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# Weak Lax pairs for lattice equations

Jarmo Hietarinta<sup>1,2\*</sup> and Claude Viallet<sup>1†</sup>

<sup>1</sup> *LPTHE / CNRS / UPMC, 4 place Jussieu 75252 Paris CEDEX 05, France*

<sup>2</sup> *Department of Physics and Astronomy, University of Turku, FIN-20014 Turku, Finland*

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## Abstract

We consider various 2D lattice equations and their integrability, from the point of view of 3D consistency, Lax pairs and Bäcklund transformations. We show that these concepts, which are associated with integrability, are not strictly equivalent. In the course of our analysis, we introduce a number of black and white lattice models, as well as variants of the functional Yang-Baxter equation.

## 1 Introduction

Recent progress in the description of integrable partial difference equations is to a great part due to the consistency approach [1, 2, 3, 4], in particular in the form of 3 dimensional Consistency-Around-a-Cube (CAC). One of the highlights of this approach is the immediate existence of a Lax pair and Bäcklund transforms (BT), which can be directly constructed from the “side-equations” of the cube [2, 5]. One can say, in effect, that the set of side equations yields the BT and the Lax pair. One situation where side equations have been used effectively is in constructing soliton solutions to the lattice equations [6, 7, 8].

Originally it was assumed that on all faces of the cube the equations were the same in form, depending on the relevant corner variables (one component at each corner) and spectral parameters. Recently, 3D consistent sets have appeared, with different equations on the faces [9, 10, 11, 12, 13]. In this more general context it is of interest to take a closer look at the Lax pairs, Bäcklund transforms and consistency, and investigate what they are good for. It should be noted that in the context of partial differential equations the existence of trivial Lax pairs is well known [14] and similar examples have also been noted for some discrete equations (see [15], Chapt. 6).

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\*E-mail: jarmo.hietarinta@utu.fi

†E-mail: claude.viallet@upmc.fr

We will first recall how consistency around the cube, existence of Lax pairs and Bäcklund transformations are intimately related for lattice maps on the square lattice given by multi-affine relations (sections 2 and 3). These considerations apply to the elementary cells, and are *local*. In section 4, we describe specific examples having the CAC property. We detail in particular the already known explicit forms of the equations and Lax pairs which we use in the rest of this paper. In particular, we find that in some cases the zero curvature condition (ZCC) yields two different equations that can be used to define rational evolution in the lattice. We then address the *global* problem of defining equations over the whole lattice, with the guideline given by the 3 dimensional structure coming from the CAC construction (section 5), and check integrability with the calculation of algebraic entropy. In section 6, we push the use of  $2 \times 2$ -matrix Lax pairs to its limits, by constructing discrete systems over a larger sub-lattice of the original lattice. This brings to the light interesting (integrable) structures related to a generalized form of the functional Yang-Baxter equations [16].

## 2 3D consistency, a reminder

The starting point is a regular 2D square lattice, with vertices labeled by integers  $n, m$ . Functions  $x_{n,m}$  are associated to the vertices, and they are subject to a constraint at all elementary cells. This constraint is expressed by an equation  $Q(x_{n,m}, x_{n+1,m}, x_{n,m+1}, x_{n+1,m+1}) = 0$ , assumed to be multi-affine in the four vertex variables. It should depend on all 4 vertex variables, and it should not factorize. It may also depend on some parameters. Sometimes the parameters can be associated to specific directions of the lattice, in which case they appear as “spectral parameters”. To ease the notation, one usually denotes the running value  $x_{n,m}$  by  $x_{n,m} = x$ , and for neighboring values one only indicates the shifts:  $x_{n,m+1} = x_2$ , or in 3D setting,  $x_1 = x_{n+1,m,k}$ ,  $x_{113} = x_{n+2,m,k+1}$  etc.

For multidimensional consistency one needs to build a cube on top of a square and give equations on all six faces of the cube, see Figure 1, the bottom equation being the original one.

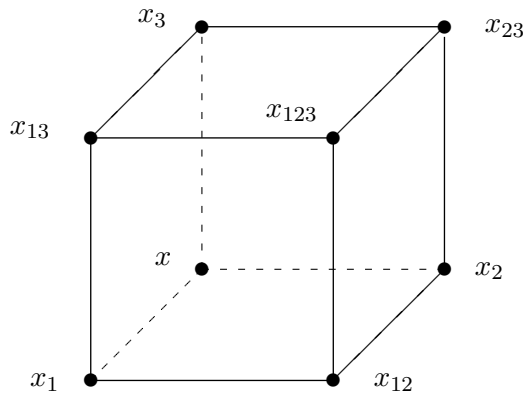


Figure 1: The consistency cube

Supposing  $x, x_1, x_2, x_3$  are given, one can compute  $x_{12}$  using the bottom equation,  $x_{13}$

using the left side equation, and  $x_{23}$  using the back side equation. We may then calculate the final value  $x_{123}$  in three different ways, using the front-, right- and top-equations, respectively. The equations are said to be Consistent-Around-a-Cube (CAC) if the three ways yield the same value for  $x_{123}$ .

Note that it is also possible to check CAC with other initial values that allow full evolution, for example,  $x_3, x, x_1, x_{12}$ .

### 3 Lax $\simeq$ Bäcklund

The Lax/BT approaches differ from CAC in that only the side equations are used as input, while the top and bottom equations are supposed to be derived from them.

#### 3.1 Construction of the Lax pair

The Lax pair for the bottom equation is constructed from the side equations by isolating the variables  $x_3, x_{13}, x_{23}, x_{123}$ , and writing the equations as:

$$\text{left:} \quad x_{13}x_3 c_1(x_1, x) + x_{13} c_2(x_1, x) + x_3 c_3(x_1, x) + c_4(x_1, x) = 0, \quad (1a)$$

$$\text{right} \quad x_{123}x_{23} \bar{c}_1(x_{12}, x_2) + x_{123} \bar{c}_2(x_{12}, x_2) + x_{23} \bar{c}_3(x_{12}, x_2) + \bar{c}_4(x_{12}, x_2) = 0, \quad (1b)$$

$$\text{back} \quad x_{23}x_3 b_1(x_2, x) + x_{23} b_2(x_2, x) + x_3 b_3(x_2, x) + b_4(x_2, x) = 0, \quad (1c)$$

$$\text{front} \quad x_{123}x_{13} \bar{b}_1(x_{12}, x_1) + x_{123} \bar{b}_2(x_{12}, x_1) + x_{13} \bar{b}_3(x_{12}, x_1) + \bar{b}_4(x_{12}, x_1) = 0, \quad (1d)$$

The coefficient functions  $b_i, \bar{b}_i, c_i, \bar{c}_i$  will be affine linear in their arguments.

We now introduce the homogeneous coordinates  $f, g$  for  $x_3$  and its shifts by

$$x_3 = f/g, \quad x_{23} = f_2/g_2, \quad x_{13} = f_1/g_1, \quad x_{123} = f_{12}/g_{12}.$$

This amounts to considering  $x_3$  as belonging to the projective space  $CP^1$ . We denote by  $\psi$  the pair

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}$$

which is defined up to a global factor. Equations (1) can then be written as

$$\psi_1 \simeq L\psi, \quad (\psi_2)_1 \simeq \bar{L}\psi_2, \quad \psi_2 \simeq M\psi, \quad (\psi_1)_2 \simeq \bar{M}(\psi_1),$$

where

$$L(x_1, x) \simeq \begin{pmatrix} -c_3(x_1, x) & -c_4(x_1, x) \\ c_1(x_1, x) & c_2(x_1, x) \end{pmatrix}, \quad M(x_2, x) \simeq \begin{pmatrix} -b_3(x_2, x) & -b_4(x_2, x) \\ b_1(x_2, x) & b_2(x_2, x) \end{pmatrix},$$

and similarly for the bar-quantities. Since we are working in  $CP^1$  all equalities are projective equalities, and as a reminder of this we use the symbol  $\simeq$  to indicate that two matrices are equivalent if their entries are proportional, in other words,  $L$  and  $M$  belong to  $PGL(2, C)$ .

The Lax matrices provide parallel transport of  $\psi$  along the bonds of the lattice. The zero curvature condition means that the parallel transport along any closed path on the lattice is trivial. It is necessary and sufficient to ensure that taking  $\psi$  from position  $(0, 0)$  to  $(1, 1)$  via the two routes  $(0, 0) \rightarrow (0, 1) \rightarrow (1, 1)$  and  $(0, 0) \rightarrow (1, 0) \rightarrow (1, 1)$  gives the same result:

$$(\psi_1)_2 \simeq (\psi_2)_1, \quad \text{i.e.,} \quad \bar{L}(x_{12}, x_2)M(x_2, x) \simeq \bar{M}(x_{12}, x_1)L(x_1, x). \quad (2)$$

This matrix relation yields three scalar equations.<sup>1</sup> These equations are written in terms of the variables  $x, x_1, x_2, x_{12}$ , and in order to satisfy the ZCC we must impose a constraint on these variables. In standard cases, this constraint will just be the bottom equation. Indeed, in integrable cases the three equations in (2) have a common factor. It may also happen that the ZCC is satisfied automatically, if the side equations are simple enough, or sometimes the common factor may factorize, yielding multiple choices. We will examine specific examples of this phenomenon below.

### 3.2 Direct approach: Bäcklund transformation

A Bäcklund transformation (BT) is generated by a set of equations  $\mathcal{E}(X, Y) = 0$  on two sets of functions  $X$  and  $Y$ , such that eliminating one set of functions say  $X$  (resp.  $Y$ ) one gets an equations  $E(Y) = 0$  (resp.  $E'(X) = 0$ ) on the other set (for details, see Proposition 4.1 in [11].)

For a 3D consistent set of equations the side equations provide a BT between the top and bottom equations. In order to derive the bottom equation from the side equations one proceeds as follows:

1. Solve  $x_{13}$  from the left equation (1a).
2. Solve  $x_{23}$  from the back equation (1c).
3. Solve  $x_{123}$  from the front equation (1d).
4. After this the right equation (1b) can be written as  $x_3^2 \mathcal{R}_2 + x_3 \mathcal{R}_1 + \mathcal{R}_0 = 0$ , where  $\mathcal{R}_i$  are polynomials in  $x, x_1, x_2, x_{12}$ . The greatest common divisor (GCD) of  $\mathcal{R}_i$  (or one of its factors if it factorizes) will be the bottom equation.

One can find the following explicit forms for the  $\mathcal{R}_i$ :

$$\mathcal{R}_2 = \det \begin{vmatrix} c_1 & 0 & \bar{b}_3 & \bar{b}_1 \\ 0 & b_1 & \bar{c}_3 & \bar{c}_1 \\ c_3 & 0 & \bar{b}_4 & \bar{b}_2 \\ 0 & b_3 & \bar{c}_4 & \bar{c}_2 \end{vmatrix}, \quad \mathcal{R}_0 = \det \begin{vmatrix} c_2 & 0 & \bar{b}_3 & \bar{b}_1 \\ 0 & b_2 & \bar{c}_3 & \bar{c}_1 \\ c_4 & 0 & \bar{b}_4 & \bar{b}_2 \\ 0 & b_4 & \bar{c}_4 & \bar{c}_2 \end{vmatrix} \quad (3)$$

$$\mathcal{R}_1 = \det \begin{vmatrix} c_2 & 0 & \bar{b}_3 & \bar{b}_1 \\ 0 & b_1 & \bar{c}_3 & \bar{c}_1 \\ c_4 & 0 & \bar{b}_4 & \bar{b}_2 \\ 0 & b_3 & \bar{c}_4 & \bar{c}_2 \end{vmatrix} + \det \begin{vmatrix} c_1 & 0 & \bar{b}_3 & \bar{b}_1 \\ 0 & b_2 & \bar{c}_3 & \bar{c}_1 \\ c_3 & 0 & \bar{b}_4 & \bar{b}_2 \\ 0 & b_4 & \bar{c}_4 & \bar{c}_2 \end{vmatrix}. \quad (4)$$

---

<sup>1</sup>One could do the computations in some particular representative of the equivalence class, e.g., by requiring the matrices to be uni-modular. However this is not necessary, and may in fact be cumbersome, if it introduces square roots.

That the Lax and the BT approaches yield the same equations can be seen as follows: Let us denote  $V := \overline{M}L$ ,  $U := \overline{L}M$ , then the matrix elements of  $V, U$  are related to  $\mathcal{R}_i$  as follows (here subscripts indicate the matrix element):

$$\frac{U_{21}}{U_{11}} - \frac{V_{21}}{V_{11}} \simeq \frac{\mathcal{R}_2}{V_{11}U_{11}}, \quad \frac{U_{22}}{U_{12}} - \frac{V_{22}}{V_{12}} \simeq \frac{\mathcal{R}_0}{V_{12}U_{12}}, \quad (5)$$

$$\frac{U_{21}V_{12}}{U_{11}V_{11}} - \frac{U_{12}V_{21}}{U_{11}V_{11}} + \frac{U_{22}}{U_{11}} - \frac{V_{22}}{V_{11}} \simeq \frac{\mathcal{R}_1}{U_{11}V_{11}}. \quad (6)$$

Thus  $U \simeq V$  iff  $\mathcal{R}_i = 0$ .

Of course one can equally well use the side equations to solve for the variables  $x_1, x_2, x_{12}$  from the left-, back-, and front-equations, after which the right-equation will be a polynomial in  $x$  with coefficient depending on  $x_3, x_{13}, x_{23}, x_{123}$ , with their GCD yielding the top equation.

## 4 Examples

### 4.1 Linear side equations

As a first example we consider the case where all the side equations are linear, i.e., the left side equation is:

$$x_{13} - x_1 - x_3 + x = 0 \quad (7)$$

and we have the same equation with suitable subscript changes on the other vertical sides. One then finds that neither the Lax nor the BT approach yields anything, The conditions are satisfied without reference to the bottom equation.

What about CAC? It will involve both the bottom and the top equations, so it should give some conditions. Indeed, if one tries CAC with (7) and a completely general multi-affine bottom and top equations, related by a shift (but with same parameters) one finds a consistent set with the bottom equation

$$a(x - x_1)(x_2 - x_{12}) + b(x - x_2)(x_1 - x_{12}) + c(x - x_1 - x_2 + x_{12}) + d = 0. \quad (8)$$

This is a combination of the lattice modified KdV equation (lmKdV) (aka Q1 in the ABS list [4]), and the linear equation.

A similar analysis can be done starting from the linearizable side equation

$$x_{13}x = x_1x_3. \quad (9)$$

The Lax matrix for this system is diagonal and the ZCC is automatically satisfied. From CAC analysis we find that (9) is compatible with the six parameter family of homogeneous bottom/top equations of degree 2

$$a_1 xx_1 + a_2 xx_2 + a_3 xx_{12} + a_4 x_1 x_2 + a_5 x_1 x_{12} + a_6 x_2 x_{12} = 0. \quad (10)$$

We will examine the integrability of these equations in section (5.1).

## 4.2 H1

The lattice potential KdV (lpKdV), which describes the permutability property of continuous KdV, is a paradigm of integrable lattice equations (aka H1 in the ABS list [4]). For this system everything works well. [We will return to this model in a new context in section 6.2.]

The model is given by

$$\text{H1} := (x_1 - x_2)(x - x_{12}) - p + q = 0. \quad (11)$$

After imposing this equation on the sides of the CAC cube, with suitable variable changes, i.e.,

$$\text{left: } (x - x_{13})(x_1 - x_3) - (p - r) = 0, \quad (12a)$$

$$\text{right: } (x_2 - x_{123})(x_{12} - x_{23}) - (p - r) = 0, \quad (12b)$$

$$\text{back: } (x - x_{23})(x_2 - x_3) - (q - r) = 0, \quad (12c)$$

$$\text{front: } (x_1 - x_{123})(x_{12} - x_{13}) - (q - r) = 0, \quad (12d)$$

one easily finds

$$\mathcal{R}_2 = (x_{12} - x) \cdot \text{H1}, \quad (13a)$$

$$\mathcal{R}_1 = [(x - x_{12})(x_1 + x_2) - \alpha - \beta + 2\gamma] \cdot \text{H1}, \quad (13b)$$

$$\mathcal{R}_0 = [x_1 x_2 (x_{12} - x) + (\beta - \gamma)x_1 + (\alpha - \gamma)x_2] \cdot \text{H1}, \quad (13c)$$

with H1, as given in (11), as the GCD. Similarly by working on the bottom variables  $x, x_1, x_2, x_{12}$  one obtains the 3-shifted H1 as the top equation.

The Lax matrices in this case are

$$L(x_1, x) = \begin{pmatrix} x & p - r - x x_1 \\ 1 & -x_1 \end{pmatrix}, \quad M(x_2, x) = \begin{pmatrix} x & q - r - x x_2 \\ 1 & -x_2 \end{pmatrix}, \quad (14)$$

and one easily finds that

$$M(x_{12}, x_1) L(x_1, x) - L(x_{12}, x_2) M(x_2, x) = \text{H1} \times \begin{pmatrix} 1 & -(x_1 + x_2) \\ 0 & -1 \end{pmatrix}.$$

## 4.3 $\text{H1}_\epsilon$ : a deformed version of H1

This is an asymmetric deformation of (12) [9, 11]

$$\text{left: } (x - x_{13})(x_1 - x_3) - (p - r)(1 + \epsilon x x_{13}) = 0, \quad (15a)$$

$$\text{right: } (x_2 - x_{123})(x_{12} - x_{23}) - (p - r)(1 + \epsilon x_{12} x_{23}) = 0, \quad (15b)$$

$$\text{back: } (x - x_{23})(x_2 - x_3) - (q - r)(1 + \epsilon x x_{23}) = 0, \quad (15c)$$

$$\text{front: } (x_1 - x_{123})(x_{12} - x_{13}) - (q - r)(1 + \epsilon x_{13} x_{12}) = 0. \quad (15d)$$

The BT or Lax computations give as GCD the bottom equation

$$(x - x_{12})(x_1 - x_2) - (p - q)(1 + \epsilon x x_{12}) = 0 \quad (15e)$$

and similarly for the top equation we get

$$(x_3 - x_{123})(x_{13} - x_{23}) - (p - q)(1 + \epsilon x_{13}x_{23}) = 0. \quad (15f)$$

If we draw a line connecting the corners appearing in the deformation term as a product, the lines form a tetrahedron inside the cube, see Figure 2.

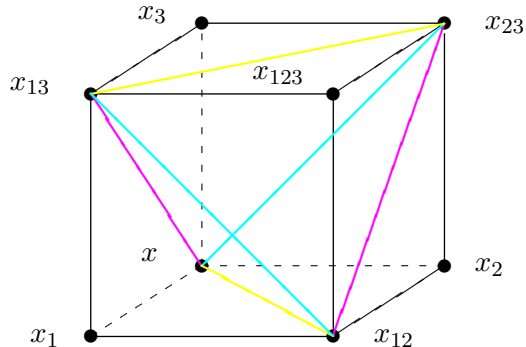


Figure 2: The cube of H1e

The special thing about these equations is that the parallel sides are *not* obtained by shifts, but the shift must be accompanied by a  $90^\circ$  rotation, as described in Figure 2. The fact that the parallel sides are not identical implies that the Lax matrices associated to parallel bonds come in two different forms. Letting

$$L(x_1, x) = \begin{pmatrix} x + \epsilon(p - r)x_1 & p - r - xx_1 \\ 1 & -x_1 \end{pmatrix}, \quad M(x_2, x) = L|_{p \rightarrow q, x_1 \rightarrow x_2}, \quad (16)$$

and

$$L'(x_1, x) = \begin{pmatrix} x & p - r - xx_1 \\ 1 & -x_1 - \epsilon(p - r)x \end{pmatrix}, \quad M'(x_2, x) = L'|_{p \rightarrow q, x_1 \rightarrow x_2}, \quad (17)$$

we have

$$M'(x_{12}, x_1) L(x_1, x) - L'(x_{12}, x_2) M(x_2, x) = (18e) \times \begin{pmatrix} 1 - \epsilon x_1 x_2 & -(x_1 + x_2) \\ -\epsilon x_1 x_2 & -1 + \epsilon x_1 x_2 \end{pmatrix}.$$

Since we have two different parallel sides we can have two different bottom equations, one derived from  $M'(x_{12}, x_1) L(x_1, x) - L'(x_{12}, x_2) M(x_2, x)$  as above, the other possible choice being

$$M(x_{12}, x_1) L'(x_1, x) - L(x_{12}, x_2) M'(x_2, x) = (15e) \times \begin{pmatrix} 1 & -(x_1 + x_2) \\ 0 & -1 \end{pmatrix}.$$

The totality of equations on this cube, obtained from the previous one with  $L \leftrightarrow L'$  etc actually corresponds to an inversion of the cube by

$$x \leftrightarrow x_{123}, x_1 \leftrightarrow x_{23}, x_2 \leftrightarrow x_{13}, x_3 \leftrightarrow x_{12},$$



which yields

$$\text{left: } (x - x_{13})(x_1 - x_3) - (p - r)(1 + \epsilon x_1 x_3) = 0, \quad (18a)$$

$$\text{right: } (x_2 - x_{123})(x_{12} - x_{23}) - (p - r)(1 + \epsilon x_2 x_{123}) = 0, \quad (18b)$$

$$\text{back: } (x - x_{23})(x_2 - x_3) - (q - r)(1 + \epsilon x_2 x_3) = 0, \quad (18c)$$

$$\text{front: } (x_1 - x_{123})(x_{12} - x_{13}) - (q - r)(1 + \epsilon x_1 x_{123}) = 0, \quad (18d)$$

$$\text{bottom: } (x - x_{12})(x_1 - x_2) - (p - q)(1 + \epsilon x_1 x_2) = 0, \quad (18e)$$

$$\text{top: } (x_3 - x_{123})(x_{13} - x_{23}) - (p - q)(1 + \epsilon x_3 x_{123}) = 0. \quad (18f)$$

These are also consistent. We thus have two kinds of cubes, one defined with (15), and one with (18). These cubes can be glued together: note that the 2-shift of (18a) is equal to (15b). In fact under the central reflection the diagonal lines in Figure 2 would change to the other diagonals.

We will return later to the question of gluing the cubes to fill the space.

#### 4.4 Flipped $H1_\epsilon$

This model was proposed in [13] (see Sec. 3.1, Case  $(\epsilon, 0, 0, \epsilon)$ ). In that paper the model was given in a cube with flipped coordinates  $x_2 \leftrightarrow x_{23}$ ,  $x_1 \leftrightarrow x_{13}$ . In the coordinates of Figure 1 the side equations of this model are given by

$$\text{left: } (x - x_1)(x_{13} - x_3) - (p - r)(1 + \epsilon x x_1) = 0, \quad (19a)$$

$$\text{right: } (x_2 - x_{12})(x_{123} - x_{23}) - (p - r)(1 + \epsilon x_2 x_{12}) = 0, \quad (19b)$$

$$\text{back: } (x - x_2)(x_{23} - x_3) - (q - r)(1 + \epsilon x x_2) = 0, \quad (19c)$$

$$\text{front: } (x_1 - x_{12})(x_{123} - x_{13}) - (q - r)(1 + \epsilon x_1 x_{12}) = 0. \quad (19d)$$

**BT derivation of bottom and top-equations.** Now computing the values of  $x_{13}$ ,  $x_{23}$ ,  $x_{123}$  from left-, back- and right-equations, respectively, gives for the front equation an expression that does not contain  $x_3$  at all. This expression factorizes into *two* factors and thus we could have two different bottom-equations:

$$\text{bottom1 } \epsilon x x_1 x_2 x_{12} (1/x - 1/x_1 - 1/x_2 + 1/x_{12}) - x + x_1 + x_2 - x_{12} = 0, \quad (20a)$$

$$\text{bottom2 } p(x - x_2)(x_1 - x_{12}) + q(x - x_1)(x_{12} - x_2) + r(x - x_{12})(x_2 - x_1) = 0. \quad (20b)$$

Similarly, working with the values at the bottom square we get two candidates for the top equations:

$$\text{top1 } x_3 - x_{13} - x_{23} + x_{123} = 0, \quad (21a)$$

$$\begin{aligned} \text{top2 } & p(x_3 - x_{23})(x_{13} - x_{123}) + q(x_3 - x_{13})(x_{123} - x_{23}) + r(x_3 - x_{123})(x_{23} - x_{13}) \\ & + \epsilon(p - q)(q - r)(r - p) = 0. \end{aligned} \quad (21b)$$

Notice that in down2 and top2, there is an explicit dependence on  $r$  even though these are 2D equations. In fact, one should now consider  $r$  as a global parameter, although it was associated with the third dimension in the above derivation.

Performing the usual CAC computations with the given sides and different top and bottom equations reveals that the set of equations is consistent in two cases: with the pair (bottom1,top1), or with (bottom2,top2), which was given in [9]. In other words, given the side equations (19) there are two consistent ways of completing the cube. Continuing further, we can also interpret the cube to provide a BT between the left and right equations. Indeed, if we do the above BT construction on the corners of the left side equation the result is identically zero with (bottom1,top1), while (bottom2,top2) produces the right-equation.

**The Lax matrices.** The standard procedure gives

$$L(x_1, x) = \begin{pmatrix} 1 & \lambda(p, x, x_1) \\ 0 & 1 \end{pmatrix}, \quad M(x_2, x) = \begin{pmatrix} 1 & \lambda(q, x, x_2) \\ 0 & 1 \end{pmatrix}, \quad \lambda(a, x, y) := \frac{(a-r)(1+\epsilon xy)}{x-y} \quad (22)$$

Since the matrices are upper triangular the ZCC implies

$$\Sigma := \lambda(p, x_2, x_{12}) + \lambda(q, x, x_2) - \lambda(q, x_1, x_{12}) - \lambda(p, x, x_1) = 0. \quad (23)$$

Remarkably enough, the above sum factorize as

$$\Sigma = \text{bottom1} \cdot \text{bottom2} / [(x-x_1)(x-x_2)(x_1-x_{12})(x_2-x_{12})].$$

Note that we can write (23) also in the form

$$\Sigma = (T-1)\lambda(p, x, x_1) - (S-1)\lambda(q, x, x_2) = 0,$$

where  $T$  is a shift in  $m$  and  $S$  a shift in  $n$ . This of course is in the form of a conservation law.

## 5 Filling the space with consistent cubes

So far we have only considered a single cube and its CAC/Lax/BT. But as the name indicates, lattice equations should be defined over the whole lattice. This brings further complications, for example with one cube we could freely do different Möbius transformations in each corner of a cube, but when the cube is part of lattice such seemingly innocuous actions will affect neighboring cubes as well and can destroy the lattice structure.

The rule is simple: cover the two dimensional lattice with consistent cubes, with the condition that adjacent vertical faces coincide exactly, that is to say their four corner values satisfy the same equation. This is expected to produce integrable lattice equations.

We will follow this guideline for various models mentioned before, and systematically check integrability of the lattice equations so obtained, by calculating their algebraic entropy [17, 18]: the vanishing of the entropy is a yes/no test which gives a clear cut separation between integrable and non integrable cases.

We may briefly recall how to calculate the entropy. The local equation determines an evolution, starting from initial conditions given for example on a diagonal staircase (lattice

points of coordinates  $(m, n)$  with  $m + n = 0$  or  $1$ ). The solution is then calculated on diagonals moving away from the diagonal of initial conditions, explicitly in terms of these initial conditions. The algebraic entropy is defined as the rate of growth of the degrees on these diagonals. Exponential growth is generic, and polynomial growth is characteristic of integrability, while linear growth is associated with linearizable equations [17, 18, 19].

The exact shape of the diagonal line on which the initial values are given is not important. If one modifies this shape locally, the sequence of degrees will change, but not its asymptotic rate of growth. This should be kept in mind for some of the models studied below. For example one could very well change the initial diagonal (steps of height 1 and width 1) to a diagonal with bigger steps, for example with height 2 and width 2, as this will not affect the calculation of the entropy.

## 5.1 Equations consistent with linear sides

We can check the integrability of the two quad equations given in section (4.1). The first one, i.e., equation (8), leads to the sequence of degrees:

$$\{d_n\} = 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 67, 79, \dots$$

that is to say  $d_n = 1 + (n^2 + n)/2$ . This quadratic growth confirms integrability.

**Remark:** Since it is integrable, it should also be consistent with a nontrivial set of side-equations, such that one can produce it via Lax/BT computations, but we will leave this open.

For the second equation of section (4.1), that is to say the general homogeneous relation (10) of degree 2, the result is different. We get the following sequence of degrees:

$$\{d_n\} = 1, 2, 4, 9, 21, 50, 120, 289, 697 \dots$$

which may be fitted with the generating function

$$g(s) = \sum_n d_n s^n = \frac{s^2 + s - 1}{(1 - s)(s^2 + 2s - 1)} \tag{24}$$

indicating a non vanishing entropy  $\epsilon = \log(1 + \sqrt{2})$ , showing non-integrability. Thus even CAC is not sharp in this case. It was actually shown in [20] that the simple additional condition  $a_3 = a_4$ , renders (9) integrable.

## 5.2 H1 $\epsilon$ : Checkerboard lattice

We have already noted that in the H1 $\epsilon$  model there are two different cubes related by inversion, and that these cubes can be glued together in a unique way, providing a black-white lattice. This problem has been discussed in detail in [11].

Since the gluing process is completely fixed and each cube has the CAC property it is expected that the composite lattice is integrable. We have calculated the sequence of degrees  $\{d_n\}$  for the evolution defined by this model, with initial conditions on a diagonal as prescribed above. The outcome is the sequence

$$\{d_n\} = 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, \dots$$

fitted by the generating function

$$g(s) = \sum_n d_n s^n = \frac{(1+s)^2}{(1-s)^3}$$

The above sequence has quadratic growth. Thus the entropy is vanishing and the model is integrable.

### 5.3 Flipped H1 $\epsilon$ : More black and white lattices

In the flipped H1 $\epsilon$  case (Sec. 4.4), the side equations are the same, allowing simple gluing together of the cubes, but, as we saw in section (4.4), the side-equations are somewhat weak and allow two different compatible pairs of bottom/top equations. We may then cover the two dimensional lattice with consistent cubes, assigning either equation bottom1 (top1), which we call *white*, or equation bottom2(top2) which we call *black* to each elementary cell. This can be done in an arbitrary way if one just insists on having a compatible 3D structure of cubes over the 2D lattice. It is then natural to ask which of the configurations obtained in this way are integrable.

Let us consider periodic distributions. The lattice is divided into rectangular groups of cells of width  $h$  and height  $v$ . Within such a rectangle, a fixed assignment is made, and the pattern is repeated periodically in both directions. (A pattern with  $v = 1$  and  $h = 1$  gives a uni-colored assignment.)

Consider for example  $(h, v) = (2, 2)$ . There are a priori  $2^4$  possible patterns of that size, but only 3 inequivalent ones which cannot be reduced to configurations having smaller periods (see Figure 3). The naming convention is to list the colors starting from the lower left corner onwards, denoting bottom1/top1 with 0, alias white, bottom2/top2 with 1, alias black. The equivalence of patterns comes from the fact that we have to look at the lattice globally. It is easy to see, for example, that in the case  $(h, v) = (2, 2)$  we have the equivalences  $[0100] \simeq [0010] \simeq [1000] \simeq [0001]$ ,  $[1011] \simeq [1101] \simeq [0111] \simeq [1110]$ , and  $[1010] \simeq [0101]$  (checkerboard lattice). Moreover  $[0000]$  and  $[1111]$  have periods  $(1, 1)$ ,  $[0101]$  and  $[1010]$ ,  $[0011]$ ,  $[1100]$  have periods  $(2, 1)$  and  $(1, 2)$ .

**Claim:** *Some of the distributions are integrable, and some are not.* Although the pattern is 3D consistent, the Lax pair is weak and cannot precisely fix the bottom and top equations.

**$1 \times 1$  patterns (unicolor distributions).** Both unicolor distributions  $(h, v) = (1, 1)$  have vanishing entropy. The purely white one is linear. The purely black one is non-trivially

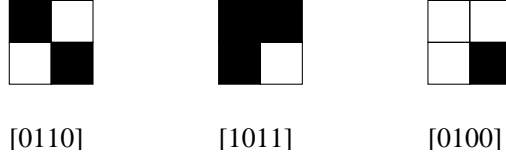


Figure 3: The three inequivalent  $(2, 2)$  patterns.

integrable, showing quadratic growth of the sequence of degrees

$$\{d_n\} = 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172 \dots \quad (25)$$

**$1 \times 2$  patterns.** Both  $1 \times 2$  patterns  $(h, v) = (1, 2)$  or  $(h, v) = (2, 1)$ , that is to say alternating black and white stripes, are integrable, with quadratic growth of the degrees.

**$2 \times 2$  patterns.** For  $(h, v) = (2, 2)$  we have different results for the different patterns in Figure 3.

- Both [1010] and [0100] are integrable, with quadratic growth of the degrees.
- The calculation of the degrees for [1011] yields the sequence

$$\begin{aligned} \{d_n\} = & 1, 2, 4, 8, 18, 41, 93, 215, 493, 1132, 2600, 5970, 13710, \\ & 31487, 72313, 166077, 381417, 875974, 2011788, 4620332 \dots \end{aligned} \quad (26)$$

This sequence is fitted by the rational generating function

$$g(s) = \sum_n d_n s^n = \frac{1 - s^2 - 2s^3 + s^5 - s^6 - s^8 + s^9}{(1 - s)(s + 1)(s^4 - 2s^3 - 2s + 1)(s^4 + 1)}, \quad (27)$$

and gives a non vanishing entropy  $\epsilon = \log(s)$  with  $s$  the largest root of  $s^4 - 2s^3 - 2s + 1$ , approximately  $\epsilon = \log(2.29663)$ .

**Caveat:** When computing sequences of degrees, one should in principle consider iterations of the whole pattern, but that tends to make the calculations heavier. For the sequence (26), this would mean considering only the subsequence formed by odd terms, leading to a growth given by the maximal root  $\tau$  of  $t^4 - 4t^3 - 6t^2 - 4t + 1$ . Of course  $\tau = \sigma^2$ .

**$2 \times 3$  patterns.** We have examined all the period  $(2, 3)$  patterns. The various nonequivalent patterns are depicted in Figure 4.

The computations show that what matters is not just the proportion of black and white cells, but the actual conformation of the pattern. For example the period  $(2, 3)$  patterns [010110] and [001011] have an equal number of black and white cells. The first one is

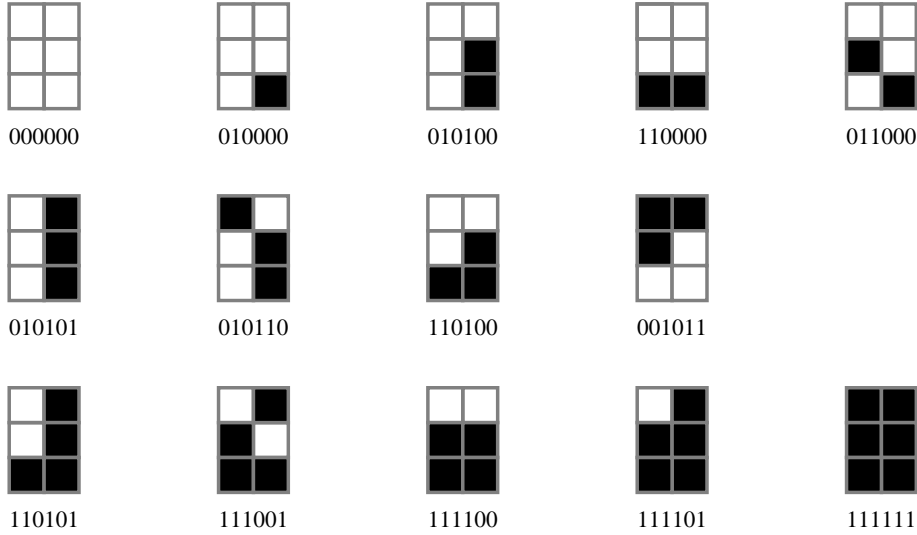


Figure 4: Nonequivalent period (2, 3) patterns.

integrable (quadratic growth of the degrees) while the latter is not, as may be seen from the sequence:

$$\{d_n\} = 1, 1, 2, 3, 5, 9, 19, 84, 169, 329, 631, 1199, 2287, 4412, 8627, 17059, 33941, 67573, 134071, 264576, 519343, 1015531, 1982461, 3871597, 7574863, 14855790 \dots (28)$$

This sequence has exponential growth, but is not long enough to determine an exact value of the entropy. The approximate value is  $\log(1.96)$ .

Out of the 14 period (2, 3) nonequivalent patterns, we have one linear case (all white [000000]), eight integrable cases ([010000], [010100], [110000], [011000], [010101], [010110], [111100], [111111]), and four non integrable ones ([110100], [001011], [110101], [111001]). The following pictures show the aspect of two integrable cases and two non integrable ones.

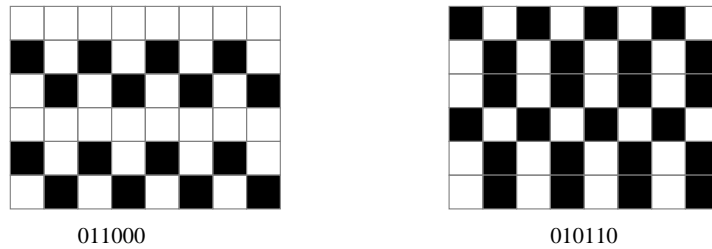


Figure 5: Two integrable (2, 3) patterns.

**Remark:** The entropy calculations for these patterns can be made equally well with the relations { white = bottom1, black=bottom2 } or with { white = top1, black=top2}. Both would give the same results.

From the above results, one may already conclude that random distributions are expected to be non-integrable.

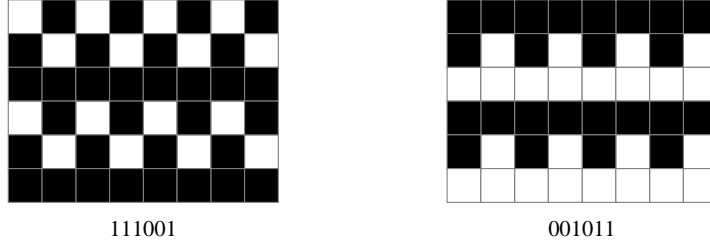


Figure 6: Two nonintegrable  $(2, 3)$  patterns.

## 6 Lax pair for a $2 \times 2$ sublattice

We will next push the Lax concept and ZCC to a  $2 \times 2$  sublattice described in Figure 7. (Such sub-lattices have been discussed previously, e.g., in [11].) To determine the evolution we need 5 initial values, marked with black disks and expect to get values for the vertices at the open circles. Since the Lax matrices belong to  $PGL(2, C)$  the zero curvature conditions can provide at most three equations, and thus if everything works well the evolution is determined. It is also clear that this will not work for bigger sub-lattices as we would then need to provide more than 3 values.

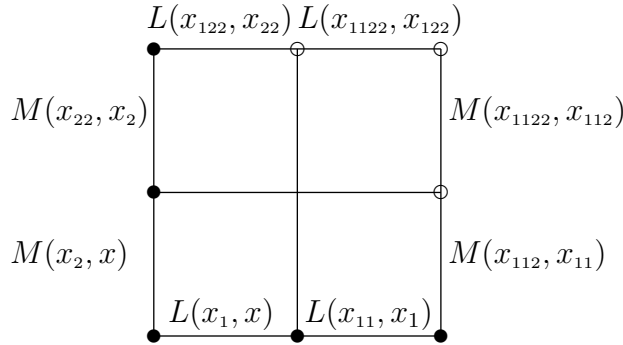


Figure 7: The  $2 \times 2$  configuration. Values at black discs are initial data  $(x, x_1, x_{11}, x_2, x_{22})$ , and values at open circles  $(x_{112}, x_{122}, x_{1122})$  should be determined by the evolution.

The zero curvature condition for the  $2 \times 2$  sublattice is given by

$$\begin{aligned}
 M(x_{1122}, x_{112}) M(x_{112}, x_{11}) L(x_{11}, x_1) L(x_1, x) \simeq \\
 L(x_{1122}, x_{122}) L(x_{122}, x_{22}) M(x_{22}, x_2) M(x_2, x).
 \end{aligned}
 \tag{29}$$

## 6.1 Flipped H1 $\epsilon$

In the flipped H1 $\epsilon$  case the Lax matrices were given in Eq. (22) Now using this on a  $2 \times 2$  sub-lattice we get just one condition, namely

$$\begin{aligned} & \lambda(p, x_{122}, x_{1122}) + \lambda(p, x_{22}, x_{122}) + \lambda(q, x_2, x_{22}) + \lambda(q, x, x_2) = \\ & \lambda(q, x_{112}, x_{1122}) + \lambda(q, x_{11}, x_{112}) + \lambda(p, x_1, x_{11}) + \lambda(p, x, x_1). \end{aligned}$$

From this one can in principle solve  $x_{1122}$  in terms of the other variables. However, this equation does not determine the values for  $x_{122}$  or  $x_{112}$  and therefore these Lax matrices fail to give the evolution.

## 6.2 H1

For this basic model the Lax matrices were given in Eq. (14). Condition (29) leads to equations that have *two rational* solutions: The regular one

$$x_{112} = x_1 + \frac{(p-q)(x_1-x_2)}{(x_1-x_2)(x-x_{11})-(p-q)}, \quad (30a)$$

$$x_{122} = x_2 + \frac{(q-p)(x_2-x_1)}{(x_2-x_1)(x-x_{22})-(q-p)}, \quad (30b)$$

$$x_{1122} = x + (p-q) \frac{(p-q)(x_{11}+x_{22}-2x) + 2(x-x_{11})(x-x_{22})(x_1-x_2)}{(p-q)^2 - (x-x_{11})(x-x_{22})(x_1-x_2)^2}, \quad (30c)$$

and an exotic solution

$$x_{1122} = x_{11} + x_{22} - x, \quad (31a)$$

$$x_{122} = x_{112} - x_1 + x_2, \quad (31b)$$

$$x_{112} = x_1 - \frac{(x_1-x_2)[(p-r)(x-x_{22}) + (q-r)(x-x_{11})]}{(p-r)(x-x_{22}) - (q-r)(x-x_{11}) - (x-x_{11})(x-x_{22})(x_1-x_2)}, \quad (31c)$$

The regular solution could also be obtained using the evolution on the original lattice, first solving for  $x_{12}$ . As a consequence  $x_{122}$  depends only on  $x, x_1, x_2, x_{22}$  and  $x_{112}$  only on  $x, x_1, x_2, x_{11}$ . The exotic solution is different, as  $x_{122}$  and  $x_{112}$  both depend on *all* initial values. Furthermore, it depends on  $p-r$  and  $q-r$ , and not solely on  $p-q$ , as is the case for the regular solution.

We may view the variables  $x, x_{11}, x_{22}, x_{1122}$  as associated to the vertices and  $x_1, x_2, x_{112}, x_{122}$  as associated to the bonds of the  $2 \times 2$  sublattice.

In the algebraic entropy analysis the vertex variables are linear and for the bond variables we find the sequence of degrees

$$\{d_n\} = 1, 4, 13, 28, 49, 76, 109, 148, 193, 244, 301, 364, 433, \dots \quad (32)$$

This sequence can be fitted with the generating function

$$\zeta(s) = \sum_n d_n s^n = \frac{1 + 4s^2 + s}{(1-s)^3} \quad (33)$$



The sequence has quadratic growth, signaling integrability.

What is the nature of the exotic solution? Since the vertex variables have independent linear evolution we can solve the equation with  $x_{2n,2m} = F(n) + G(m)$ . When this is substituted into the bond equations they give a non-autonomous generalization of a Yang-Baxter map: using the coarse grained indexing  $w_{n,m} = x_{2n,2m}$ ,  $X_{n,m} = x_{2n+1,2m}$ ,  $Y_{n,m} = x_{2n,2m+1}$ , i.e.,  $x_1 = X$ ,  $x_2 = Y$ ,  $x_{112} = Y_1$ ,  $x_{122} = X_2$  we have

$$Y_1 - X = P(X, Y), \quad X_2 - Y = P(X, Y). \quad (34a)$$

The solution  $w = \text{constant}$  is not allowed and if either  $F$  or  $G$  is constant,  $P$  collapses to  $P = \pm(x - y)$ . In the generic case, denoting

$$f(n) := \frac{p - r}{F(n) - F(n + 1)}, \quad g(m) := \frac{q - r}{G(m) - G(m + 1)},$$

we get

$$P = \frac{(X - Y)[f(n) + g(m)]}{X - Y - f(n) + g(m)}.$$

After the further translation  $X \mapsto X + f(n) + T(n, m)$ ,  $Y \mapsto Y + g(n) + T(n, m)$ , where  $T$  is a solution of  $T(n, m + 1) - T(n, m) = 2g(m)$ ,  $T(n + 1, m) - T(n, m) = 2f(n)$  we finally get

$$P = \frac{f(n)^2 - g(m)^2}{X - Y}, \quad (34b)$$

which is a non-autonomous version of the Adler map [21] (aka  $F_V$  in the classification [22]). The situation can be described by the following diagram:

$$\begin{array}{ccc} \text{standard solution} & \longleftarrow \text{H1 Lax for } 2 \times 2 \text{ sublattice} & \longrightarrow \text{exotic solution (31)} \\ & \uparrow & \downarrow \\ & \text{H1} & \text{non-autonomous } F_V \text{ (34)} \end{array}$$

### 6.3 H3

The phenomenon described in the previous section is not generic. Indeed, applying the same coarse-graining to an arbitrary integrable quad-equation will lead to a system having only one rational solution (the regular one coming from the original lattice).

We have, however, found more examples where an exotic rational solution exists. Here is one, provided by the lattice modified KdV (lmKdV) (aka  $H3_{\delta=0}$ ). In that case the defining relations of the exotic solution are:

$$x_{1122} x = x_{11} x_{22}, \quad (35a)$$

$$x_{122} x_1 = x_{112} x_2, \quad (35b)$$

$$\frac{x_{122}}{x_2} = \frac{x_{112}}{x_1} = \frac{(q^2 x_{22} - r^2 x)(x - x_{11}) p x_1 - (p^2 x_{11} - r^2 x)(x - x_{22}) q x_2}{(r^2 x_{22} - q^2 x)(x_{11} - x) p x_2 - (r^2 x_{11} - p^2 x)(x_{22} - x) q x_1} \quad (35c)$$

In the algebraic entropy analysis the sequence of degrees for the vertex variables has linear growth as expected, while the sequence for the bonds is the same as for H1 (see above).

Now the equation on the vertex variables (35a) can be solved with

$$x_{2n,2m} = F(n)G(m),$$

and if we introduce

$$f(n) := \frac{r^2 F(n+1) - p^2 F(n)}{F(n+1) - F(n)}, \quad g(m) := \frac{r^2 G(m+1) - q^2 G(m)}{G(m+1) - G(m)},$$

we obtain the bond equations in the form

$$\frac{X_2}{Y} = \frac{Y_1}{X} = \frac{(q^2 + r^2 - g(m))pX - (p^2 + r^2 - f(n))qY}{f(n)qX - g(m)pY}, \quad (36)$$

using the previously introduced notation. With the further scaling

$$X(n, m) \mapsto p T(n, m) X(n, m)/f(n), \quad Y(n, m) \mapsto q T(n, m) Y(n, m)/g(m),$$

where  $T$  solves

$$T(n, m+1)/T(n, m) = (q^2 + r^2 - g(m))/g(m), \quad T(n+1, m)/T(n, m) = (p^2 + r^2 - f(n))/f(n),$$

equation (36) reduces to

$$X_2 = \frac{Y}{\alpha(n)} P, \quad Y_1 = \frac{X}{\beta(m)} P, \quad P = \frac{\alpha(n) X - \beta(m) Y}{X - Y}, \quad (37)$$

where

$$\alpha(n) = \lambda p^2 / [f(n)(f(n) - p^2 - r^2)], \quad \beta(m) = \lambda q^2 / [g(m)(g(m) - q^2 - r^2)].$$

This is nothing but a non-autonomous version of  $F_{III}$  in the classification of [22]. The diagram presented for H1 works also for H3( $\delta = 0$ ).

We have found that this phenomenon occurs also for the lattice modified KdV and for the lattice Schwarzian KdV. We have examined in some detail the properties of the models defined in this way and we will present these results elsewhere [16].

## 7 Discussion

We have discussed the strength of the Lax pairs (or the zero curvature condition) and BT through various examples. We have found several cases where the ZCC does not uniquely determine the evolution but allows two possibilities. This happens, e.g., in the flipped H1 $\epsilon$  model. If one then builds an infinite lattice by arbitrarily choosing for each cell one of the two allowed relations, the result is sometimes integrable and sometimes shows nonzero entropy.

If the ZCC is pushed to a  $2 \times 2$  sublattice we get more examples where it is ambiguous and yields both the regular solution as well as an exotic one. The latter cannot be generated by some equation in the sublattice, because in the exotic solution the variables  $x_{122}$  and  $x_{112}$  depend on both  $x_{11}$  and  $x_{22}$  which is not possible using the equations on the elementary squares.

The equations we have obtained in this way can be interpreted as having vertex and edge variables: the variables with even number of indices live at the vertices while the ones with an odd number of indices live on the edges. The edge variables evolve as in a non-autonomous functional Yang-Baxter equation. For more details see [16].

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