking values in complete fields of positive characteristic. Our aim is to study a sequence $(g_k^\star(z,t))_{k\geq 0}$ of families of Drinfeld modular forms that produces, for certain values of the parameter t, several kinds of Eisenstein series considered by Gekeler. We obtain formulas involving these functions depending on the parameter t. To obtain our results, we introduce and discuss *τ-linear recurrent sequences* and *deformations of vectorial modular forms*, and we give on the way some applications to the study of *extremal quasi-modular forms*.

Contents

1 Introduction, results

The present paper deals with the following loosely question: *why do there exist two kinds of "Eisenstein series" in the theory of Drinfeld modular forms for $\mathbf{GL}_2(\mathbb{F}_q[\theta])$, and not just one as in the theory for $\mathbf{SL}_2(\mathbb{Z})$?* Here, we give a tentative of answer: *the theories are not too different, the two kinds of the drinfeldian framework indeed come from a unique family of functions.* We will study this family.

let $q=p^e$ be a power of a prime number p with $e>0$ an integer, let $\mathbb{F}_q$ be the finite field with q elements. We consider the polynomial ring $A=\mathbb{F}_q[\theta]$ and its fraction field $K=\mathbb{F}_q(\theta)$, with θ an indeterminate over $\mathbb{F}_q$. On K, we will consider the absolute value $|\cdot|$ defined by $|a|=q^{\deg_\theta a}$, a being in K, so that $|\theta|=q$. Let $K_\infty:=\mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let $K_\infty^{\mathrm{alg.}}$ be an algebraic closure of K_∞, let C be the completion of $K_\infty^{\mathrm{alg.}}$ for the unique extension of $|\cdot|$ to $K_\infty^{\mathrm{alg.}}$, and let $K^{\mathrm{alg.}}$ be the algebraic closure of K in C. The presentation

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of our results requires that we first introduce some of the tools that will be used all along the paper.

1. Drinfeld modular and quasi-modular forms. Following Gekeler in [9], we denote by Ω the set $C \setminus K_\infty$, which has a structure of rigid analytic space. The group

$$\Gamma = \mathbf{GL}_2(A)$$

acts discontinuously on Ω by homographies; for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \Omega$, we denote by $\gamma(z) = (az+b)/(cz+d)$ the action of γ on z. Gekeler considered three algebraically independent functions

$$E, g, h : \Omega \to C$$

such that, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \Omega$:

$$\begin{aligned} g(\gamma(z)) &= (cz+d)^{q-1} g(z), \\ h(\gamma(z)) &= (cz+d)^{q+1} \det(\gamma)^{-1} h(z), \\ E(\gamma(z)) &= (cz+d)^2 \det(\gamma)^{-1} \left(E(z) - \frac{c}{\widetilde{\pi}(cz+d)} \right). \end{aligned} \tag{1}$$

These functions are holomorphic in the sense of [8, Definition 2.2.1]. Here, $\widetilde{\pi}$ is a fundamental period of the *Carlitz exponential function* e_{Car} defined, for all $\zeta \in C$, by the sum of the converging series:

$$e_{\mathrm{Car}}(\zeta) = \sum_{n \geqslant 0} \frac{\zeta^{q^n}}{d_n}, \tag{2}$$

where $d_0 := 1$ and $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$, with $[i] = \theta^{q^i} - \theta$ if $i > 0$.

It is possible to show that $\widetilde{\pi}$ is equal, up to a choice of a $(q-1)$-th root of $-\theta$, to the (value of the) convergent product:

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_\infty((-\theta)^{\frac{1}{q-1}}) \setminus K_\infty.$$

According to Gekeler in [9] (but we will prefer to borrow notations from Gerritzen-van der Put in [12]), the "local parameter at infinity" of the quotient space $\Gamma\backslash\Omega$ can be defined as a map $\Omega \to C$ by:

$$u(z) = \frac{1}{e_{\mathrm{Car}}(\widetilde{\pi} z)}.$$

In [9], it is proved that E, g, h have, locally at $u = u(z) = 0$, convergent u-expansions in $A[[u]]$. The functional equations above and the "nice local behavior at infinity" indicate that g, h are *Drinfeld modular forms*, of *weights* $q-1$, $q+1$ and *types* $0, 1$ respectively.

After (1) it is apparent that E is not a Drinfeld modular form. In [9], Gekeler calls it a "false Eisenstein series" of weight 2 and type 1. Nevertheless, it is the prototype of *Drinfeld quasi-modular form*, of weight 2, type 1 and depth 1 (see [3]).

In the classical theory of modular forms for $\mathbf{SL}_2(\mathbb{Z})$, Eisenstein series

$$G_{2k}(z) = \sum_{c,d \in \mathbb{Z}}{}' (cz+d)^{-2k}, \quad (k \geq 2) \tag{3}$$

appear in the coefficients of the Laurent expansion at zero of Weierstrass $\wp$-functions

$$\wp(z\mathbb{Z}+\mathbb{Z};\zeta)=\zeta^{-2}+\sum_{k=2}^{\infty}\zeta^{2k-2}\lambda_k G_{2k}(z), \tag{4}$$

where the sequence of complex numbers $(\lambda_k)_{k\geq 2}$ is explicit and independent on z,ζ. The differential equation of $\wp(z\mathbb{Z}+\mathbb{Z};\cdot)$ then yields an explicit quadratic recursive relation

$$G_{2k+4}=\sum_{i=2}^{k}\mu_i G_{2k-2i}G_{2i}, \tag{5}$$

where the coefficients μ_i can be computed with the help of, for example, [23, p. 19].

Goss was the first, in [13], to consider a similar notion of Eisenstein series for the group Γ:

$$G_{k(q-1)}=\sum_{c,d\in A}{}'(cz+d)^{-k(q-1)},\quad (k\geq 1) \tag{6}$$

(the dash $'$ means that we avoid the couple $(c,d)=(0,0)$ in the sum, and we allow an abuse of notation here). These series no longer occur in the series expansion of the "analogues of $\wp$": the *exponential functions* of rank 2 *Drinfeld modules*.

For $z\in\Omega$, we denote by Λ_z the A-module $A+zA$, free of rank 2. The evaluation at $\zeta\in C$ of the exponential function e_{Λ_z} associated to the lattice Λ_z is given by the series

$$e_{\Lambda_z}(\zeta)=\sum_{i=0}^{\infty}\alpha_i(z)\zeta^{q^i}, \tag{7}$$

for functions $\alpha_i:\Omega\to C$ with $\alpha_0=1$, this series expansion can be understood in many ways as the analogue of (4) in the Drinfeldian theory. For all i, α_i is a Drinfeld modular forms of weight q^i-1 and type 0, but in general, it is not proportional to G_{q^i-1}. On the other hand, no general recursive formula is known to compute all the series $G_{k(q-1)}$, like (5).

At least, recursively defined sequences can be extracted from the sequence $(G_{k(q-1)})_{k\geq 1}$. As Gekeler points out, for example in [11], we must then distinguish between *ortho*-Eisenstein ([1]) series and *para*-Eisenstein series, belonging to sequences that we recall now.

The first sequence, $(g_k)_{k\geq 0}$, introduced by Gekeler in [9], is determined by setting $g_0=1, g_1=g$ and then, inductively, by defining:

$$g_k=g_{k-1}g^{q^{k-1}}-[k-1]g_{k-2}\Delta^{q^{k-2}},\quad (k\geqslant 2),$$

where $\Delta=-h^{q-1}$. For fixed k, this is the normalisation ([2]) of the Eisenstein series of weight q^k-1:

$$g_k=(-1)^{k+1}\widetilde{\pi}^{1-q^k}(\theta^{q^k}-\theta)\cdots(\theta^q-\theta)\sum_{c,d\in A}{}'(cz+d)^{1-q^k}. \tag{8}$$

Together with $(g_k)_{k\geq 0}$, we have the second sequence $(m_k)_{k\geq 0}$ of Drinfeld modular forms, called *para-Eisenstein series*, discussed by Gekeler in [11], possessing the same sequence of weights and types. This can be defined inductively in the following way:

$$m_0=1,\quad m_1=g,\quad m_k=gm_{k-1}^q+[k-1]^q\Delta m_{k-2}^{q^2},\quad (k\geq 2).$$

[1] We drop the suffix "ortho" henceforth.

[2] A formal Laurent series in powers of u is said to be *normalised* if the monomial of lowest order in u appearing in it is monic. By abuse of language, we will say that a non-zero modular form is normalised if its u-expansion is normalised.

It is easily shown that $\alpha_k = d_k m_k$ for all k.

For w integer and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, we denote by $M_{w,m}$ the C-vector space of Drinfeld modular forms of weight w and type m and we denote by $M = \oplus_{w,m} M_{w,m}$ the C-algebra generated by Drinfeld modular forms. Gekeler [9, Theorem (5.13)] proved that $M = C[g,h]$.

Let

$$\tau : M[z] \otimes \mathbb{F}_q((t)) \to M[z] \otimes \mathbb{F}_q((t))$$

be the unique $\mathbb{F}_q((t))$-linear map extending the Frobenius map $f \mapsto f^q$ on $M[z]$; for example, $\tau t = t, \tau g = g^q, \tau z = z^q$. It is easy to show that τ induces an injective linear map from $M_{w,m} \otimes \mathbb{F}_q((t))$ to $M_{qw,m} \otimes \mathbb{F}_q((t))$.

We introduce here a third new sequence $(g_k^\star)_{k\geq 0}$ recursively as follows: we take once again the initial data $g_0^\star = 1, g_1^\star = g$ and then we set:

$$g_k^\star = g(\tau g_{k-1}^\star) + (t - \theta^q)\Delta(\tau^2 g_{k-2}^\star), \quad (k \geq 2). \tag{9}$$

For example,

$$\begin{aligned} g_2^\star &= g^{1+q} - \Delta(t - \theta^q), \\ g_3^\star &= g^{1+q+q^2} + \Delta g^{q^2}(t - \theta^q) + \Delta^q g(t - \theta^{q^2}), \end{aligned}$$

and in general, $g_k^\star \in M_{q^k-1,0} \otimes \mathbb{F}_q[t]$. The recursion process involved in this definition is typical of what we study in this paper; we will refer to such a kind of sequence as to a *τ-linear recurrent sequence* (of order 2, because $g_k^\star = X(\tau g_{k-1}^\star) + Y(\tau^2 g_{k-2}^\star)$ with $X = g$ and $Y = (t - \theta^q)\Delta$).

It is easy to show (by induction, but read Section 2) that the sequence $(g_k^\star)_{k\geq 0}$ can be defined alternatively, by choosing the same initial data and setting:

$$g_k^\star = g^{q^{k-1}} g_{k-1}^\star + \Delta^{q^{k-2}}(t - \theta^{q^{k-1}}) g_{k-2}^\star, \quad (k \geq 2). \tag{10}$$

This time, we speak about a *τ-linearised recurrent sequence* (of order 2, because we can write $g_k^\star = (\tau^k X) g_{k-1}^\star + (\tau^k Y) g_{k-2}^\star$ with $X = g^{1/q}$ and $Y = \Delta^{1/q^2}(t - \theta^{1/q})$). The rudiments of the theory of such sequences are established in Section 2 below.

As the example of the sequence $(g_k^\star)_k$ shows, a given recurrent sequence can be at once τ-linear and τ-linearised; although this is not necessarily true in general (the tools developed in the next section will tell us when this precisely occurs), it has particular benefit for us in our specific case. Indeed, from (9), it follows that

$$g_k^\star(z, \theta^{q^k}) = m_k(z), \quad (k \geq 0),$$

while from (10) we obtain that

$$g_k^\star(z, \theta) = g_k(z), \quad (k \geq 0).$$

Summing up everything together, we understand that the sequence $(g_k^\star)_{k\geq 0}$ somewhat encompasses the two different kinds of Eisenstein series of weights $(q^k - 1)_{k\geq 0}$ considered by Gekeler and is an object that needs to be studied on its own. This we do in the present paper, but, before arriving at our principal results, we need to introduce further tools.

The functions $s_{Car}, \boldsymbol{d}_1, \boldsymbol{d}_2$. We need to recall the definitions of some now classical functions that naturally arise in the theory of Anderson's t-motives, also considered in [22] (we will use notations introduced in the latter paper). The first function is defined by the series:

$$s_{\mathrm{Car}}(t) := \sum_{i=0}^{\infty} e_{\mathrm{Car}}\left(\frac{\widetilde{\pi}}{\theta^{i+1}}\right) t^i, \tag{11}$$

which converges for $|t| < q$. This is the *canonical rigid analytic trivialisation* of the so-called *Carlitz's motive*. We refer to [21] for a description of the main properties of it, or to the papers [1, 2, 20], where it was originally introduced and appears with different notations.

The series (introduced and studied in [22])

$$\begin{aligned} \boldsymbol{s}_1(z,t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t}, \\ \boldsymbol{s}_2(z,t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}, \end{aligned}$$

converge on $\Omega \times B_q$ and define two functions $\Omega \to C[[t]]$ with the series in the image converging on B_q (B_r denotes the open disk of center 0 and radius $r > 0$). We point out that for a fixed choice of $z \in \Omega$, the matrix function $(\boldsymbol{s}_1(z,t), \boldsymbol{s}_2(z,t))$ is the *canonical rigid analytic trivialisation* of the *t-motive associated* to the lattice Λ_z. We recall that we have set, for $i = 1, 2$ (in [22]):

$$\boldsymbol{d}_i(z,t) := \widetilde{\pi} s_{\mathrm{Car}}(t)^{-1} \boldsymbol{s}_i(z,t)$$

and we point out that, in the notations of [22], $\boldsymbol{d} = \boldsymbol{d}_2$. The advantage of using these functions, comparing with the $\boldsymbol{s}_i$'s, is that evaluation at $t = \theta$ makes sense, and we can check:

$$\boldsymbol{d}_1(z,\theta) = z, \quad \boldsymbol{d}_2(z,\theta) = 1. \tag{12}$$

Moreover, as it was pointed out in [22], $\boldsymbol{d}_2$ has a u-expansion defined over $\mathbb{F}_q[t,\theta]$ (see later, Proposition 19).

In this paper, we will obtain the following theorem.

Theorem 1 *For all $k \geq 0$, we have:*

$$g_k^\star = h^{q^k}(\tau^{k+1} s_{Car})\{\boldsymbol{d}_1 \tau^{k+1}(\boldsymbol{d}_2) - \boldsymbol{d}_2 \tau^{k+1}(\boldsymbol{d}_1)\}.$$

Just as classical recurrent sequences, τ-recurrent sequences have *characteristic roots* (see Section 2). The interest of Theorem 1 relies in that the characteristic roots of the τ-recurrent sequence $(g_k^\star)_{k\geq 0}$ are explicitly computed, and turn out to be the functions

$$h\tau(s_{\mathrm{Car}} \boldsymbol{d}_2), \quad -h\tau(s_{\mathrm{Car}} \boldsymbol{d}_1).$$

We will also compute series expansions of "Eisenstein type", like (8), for the forms $g_k^\star$. To ease the next discussion, we mildly modify the aspect of the g_k's:

$$g_k(z) = (-1)^{k+1} \widetilde{\pi}^{1-q^k} [k] \cdots [1] \left(z \sum_{c,d \in A}{}' \frac{c}{(cz+d)^{q^k}} + \sum_{c,d \in A}{}' \frac{d}{(cz+d)^{q^k}} \right). \tag{13}$$

Let t be an element of C. We have the "evaluating at t" ring homomorphism

$$\chi_t : A \to \mathbb{F}_q[t]$$

defined by $\chi_t(a) = a(t)$. In other words, $\chi_t(a)$ is the image of the polynomial map $a(t)$ obtained by substituting, in $a(\theta)$, θ by t. For example, $\chi_t(1) = 1$ and $\chi_t(\theta) = t$. The notation is motivated by the fact that if we choose $t \in \mathbb{F}_q^{\mathrm{alg.}}$ then χ_t factors through a Dirichlet character modulo the ideal generated by the minimal polynomial of t in A.

Let α be a positive integer. We consider the value of L-series:

$$L(\chi_t, \alpha) = \sum_{a \in A^+} \chi_t(a) a^{-\alpha} = \prod_{\mathfrak{p}} (1 - \chi_t(\mathfrak{p}) \mathfrak{p}^{-\alpha})^{-1} \in K_\infty[[t]],$$

where A^+ denotes the set of monic polynomials of A, converging for all $t \in B_q$ and not identically zero, and where the eulerian product runs over the monic irreducible polynomials of A.

In this paper, we shall prove the following theorem.

Theorem 2 *For all $z \in \Omega$, $t \in C$ with $|t|$ small enough, and $k \geq 0$, the following series expansion holds:*

$$L(\chi_t, q^k) g_k^\star(z,t) = -\boldsymbol{d}_1(z,t) \sum_{c,d \in A}{}' \frac{\chi_t(c)}{(cz+d)^{q^k}} - \boldsymbol{d}_2(z,t) \sum_{c,d \in A}{}' \frac{\chi_t(d)}{(cz+d)^{q^k}}.$$

In particular, for $t = \theta^{q^k}$, we find a series expansion of "Eisenstein type" for the modular forms m_k, in the style of (13), while taking the limit $t \to \theta$ one recovers (13) after some calculation.

The functions $\boldsymbol{f}_1, \boldsymbol{f}_2$. Both the series

$$\boldsymbol{f}_1(z,t) = \sum_{c,d \in A}{}' \frac{\chi_t(c)}{cz+d}, \quad \boldsymbol{f}_2(z,t) = \sum_{c,d \in A}{}' \frac{\chi_t(d)}{cz+d}$$

play a special role in Theorem 2, as they also are, according to Theorem 1, the characteristic roots of the τ-linear recurrent sequence (9). They converge for $(z,t) \in \Omega \times B_q$ and define functions $\Omega \to C[[t]]$ such that all the series in the images converge over B_q, and will also be in the center of interest of this paper.

We will treat the vector functions ${}^t\mathcal{E}$ with $\mathcal{E} = L(\chi_t, 1)^{-1}(\boldsymbol{f}_1, \boldsymbol{f}_2)$ and $\mathcal{F} = \binom{\boldsymbol{d}_1}{\boldsymbol{d}_2}$ as examples of *deformations of vectorial modular forms*. Thanks to this interpretation and τ-linear recurrent sequences, we will prove:

Theorem 3 *The following identity holds in the domain $\Omega \times B_q$:*

$$\mathcal{E} = (\tau s_{Car}) h(\tau \boldsymbol{d}_2, -\tau \boldsymbol{d}_1). \tag{14}$$

The function $\boldsymbol{E}$. This was introduced in [22] and is just the product:

$$\boldsymbol{E} = -h\tau \boldsymbol{d}_2,$$

see also [5]. This function is defined over $\Omega \times C$ and defines a map $\Omega \to C[[t]]$ such that the series in the image have infinite radius of convergence; it has the property that $\boldsymbol{E}(z, \theta) = E(z)$. Theorem 3 says that

$$\boldsymbol{f}_1(z,t) = -\widetilde{\pi} \boldsymbol{E}(z,t). \tag{15}$$

The limit $t \to \theta$ of the left-hand side of (15) exists:

$$\lim_{t \to \theta} \sum_{c,d \in A}{}' \frac{\chi_t(c)}{cz+d} = -\sum_{c \in A^+} \sum_{d \in A} \frac{c}{cz+d} = -\widetilde{\pi} E(z), \tag{16}$$

where the convergence of the series is conditional.

As for the right-hand side of (15), this has limit $\widetilde{\pi}^q h(z) F_z^{(1)}(1)$ in the notations of [10]; by using (7.1) of loc. cit. the value is shown to be $\eta_2(z)$, the second *quasi-period* of the lattice Λ_z. Therefore, for $t = \theta$, (15) becomes Theorem 7.10 of [10], which states that

$$\eta_2 = \widetilde{\pi}^{1-q} E(z)/h(z).$$

Hence, for $t \to \theta$, Theorem 3 yields a well known result on the relation between the second quasi-period of the lattice Λ_z and quasi-modular forms. The interest of Theorem 3 is obviously not confined to this remark; it is that the identity holds for any choice of $t \in B_q$. We then obtain the following result, providing a pleasant series expansion of $\boldsymbol{E}$ "near infinity":

Corollary 4 *We have the following identity, valid for all $z \in \Omega$ and $t \in C$ such that $|t| < q^q$:*

$$\boldsymbol{E}(z,t) = \sum_{c \in A^+} \chi_t(c) u_c(z),$$

where we have used the functions $u_c(z) := e_{\mathrm{Car}}(c\widetilde{\pi}z)^{-1}$, with e_{Car} Carlitz's exponential. For $t = \theta$, this reduces to [9, (8.2)].

Since for any w, an element f of $M_{w,0} \otimes \mathbb{F}_q[t]$ has u-expansion

$$f(z) = \sum_{i \geq 0} c_i(t) u(z)^i$$

with the c_i's in $C[t]$, converging locally at $u = 0$ for every fixed t, it is then very natural to try to write down the u-expansions of the $g_k^\star$'s.

This paper contains a result, Theorem 31, providing a simple way to compute the u-expansions of the series $g_k^\star$'s from the u-expansions of $\boldsymbol{d}_2$ and a "mysterious" function Υ which allows u-expansion, introduced in Section 5; the paper also presents simple algorithms to compute the u-expansion of Υ, as we did for $\boldsymbol{d}_2$ in [22]. We do not state Theorem 31 here (the statement requires some further preparation), but we mention a simple corollary of it.

Corollary 5 *The truncation of the u-expansion of $g_k^\star$ to the order $q^k + 2q - 3$ is given, for $q \neq 2$, by the truncation to the same order of the series:*

$$\boldsymbol{d}_2 \left(1 + \sum_{i=0}^{k-1} (t - \theta^{q^k}) \cdots (t - \theta^{q^{i+1}}) u^{q^k - q^i} \right) - (t - \theta) u^{q^k + q - 2}.$$

The case $q = 2$ is more involved, but can also be handled with the methods described here. Choosing $t = \theta$ and using (12), we get the well known truncation to the order q^k of the u-expansion of g_k first computed by Gekeler in [9]:

$$1 + \sum_{i=0}^{k-1} (\theta - \theta^{q^k}) \cdots (\theta - \theta^{q^{i+1}}) u^{q^k - q^i} + \cdots .$$

If on the other side we replace $t = \theta^{q^k}$ in the above expression, we obtain some coefficients of small order for the para-Eisenstein series m_k provided we have knowledge of the u-expansion of $\boldsymbol{d}_2$ up to a certain order. Observe that then, the sum over $i = 0, \ldots, k-1$ vanishes. In [22], we gave algorithms to perform these computations.

The proof of Theorem 3 that we present here will yield, as a by-product:

Corollary 6 *The following identity holds:*

$$L(\chi_t, 1) = -\frac{\widetilde{\pi}}{(t - \theta) s_{Car}}.$$

According to Corollary 6, the product $L(\chi_t,1)(t-\theta)s_{\mathrm{Car}}$ does not depend on t and the inverse of $(t-\theta)s_{\mathrm{Car}}$ (that is, the function $\mathbf{\Omega}$ of [2]) is proportional to an L-value thus allowing entire analytic continuation in terms of the parameter t ([3]). It follows that

$$\lim_{t\to\theta} L(\chi_t,1)=1. \tag{17}$$

Link with extremal quasi-modular forms. To complete our paper, we will describe some links between the present work, [22], and the joint work [4]. For l,w non-negative integers and m a class of $\mathbb{Z}/(q-1)\mathbb{Z}$, we introduce the C-vector space of *Drinfeld quasi-modular forms of weight w, type m and depth $\leq l$*:

$$\widetilde{M}_{w,m}^{\leq l}=M_{w,m}\oplus M_{w-2,m-1}E\oplus\cdots\oplus M_{w-2l,m-l}E^l.$$

In [4], we have introduced the sequence of Drinfeld quasi-modular forms $(x_k)_{k\geq 0}$ with $x_k\in \widetilde{M}_{q^k+1,1}^{\leq 1}\setminus M$, defined by $x_0=-E$, $x_1=-Eg-h$ and by the recursion formula

$$x_k=x_{k-1}g^{q^{k-1}}-[k-1]x_{k-2}\Delta^{q^{k-2}},\quad k\geq 2,$$

where we recall that $\Delta=-h^{q-1}$. The spaces $\widetilde{M}_{w,m}^{\leq l}$ embed in $C[[u]]$. In [4, Theorem 1.2], we have showed that for all $k\geq 0$, x_k is *extremal*, in the sense that its order of vanishing at $u=0$, denoted by $\nu_\infty(x_k)$, is the biggest possible value for $\nu_\infty(f)$, if $f\in \widetilde{M}_{q^k+1,1}^{\leq 1}\setminus\{0\}$. We also computed the order of vanishing: $\nu_\infty(x_k)=q^k$ for all k. After [4, Proposition 2.3], the series expansion of

$$E_k=(-1)^{k+1}\frac{x_k}{[1][2]\cdots[k]}$$

of x_k begins with u^{q^k} (where the empty product is 1 by definition). Hence, with $E_0=E$, E_k is the unique normalised extremal quasi-modular form in $\widetilde{M}_{q^k+1,1}^{\leq 1}$ for all $k\geq 0$.

We also recall, from [3], the derivation $D_1=u^2d/du$ on $C[[u]]$, which yields a C-linear map $\widetilde{M}_{w,m}^{\leq l}\to\widetilde{M}_{w+2,m+1}^{\leq l+1}$.

We will obtain the following result.

Theorem 7 *For $k\geq 0$, we have*

$$E_k(z) \quad = \quad (\tau^k\boldsymbol{E})(z,\theta). \tag{18}$$

In particular, we have the series expansions

$$E_k=\sum_{c\in A^+} cu_c^{q^k}, \tag{19}$$

from which it is apparent that E_k has u-expansion defined over A.

Remarks. 1. The integrality of the coefficients of the normalised extremal quasi-modular form of weight q^k+1 and type 1 supports Conjecture 2 of Kaneko and Koike in [16], asserting that if $f_{l,w}\in\mathbb{Q}[[q]]$ is the q-expansion of the normalised extremal quasi-modular form of weight w and depth $l\leq 4$, then $f_{l,w}\in\mathbb{Z}_p[[x]]$ for all $p>w$.

[3] For $t=0$, this implies the case $s=1$ of [7, Theorem 2]. I am thankful to Vincent Bosser for pointing out this remark. It is possible, modifying our arguments, to obtain the other cases of the above-mentioned result but we refrain from writing this here.

2. The u-expansion (19) follows from Corollary (4) applying (18); hence, the integrality result is a consequence of our Theorem 3. It is easy to show that, for all $k \geq 0$, g_k is the *extremal modular form* in $M_{q^k-1,0}$ ([4]). After Corollary 5, the normalisation of $D_1 g_k$ is the extremal quasi-modular form E_k:

$$E_k = (-1)^k \frac{D_1 g_k}{[1][2]\cdots[k]}. \tag{20}$$

Roughly speaking, in the classical theory of modular forms for the group $\mathbf{SL}_2(\mathbb{Z})$ we have only one analogue of this striking situation, which is related to the *theta series* associated to the *Leech lattice* (of weight 12):

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta$$

where now, E_{12} denotes the classical normalised classical Eisenstein series of weight 12 and Δ is the normalised cusp form of weight 12.

Let $f_{1,14}$ be the normalised extremal quasi-modular form of weight 14 and depth 1 in the sense of [16]. The only analogue of the formula (20) at $t = \theta$ in the classical framework is:

$$f_{1,14} = \frac{1}{393120} D\Theta_{\Lambda_{24}},$$

where D denotes *Ramanujan's derivation* $(2\pi i)^{-1}d/dz$. This agrees with the above mentioned conjecture of Kaneko and Koike because no prime exceeding 13 divides 393120. Numerical inspection suggests that $f_{1,14}$ is defined over $\mathbb{Z}$ but this property does not seem to be easy to prove ([5]). In the Drinfeldian case, the integrality of the coefficients of E_k is an ultimate consequence of our formula (15). So far, we do not know about an analogue of this formula in the classical framework.

2 τ-recurrent sequences

This section is devoted to the basic elements of the theory of τ-recurrent sequences; the presentation is made in a mild setting, yet more general than required by the rest of the paper. In this section, $\mathcal{K}$ denotes any field endowed with an automorphism $\tau : \mathcal{K} \to \mathcal{K}$ of infinite order. We will refer to the couple $(\mathcal{K}, \tau)$ as to a *difference field*. We denote by $\mathcal{K}^\tau$ the constant subfield of $\mathcal{K}$, that is, the subfield whose elements x satisfy $\tau x = x$.

Let $x_1, \ldots, x_s$ be elements of $\mathcal{K}$. Their τ-*wronskian* (sometimes called "casoratian") is the determinant:

$$W_\tau(x_1, \ldots, x_s) = \det \begin{pmatrix} x_1 & \tau x_1 & \cdots & \tau^{s-1} x_1 \\ x_2 & \tau x_2 & \cdots & \tau^{s-1} x_2 \\ \vdots & \vdots & & \vdots \\ x_s & \tau x_s & \cdots & \tau^{s-1} x_s \end{pmatrix}.$$

Lemma 8 *The elements $x_1, \ldots, x_s$ are $\mathcal{K}^\tau$-linearly independent if and only if $W_\tau(x_1, \ldots, x_s) \neq 0$.*

[4] This means that g_k is the unique normalised form in $M_{q^k-1,0} \setminus \{0\}$ with the maximal order of vanishing of $g_k - 1$.

[5] Notice however, that $f_{1,14} \in \mathbb{Z}_{11}[[q]]$ because 11 does not divide 393120. I am thankful to Gabriele Nebe for having observed that we also have $f_{1,14} \in \mathbb{Z}_p[[q]]$ for $p = 5, 7, 13$ by using some properties of the action of the double cover of the Conway group $2\mathrm{Co}_1$ over Λ_{24}.

Proof. This is a classical result that can be easily proved by induction on $s \geq 0$; we recall the proof here for convenience of the reader. First of all, we notice that $W_\tau(x_1, \ldots, x_s) = 0$ if and only if there exist elements $\lambda_1, \ldots, \lambda_s \in \mathcal{K}$, not all zero, such that

$$x_1(\tau^n \lambda_1) + \cdots + x_s(\tau^n \lambda_s) = 0, \quad n \in \mathbb{Z}. \tag{21}$$

Obviously, the lemma is true for $s = 1$ so we consider $s > 1$ and $x_1, \ldots, x_s$ such that $W_\tau(x_1, \ldots, x_s) = 0$; there exist $\lambda_1, \ldots, \lambda_s \in \mathcal{K}$ such that (21) holds.

If $\lambda_1, \ldots, \lambda_s$ are all in $\mathcal{K}^\tau$, we are done. Hence, we can assume that $\lambda_1 \notin \mathcal{K}^\tau$, so that

$$x_2(\tau^n \gamma_2) + \cdots + x_s(\tau^n \gamma_s) = 0, \quad n \in \mathbb{Z},$$

with $\gamma_i = \frac{\lambda_i}{\lambda_1} - \frac{\tau\lambda_i}{\tau\lambda_1}$ for $i = 2, \ldots, s$ and the lemma follows by induction on s. □

Remark. The proof is effective in the sense that the space generated by $(\lambda_1, \ldots, \lambda_s)$ can be explicitly computed in terms of the x_i's following the inductive process step by step. Moreover, it sometimes (but not always) happens that limit processes, such as taking $n \to \infty$, furnish explicit $\mathcal{K}^\tau$-linear dependence relations skipping the induction process.

We review now some elementary facts about *τ-linear recurrent sequences* and their associated *τ-linear equations*.

Let L be a τ-linear operator in the skew polynomial ring $\mathcal{K}[\tau]$. If $L = A_0\tau^0 + \cdots + A_s\tau^s$ with $A_s \neq 0$, we will say that L has order s. We will also say that the operator L is *simple* if $A_0 \neq 0$. From now on, we will only consider simple such operators, unless otherwise specified.

Let $\mathcal{G} : \mathbb{Z} \to \mathcal{K}$ be a sequence (this will be often denoted by $(\mathcal{G}_k)_{k\in\mathbb{Z}}$) and $\lambda \in \mathcal{K}$. We will write $\lambda * \mathcal{G}$ for the sequence

$$(\lambda * \mathcal{G})_{k\in\mathbb{Z}} = ((\tau^k\lambda)\mathcal{G}_k)_{k\in\mathbb{Z}}.$$

With this action of $\mathcal{K}$, the set of the sequences $\mathbb{Z} \to \mathcal{K}$ is a vector space over $\mathcal{K}$.

Let $L = A_0\tau^0 + \cdots + A_s\tau^s \in \mathcal{K}[\tau]$ be an operator as above, and let $\mathcal{G}$ be a sequence $\mathbb{Z} \to \mathcal{K}$. We will write $L(\mathcal{G})$ for the sequence

$$L(\mathcal{G}) := (A_0\tau^0\mathcal{G}_k + \cdots + A_s\tau^s\mathcal{G}_{k-s})_{k\in\mathbb{Z}}.$$

We will say that $\mathcal{G} = (\mathcal{G}_k)_{k\in\mathbb{Z}}$ is a *τ-linear recurrent sequence* with coefficients in $\mathcal{K}$ associated to L if

$$L(\mathcal{G}) \equiv 0. \tag{22}$$

We will also say that $\mathcal{G}$ is of order s, if for any non-zero operator $L' \in \mathcal{K}[\tau]$ of order $< s$, $L'(\mathcal{G}) \not\equiv 0$.

Let $V = V(L)$ be the set of all the τ-recurrent sequences satisfying (22) for a given non-zero operator L in $\mathcal{K}[\tau]$. Since $L(\lambda * \mathcal{G}) = \lambda * L(\mathcal{G})$, V has a structure of $\mathcal{K}$-vector space; the dimension is finite, equal to s.

Assume that V contains some constant sequence $(x)_{k\in\mathbb{Z}}$. Then, x is a solution of the *associated linear τ-difference equation*

$$Lx = 0. \tag{23}$$

In other words, with L as above, we have $A_0x + A_1\tau x + \cdots + A_s\tau^s x = 0$. The set $V^\tau = V^\tau(L)$ of solutions of (23) has a natural structure of $\mathcal{K}^\tau$-vector space.

Lemma 9 *Let L be simple of order s and let $V = V(L), V^\tau = V^\tau(L)$ be as above. We have $\dim V^\tau \geq s$ if and only if $\dim V^\tau = s$. In the latter case, choose a basis $(x_1, \ldots, x_s)$ of V^τ. If $\mathcal{F} \in \mathbf{Mat}_{s\times 1}(\mathcal{K})$ is defined by ${}^t\mathcal{F} = (x_1, \ldots, x_s)$ (transpose), then the map*

$$\mathcal{E} : V \to \mathbf{Mat}_{1\times s}(\mathcal{K}) \cong \mathcal{K}^s$$

defined by

$$\mathcal{G} = (\mathcal{G}_k)_{k\in\mathbb{Z}} \mapsto \mathcal{E}(\mathcal{G}) := (\mathcal{G}_0, \tau^{-1}\mathcal{G}_1, \ldots, \tau^{-s+1}\mathcal{G}_{s-1}) \cdot M^{-1} \tag{24}$$

is an isomorphism of $\mathcal{K}$-vector spaces.

Proof. Let us assume that the dimension of V^τ is not smaller than s. Then, there exist $\mathcal{K}^\tau$-linear elements $x_1, \ldots, x_r$ of $\mathcal{K}$ solutions of (23) with $r \geq s$.

Let $\mathcal{F} \in \mathbf{Mat}_{r\times 1}(\mathcal{K})$ be such that ${}^t\mathcal{F} = (x_1, \ldots, x_r)$. By Lemma 8, $W_\tau(x_1, \ldots, x_r) \neq 0$ and the matrix $M = (\mathcal{F}, \tau^{-1}\mathcal{F}, \ldots, \tau^{-r+1}\mathcal{F})$ is invertible. The map

$$V \to \mathbf{Mat}_{1\times r}(\mathcal{K}) \cong \mathcal{K}^r \tag{25}$$

defined by

$$\mathcal{G} = (\mathcal{G}_k)_{k\in\mathbb{Z}} \mapsto (\mathcal{G}_0, \tau^{-1}\mathcal{G}_1, \ldots, \tau^{-r+1}\mathcal{G}_{r-1}) \cdot M^{-1}$$

is then an isomorphism of $\mathcal{K}$-vector spaces and $r = s$. Therefore, $\dim_{\mathcal{K}^\tau} V^\tau = s$. □

An operator $L = A_0\tau^0 + \cdots + A_s\tau^s \in \mathcal{K}[\tau]$ of order s is said to be *split* if $\dim V^\tau(L) = s$. A split operator is also simple. This definition obviously depends on the field $\mathcal{K}$.

Proposition 10 *Let $L \in \mathcal{K}[\tau]$ be a split operator of order s and choose a basis $(x_1, \ldots, x_s)$ of $V^\tau(L)$. Let V the $\mathcal{K}$-vector space of the τ-recurrent sequences $\mathcal{G}$ such that $L(\mathcal{G}) = 0$. Then, for all $\mathcal{G} \in V$ there exists one and only one element $\mathcal{E} \in \mathbf{Mat}_{1\times s}(\mathcal{K})$ such that for all $k \in \mathbb{Z}$,*

$$\mathcal{G}_k = (\tau^k\mathcal{E}) \cdot \mathcal{F}. \tag{26}$$

Proof. This follows from Lemma 9, taking $\mathcal{E} = \mathcal{E}(\mathcal{G})$ as in (25). □

Proposition 11 *Let $x_1, \ldots, x_s$ be elements of $\mathcal{K}$. Define, for $k = 0, \ldots, s$,*

$$A_k = A_k^\tau(x_1, \ldots, x_s) := (-1)^{s+k} \det \begin{pmatrix} \tau^0 x_1 & \cdots & \widehat{\tau^k x_1} & \cdots & \tau^s x_1 \\ \vdots & & \vdots & & \vdots \\ \tau^0 x_s & \cdots & \widehat{\tau^k x_s} & \cdots & \tau^s x_s \end{pmatrix},$$

where the hats mean that the corresponding column must be discarded. Denote by $V^\tau(x_1, \ldots, x_s)$ the $\mathcal{K}^\tau$-vector space generated by the x_i and let us consider the operator

$$L = L(x_1, \ldots, x_s) = A_0\tau^0 + \cdots + A_s\tau^s. \tag{27}$$

Then,

$$V^\tau(L) = V^\tau(x_1, \ldots, x_s).$$

If the x_i's are $\mathcal{K}^\tau$-linearly independent, then L is split of order s.

Let $\mathcal{F} = {}^t(x_1, \ldots, x_s)$ be a matrix of $\mathbf{Mat}_{s\times 1}(\mathcal{K})$ whose entries are $\mathcal{K}^\tau$-linearly independent. For all $\mathcal{E} \in \mathbf{Mat}_{1\times s}(\mathcal{K})$, the sequence $\mathcal{G} = (\mathcal{G}_k)_{k\in\mathbb{Z}}$ defined by

$$\mathcal{G}_k = (\tau^k\mathcal{E}) \cdot \mathcal{F}$$

belongs to $V(L)$ with L as in (27), and every sequence of $V(L)$ can be expressed as above for some $\mathcal{E}$.

Proof. The existence of the operator L follows easily by solving the τ-difference equation

$$W_\tau(x_1, \dots, x_s, X) = 0.$$

Indeed, by Lemma 8, we have that $W_\tau(x_1, \dots, x_s, x) = 0$ with $x \in \mathcal{K}$ if and only if x belongs to $\mathbf{Vect}_{\mathcal{K}^\tau}(x_1, \dots, x_s)$. The non-vanishing of A_0 is also obvious as $A_0 = (-1)^s \tau A_s = \tau W_\tau(x_1, \dots, x_s)$. The final part of the proposition follows from a simple application of Proposition 11 which provides the operator L. □

The entries of $\mathcal{E}$ in Proposition 11 are called the *characteristic roots* of the τ-linear recurrent sequence $\mathcal{G}_k$.

Let us consider a sequence $\mathcal{G}$ as in (26), with $\mathcal{E}, \mathcal{F}$ two matrices with entries in $\mathcal{K}$. Then, with the above notations, we can introduce the *adjoint sequence* $\mathcal{H} = (\mathcal{H}_k)_{k\in\mathbb{Z}}$ defined by

$$\mathcal{H}_k = \tau^{-k}\mathcal{G}_k = \mathcal{E} \cdot (\tau^{-k}\mathcal{F}).$$

Let us assume that the entries of the matrix $\mathcal{E} \in \mathbf{Mat}_{1\times s}(\mathcal{K})$ are $\mathcal{K}^\tau$-linearly independent. Then, $W_{\tau^{-1}}(\mathcal{E}) \neq 0$ and the above arguments with τ replaced by τ^{-1} ensure that $\mathcal{H}$ is a τ^{-1}-recurrent sequence of order s so that there exists a split operator $L' \in \mathcal{K}[\tau^{-1}]$ of order s such that $L'(\mathcal{H}) = 0$. If $L' = A'_0\tau^0 + \cdots + A'_s\tau^s$ (so that $A'_i = A_i^{\tau^{-1}}(\mathcal{E})$), then, for all $k \in \mathbb{Z}$

$$A'_0\mathcal{H}_k + A'_1\tau^{-1}\mathcal{H}_{k-1} + \cdots + A'_s\tau^{-s}\mathcal{H}_{k-s} = 0.$$

Applying τ^s to the previous identities implies that the sequence $\mathcal{G}$ satisfies the following τ-*linearised recurrent sequence* of order s:

$$(\tau^k A'_0)\mathcal{G}_k + (\tau^k A'_1)\mathcal{G}_{k-1} + \cdots + (\tau^k A'_s)\mathcal{G}_{k-s} = 0, \quad k \in \mathbb{Z}.$$

We will say that a sequence $\mathcal{G}$ of $\mathcal{K}$ is *generic* if there exist matrices $\mathcal{E} \in \mathbf{Mat}_{1\times s}(\mathcal{K})$ and $\mathcal{F} \in \mathbf{Mat}_{s\times 1}(\mathcal{K})$, both with $\mathcal{K}^\tau$-linearly independent entries, such that for all $k \in \mathbb{Z}$,

$$\mathcal{G}_k = (\tau^k\mathcal{E}) \cdot \mathcal{F}.$$

Then, we obviously have the following proposition, containing all the properties encountered so far; later, we will use it for a specific generic sequence of modular forms.

Proposition 12 *Let $\mathcal{G} = \mathcal{G}_k = (\tau^k\mathcal{E}) \cdot \mathcal{F}$ be a generic sequence. If $L = A_0\tau^0 + \cdots + A_s\tau^s$ is the split operator of $\mathcal{K}[\tau]$ associated to $\mathcal{F}$ and if $L' = A'_0\tau^0 + \cdots + A'_s\tau^{-s}$ is the split operator of $\mathcal{K}[\tau^{-1}]$ associated to ${}^t\mathcal{E}$ (by Proposition 11), then $\mathcal{G}$ is at once τ-linear recurrent and τ-linearised recurrent (in both ways of order s). More precisely, for all $k \in \mathbb{Z}$,*

$$A_0\tau^0\mathcal{G}_k + A_1\tau^1\mathcal{G}_{k-1} + \cdots + A_s\tau^s\mathcal{G}_{k-s} = 0, \tag{28}$$

$$(\tau^k A'_0)\mathcal{G}_k + (\tau^k A'_1)\mathcal{G}_{k-1} + \cdots + (\tau^k A'_s)\mathcal{G}_{k-s} = 0. \tag{29}$$

2.1 Extending to existentially closed fields

It is helpful, in some points of this paper, notably before computing solutions of certain τ-difference equations, to first justify their existence in some simple way. In this subsection we explain how to do it extending the difference field $(\mathcal{K}, \tau)$. In practice, the reader will not see further references to this subsection elsewhere in the paper. Furthermore, its use can be avoid in each specific case, but we found appropriate to mention this aspect of the theory here.

By the so-called "ACFA" theory of Chatzidakis and Hrushowski [6], there is an *existentially closed field* $\mathbb{K}$ containing $\mathcal{K}$ (more precisely, one speaks of the couple $(\mathbb{K}, \tau)$ as being existentially closed). This means that there exists a field $\mathbb{K}$ with an automorphism which extends τ (again denoted with τ), such that the constant subfield of $\mathbb{K}$ for this automorphism is $\mathcal{K}^\tau$, and every linear τ-difference equation of positive order has at least a non-zero solution $x \in \mathbb{K}$.

Lemma 13 *Let us assume that $(\mathcal{K}, \tau)$ is existentially closed. $f\, L = A_0\tau^0 + \cdots + A_s\tau^s \in \mathcal{K}[\tau]$ is such that $A_sA_0 \neq 0$ as above, then* $\dim V^\tau(L) = s$.

Proof. The proof requires that we solve non-homogeneous equations as well. We proceed by induction on $s \geq 0$. If $s = 0$, the statement of the lemma is trivial. Let us assume now that $s > 0$. Since $\mathcal{K}$ is existentially closed, there exists a solution $x_0 \neq 0$ of $Lx = 0$. Right division algorithm holds in $\mathcal{K}[\tau]$, so that there exists $\tilde{L} \in \mathcal{K}[\tau]$ unique, with $L = \tilde{L}L_{x_0}$, where, for $x \in \mathcal{K}^\times$, we have written $L_x = \tau - (\tau x)/x$. Since the order of $\tilde{L}$ is $s-1$, there exist $y_1, \ldots, y_{s-1}$ $\mathcal{K}^\tau$-linearly independent elements of $\mathcal{K}$ such that $\tilde{L}y_i = 0$ for all i. Now, for all $i \geq 1$, let x_i be a solution of $L_{x_0}x_i = y_i$ (they exist, again because $\mathcal{K}$ is existentially closed). Then, $x_0, x_1, \ldots, x_{s-1}$ are s linearly independent elements of $\mathcal{K}$, solutions of $Lx = 0$ so that $\dim V^\tau(L) \geq s$. By Lemma 9, $\dim V^\tau(L) = s$. □

An example of $\mathcal{K}$ which is not existentially closed. We take $\mathcal{K} = C((t))$, τ being $\mathbb{F}_q((t))$-linear, defined by $\tau c = c^q$ if $c \in C$. Then, $\mathcal{K}^\tau = \mathbb{F}_q((t))$. Consider now the equation

$$\tau X = (t - T)X, \tag{30}$$

with $T \in C^\times$. The set of solutions of this equation is a $\mathcal{K}^\tau$-vector space of dimension 1, and its elements are the series $\sum_{n \geq n_0} c_n t^n$ with $n_0 \in \mathbb{Z}$, $c_n = 0$ if $n < n_0$ and, for all n,

$$c_n^q = -Tc_n + c_{n-1}.$$

If we take $T = \theta$ in (30), then we are left with the equation

$$\tau X = (t - \theta)X. \tag{31}$$

In this case, it is well known that the equation can be even solved in the subfield $\mathbb{L} = \mathbf{Frac}(\mathbb{T}) \subset \mathcal{K}$, where $\mathbb{T}$ is the Tate ring of series $\sum_{i \geq 0} c_i t^i$ converging for $t \in C$ with $|t| \leq 1$ (over which τ acts and induces an automorphism), see, for example [2]. Also, from Theorem 2.2.9 of [8] or Lemma 14, one deduces that $\mathbb{L}^\tau = \mathbb{F}_q(t)$.

Solutions of the equation (31) can be constructed by using Carlitz's exponential function. Indeed, if $\widetilde{\pi} \in C$ is a fundamental period of e_{Car}, a solution of (31) is given by the series s_{Car} defined in (11).

If $T = 0$, equation (30) has no non-zero solutions in $\mathcal{K} = C((t))$; hence $(\mathcal{K}, \tau)$ is not existentially closed.

Let us consider the (integral) ring R of series $\sum_{n \in \mathbb{Z}} c_n t^n$ converging in the annulus $t \in C$, $|t| > 1$. Examples of such series are the $f_a = \sum_{n \in \mathbb{Z}} a^{q^n} t^n$, with $a \in C^\times$ such that $|a| < 1$. If $\mathcal{K}$ is extended to the fraction field of R, τ extends to an automorphism $\mathcal{K}$ and $\mathcal{K}^\tau = \mathbb{F}_q((1/t))$. For all $a \in C$ with $0 < |a| < 1$, f_a is solution of $\tau X = tX$. The vector space V^τ is in this case of dimension 1, generated by anyone of these series, but the difference field $\mathcal{K}$ together with the above extension of τ so constructed certainly is not the existentially closed field $\mathbb{K}$ mentioned above; indeed, the constant field $\mathcal{K}^\tau$ is now strictly bigger than $\mathbb{F}_q(t)$.

3 Deformations of vectorial modular forms

After having described some basic facts of the theory of τ-linear recurrent sequences, we come back to our modular forms and we now start dealing with vectorial modular forms and their deformations. For this, we are making again specific choice of $\mathcal{K}, \tau$ etc.

3.1 Notation, tools

Let t be an indeterminate transcendental over $K = \mathbb{F}_q(\theta)$. Often in this paper, t will be also a parameter varying in C and we will freely switch from formal series to functions.

For a positive real number r, we denote by $\mathbb{T}_{<r}$ the sub-C-algebra of $C[[t]]$ whose elements are formal series $\sum_{i\geq 0} c_i t^i$ that converge for any $t \in C$ with $|t| < r$. We also denote by $\mathbb{T}_\infty$ the sub-C-algebra of series that converge everywhere in C. If $r_1 > r_2 > 0$, we have

$$\mathbb{T}_{<r_2} \supset \mathbb{T}_{<r_1} \supset \mathbb{T}_\infty.$$

The *Tate algebra* of formal series of $C[[t]]$ converging for all t such that $|t| \leq 1$ will be denoted by $\mathbb{T}_1$ or $\mathbb{T}$; it is contained in $\mathbb{T}_{<1}$ and contains $\mathbb{T}_{<1+\epsilon}$ for all $\epsilon > 0$; clearly, $C[[t]] \supset \mathbb{T}_1 \supset \mathbb{T}_\infty$.

The ring $C[[t]]$ is endowed with the $\mathbb{F}_q[[t]]$-linear automorphism τ acting on formal series as follows:

$$\tau \sum_i c_i t^i = \sum_i c_i^q t^i.$$

This automorphism induces automorphisms of $\mathbb{T}_1, \mathbb{T}_\infty$.

We will work with certain functions $f : \Omega \times B_r \to C$ with the property that for all $z \in \Omega$, $f(z,t)$ can be identified with and element of $\mathbb{T}_{<r}$. For such functions we will then also write $f(z)$ to stress the dependence on $z \in \Omega$ when we want to consider them as functions $\Omega \to \mathbb{T}_{<r}$ for some r. Sometimes, we will not specify the variables z, t and just write f instead of $f(z,t)$ or $f(z)$ to lighten our formulas.

In all the following, $\mathbf{Hol}(\Omega)$ denotes the ring of holomorphic functions on Ω and $\mathbf{Me}(\Omega)$ its fraction field. For r a positive real number, let us denote by $\mathcal{R}_{<r}$ (resp. $\mathcal{R}$ or $\mathcal{R}_1$) the (integral) ring whose elements are the formal series $f = \sum_{i\geq 0} f_i t^i$, such that

1. For all i, f_i is a map $\Omega \to C$.
2. For all $z \in \Omega$, $\sum_{i\geq 0} f_i(z) t^i$ is an element of $\mathbb{T}_{<r}$ (resp. $\mathbb{T}$).
3. For all i, f_i belongs to $\mathbf{Hol}(\Omega)$.

We shall write

$$\mathcal{R}_\infty = \bigcap_{r>0} \mathcal{R}_{<r}$$

and allow r to vary in $\mathbb{R}_{>0} \cup \{\infty\}$. The rings $\mathcal{R}$ and $\mathcal{R}_\infty$ are endowed with injective endomorphisms τ acting on formal series as follows:

$$\tau \sum_{i\geq 0} f_i(z) t^i = \sum_{i\geq 0} f_i(z)^q t^i.$$

Let $\mathbf{Me}^{1/p^\infty}(\Omega)$ be the perfect closure of $\mathbf{Me}(\Omega)$. The operator τ extends in an unique way to an automorphism of the integral domain $\mathbf{Me}^{1/p^\infty}(\Omega)[[t]]$, in which we can embed the integral domains

$$\bigcup_{k\in\mathbb{Z}} \tau^k \mathcal{R}, \quad \bigcup_{k\in\mathbb{Z}} \tau^k \mathcal{R}_\infty,$$

and we denote by $\mathcal{K}$ and $\mathcal{K}_\infty$ the respective fraction fields. It is easy to show that τ induces automorphisms of these fields as well. From now on, the couples $(\mathcal{K},\tau),(\mathcal{K}_\infty,\tau)$ will play the role of the difference field $(\mathcal{K},\tau)$ of Section 2.

Lemma 14 *We have* $\mathcal{K}^\tau = \mathcal{K}_\infty^\tau = \mathbb{F}_q(t)$.

Proof. It suffices to compute $\mathcal{K}^\tau$. Let f be in $\mathcal{K}^\tau$ and let us choose $z \in \Omega$. Evaluation of f at z is meaningful and yields a formal series ϕ_z in $C((t))^\tau = \mathbb{F}_q((t))$. Therefore, we can write

$$f(z,t) = \sum_{i \geq i_0} c_i(z) t^i$$

with $c_i^{q^n} \in \mathbf{Me}(\Omega)$ such that, when defined, $c_i(z) \in \mathbb{F}_q$. Hence, $f \in \mathbb{F}_q((t))$ and we can use Theorem 2.2.9 of [8] to show that f lies in $\mathbb{F}_q(t)$. □

We end this preparatory section with some conventions on u-expansions. We will say that a series $\sum_{i \geq i_0} c_i u^i$ (with the coefficients c_i in some ring) is *normalised*, if $c_{i_0} = 1$. We will also say that the series is *of type* $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ if $i \not\equiv m \pmod{q-1}$ implies $c_i = 0$. This definition is obviously compatible with the notion of type of a Drinfeld modular form already discussed in the introduction.

3.2 Basic properties of vectorial modular forms.

In this subsection we introduce *deformations of vectorial modular forms*. This part is largely inspired by a conspicuous collection of papers about vectorial modular forms for $\mathbf{SL}_2(\mathbb{Z})$ notably by Knopp, Mason.

To make our list of references self contained, we only mention [18, 19], leaving the reader to further explore the literature. In particular, we learned from [18, Section 3] how to construct *vectorial Poincaré series*, of which we propose a Drinfeldian counterpart in Subsection 3.3.2. It should be noticed, however, that our construction is not a complete adaptation of Knopp and Mason's constructions and the analogy is superficial.

The main differences are two. Firstly, these authors associate vectorial modular forms and vectorial Poincaré series to *general* representations of $\mathbf{SL}_2(\mathbb{Z})$, while we restrict our attention to a very special class of representations $\rho_{t,l}$. Secondly, Knopp and Mason's representations act on $\mathbb{C}^s$, while ours, act on $\mathbb{F}_q(t)^s$, namely, they depend on the parameter t, not present in Knopp and Mason's papers.

These differences are motivated by a fundamental gap between the theories. While symmetric powers of two-dimensional irreducible representations of $\mathbf{SL}_2(\mathbb{Z})$ are irreducible, symmetric powers of two-dimensional irreducible representations of $\mathbf{GL}_2(A)$ are the most often *not* irreducible and split along tiny irreducible sub-representations. Thence, our approach contains some novelties compared to the above mentioned works apart from dealing with the positive characteristic case.

In this subsection we consider representations

$$\rho : \Gamma \to \mathbf{GL}_s(\mathbb{F}_q((t))). \tag{32}$$

We assume that the determinant $\det(\rho)$ is the μ-th power of the determinant character, for some $\mu \in \mathbb{Z}/(q-1)\mathbb{Z}$. In all the following, given $\gamma \in \Gamma$, we denote by J_γ the associated factor of automorphy $cz+d$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 15 A *deformation of vectorial modular form* (abridged to DVMF) of weight w, dimension s, type m and radius $r \in \mathbb{R}_{>0} \cup \{\infty\}$ associated with a representation ρ as in (32) is a column matrix $\mathcal{F} \in \mathbf{Mat}_{s\times 1}(\mathcal{R}_{<r})$ such that, considering $\mathcal{F}$ as a map $\Omega \to \mathbf{Mat}_{s\times 1}(\mathbb{T}_r)$ we have, for all $\gamma \in \Gamma$,

$$\mathcal{F}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m}\rho(\gamma)\cdot\mathcal{F}(z).$$

The definition means that if the radius is ∞, then the entries of $\mathcal{F}$ are in $\mathcal{R}_\infty$.

The set of deformations of vectorial modular forms of weight w, dimension s, type m and radius r associated to a representation ρ is a $\mathbb{T}_{<r}$-module (or $\mathbb{T}_\infty$-module if $r=\infty$) that we will denote by $\mathcal{M}^s_{w,m}(\rho,r)$ or $\mathcal{M}^s_{w,m}(\rho)$ when the reference to a particular radius is clear.

In this paper, $M^!_{w,m}$ denotes the C-vector space (of infinite dimension) generated by quotients f/g with $f \in M_{w',m'}$, $g \in M_{w'',m''} \setminus \{0\}$ such that $w'-w''=w$ and $m'-m''=m$.

If $s=1$ and if $\rho=\mathbf{1}$ is the constant map, then $\mathcal{M}^1_{w,m}(\mathbf{1},r) = M^!_{w,m}\otimes\mathbb{T}_{<r}$. Therefore, for general s, we have a graded $M^!_{w,m}\otimes\mathbb{T}_{<r}$-module

$$\mathcal{M}^s(\rho,r) = \bigoplus_{w,m}\mathcal{M}^s_{w,m}(\rho,r).$$

Lemma 16 *Let k be a non-negative integer. If $\mathcal{F}$ is in $\mathcal{M}^s_{w,m}(\rho,r)$, then $\tau^k\mathcal{F} \in \mathcal{M}^s_{wq^k,m}(\rho,r^{q^k})$ and $(\tau^{-k}\mathcal{F})^{q^k} \in \mathcal{M}^s_{w,m}(\rho,r)$.*

Proof. from the definition,

$$(\tau^k\mathcal{F})(\gamma(z)) = J_\gamma^{wq^k}\det(\gamma)^{-m}\rho(\gamma)(\tau^k\mathcal{F})$$

because $\tau(\rho(\gamma)) = \rho(\gamma)$. □

Proposition 17 *Let us assume that $r>1$, let us consider $\mathcal{F}$ in $\mathcal{M}^s_{w,m}(\rho,r)$ and ${}^t\mathcal{E}$ in $\mathcal{M}^s_{w',m'}({}^t\rho^{-1},r)$, choose nonnegative integers $k_1,\dots,k_s$ and set $k=\max\{k_1,\dots,k_s\}$. Then*

$$\det(\tau^{k_1}\mathcal{F},\dots,\tau^{k_s}\mathcal{F}) \in M^!_{w(q^{k_1}+\cdots+q^{k_s}),sm+\mu}\otimes\mathbb{T}_{<r},$$

and

$$\det(\tau^{-k_1}({}^t\mathcal{E}),\dots,\tau^{-k_s}({}^t\mathcal{E}))^{q^k} \in M^!_{w'(q^{k-k_1}+\cdots+q^{k-k_s}),sm'-\mu}\otimes\mathbb{T}_{<r}.$$

In particular,

$$W_\tau(\mathcal{F}) \in M^!_{w(1+q+q^2+\cdots+q^{s-1})),sm+\mu}\otimes\mathbb{T}_{<r},$$

and

$$W_{\tau^{-1}}({}^t\mathcal{E})^{q^{s-1}} \in M^!_{w'(1+q+q^2+\cdots+q^{s-1})),sm'-\mu}\otimes\mathbb{T}_{<r}.$$

Moreover, for nonnegative k, if $\mathcal{G}_k$ denotes $(\tau^k\mathcal{E})\cdot\mathcal{F}$, then

$$\mathcal{G}_k \in M^!_{w+w'q^k,m+m'}\otimes\mathbb{T}_{<r}.$$

Proof. Most of proposition's proof is straightforward with the tools developed in Section 2, so we can omit some details. Define the matrix function:

$$\mathbf{M}_{k_1,\dots,k_s} = (\tau^{k_1}\mathcal{F},\dots,\tau^{k_s}\mathcal{F}).$$

After Lemma 16, we have, for $\gamma\in\mathbf{GL}_2(A)$:

$$\mathbf{M}_{k_1,\dots,k_s}(\gamma(z)) = \det(\gamma)^{-m}\rho(\gamma)\cdot\mathbf{M}_{k_1,\dots,k_s}(z)\cdot\mathbf{Diag}(J_\gamma^{wq^{k_1}},\cdots,J_\gamma^{wq^{k_s}}).$$

If the k_i's are all positive, the coefficients of the t-expansions of the entries of $\det(\tau^{k_1}\mathcal{F},\dots,\tau^{k_s}\mathcal{F})$ are holomorphic functions on Ω. It they are all negative, the corresponding coefficients, raised to the power q^k, are holomorphic on Ω.

The part of the proposition involving the determinant of $\mathbf{M}_{k_1,\dots,k_s}$ follows easily. There is no additional difficulty in proving the part concerning the form $\mathcal{E}$.

Also the latter property of the sequence $(\mathcal{G}_k)_k$ follows easily from Lemma 16. Indeed, by this lemma, $\tau^k({}^t\mathcal{E})$ is in $\mathcal{M}^s_{wq^k,m'}({}^t\rho^{-1},r)$. Let γ be in $\mathbf{GL}_2(A)$. We know that

$$(\tau^k\mathcal{E})(\gamma(z)) = J_\gamma^{wq^k}\det(\gamma)^{-m}\,{}^t\mathcal{E}(z)\cdot\rho^{-1}(\gamma)$$

and

$$\mathcal{F}(\gamma(z)) = J_\gamma^{w'}\det(\gamma)^{-m'}\rho(\gamma)\cdot\mathcal{F}(z).$$

Hence,

$$\mathcal{G}_k(\gamma(z)) = J_\gamma^{wq^k+w'}\det(\gamma)^{-m-m'}\mathcal{G}_k(z),$$

from which we deduce that $\mathcal{G}_k \in M^!_{wq^k+w',m+m'}\otimes\mathbb{T}_{<r}$. □

The next proposition is a mere reproduction of the main properties described in Section 2 in the framework of deformations of vectorial modular forms.

Proposition 18 *Assuming that $r>1$, let us consider $\mathcal{F}$ in $\mathcal{M}^s_{w,m}(\rho,r)$ and let $\mathcal{E}$ be such that ${}^t\mathcal{E}$ is in $\mathcal{M}^s_{w',m'}({}^t\rho^{-1},r)$. For $k\in\mathbb{Z}$, let us write $\mathcal{G}_k=(\tau^k\mathcal{E})\cdot\mathcal{F}$.*

Then, for all $k=0,\dots,s$, we have

$$A_k = A_k^\tau(\mathcal{F}) \in M^!_{(1+q+\cdots+\widehat{q^k}+\cdots+q^s)w,sm+\mu}\otimes\mathbb{T}_{<r}$$

(the hat means that we skip the corresponding term in the sum). Let L be the operator $A_0\tau^0+\cdots+A_s\tau^s$. If $r>1$ and if the components of $\mathcal{F}$ are $\mathbb{F}_q(t)$-linearly independent, then L is split, for any k nonnegative integer, $\mathcal{G}_k$ is an element of $M^!_{wq^k+w',m+m'}\otimes\mathbb{T}_{<r}$ and we have the relations

$$A_0\mathcal{G}_k + A_1\tau\mathcal{G}_{k-1}+\cdots+A_s\tau^s\mathcal{G}_{k-s} \;=\; 0.$$

For all $k=0,\dots,s$, $A'_k=A_k^{\tau^{-1}}(\mathcal{E})$ is such that

$$(A'_k)^{q^s}\in M^!_{(1+q+\cdots+\widehat{q^k}+\cdots+q^s)w,sm-\mu}\otimes\mathbb{T}_{<r}.$$

If the entries of $\mathcal{E}$ are $\mathbb{F}_q(t)$-linearly independent and $r>1$, then the operator $L'=A'_0\tau^0+\cdots+A'_s\tau^{-s}$ is split and we also have the relations:

$$(\tau^kA'_0)\mathcal{G}_k+(\tau^kA'_1)\mathcal{G}_{k-1}+\cdots+(\tau^kA'_s)\mathcal{G}_{k-s} \;=\; 0.$$

Proof. By Lemma 14 and Lemma 8, the τ-wronskian of $\mathcal{F}$ is non-zero. We apply Proposition 11 to obtain that L is split of order s and if the components of $\mathcal{E}$ are $\mathbb{F}_q(t)$-linearly independent, also L' is split. By Proposition 17, the coefficients A_i are modular as claimed. The part of the proposition involving properties of the form $\mathcal{E}$ is similar and left to the reader. Then, the proposition completely follows from Proposition 12. □

3.3 Examples.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we write $\chi_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix}$. From now on, we will use the representation $\rho = \rho_{t,1} : \mathbf{GL}_2(A) \to \mathbf{GL}_2(\mathbb{F}_q[t])$ defined by

$$\rho_{t,1}(\gamma) = \chi_t(\gamma)$$

and its *symmetric powers* of order l for $l \geq 1$

$$\rho_{t,l} = S^l(\rho_{t,1}) : \mathbf{GL}_2(A) \to \mathbf{GL}_{l+1}(\mathbb{F}_q[t]),$$

realised in the space of polynomial homogeneous of degree $s = l + 1$ with coefficients in $\mathbb{F}_q[t]$:

$$\rho_{t,l}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X^{s-r}Y^r) = (\chi_t(a)X + \chi_t(c)Y)^{s-r}(\chi_t(b)X + \chi_t(d)Y)^r.$$

The determinant of $\rho_{t,l}$ is the $l(l+1)/2$-th power of the determinant character:

$$\det(\rho_{t,l}(\gamma)) = \det(\gamma)^{\frac{l(l+1)}{2}}.$$

Together with $\rho_{t,l}$ we will also use the representation ${}^t\rho_{t,l}^{-1}$ (transpose of the inverse) and we set $\rho_{t,0}(\gamma) = 1$ for all γ.

3.3.1 First example: the functions Φ_l

We first discuss again the functions $\boldsymbol{d}_1, \boldsymbol{d}_2$ mentioned in the introduction.

For $z \in \Omega$, we have denoted by Λ_z the A-module $A + zA$, and we have the expression (7) for the exponential function e_{Λ_z}. We recall that:

$$\begin{aligned} \boldsymbol{s}_1(z,t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t} \\ \boldsymbol{s}_2(z,t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}. \end{aligned}$$

These are functions $\Omega \times B_q \to C$. From [22], we deduce that $\boldsymbol{s}_1, \boldsymbol{s}_2$ lie in $\mathcal{R}_{<q}$.

At θ, the functions $\boldsymbol{s}_i(z, \cdot)$ have simple poles. Their respective residues are $-z$ for the function $\boldsymbol{s}_1(z, \cdot)$ and -1 for $\boldsymbol{s}_2(z, \cdot)$. Moreover, we have $\boldsymbol{s}_1^{(1)}(z, \theta) = \eta_1$ and $\boldsymbol{s}_2^{(1)}(z, \theta) = \eta_2$, where η_1, η_2 are the *quasi-periods* of Λ_z (see [21, Section 4.2.4] and [10, Section 7]). We set, for $i = 1, 2$:

$$\boldsymbol{d}_i(z,t) := \widetilde{\pi} s_{\mathrm{Car}}(t)^{-1} \boldsymbol{s}_i(z,t),$$

with s_{Car} defined in (11). We point out that, in the notations of [22], $\boldsymbol{d} = \boldsymbol{d}_2$. At first sight, we only have $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathcal{R}_{<q}$. However, one sees easily that $s_{\mathrm{Car}}^{-1} \in \mathbb{T}_\infty$ from which it follows that $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathcal{R}_\infty$.

The functions $\boldsymbol{d}_1, \boldsymbol{d}_2$ enjoy several properties that can be easily deduced from [22]. Here, we are concerned with a τ-difference linear equation, a *deformation of Legendre's identity*, the quality of being a deformation of vectorial modular form and a u-expansion for $\boldsymbol{d}_2$. These properties where obtained in [22] for the functions $\boldsymbol{s}_1, \boldsymbol{s}_2$. Here we collect them in the following proposition, in terms of the functions $\boldsymbol{d}_1, \boldsymbol{d}_2$ (the deduction of the proposition from [22] is immediate).

Proposition 19 *We have five properties for the $\boldsymbol{d}_i$'s.*

1. $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathcal{R}_\infty$.

2. *Let us write* $\Phi_1 = \binom{\boldsymbol{d}_1}{\boldsymbol{d}_2}$. *We have* $\Phi_1 \in \mathcal{M}^2_{-1,0}(\rho_{t,1}, \infty)$.

3. *The following* τ*-linear difference equations hold:*

$$\boldsymbol{d}_i = (t - \theta^q)\Delta\boldsymbol{d}_i^{(2)} + g\boldsymbol{d}_i^{(1)}, \quad i = 1, 2. \tag{33}$$

4. *Let us consider the matrix function:*

$$\Psi(z,t) := \begin{pmatrix} \boldsymbol{d}_1(z,t) & \boldsymbol{d}_2(z,t) \\ \boldsymbol{d}_1^{(1)}(z,t) & \boldsymbol{d}_2^{(1)}(z,t) \end{pmatrix}.$$

   For all $z \in \Omega$ *and* t *with* $|t| < q$*:*

$$\det(\Psi) = (t-\theta)^{-1}h(z)^{-1}s_{Car}(t)^{-1}. \tag{34}$$

5. *We have the series expansion*

$$\boldsymbol{d}_2 = \sum_{i \geq 0} c_i(t)u^{(q-1)i} \in 1 + u^{q-1}\mathbb{F}_q[t,\theta][[u^{q-1}]], \tag{35}$$

   convergent for t, u *sufficiently close to* $(0,0)$.

More precisely, we showed in [22] that the series expansion in powers of u of $\boldsymbol{d}_2$ is as follows:

$$\boldsymbol{d}_2 = 1 + (\theta - t)u^{q-1} + (\theta - t)u^{(q^2-q+1)(q-1)} + \cdots \in 1 + (t-\theta)\mathbb{F}_q[t,\theta][[u^{q-1}]], \tag{36}$$

where the dots $\cdots$ stand for terms of higher degree in u.

For $l \geq 1$ fixed, let us consider the function:

$$\Phi_l = \begin{pmatrix} \boldsymbol{d}_1^l \\ \boldsymbol{d}_1^{l-1}\boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_1\boldsymbol{d}_2^{l-1} \\ \boldsymbol{d}_2^l \end{pmatrix} : \Omega \to \mathbf{Mat}_{l+1\times 1}(\mathbb{T}_\infty),$$

so that $\Phi_l \in \mathbf{Mat}_{l+1\times 1}(\mathcal{R}_\infty)$.

Lemma 20 *We have* $\Phi_l \in \mathcal{M}^{l+1}_{-l,0}(\rho_{t,l}, \infty)$ *and the components of* Φ_l *are* $\mathbb{F}_q(t)$*-linearly independent.*

Proof. The first property is obvious after Proposition 19. Assume that we have a non-trivial linear dependence relation with the c_i's in $\mathbb{F}_q(t)$:

$$\sum_{i=0}^{l} c_i \boldsymbol{d}_1^i \boldsymbol{d}_2^{l-i} = 0.$$

Then, replacing $t = \theta$ and using (12), we find

$$\sum_{i=0}^{l} c_i(\theta) z^i = 0$$

which is impossible. □

By Proposition 17 and Lemma 20, there is a split operator of order $l+1$:

$$L_l = A_{l,0}\tau^0 + \cdots + A_{l,l+1}\tau^{l+1} \tag{37}$$

such that $L_l\Phi_l = 0$. In particular, $A_{l,l+1} \neq 0$. Moreover, it is easy to check that for all l and $0 \leq i \leq l+1$, there exists an integer $\mu = \mu(i,l)$ such that $h^\mu A_{l,i} \in M_{*,*} \otimes \mathbb{T}$ (use (34)).

More specifically, if $l = 1$, we find

$$A_{1,2} = \det(\Psi), \quad A_{1,1} = -\frac{\det(\Psi)g}{\Delta(t-\theta^q)}, \quad A_{1,0} = -\frac{\det(\Psi)}{\Delta(t-\theta^q)},$$

implying (33), and

$$L_1 = -\tau^0 - g\tau + \Delta(t-\theta^q)\tau^2. \tag{38}$$

If $l = 2$, we find, after some rather heavy computation using (33) and (34):

$$\begin{aligned} W_\tau(\Phi_2) = A_{2,3} &= \frac{\det(\Psi)^3 g}{\Delta^2(\theta^q - t)^2}, \\ A_{2,2} &= \frac{\det(\Psi)^3 g^q (g^{1+q} + \Delta(t-\theta^q))}{\Delta^{2+2q}(\theta^q - t)^2(\theta^{q^2} - t)^2} \\ A_{2,1} &= -\frac{\det(\Psi)^3 g (g^{1+q} + \Delta(t-\theta^q))}{\Delta^{3+2q}(\theta^q - t)^3(\theta^{q^2} - t)^2}, \\ A_{2,0} &= -\frac{\det(\Psi)^3 g^q}{\Delta^{3+2q}(\theta^q - t)^3(\theta^{q^2} - t)^2}, \end{aligned}$$

and

$$L_2 = -\tau^0 - g^{1-q}(g^{1+q} + \Delta(t-\theta^q))\tau + (g^{1+q} + \Delta(t-\theta^q))\Delta(\theta^q - t)\tau^2 + g^{1-q}\Delta^{1+2q}(\theta^q - t)(\theta^{q^2} - t)^2\tau^3. \tag{39}$$

The explicit determination of the coefficients of the operator (37) for the vectorial forms Φ_l for general l looks like a difficult computational problem.

3.3.2 Second example: Deformations of vectorial Poincaré series

Following [9], let us consider the subgroup $H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$ of $\Gamma = \mathbf{GL}_2(A)$ and its left action on Γ.

For $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the map $\delta \mapsto (c,d)$ induces a bijection between the orbit set $H\backslash\Gamma$ and the set of $(c,d) \in A^2$ with c, d relatively prime. For $l \geq 0$, let $V_l(\delta)$ be the row matrix

$$(\chi_t(c)^l, \chi_t(c)^{l-1}\chi_t(d), \ldots, \chi_t(c)\chi_t(d)^{l-1}, \chi_t(d)^l).$$

We consider the factor of automorphy

$$\mu_{\alpha,m}(\delta, z) = \det(\delta)^{-m}(cz+d)^\alpha,$$

where m and α are positive integers (later, m will also determine a type, that is, a class modulo $q-1$).

It is easy to show that the quantity

$$\mu_{\alpha,m}(\delta,z)^{-1}u^m(\delta(z))V_l(\delta)$$

only depends on the class of $\delta \in H\backslash\Gamma$, so that we can consider the following series:

$$\mathcal{E}_{\alpha,m,l}(z) = \sum_{\delta\in H\backslash\Gamma} \mu_{\alpha,m}(\delta,z)^{-1}u^m(\delta(z))V_l(\delta).$$

This is a row matrix whose $l+1$ entries are formal series.

Let $\mathcal{V}$ be a space of functions $\Omega \to \mathbf{Mat}_{1\times l+1}(C[[t]])$. We introduce, for α, m integers, $f \in V$ and $\gamma \in \Gamma$, the Petersson slash operator defined by:

$$f|_{\alpha,m}\gamma = \det(\gamma)^m(cz+d)^{-\alpha}f(\gamma(z))\cdot\rho_{t,l}(\gamma).$$

This will be used in the next proposition, where $\log_q^+(x)$ denotes the maximum between 0 and $\log_q(x)$, the logarithm in base q of $x > 0$.

Proposition 21 *Let α, m, l be non-negative integers with $\alpha \geq 2m + l$, and write $r(\alpha, m, l) = (\alpha - 2m - l)/l$ if $l \neq 0$, and $r(\alpha, m, l) = \infty$ if $l = 0$. We have the following properties.*

1. *For $\gamma \in \Gamma$, the map $f \mapsto f|_{\alpha,m}\gamma$ induces a permutation of the subset of $\mathcal{V}$:*

$$\mathcal{S} = \{\mu_{\alpha,m}(\delta,z)^{-1}u^m(\delta(z))V_l(\delta); \delta \in H\backslash\Gamma\}.$$

2. *If $t \in C$ is chosen so that $r(\alpha, m, l) > \log_q^+ |t|$, then the components of $\mathcal{E}_{\alpha,m,l}(z,t)$ are series of functions of $z \in \Omega$ which converge absolutely and uniformly on every compact subset of Ω to holomorphic functions.*

3. *If $\log_q |t| < 0$, then the components of $\mathcal{E}_{\alpha,m,l}(z,t)$ converge absolutely and uniformly on every compact subset of Ω also if $\alpha - 2m > 0$.*

4. *For any choice of α, m, l, t submitted to the convergence conditions above, the matrix function ${}^t\mathcal{E}_{\alpha,m,l}(z,t)$ belongs to the space $\mathcal{M}^{l+1}_{\alpha,m}({}^t\rho^{-1}_{t,l}, r(\alpha,m,l))$.*

5. *If $\alpha - l \not\equiv 2m \pmod{(q-1)}$, the matrix function $\mathcal{E}_{\alpha,m,l}(z,t)$ is identically zero.*

6. *If $\alpha - l \equiv 2m \pmod{(q-1)}$ and $\alpha \geq (q+1)m$, then $\mathcal{E}_{\alpha,m,l}$ is not identically zero in its domain of convergence.*

Proof. 1. We choose $f_\delta = \mu_{\alpha,m}(\delta,z)^{-1}u^m(\delta(z))V_l(\delta) \in \mathcal{S}$, for some $\delta \in H\backslash\Gamma$ corresponding to a couple $(c,d) \in A^2$ with c, d relatively prime. We have

$$\begin{aligned}
f_\delta(\gamma(z)) &= \mu_{\alpha,m}(\delta,\gamma(z))^{-1}u^m(\delta(\gamma(z)))V_l(\delta)\\
&= \mu_{\alpha,m}(\gamma,z)\mu_{\alpha,m}(\delta\gamma,z)^{-1}u^m(\delta\gamma(z)))V_l(\delta),\\
&= \mu_{\alpha,m}(\gamma,z)\mu_{\alpha,m}(\delta\gamma,z)^{-1}u^m(\delta\gamma(z)))V_l(\delta\gamma)\cdot\rho_{t,l}(\gamma)^{-1},\\
&= \mu_{\alpha,m}(\gamma,z)\mu_{\alpha,m}(\delta',z)^{-1}u^m(\delta'(z))V_l(\delta')\cdot\rho_{t,l}(\gamma)^{-1},\\
&= \mu_{\alpha,m}(\gamma,z)f_{\delta'}\cdot\rho_{t,l}(\gamma)^{-1},
\end{aligned}$$

with $\delta' = \delta\gamma$ and $f_{\delta'} = \mu_{\alpha,m}(\delta',z)^{-1}u^m(\delta'(z))V_l(\delta')$, from which part 1 of the proposition follows.

2. Convergence and holomorphy are ensured by simple modifications of [9, (5.5)], or by the arguments in [12, Chapter 10]. More precisely, let us choose an integer $0 \leq s \leq l$ and look at the component at the place s

$$\mathcal{E}_s = \sum_{\delta \in H\backslash\Gamma} \mu_{\alpha,m}(\delta, z)^{-1} u(\delta(z))^m \chi_t(c^s d^{l-s})$$

of the vector series $\mathcal{E}_{\alpha,m,l}$. Writing $\alpha = n(q-1)+2m+l$ with n positive integer, we see, following [12, pp. 304-305], that the term of the series $\mathcal{E}_s$:

$$\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{l-s})$$

(where δ corresponds to (c, d)) has absolute value bounded from above by a constant depending on z only (compare with the constant $d(z)$ on top of p. 305 of [12]), multiplied by the quantity

$$\left| \frac{\chi_t(c^s d^{l-s})}{(cz+d)^{n(q-1)+l}} \right|.$$

We now look for upper bounds for the terms of the series above separating several cases just as Gerritzen and van der Put do in loc. cit.

We notice that

$$\mathcal{E}_{\alpha,m,l}(z) = \sum_{f_\delta \in \mathcal{S}} f_\delta.$$

Hence, taking into account the first part of the proposition, to check convergence, we can freely substitute z with $z + a$ with $a \in A$ and we may assume, without loss of generality, that $\deg_\theta z = \lambda \notin \mathbb{Z}$. In this case, for all c, d, $|cz + d| = \max\{|cz|, |d|\}$. If

$$\deg_\theta c + \lambda \leq \deg_\theta d,$$

then

$$\left| \frac{\chi_t(c^s d^{l-s})}{(cz+d)^{n(q-1)+l}} \right| \leq \kappa |\chi_t(d)^l / d^{n(q-1)+l}| \leq \kappa q^{\deg_\theta d(l \log_q^+ |t| - n(q-1) - l)}$$

where κ is a constant depending on z, and the corresponding sub-series converges with the given constraints on the parameters, because $l \log_q^+ |t| - n(q-1) - l < 0$. If

$$\deg_\theta c + \lambda > \deg_\theta d,$$

then

$$\left| \frac{\chi_t(c^s d^{l-s})}{(cz+d)^{n(q-1)+l}} \right| \leq \kappa' |\chi_t(c)^l / c^{n(q-1)+l}| \leq \kappa' q^{\deg_\theta c(l \log_q^+ |t| - n(q-1) - l)},$$

with a constant κ' depending on z, again because $l \log_q^+ |t| - n(q-1) - l < 0$. This ends the proof of the second part of the Proposition.

3. This follows from the inequalities of the proof of the second part because if $\log_q |t| < 0$, then $|\chi_t(c^s d^{l-s})| \leq 1$.

4. The property is obvious by the first part of the proposition, because $\mathcal{E}_{\alpha,m,l}(z) = \sum_{f \in \mathcal{S}} f$.

5. We consider $\gamma = \mathbf{Diag}(1, \lambda)$ with $\lambda \in \mathbb{F}_q^\times$; the corresponding homography, multiplication by λ^{-1}, is the same as that of $\mathbf{Diag}(\lambda^{-1}, 1)$. Hence, we have:

$$\begin{aligned} \mathcal{E}_{\alpha,m,l}(\gamma(z)) &= \lambda^{\alpha-m} \mathcal{E}_{\alpha,m,l}(z) \cdot \mathbf{Diag}(1, \lambda^{-1}, \ldots, \lambda^{-l}) \\ &= \lambda^m \mathcal{E}_{\alpha,m,l}(z) \cdot \mathbf{Diag}(\lambda^l, \lambda^{l-1}, \ldots, 1), \end{aligned}$$

from which it follows that $\mathcal{E}_{\alpha,m,l}$ is identically zero if $\alpha - l \not\equiv 2m \pmod{q-1}$.

6. It is easy to modify the arguments in the proof of [12, Proposition 10.5.2], where the case $l = 0$ is handled. Indeed, let us choose the value $t = 0$ and consider any component of the vector $\mathcal{E}_{\alpha,m,l}|_{t=\theta}(\sqrt{\theta})$.

Just as in [12], the sum can be again decomposed into three terms A, B, C, submitted to the same estimates as on pp. 305-306 of loc. cit., from which we deduce right away that with the constraints above on α, m, the function $\mathcal{E}_{\alpha,m,l}$ is not identically zero. □

3.3.3 A special case: vectorial Eisenstein series

After Proposition 21, if $\alpha > 0$ and $\alpha \equiv l \pmod{q-1}$, then $\mathcal{E}_{\alpha,0,l} \neq 0$. We are going to use these series with $m = 0$ (called *deformations of vectorial Eisenstein series*), especially when $l = 1$. In this subsection, we give some lemmas that will be helpful later.

Let α, l be non-negative integers with $\alpha > 0$ and let us consider the following value of L-series:

$$L(\chi_t^l, \alpha) = \sum_{a \in A^+} a^{-\alpha} \chi_t(a)^l \in K_\infty[[t]],$$

where as usual, A^+ denotes the set of monic polynomials of A.

Lemma 22 *The series $L(\chi_t^l, \alpha)$ converges for all t such that $\log_q |t| < \alpha/l$. In particular, if $\alpha > l + 1$, the series converges at $t = \theta$ to the Carlitz's-Goss zeta value $\zeta(\alpha - l) = \sum_{a \in A^+} a^{\alpha - l}$. Moreover, we have the following relation:*

$$\tau L(\chi_t^l, \alpha) = L(\chi_t^l, q\alpha). \tag{40}$$

Proof. The convergence properties all follow directly from the identity $|\chi_t(a)^l / a^\alpha| = q^{\deg_\theta a(l \log_q |t| - \alpha)}$. As for the τ-difference relation, this is obvious and does not need any further explanation. □

We now observe the following computation, where we recall that we have written

$$V_l(c,d) = (\chi_t(c)^l, \chi_t(c)^{l-1}\chi_t(d), \ldots, \chi_t(d)^l).$$

$$\begin{aligned} \sum_{c,d}{}' (cz+d)^{-\alpha} V_l(c,d) &= \sum_{(c',d')=1} \sum_{a \in A^+} a^{-\alpha}(c'z+d')^{-\alpha} V_l(ac', ad') \\ &= L(\chi_t^l, \alpha)\mathcal{E}_{\alpha,0,l}, \end{aligned}$$

where the first sum is over couples of A^2 distinct from $(0,0)$, while the second sum is over the couples (c', d') of relatively prime elements of A^2. This yields the following lemma.

Lemma 23 *With α, l as above, the following identity holds:*

$$\mathcal{E}_{\alpha,0,l} = L(\chi_t^l, \alpha)^{-1} \sum_{c,d}{}' (cz+d)^{-\alpha} V_l(c,d),$$

and $\mathcal{E}_{\alpha,0,l}$ is not identically zero if and only if $\alpha \equiv l \pmod{q-1}$.

In particular, if $l = 0$, we obtain classical Eisenstein series up to a factor of proportionality:

$$\mathcal{E}_{\alpha,0,0}(z,t) = L(1,\alpha)^{-1} \sum_{c,d}{}' (cz+d)^{-\alpha}.$$

4 Proof of the main results

Let α, m, l be non-negative integers such that $\alpha - 2m > l$ and $\alpha - l \equiv 2m \pmod{(q-1)}$. The series $\mathcal{E}_{\alpha,m,l}, \Phi_l$ determine functions:

$$\begin{aligned} \mathcal{E}_{\alpha,m,l} : \Omega &\to \mathbf{Mat}_{1\times l+1}(\mathcal{R}_{<r}), \\ \Phi_l : \Omega &\to \mathbf{Mat}_{l+1\times 1}(\mathcal{R}_\infty), \end{aligned}$$

with $r = r(\alpha, m, l)$ as in Proposition 21. We consider the functions

$$\mathcal{G}_{\alpha,m,l,k} = (\tau^k \mathcal{E}_{\alpha,m,l}) \cdot \Phi_l = \mathcal{E}_{q^k\alpha,m,l} \cdot \Phi_l : \Omega \to \mathbb{T}_{<r}.$$

We will study $\mathcal{G}_{\alpha,0,l,0}$ in more detail in the case $l = 1$, by using Lemma 23 and rather quickly, we will restrict our attention to the case $\alpha = 1$. We make this choice to ease the reading of this text as much as possible. However, up to certain technical complications, it is possible to extend most of the investigations developed here to $l = 1$ and general α, and even to certain higher values of l.

To begin with, we will be mainly concerned with the following proposition.

Proposition 24 *Let α be positive, such that $\alpha \equiv 1 \pmod{q-1}$. Then, the sequence*

$$(\mathcal{G}_{\alpha,m,1,k})_{k\in\mathbb{Z}}$$

is generic for the difference field $(\mathcal{K}, \tau)$. Moreover, if $\alpha < q^2$, then:

$$\mathcal{G}_{\alpha,0,1,0} = -E_{\alpha-1},$$

where $E_{\alpha-1}$ is the normalised Eisenstein series of weight $\alpha - 1$.

See [9, (6.3)] for the definition of (the non-normalised) Eisenstein series. We may remark that although the variable t is involved in the construction of $\mathcal{G}_{\alpha,0,1,0}$ for such a choice of the parameters, the function ultimately does not depend on it. The proof of Proposition 24 will occupy the next subsection, and several lemmas obtained there will be again used in the text.

4.1 Proof of Proposition 24

Following Gekeler [9, Section 3], we recall that for all $\alpha > 0$ there exists a polynomial $G_\alpha(u) \in C[u]$, called the *α-th Goss polynomial*, such that, for all $z \in \Omega$, $G_\alpha(u(z))$ equals the sum of the convergent series

$$\widetilde{\pi}^{-\alpha} \sum_{a\in A} \frac{1}{(z+a)^\alpha}.$$

Several properties of these polynomials are collected in [9, Proposition (3.4)]. We highlight that for all α, G_α is of type α as a formal series of $C[[u]]$. Namely:

$$G_\alpha(\lambda u) = \lambda^\alpha G_\alpha(u), \quad \text{for all } \lambda \in \mathbb{F}_q.$$

We also recall, for $a \in A$, the function

$$u_a(z) := u(az) = e_{\mathrm{Car}}(\widetilde{\pi} a z)^{-1} = u^{|a|} f_a(u),$$

where $f_a \in A[[u]]$ is the *a-th inverse cyclotomic polynomial* defined in [9, (4.6)]. Obviously, we have

$$u_{\lambda a} = \lambda^{-1} u_a \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

Lemma 25 *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. We have, for all $t \in C$ such that $|t| < q$ and $z \in \Omega$, convergence of both the series below, and equality:*

$$\sum_{c,d\in A}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} = -\widetilde{\pi}^\alpha \sum_{c\in A^+} \chi_t(c) G_\alpha(u_c(z)).$$

Moreover, for $\alpha > 1$, convergence holds for $|t| \leq q$.

Proof. Convergence features are easy to check. We then compute:

$$\begin{aligned}
\sum_{c,d}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} &= \sum_{c\neq 0} \chi_t(c) \sum_{d\in A} \frac{1}{(cz+d)^\alpha} \\
&= \widetilde{\pi}^\alpha \sum_{c\neq 0} \chi_t(c) \sum_{d\in A} \frac{1}{(c\widetilde{\pi}z+d\widetilde{\pi})^\alpha} \\
&= \widetilde{\pi}^\alpha \sum_{c\neq 0} \chi_t(c) G_\alpha(u_c) \\
&= \widetilde{\pi}^\alpha \sum_{c\in A^+} \chi_t(c) \sum_{\lambda\in\mathbb{F}_q^\times} \lambda^{1-\alpha} G_\alpha(u_c) \\
&= -\widetilde{\pi}^\alpha \sum_{c\in A^+} \chi_t(c) G_\alpha(u_c).
\end{aligned}$$

□

Let n be a non-negative integer. For a formal series $f = \sum_n c_n t^n$ of $\mathcal{R}[[t]]$ (with $\mathcal{R}$ any ring) we denote by $[f]_n$ the *truncation* of f to the n-th power (do not mix up with a similar definition for u-expansions, appearing in this paper), that is, the unique polynomial P of degree n such that $P \equiv f \pmod{(t)^{n+1}}$:

$$[f]_n = \sum_{i=0}^{n} c_i t^i.$$

Lemma 26 *Let α be such that $\alpha \equiv 1 \pmod{q-1}$. We have, for $\alpha > 0$ and $t \in C$ such that $|t| < q$, or for $\alpha > 1$ and $t \in C$ such that $|t| \leq q$, regardless to the choice of $z \in \Omega$, convergence of the series*

$$\sum_{c,d\in A}{}' \frac{\chi_t(d)}{(cz+d)^\alpha}.$$

We can identify this series with a formal series of $\mathbf{Hol}(\Omega)[[t]]$, and for all $n \geq 0$ and $z \in \Omega$, we have convengence and equality of both the series below:

$$\left[\sum_{c,d\in A}{}' \frac{\chi_t(d)}{(cz+d)^\alpha} + L(\chi_t,\alpha)\right]_n = \widetilde{\pi}^\alpha \theta^{-\alpha(n+1)} \sum_{a\in A, \deg_\theta a \leq n} \chi_t(a) \sum_{c\in A\setminus\{0\}} G_\alpha\left(u((cz+a)\theta^{-1-n})\right).$$

Proof. Again, all the convergence properties are easy to check so we skip the details of the corresponding verifications. Separating the couples (c,d) with $c = 0$ from those with $c \neq 0$, we split the sum:

$$\sum_{c,d\in A}{}' \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t,\alpha) + \sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha},$$

and we need to compute the last series. Let $c \in A$ be non-zero. Writing $d = \theta^{n+1}r + a$ with $d, r, a \in A$ and $\deg_\theta a \leq n$, we have:

$$\begin{aligned}\sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} &= \sum_{a\in A,\deg_\theta a\leq n} \sum_{d\equiv a \pmod{\theta^{n+1}}} \frac{\chi_t(d)}{(cz+d)^\alpha} \\ &= \sum_{a\in A,\deg_\theta a\leq n} \sum_{r\in A} \frac{\chi_t(r)t^{n+1}+\chi_t(a)}{(cz+r\theta^{n+1}+a)^\alpha} \\ &= \sum_{a\in A,\deg_\theta a\leq n} \chi_t(a) \sum_{r\in A} \frac{1}{(cz+r\theta^{n+1}+a)^\alpha} + F_n(z),\end{aligned}$$

where $F_n(z)$ is an element of $t^{n+1}\mathbf{Hol}(\Omega)[[t]]$.

Just as in the proof of Lemma 25, we obtain:

$$\sum_{a\in A,\deg_\theta a\leq n} \chi_t(a) \sum_{r\in A} \frac{1}{(cz+r\theta^{n+1}+a)^\alpha} = \widetilde{\pi}^\alpha \theta^{-\alpha(n+1)} \sum_{a\in A,\deg_\theta a\leq n} \chi_t(a) G_\alpha\left(u((cz+a)\theta^{-1-n})\right),$$

and the lemma follows. □

To prove Proposition 24, we will need two more lemmas. In the next lemma, $|z|_i$ denotes, for $z \in C$, the infimum $\inf_{a\in K_\infty}\{|z-a|\}$.

Lemma 27 *Let $\alpha > 0$ be an integer. For all $t \in C$ such that $|t| < q$, we have*

$$\lim_{|z|_i=|z|\to\infty} \boldsymbol{d}_1(z) \sum_{c,d}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} = 0.$$

Proof. We recall from [22] the series expansion

$$\boldsymbol{d}_1(z) = \frac{\widetilde{\pi}}{s_{\mathrm{Car}}(t)} \boldsymbol{s}_2(z) = \frac{\widetilde{\pi}}{s_{\mathrm{Car}}(t)} \sum_{n\geq 0} e_{\Lambda_z}\left(\frac{z}{\theta^{n+1}}\right) t^n,$$

converging for all t such that $|t| < q$ and all $z \in \Omega$.

By a simple modification of the proof of [10, Lemma 5.9 p. 286], we have

$$\lim_{|z|_i=|z|\to\infty} u(z) t^n e_{\Lambda_z}(z/\theta^{n+1})^q = 0$$

uniformly in $n > 0$, for all t such that $|t| \leq q$.

Moreover, it is easy to show that

$$\lim_{|z|_i=|z|\to\infty} u(z) e_{\Lambda_z}(z/\theta)^q = \widetilde{\pi}^{-q} \lim_{|z|_i=|z|\to\infty} e^q_{\mathrm{Car}}(\widetilde{\pi}z/\theta)/e_{\mathrm{Car}}(\widetilde{\pi}z) = 1.$$

This suffices to show that

$$\lim_{|z|_i=|z|\to\infty} \boldsymbol{d}_1(z) G_\alpha(u_c(z)) = 0$$

uniformly for $c \in A^+$, for all t such that $|t| < q$. The lemma then follows from the use of Lemma 25. □

Lemma 28 *Let $\alpha > 0$ be an integer. For all $t \in C$ such that $|t| < q$, we have*

$$\lim_{|z|_i=|z|\to\infty} \sum_{c,d}{}' \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t, \alpha).$$

Proof. Taking into account Lemma 26, to complete our proof, it suffices to show that

$$\lim_{|z|_i=|z|\to\infty} \sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = 0.$$

But Lemma 26 again tells us that for all $n \geq 0$,

$$\lim_{|z|_i=|z|\to\infty} \left[\sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} \right]_n = 0,$$

and the convergence is uniform for $n \geq 0$. □

Proof of Proposition 24. By definition, $\mathcal{G}_{\alpha,m,1,k} = (\tau^k \mathcal{E}_{\alpha,m,1}) \cdot \Phi_1$ and we know that the components of Φ_1 are $\mathbb{F}_q(t)$-linearly independent (Lemma 20). The $\mathbb{F}_q(t)$-linear independence of the components of $\mathcal{E}_{\alpha,m,1}$ follows from analysing the behaviour at $u = 0$ described by Lemmas 27 and 28. This means that the sequence $(\mathcal{G}_{\alpha,m,1,k})_{k\in\mathbb{Z}}$ is generic hence proving the first part of the proposition.

According to Lemma 23, we need, to finish the proof of the proposition, to compute the sum of the series:

$$F_\alpha(z) := \boldsymbol{d}_1(z) \sum_{c,d}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} + \boldsymbol{d}_2(z) \sum_{c,d}{}' \frac{\chi_t(d)}{(cz+d)^\alpha},$$

which converges in Ω, noticing that this corresponds to the case $\alpha < q^2$ in Theorem 2.

After (35), we have that for all t with $|t| < q$, $\lim_{|z|_i=|z|\to\infty} \boldsymbol{d}_2(z) = 1$. Hence we have, after Lemmas 27 and 28, the existence of a limit for $|z| = |z|_i \to \infty$ for F_α and

$$\lim_{|z|_i=|z|\to\infty} F_\alpha(z) = -L(\chi_t, \alpha).$$

In particular, $F_\alpha(z)$ is a modular form of $M_{\alpha-1,0} \otimes \mathbb{T}_{<q}$. Since for the selected values of α, $M_{\alpha-1,0} = \langle E_{\alpha-1} \rangle$, we obtain that $F_\alpha = -L(\chi_t, \alpha) E_{\alpha-1}$. After Lemma 23, the proposition follows. □

4.2 Proofs of the main theorems

We prove Theorem 3 here and Theorems 1, 2 are simple consequences of it. From the proof, we will also deduce Corollaries 4, 6. At the end of the subsection, there is a proof of Theorem 7.

Let us write:

$$\mathcal{E} = \mathcal{E}_{1,0,1} = L(\chi_t, 1)^{-1} \sum_{c,d\in A}{}' \left(\frac{\chi_t(c)}{cz+d}, \frac{\chi_t(d)}{cz+d} \right), \quad \mathcal{F} = \Phi_1 = \begin{pmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{pmatrix}, \quad \mathcal{G}_k = \mathcal{G}_{1,0,1,k} = (\tau^k \mathcal{E}) \cdot \mathcal{F}.$$

With the notations of the introduction, we have

$$\mathcal{E} = L(\chi_t, 1)^{-1} (\boldsymbol{f}_1, \boldsymbol{f}_2).$$

We know by (33) that the entries of $\mathcal{F}$ span the $\mathbb{F}_q(t)$-vector space of solutions in $\mathcal{K}_{<q}$ of the τ-difference equation

$$Lx = 0,$$

where $L = L_1$ is the operator defined in (38), and Proposition 18 implies that $L(\mathcal{G}_k) \equiv 0$. The same proposition states the existence of $L' = A_0\tau^0 + A_1'\tau^{-1} + A_2'\tau^{-2} \in \mathcal{K}_{<q}[\tau^{-1}]$ such that $L'(\mathcal{G}_k) \equiv 0$ and such that the entries of $\mathcal{E}$ span the $\mathbb{F}_q(t)$-vector space of solutions in $\mathcal{K}_{<q}$ of

$$L'x = 0.$$

We first explicitly compute L' (the computation is possible for other values of α).

By Proposition 24, we have $\mathcal{G}_0 = -1$ and $\mathcal{G}_1 = -g$ and since $L(\mathcal{G}_k) \equiv 0$, we have, in particular,

$$\mathcal{G}_2 = -(g^{1+q} - \Delta(t - \theta^q)), \tag{41}$$

$$\mathcal{G}_3 = -(g^{1+q+q^2} + \Delta g^{q^2}(t - \theta^q) + \Delta^q g(t - \theta^{q^2})). \tag{42}$$

From $L'(\mathcal{G}_k) \equiv 0$ we deduce that there exist U_1, U_2 such that, for all $k \in \mathbb{Z}$,

$$\mathcal{G}_k = (\tau^k U_1)\mathcal{G}_{k-1} + (\tau^k U_2)\mathcal{G}_{k-2},$$

and we want to determine U_1, U_2. This computation reduces to the solution of a linear system in indeterminates V_1, V_2:

$$\begin{aligned} V_1(\tau\mathcal{G}_1) + V_2(\tau\mathcal{G}_0) &= \tau\mathcal{G}_2, \\ V_1\mathcal{G}_2 + V_2\mathcal{G}_1 &= \mathcal{G}_3, \end{aligned}$$

that, together with (41), (42) and the relations $V_i = \tau^3 U_i$, yields:

$$U_1 = g^{1/q}, \quad U_2 = \Delta^{1/q^2}(t - \theta^{1/q}).$$

In other words, we can take

$$L' = -\tau^0 + g^{1/q}\tau^{-1} + \Delta^{1/q^2}(1 - \theta^{1/q})\tau^{-2}.$$

This is the τ^{-1}-form of the *adjoint* of L of [14, Goss, Section 1.7], denoted by L^* there. Keeping the notations of Goss, we then have the τ-form of the adjoint, $L^{\mathrm{ad}} = \tau^2 L' \in \mathcal{K}[\tau]$:

$$L^{\mathrm{ad}} = -\tau^2 + g^q\tau + \Delta(t - \theta^q)\tau^0.$$

We have proved that $L^{\mathrm{ad}}\mathcal{E} = 0$ and since the components of $\mathcal{E}$ are $\mathbb{F}_q(t)$-linearly independent, $\mathcal{E}$ spans the vector space V_0^{ad} of solutions of

$$L^{\mathrm{ad}}x = 0. \tag{43}$$

But after [22], the function $s^{(1)}_{\mathrm{Car}}\boldsymbol{E}$ is also solution of (43). Hence, $s^{(1)}_{\mathrm{Car}}\boldsymbol{E}$ is in V_0^{ad} and we can find $\lambda, \mu \in \mathbb{F}_q(t)$ such that

$$s^{(1)}_{\mathrm{Car}}\boldsymbol{E} = \mathcal{E} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.$$

In the next step of the proof, we compute λ, μ. To this purpose, we look again at the behaviour for $|z|_i = |z| \to \infty$. We recall from [22] that the u-expansion of $\boldsymbol{E}$ is:

$$\boldsymbol{E} = u(1 + u^{(q-1)^2} - (t - \theta)u^{(q-1)q} + \cdots). \tag{44}$$

By Lemma 26, we then find $\mu = 0$.

Therefore,

$$L(\chi_t, 1)^{-1} \boldsymbol{f}_1 = \lambda s^{(1)}_{\mathrm{Car}} \boldsymbol{E}.$$

By Lemma 25,

$$-\widetilde{\pi} L(\chi_t, 1)^{-1} \sum_{c \in A^+} \chi_t(c) u_c = \lambda s^{(1)}_{\mathrm{Car}} \boldsymbol{E}. \tag{45}$$

The left- and right-hand sides of this equality are formal series in powers of u. We compute the coefficients of u (thanks to (44) and to the well known properties of the functions u_a, [9, p. 685]) and we find the identity:

$$L(\chi_t, 1) = -\frac{\widetilde{\pi}}{\lambda s^{(1)}_{\mathrm{Car}}}. \tag{46}$$

To compute λ, we would only need to replace t by θ if both the left- and right-hand side of the equality above were meaningful. Unfortunately, the series $L(\chi_t, 1)$ diverges at $t = \theta$. However, we know from (46) that this function has a limit for $t \to \theta$, and this limit must be an element of $\mathbb{F}_q(\theta)$.

To compute it, it suffices to apply τ to both the left- and right-hand sides of (46) and use (40); unfortunately, the argument needs to be modified if $q = 2$ so we assume from now on that $q > 2$, leaving the reader the task of completing the proof when $q = 2$. We obtain:

$$\tau L(\chi_t, 1) = L(\chi_t, q) = \frac{-\widetilde{\pi}^q}{\lambda s^{(2)}_{\mathrm{Car}}} = \frac{-\widetilde{\pi}^q}{\lambda s^{(2)}_{\mathrm{Car}}} = \frac{-\widetilde{\pi}^q}{\lambda (t-\theta) s^{(1)}_{\mathrm{Car}}}.$$

The limit $\lim_{t \to \theta} L(\chi_t, q)$ is the well known value of the Carlitz-Goss zeta function:

$$\lim_{t \to \theta} L(\chi_t, q) = \zeta(q-1) = -\frac{\widetilde{\pi}^{q-1}}{[1]}.$$

Since $\lim_{t \to \theta} s^{(1)}_{\mathrm{Car}} = -\widetilde{\pi}$, we obtain that $\lambda(\theta) = 1$. But now, $\lambda = 1$.

As a by-product of our computations, we have deduced Corollary 6. We finish the proof of Theorem 3. Corollary 6 (condensing the above discussion) and (45), deliver the following equality:

$$\boldsymbol{E} = \sum_{c \in A^+} \chi_t(c) u_c, \tag{47}$$

that is, Corollary 4.

Now, equation (47) is equivalent to

$$L(\chi_t, 1)^{-1} \boldsymbol{f}_1 = s^{(1)}_{\mathrm{Car}} \boldsymbol{E}.$$

By Proposition 21, we have that

$$\boldsymbol{f}_1(-1/z) = z^{-1} \boldsymbol{f}_2(z).$$

By the functional equation $\boldsymbol{E}(-1/z) = z(\boldsymbol{d}^{(1)}_1 / \boldsymbol{d}^{(1)}_2) \boldsymbol{E}(z)$ of $\boldsymbol{E}$ described in [22] and using that $\boldsymbol{E} = -h \boldsymbol{d}^{(1)}_2$ (as in [22]), we then obtain the equality (14). This completes the proof of our Theorem 3. □

Proof of Theorems 1 and 2. For $k = 0, 1$, Theorems 1 and 2 agree with Proposition 24. Since $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$ satisfies

$$L_1(\mathcal{G}) = 0$$

where L_1 is the operator defined in (38) and by definition, $L_1((g_k^\star)_{k\in\mathbb{Z}}) = 0$, the two sequences $\mathcal{G}$ and $(g_k^\star)_{k\in\mathbb{Z}}$ have the same initial data, so they are equal. □

Proof of Theorem 7. We recall that after [22, 5], for all $k \geq 0$, $\boldsymbol{E}$ is a *deformation of Drinfeld quasi-modular form* of weight $(q^k, 1)$ and type 1 and the function $\phi_k(z) := \boldsymbol{E}^{(k)}(z, \theta)$ is a well defined Drinfeld quasi-modular form in the space $\widetilde{M}^{\leq 1}_{q^k+1,1}$. By (44),

$$\boldsymbol{E}^{(k)} = u^{q^k} + \cdots,$$

and again by [4, Theorem 1.2] $\boldsymbol{E}^{(k)}(z, \theta)$ is normalised, extremal, therefore proportional to x_k for all k. By [4, Proposition 2.3],

$$x_k = (-1)^{k+1} L_k u^{q^k} + \cdots,$$

where $L_k = [k][k-1]\cdots[1]$ if $k > 0$ and $L_0 = 1$. From Corollary 4, for all $k \geq 0$, $\boldsymbol{E}^{(k)} \in \mathbb{F}_q[t, \theta][[u]]$. Therefore, $E_k \in A[[u]]$. □

Remarks. 1. From (14) we find $\mathcal{E}_{1,0,1} \cdot \Phi_1 = \mathcal{G}_0 = -1$, which is just our deformation of Legendre's identity (34).

2. The fact that, simultaneously, $L(\Phi_1) = 0$ and $L^{\mathrm{ad}}(\mathcal{E}_{1,0,1}) = 0$ ($\alpha = 1$), does not hold for general values of α. It would be interesting to understand when this takes place.

5 Computing u-expansions

Let μ be an element of C and let us consider the function:

$$s_{\mathrm{Car},\mu}(t) := \sum_{i=0}^{\infty} e_{\mathrm{Car}}\left(\frac{\mu}{\theta^{i+1}}\right) t^i.$$

The function $\mu \mapsto s_{\mathrm{Car},\mu}$ is well defined with image in $\mathbb{T}_{<q}$.

By [21, Equation (10) p. 220] we have the functional equation:

$$s^{(1)}_{\mathrm{Car},\mu}(t) = (t-\theta) s_{\mathrm{Car},\mu}(t) + e_{\mathrm{Car}}(\mu). \tag{48}$$

For fixed μ, the function $s_{\mathrm{Car},\mu}(t)$ has a simple pole in $t = \theta$ with residue $-\mu$. We point out that $s_{\mathrm{Car}} = s_{\mathrm{Car},\widetilde{\pi}}$.

We now consider the function

$$F^\star : C \to \mathbb{T}_{<q}$$

defined by $F^\star(z) = s_{\mathrm{Car},\widetilde{\pi} z}(t) \in \mathbb{T}_{<q}$ (so that $F^\star(1) = s_{\mathrm{Car}}$ and $F^\star \in \mathcal{R}_{<q}$) and the function $F : \Omega \to \mathbb{T}_{<q}$ defined by $F(z) = F^\star(z)/s_{\mathrm{Car}}$. We have $F \in \mathcal{R}_\infty$ and we can write:

$$F(z)|_{t\mapsto\theta} = z. \tag{49}$$

We have the functional equations

$$F^{(1)} = F + \frac{1}{(t-\theta) u s_{\mathrm{Car}}(t)}, \quad F^{\star(1)} = (t-\theta) F^\star + \frac{1}{u}. \tag{50}$$

In the next two propositions, we introduce the functions $\psi, \Upsilon, \psi^\star, \Upsilon^\star$. In fact, we set $\psi^\star = s^{(1)}_{\mathrm{Car}} \psi$ and $\Upsilon^\star = s^{(1)}_{\mathrm{Car}} \Upsilon$ so that we only need to define ψ and Υ, but we will discuss properties of all the four functions.

Proposition 29 *Let us define the function* $\psi = R\boldsymbol{d}_2 + R^{(1)}(\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)})$ *with* $R = 1/((t-\theta)us_{Car}) = 1/(us_{Car}^{(1)})$ *and let* $\psi^\star$ *be the function*

$$s_{Car}^{(1)}\psi = \frac{\boldsymbol{d}_2}{u} + \frac{\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)}}{(t-\theta^q)u^q} = \frac{\boldsymbol{d}_2}{u} + \frac{\boldsymbol{d}_2^{(2)}\Delta}{u^q}.$$

We have the following properties.

1. *The function* ψ *belongs to* $\mathcal{R}_{<q^q}$.
2. *The function* $\psi^\star$ *can be identified, for* $|u|, |t|$ *small, with the sum of a converging* u*-expansion*
$$\psi^\star \in u^{q-2}\mathbb{F}_q[t,\theta][[u^{q-1}]]$$
of type -1.
3. *The first few terms of the* u*-expansion of* $\psi^\star$ *read as follows:*
$$u^{q-2}(\theta - t + u^{(q-1)(q-2)} + (\theta-\theta^q)u^{(q-1)^2} + \cdots). \tag{51}$$
4. *We have, for all* u *with* $|u|$ *small enough,*
$$\lim_{t\to\theta}\psi^\star = \frac{1}{u} + \frac{Eg+h}{(\theta-\theta^q)hu^q}.$$
5. *If* $q \neq 2$, *we have* $\lim_{u\to 0}\psi^\star = 0$, *while if* $q = 2$, *we have* $\lim_{u\to 0}\psi^\star = 1 + \theta - t$.

Proof. 1. This is clear as $\boldsymbol{d}_2, \boldsymbol{d}_2^{(2)}$ belong to $\mathcal{R}_{<q^q}$ as well as $R = 1/(us_{\mathrm{Car}}^{(1)})$.

2. Writing $v = u^{q-1}$, we have by (36):

$$\boldsymbol{d}_2 = 1 + (\theta - t)v + (\theta - t)v^{q^2-q+1} + \cdots$$

and we have the series expansion

$$g = 1 + (\theta - \theta^q)v + (\theta - \theta^q)v^{q^2-q+1} + \cdots,$$

that can be obtained with [9, Corollary (10.11) and formula for U_1 on p. 691].

Substituting into the definition of ψ, we see, from $\boldsymbol{d}_2 \in \mathbb{F}_q[t,\theta][[v]]$ that

$$\psi^\star \in u^{q-2}\mathbb{F}_q[t,\theta][[v]].$$

Moreover, we know that $\Delta, \boldsymbol{d}_2, \boldsymbol{d}_2^{(2)}$ are of type 0, and it is obvious that $R^{(k)}$ is of type -1 for all k.

3. Explicitly, we compute step by step:

$$\begin{aligned}
\boldsymbol{d}_2^{(1)} &= 1 + (\theta^q - t)v^q + (\theta^q - t)v^{q(q^2-q+1)} + \cdots \\
g\boldsymbol{d}_2^{(1)} &= 1 + (\theta - \theta^q)v + (\theta^q - t)v^q + (\theta^q - \theta)(t-\theta^q)v^{q+1} + (\theta - \theta^q)v^{q^2-q+1} + \cdots \\
\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)} &= (\theta^q - t)(v - v^q + (\theta^q - \theta)v^{q+1} + v^{q^2-q+1} + \cdots) \\
\boldsymbol{d}_2 + \frac{\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)}}{(t-\theta^q)v} &= (\theta - t)v + v^{q-1} + (\theta - \theta^q)v^q - v^{q^2-q} + (\theta - t)v^{q^2-q+1} + \cdots \\
\psi^\star &= \frac{1}{u}\left(\boldsymbol{d}_2 + \frac{\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)}}{(t-\theta^q)v}\right) \\
&= u^{-1}\{(\theta - t)v + v^{q-1} + (\theta - \theta^q)v^q - v^{q^2-q} + (\theta - t)v^{q^2-q+1} + \cdots\},
\end{aligned}$$

which gives (51) and all the properties of ψ claimed by the statement of the proposition.

4. It suffices to use the definition of $\psi^\star$, $\boldsymbol{d}_2^{(2)} = \frac{\boldsymbol{d}_2 - g\boldsymbol{d}_2^{(1)}}{(t-\theta^q)\Delta}$, and the identities $\boldsymbol{E} = -h\boldsymbol{d}_2^{(1)}$ and $\boldsymbol{E}(\theta) = E, \boldsymbol{d}_2(\theta) = 1$.

5. This follows directly from (51). □

Proposition 30 *We define the functions* $\Upsilon = \boldsymbol{d}_1 - \boldsymbol{d}_2 F$ *and* $\Upsilon^\star = s_{Car}^{(1)}\Upsilon$. *The following properties hold.*

1. *We have that* $\Upsilon \in \mathcal{R}_\infty$.
2. *The function* Υ *is solution of the non-homogeneous* τ*-difference equation:*
$$\Upsilon = (t-\theta^q)\Delta\Upsilon^{(2)} + g\Upsilon^{(1)} + \psi, \tag{52}$$
3. *The function* $\Upsilon^\star$ *can be identified, for* $|u|, |t|$ *small, with the sum of a converging series:*
$$\Upsilon^\star \in \mathbb{F}_q[t,\theta][[u]]$$
of type -1. *The* u*-expansion of* $\Upsilon^\star$ *begins, for* $q \neq 2$, *with the following terms:*
$$-u^{q-2}(t - \theta + (t-\theta)u^{q(q-1)^2} + \cdots). \tag{53}$$
If $q = 2$, *the* u*-expansion of* $\Upsilon^\star$ *begins with the following terms:*
$$t + \theta + (1 + t + \theta)u^2 + \cdots. \tag{54}$$
4. *We have the limit* $\lim_{t\to\theta}\Upsilon = 0$ *for all* $z \in \Omega$ *and* Υ *is the only solution of (52) with this property.*

Proof. 1. We have seen that $F, \boldsymbol{d}_2$ are in $\mathcal{R}_\infty$, so that the property follows for Υ.

2. According to (48), we get:
$$\begin{aligned}
\boldsymbol{d}_1^{(1)} &= \boldsymbol{d}_2^{(1)}F^{(1)} + \Upsilon^{(1)} \\
&= \boldsymbol{d}_2^{(1)}(F+R) + \Upsilon^{(1)}, \\
\boldsymbol{d}_1^{(2)} &= \boldsymbol{d}_2^{(2)}F^{(2)} + \Upsilon^{(2)} \\
&= \boldsymbol{d}_2^{(2)}(F + R + R^{(1)}) + \Upsilon^{(2)}.
\end{aligned}$$

Let L be the operator L_1 defined in (38). By (33), we have $L\boldsymbol{d}_1 = 0$. Explicitly,
$$F\boldsymbol{d}_2 + \Upsilon = (t-\theta^q)\Delta((F + R + R^{(1)})\boldsymbol{d}_2^{(2)} + \Upsilon^{(2)}) + g((F+R)\boldsymbol{d}_2^{(1)} + \Upsilon^{(1)}).$$

But again by (33), $L\boldsymbol{d}_2 = 0$ and we see that all the coefficients of F in the identity above give contribution 0 (alternatively, we can apply Lemma 32 and the fact that $\boldsymbol{d}_2$ is a formal power series in u). In other words,
$$L\Upsilon + (t-\theta^q)\Delta(R + R^{(1)})\boldsymbol{d}_2^{(2)} + Rg\boldsymbol{d}_2^{(1)} = 0.$$

Eliminating $\boldsymbol{d}_2^{(2)}$ with (33) in the above expression yields
$$L\Upsilon + \psi = 0, \tag{55}$$

that is, (52).

3, 4, 5. We proceed as in [22], where we computed the u-expansion (36). We first look at the case $q \neq 2$ and then, we consider the case $q = 2$, more involved. We begin by showing that for $q \neq 2$ equation (52) has an unique solution Y which can be expanded as a formal series in powers of u, with the property that $Y|_{t=\theta} = 0$. Then, we show that $\Upsilon = Y$.

Let f be a formal series in non-negative powers of u with coefficients, say, in $\mathbb{T}_{<r}$,

$$f = \sum_i c_i u^i,$$

and let us consider the truncation

$$[f]_n = \sum_{i \leq q^n - 1} c_i u^i$$

of the series f to the order q^n, with $n \geq 0$ (do not mix up with the truncation in powers of t also used in this paper). By convention, we also set $[f]_n = c_0$ for $n < 0$. We have, for series f, g, the following simple identities:

1. $[f + g]_n = [f]_n + [g]_n$,
2. $[fg]_n = [[f]_n [g]_n]_n$,
3. $[f^{(1)}]_n = [f]^{(1)}_{n-1}$.

For all $n \geq 2$ and any series $Y = \sum_{i \geq 0} c_i u^i$ solution of (52),

$$[Y]_n = (t - \theta^q)[[\Delta]_n [Y]^{(2)}_{n-2}]_n + [[g]_n [Y]^{(1)}_{n-1}]_n + [\psi]_n.$$

Hence, if Y exists, the whole collection of its coefficients is uniquely determined by $[Y]_1$, and the integrality of the coefficients of $[Y]_n$ follows from the same property for $[Y]_1$. We recall now that we are assuming that $q \neq 2$. In this case, ψ vanishes at $u = 0$ (Proposition 29) and for $n = 1$, we find:

$$[Y]_1 = [\psi]_1.$$

This means that there exists one and only one solution of (52) for $q \neq 2$ which is a series of powers of u, with the additional property that it vanishes at $u = 0$.

Now, we need to show that Y is the function we are looking for, but this is a simple task. The set of solutions in $\mathcal{R}_\infty$ of (52) is the translated of $\mathbb{F}_q(t)$-vector space:

$$\mathbb{F}_q(t)\boldsymbol{d}_1 + \mathbb{F}_q(t)\boldsymbol{d}_2 + Y.$$

Since $\boldsymbol{d}_1 = \boldsymbol{d}_2 F + \Upsilon$ and $\boldsymbol{d}_1|_{t=\theta} = F|_{t=\theta} = z$ and $\boldsymbol{d}_2|_{t=\theta} = 1$, we have $\Upsilon|_{t=\theta} = 0$ and we see that

$$\Upsilon = Y.$$

The u-expansion (53) can be checked after explicit computation.

Also, by induction, we may verify that $\Upsilon, \Upsilon^\star$ have type -1 and that

$$\Upsilon^\star \in u^{q-2}\mathbb{F}_q[t, \theta][[v]].$$

Let us now consider the case $q = 2$, in which types are trivial and $u = v$. Here, ψ does not vanish at $u = 0$ and this case has to be handled in slightly different way. In this case, we have, returning to the unknown series Y, the identities:

$$[Y]_1 = (t - \theta^q)[[\Delta]_1 [Y]^{(2)}_0]_1 + [[g]_1 [Y]^{(1)}_0]_1 + [\psi]_1$$

and

$$[Y]_0 = [[g]_0 [Y]_0^{(1)}]_0 + [\psi]_0.$$

Now, the truncations $[\Delta]_1$ and $[g]_1$ are easy to compute:

$$\begin{aligned} [\Delta]_1 &= -u, \\ [g]_1 &= 1 + (\theta^2 + \theta)u \end{aligned}$$

By (51), $[\psi]_1$ is:

$$s_{\mathrm{Car}}^{-1}\left(1 + \frac{1}{t+\theta} + (\theta + \theta^2)u\right).$$

Hence, the constant term $c_0 = [Y]_0$ satisfies a τ-difference "Artin-Schreier" equation:

$$c_0 = c_0^{(1)} + s_{\mathrm{Car}}^{-1}\left(1 + \frac{1}{t+\theta}\right)$$

whose set of solutions is $\{s_{\mathrm{Car}}^{-1} + \lambda\}$ with $\lambda \in \mathbb{F}_q(t)$ and we are reduced to calculate λ corresponding to our function Υ, which satisfies $\Upsilon|_{t=\theta} = 0$. We deduce that $\lim_{t\to\theta} c_0 = 0$. Therefore, $\lambda = 0$ and after some computations, we find (54). The reader can verify that all the properties of the proposition have been checked. □

Remark. It can be proved that F is, up to multiplication by a factor in $\mathbb{F}_q(t)$, the only function for which we can write $\boldsymbol{d}_1 = \boldsymbol{d}_2 F + \Upsilon$, with $\boldsymbol{d}_2, \Upsilon$ formal power series of u with non-negative exponents. Since we do not need this property in this paper, we will not give its proof. Besides all this, it would be interesting to understand the nature of the function Υ. Computer-assisted experiments are possible and generate large tables of coefficients of the functions ψ and Υ, but we will not report them here.

Theorem 31 *For all $k \geq 0$, we have the identity:*

$$g_k^\star = h^{q^k}\left\{\boldsymbol{d}_2^{(k+1)}\prod_{i=1}^{k}(t-\theta^{q^i})\Upsilon^\star - \boldsymbol{d}_2\left(\boldsymbol{d}_2^{(k+1)}\left(\frac{1}{u^{q^k}} + \sum_{i=0}^{k-1}\frac{(t-\theta^{q^k})\cdots(t-\theta^{q^{i+1}})}{u^{q^i}}\right) + \frac{(\Upsilon^\star)^{(k+1)}}{t-\theta^{q^{k+1}}}\right)\right\}.$$

Proof. First of all, we recall that, for $k \geq 0$,

$$\tau^{k+1} s_{\mathrm{Car}} = L_k^\star s_{\mathrm{Car}},$$

where $L_k^\star = (t-\theta^{q^k})\cdots(t-\theta)$. We also recall that $F^{(1)} = F + R$, so that

$$\begin{aligned} F^{(k+1)} &= F + \sum_{i=0}^{k} R^{(i)} \\ &= F + \sum_{i=0}^{k} \frac{1}{L_i^\star s_{\mathrm{Car}} u^{q^i}} \\ &= s_{\mathrm{Car}}^{-1}\left(F^\star + \sum_{i=0}^{k}\frac{1}{L_i^\star u^{q^i}}\right). \end{aligned}$$

Moreover,

$$\Upsilon = \frac{\Upsilon^\star}{L_0^\star s_{\mathrm{Car}}},$$

yielding

$$\Upsilon^{(k+1)} = \frac{(\Upsilon^\star)^{(k+1)}}{L^\star_{k+1} s_{\mathrm{Car}}}.$$

Therefore, by (57) we deduce:

$$\begin{aligned}
\tau^{k+1}(s_{\mathrm{Car}}\boldsymbol{d}_1) &= L^\star_k s_{\mathrm{Car}} \boldsymbol{d}_1^{(k+1)} \\
&= L^\star_k s_{\mathrm{Car}} \left(\boldsymbol{d}_2^{(k+1)} F^{(k+1)} + \Upsilon^{(k+1)} \right) \\
&= L^\star_k s_{\mathrm{Car}} \left(\boldsymbol{d}_2^{(k+1)} \left(s_{\mathrm{Car}}^{-1} \left(F^\star + \sum_{i=0}^{k} \frac{1}{L_i^\star u^{q^i}} \right) \right) + \frac{(\Upsilon^\star)^{(k+1)}}{L^\star_{k+1} s_{\mathrm{Car}}} \right) \\
&= L^\star_k \left(\boldsymbol{d}_2^{(k+1)} \left(F^\star + \sum_{i=0}^{k} \frac{1}{L_i^\star u^{q^i}} \right) + \frac{(\Upsilon^\star)^{(k+1)}}{L^\star_{k+1}} \right) \\
&= L^\star_k \boldsymbol{d}_2^{(k+1)} F^\star + \boldsymbol{d}_2^{(k+1)} \sum_{i=0}^{k} \frac{L^\star_k}{L_i^\star u^{q^i}} + \frac{(\Upsilon^\star)^{(k+1)}}{t - \theta^{q^{k+1}}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(\tau^{k+1} s_{\mathrm{Car}})\boldsymbol{d}_1 &= L^\star_k s_{\mathrm{Car}} \boldsymbol{d}_1 \\
&= L^\star_k s_{\mathrm{Car}} (\boldsymbol{d}_2 s_{\mathrm{Car}}^{-1} F^\star + (t-\theta)^{-1} s_{\mathrm{Car}}^{-1} \Upsilon^\star) \\
&= L^\star_k \boldsymbol{d}_2 F^\star + \prod_{i=1}^{k} (t - \theta^{q^i}) \Upsilon^\star.
\end{aligned}$$

Subtracting, the terms containing $F^\star$ cancel each-others and we obtain the formula applying Theorem 1. □

Remark. Theorem 31 allows to compute the u-expansions of $(\Upsilon^\star)^{(k)}|_{t=\theta}$ for all k. For example, we deduce from the identity of the theorem for $k = 0$,

$$\frac{-1}{h} = \frac{\boldsymbol{d}_2 \boldsymbol{d}_2^{(1)}}{u} + \frac{1}{t - \theta^q} (\Upsilon^\star)^{(1)} \boldsymbol{d}_2 - (\Upsilon^\star) \boldsymbol{d}_2^{(1)},$$

after evaluation at $t = \theta$:

$$(\Upsilon^\star)^{(1)}|_{t=\theta} = \left(\frac{E}{u} - 1 \right) \frac{\theta - \theta^q}{h}.$$

Proof of Corollary 5. We assume here that $q \neq 2$ (but the case $q = 2$ can be handled in a similar way, with slightly different results). We compute the truncation $[g_k^\star]_{k+1}$ to the order q^{k+1}, by using the following properties:

$$[\boldsymbol{d}_2^{(k+1)}]_{k+1} = 1, \quad [h^{q^k}]_{k+1} = -u^{q^k}, \quad [(\Upsilon^\star)^{(k+1)}]_{k+1} = 0$$

and we proceed as in the proof of Proposition 30. We decompose the sum in the right-hand side

of the formula of Theorem 31 in four terms:

$$\begin{aligned}
[h^{q^k}\boldsymbol{d}_2\boldsymbol{d}_2^{(k+1)}u^{-q^k}]_{k+1} &= -[\boldsymbol{d}_2]_{k+1}\\
\left[h^{q^k}\boldsymbol{d}_2\boldsymbol{d}_2^{(k+1)}\sum_{i=0}^{k-1}\frac{(t-\theta^{q^k})\cdots(t-\theta^{q^{i+1}})}{u^{q^i}}\right]_{k+1} &= -\left[\boldsymbol{d}_2\sum_{i=0}^{k-1}(t-\theta^{q^k})\cdots(t-\theta^{q^{i+1}})u^{q^k-q^i}\right]_{k+1},\\
\left[h^{q^k}\boldsymbol{d}_2\boldsymbol{d}_2^{(k+1)}\frac{(\Upsilon^\star)^{(k+1)}}{t-\theta^{q^{k+1}}}\right]_{k+1} &= 0,\\
\left[-h^{q^k}\boldsymbol{d}_2^{(k+1)}\prod_{i=1}^{k}(t-\theta^{q^i})\Upsilon^\star\right]_{k+1} &= \left[u^{q^k}\Upsilon^\star\right]_{k+1}\prod_{i=1}^{k}(t-\theta^{q^i}).
\end{aligned}$$

The corollary follows summing up everything and using (53). □

5.1 Appendix: transcendence of $F^\star$ and $\boldsymbol{d}_1$ over formal Laurent series

Although we will not need it in this paper, we prove here, for further references, the transcendence of $F^\star$ and $\boldsymbol{d}_1$ over the field $C(t)((u))$.

By [17], we can embed an algebraic closure of $C(t)((u))$ in the ring $C(t)^{\text{alg.}}\langle\langle u\rangle\rangle$ of *generalised formal series* $\sum_{i\in\mathcal{I}}c_iu^i$ (whose support, ordered with $\leq$, is a *well ordered subset* of $\mathbb{Q}$; see definition in loc. cit.). We choose such an embedding.

Lemma 32 *The function $F^\star$ is transcendental over the field $C(t)((u))$.*

Proof. The function $F^\star$ is identified in an unique way with a generalised formal series. The functional equation ensures that this series has the following u-expansion:

$$F^\star=\sum_{n\geq 0}c_nu^{-1/q^{n+1}},$$

for some $c_0,c_1,\ldots$ in $C(t)^{\text{alg.}}$.

Actually, these coefficients can be computed easily, by using the functional equation, (49) and the limit $\lim_{t\to\theta}(t-\theta)s_{\text{Car}}(t)=s^{(1)}_{\text{Car}}(\theta)=-\widetilde{\pi}$. Although we will not use them here, we give their formulas for the sake of completeness: $c_0=1$ and

$$c_n=(t-\theta)(t-\theta^{1/q})\cdots(t-\theta^{1/q^n}),\quad n>0.$$

Let us suppose by contradiction that $F^\star$ is algebraic over $C(t)((u))$. By [17, Theorem 8] (read also the discussion on top of page 3465 and [24]), there exist k and $d_0,d_1,\ldots,d_k\in C(t)^{\text{alg.}}$, not all zero, such that, for all n,

$$d_0c_n+d_1c_{n+1}^p+\cdots+d_kc_{n+k}^{p^k}=0, \tag{56}$$

where p is the prime dividing q.

This means that $F^\star$ is algebraic over $C(t,u)$. Consider now the completion $\mathcal{L}_\infty=C(t)^{\text{alg.}}((u^{-1}))$ of $C(t)^{\text{alg.}}(u)$ for the u^{-1}-valuation. Then, the image of $F^\star$ in $\mathcal{L}_\infty$ can be identified with a double formal series of $C((t))((u^{-1}))$ which converges at every (t,u) such that $|u|>1$ and $|t|<q$ to the function

$$G:u\mapsto\sum_{i\geq 0}e_{\text{Car}}\left(\frac{\log_{\text{Car}}(u^{-1})}{\theta^{i+1}}\right)t^i,$$

where $\log_{\mathrm{Car}}$ is the Carlitz's logarithmic series.

The latter function extends to the u's such that $|u| > |\widetilde{\pi}|^{-1}$ and the function $z \mapsto F^\star(z)$ factors through G. Our assumptions imply that for all $z \in C$, $F^\star(z) \in \mathbb{T}_{<q}$ is algebraic over $C(t)$. However, if $z = 1$ we find $F^\star(1) = s_{\mathrm{Car}}(t) \in C((t))$, which is a transcendental function. □

Corollary 33 *The function $s_{Car}^{-1}\boldsymbol{d}_1$ is transcendental over $C(t)((u))$.*

Proof. We have, by definition,

$$s_{\mathrm{Car}}\boldsymbol{d}_1 = \boldsymbol{d}_2 F^\star + (t-\theta)^{-1}\Upsilon^\star. \tag{57}$$

We know by Proposition 19 part 5, that $\boldsymbol{d}_2$ belongs to $\mathbb{F}_q[t,\theta][[u]]$. Moreover, by Proposition 30 part 3, we know that $\Upsilon^\star \in \mathbb{F}_q[t,\theta][[u]]$. Finally, by Lemma 32, $F^\star$ is transcendental over $C(t)((u))$. □

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