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# Consistency of maximum-likelihood and variational estimators in the Stochastic Block Model

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**Abstract:** The *stochastic block model* (SBM) is a probabilistic model designed to describe heterogeneous *directed* and *undirected* graphs. In this paper, we address the asymptotic inference on SBM by use of maximum-likelihood and variational approaches. The identifiability of SBM is proved, while asymptotic properties of maximum-likelihood and variational estimators are provided. In particular, the consistency of these estimators is settled, which is, to the best of our knowledge, the first result of this type for variational estimators with random graphs.

## 1. Introduction

In the last decade, networks have arisen in numerous domains such as social sciences and biology. They provide an attractive graphical representation of complex data. However, the increasing size of networks and their great number of connections have made it difficult to interpret *network representations of data* in a satisfactory way. This has strengthened the need for statistical analysis of such networks, which could raise latent patterns in the data.

Interpreting networks as random graphs, *unsupervised classification* (clustering) of the vertices of the graph has received much attention. It is based on the idea that vertices with a similar connectivity can be gathered in the same class. The initial graph can be replaced by a simpler one without losing too much information. This idea has been successfully applied to social and biological networks, and it is out of the scope of this work to review all of them. For instance, we refer interested readers to [Nowicki and Snijders \(2001\)](#) for social networks and [Picard \*et al.\* \(2009\)](#) for biological ones.

Mixture models are a convenient and classical tool to perform unsupervised classification in usual statistical settings. Mixture models for random graphs

were proposed by Holland *et al.* (1983) who defined what they call a *stochastic block* model (SBM), in reference to an older *non stochastic block* model, widely used in social science. Assuming each vertex belongs to only one class, a *latent variable* (called the *label*) assigns every vertex to its corresponding class. SBM is therefore a versatile means to infer underlying structures in the graph. Subsequently, several versions of SBM have been studied and it is necessary to formally distinguish between them. Three binary separations can be used to this end:

1. *latent variables/parameters*: Standard SBM is a usual mixture model with random multinomial *latent variables* (Nowicki and Snijders, 2001; Daudin *et al.*, 2008; Ambroise and Matias, 2010). In this model, vertices are sampled in a population and the concern is on the population parameters (the frequency of each class and the connectivity between them). Alternatively, there is a conditional version of SBM, called CSBM, where former latent variables (labels) are now considered as *fixed parameters*. The concern is the estimation of both between-class connectivity parameters and the unknown labels associated to every vertex. For instance, CSBM is more often considered in the physical or computer science communities to perform clustering of the sampled vertices, rather than to estimate of the population parameters (Rohe *et al.*, 2010; Choi *et al.*, 2011). Obviously these two models are not far from each other. For example once the parameters of SBM have been estimated, it is easy to predict the class of each sampled node. Alternatively once the sampled vertices have been classified, it is easy to estimate the frequency of the classes. However the two models are definitively different and have different likelihoods and different sets of parameters. A specific difficulty of the CSBM model is that each node has its own parameter (the value for the node of the fixed latent variable), which leads to an infinite number of parameters in asymptotic studies.
2. The graph may be *directed* or *undirected*.
3. The graph may be *binary* or *weighted*.

One originality of SBM is that, unlike usual statistical settings where independence is assumed, it does no longer hold with SBM. Numerous approaches have been therefore developed to overcome this challenging problem. Snijders and Nowicki (1997) have studied maximum-likelihood estimators of SBM with only two classes and *binary undirected* graphs. However due to the exponential computational complexity of their optimization algorithm, their approach cannot deal with realistic graphs. A bayesian approach based on Gibbs sampling have also been explored in the same paper, and then extended by Nowicki and Snijders (2001) to more than two classes. It enables to deal with graphs with less than 200 vertices.

To remedy this computational burden, a *variational* approach has been proposed by Daudin *et al.* (2008). It can be used with *binary directed* SBM and avoids the complexity of the likelihood and bayesian approaches (see package

Mixnet, 2009). For instance, it allows to analyse graphs with size up to 3000 vertices. Note that Mariadassou *et al.* (2010) have also recently proposed a variational method for the case of the *weighted undirected* SBM. The main drawback of the variational approach is the lack of theoretical justification for it, even if it works well in practice.

Other various strategies have been designed, but they only apply to *undirected* graphs. For instance, Bickel and Chen (2010) developed a label-switching algorithm to maximize a profile-likelihood. A Gibbs sampling algorithm is advocated by Choi *et al.* (2011) to maximize the likelihood of SBM. Rohe *et al.* (2010) propose a Spectral Clustering algorithm to estimate the parameters of CSBM, while Ambroise and Matias (2010) developed a moment-based method to work with *weighted* graphs.

Consistency results for several estimators of the SBM parameters have been already established (Ambroise and Matias, 2010; Choi *et al.*, 2011; Rohe *et al.*, 2010; Bickel and Chen, 2010). However, no such result does exist either for *maximum likelihood* or *variational* estimators in SBM. Nonetheless, empirical clues (Gazal *et al.*, 2011) can be found in favour of the consistency of variational estimators in SBM. Establishing such asymptotic properties is precisely the purpose of the present work.

In this paper, we provide the first identifiability result of *binary directed* SBM under very mild assumptions. The asymptotics of maximum-likelihood and variational estimators is also addressed by use of concentration inequalities. Variational estimators are shown to be asymptotically equivalent to maximum-likelihood ones, and then consistent. However, the adopted framework assumes the number  $Q$  of classes to be known and independent of the number of vertices. Some attempts exist to provide a data-driven choice of  $Q$  (see Daudin *et al.*, 2008), but this question is out of the scope of the present paper.

The rest of the paper is organized as follows. The main notation and assumptions are introduced in Section 2, where identifiability of SBM is settled. Section 3 is devoted to the consistency of the maximum-likelihood estimators (MLE), and Section 4 to the asymptotic equivalence between variational and maximum-likelihood estimators. In particular, the consistency of variational estimators (VE) is proved. Some concluding remarks are provided in Section 5 with some further important questions.

## 2. Model definition and identifiability

Let  $\Omega = (\mathcal{V}, \mathcal{X})$  be the set of infinite random graphs where  $\mathcal{V} = \mathbb{N}$  denotes the set of countable vertices and  $\mathcal{X} = \{0, 1\}^{\mathbb{N}^2}$  the corresponding set of adjacency matrices. The random adjacency matrix, denoted by  $X = \{X_{ij}\}_{i,j \in \mathbb{N}}$ , is given by: for  $i \neq j$ ,  $X_{ij} = 1$  if an edge exists from vertex  $i$  to vertex  $j$  and  $X_{ij} = 0$  otherwise, and  $X_{ii} = 0$  (no loop). Let  $\mathbb{P}$  denote a probability measure on  $\Omega$ .

### 2.1. Stochastic Block Model (SBM)

Let us consider a random graph with  $n$  vertices  $\{v_i\}_{i=1,\dots,n}$ . These vertices are assumed to be split into  $Q$  classes  $\{C_q\}_{q=1,\dots,Q}$  depending on their structural properties.

Set  $\alpha = (\alpha_1, \dots, \alpha_Q)$  with  $0 < \alpha_q < 1$  and  $\sum_q \alpha_q = 1$ . For every  $q$ ,  $\alpha_q$  denotes the probability for a given vertex to belong to the class  $C_q$ . For any vertex  $v_i$ , its label  $Z_i$  is generated as follows

$$\{Z_i\}_{1 \leq i \leq n} \stackrel{i.i.d.}{\sim} \mathcal{M}(n; \alpha_1, \dots, \alpha_Q) \quad .$$

where  $\mathcal{M}(n; \alpha_1, \dots, \alpha_Q)$  denotes the multinomial distribution. Let  $Z_{[n]} = (Z_1, \dots, Z_n)$  denote the random vector of the labels of  $(v_1, \dots, v_n)$ .

The observation consists of an adjacency matrix  $X_{[n]} = \{X_{i,j}\}_{1 \leq i,j \leq n}$ , where  $X_{i,i} = 0$  for every  $i$  and

$$X_{ij} \mid Z_i = q, Z_j = l \stackrel{i.i.d.}{\sim} \mathcal{B}(\pi_{q,l}), \quad \forall i \neq j \quad ,$$

where  $\mathcal{B}(\pi_{q,l})$  denotes the Bernoulli distribution with parameter  $0 \leq \pi_{q,l} \leq 1$  for every  $(q, l)$ .

The log-likelihood is given by

$$\mathcal{L}_2(X_{[n]}; \alpha, \pi) = \log \left( \sum_{z_{[n]}} e^{\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)} \mathbb{P}[Z_{[n]} = z_{[n]}] \right) \quad , \quad (1)$$

where

$$\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi) = \sum_{i \neq j} X_{i,j} \log \pi_{z_i, z_j} + (1 - X_{i,j}) \log(1 - \pi_{z_i, z_j}) \quad , \quad (2)$$

$\mathbb{P}[Z_{[n]} = z_{[n]}] = \prod_{i=1}^n \alpha_{z_i}$ . In the following, let  $\theta = (\alpha, \pi)$  denote the parameter and  $\theta^* = (\alpha^*, \pi^*)$  be the true parameter value. Notice that the  $X_{i,j}$ s are not independent. However, conditioning on  $Z_i = q, Z_j = l$  yields independence.

Recall that the number  $Q$  of classes is assumed to be known and the purpose of the present work is to efficiently estimate the parameters of SBM.

### 2.2. Assumptions

Some conditions which will be used elsewhere. They are presented in the following section with a short commentary.

**Assumption 1** (A1). *For every  $(q, l) \neq (q', l')$ ,*

$$\pi_{q,l} \neq \pi_{q',l'} \quad .$$

This condition is a strong one for obtaining the identifiability of SBM. It is not a necessary condition and (A1) can be relaxed by (A2), which is a necessary condition for the identifiability.

**Assumption 2 (A2).** For every  $q \neq q'$ , there exists  $l \in \{1, \dots, Q\}$  such that

$$\pi_{q,l} \neq \pi_{q',l} \text{ or } \pi_{l,q} \neq \pi_{l,q'} .$$

(A2) excludes the possibility that two columns are equal and that the corresponding rows are also equal. This condition is consistent with the goal of SBM, which is to define  $Q$  classes  $C_1, \dots, C_Q$  with different structural properties. For instance, the connectivity properties of vertices in  $C_q$  must be different from that of vertices in  $C_l$  with  $q \neq l$ . Therefore, settings where this assumption is violated correspond to ill-specified models with too many classes.

**Assumption 3 (A3).** There exists  $\zeta > 0$  such that

$$\forall (q, l) \in \{1, \dots, Q\}^2, \quad \pi_{q,l} \in ]0, 1[ \Rightarrow \pi_{q,l} \in [\zeta, 1 - \zeta] .$$

The SBM can deal with null probabilities of connection between vertices. However, the use of  $\log \pi_{q,l}$  implies a special treatment for the case  $\pi_{q,l} = 0$ . Therefore we will analyze separately the two cases  $\pi_{q,l} = 0$  and  $\pi_{q,l} > 0$ . This implies that the two cases are well separated and that the possibility that  $\pi_{q,l} \rightarrow 0$  when  $n \rightarrow \infty$  is excluded. That is the role of condition (A3) which also includes the symmetric case  $\pi_{q,l} = 1$ . Note that (A3) is always true in the framework of this paper, with  $\pi$  not depending on  $n$  and  $Q$  finite.

**Assumption 4 (A4).** There exists  $\gamma > 0$  such that

$$\forall q \in \{1, \dots, Q\}, \quad \alpha_q \in [\gamma, 1 - \gamma] .$$

This condition, which implies that no class is drained, is necessary for the identifiability of SBM. Note that  $\alpha_q > 0 \forall q \in (1, Q)$  implies (A4) in the framework of this paper, with  $\alpha$  not depending on  $n$  and  $Q$  finite.

### 2.3. Identifiability

The identifiability of the parameters in SBM have been first obtained by [Allman et al. \(2009\)](#) for *undirected* graphs ( $\pi$  is symmetric): if  $Q = 2$ ,  $n \geq 16$ , and the coefficients of  $\pi$  are all different, the parameters are identifiable up to label switching. [Allman et al. \(2010\)](#) also established that for  $Q > 2$ , if  $n$  is even and  $n \geq (Q - 1 + \frac{(Q+2)^2}{4})^2$  (with a similar condition if  $n$  is odd), the parameters of SBM are generically identifiable, that is, identifiable except on a set with null Lebesgue measure.

First, generic identifiability (up to label switching) of the parameters of SBM is proved for *directed or undirected* graphs as long as  $n \geq 2Q$ .

**Theorem 2.1.** Let  $n \geq 2Q$  and assume that for any  $1 \leq q \leq Q$ ,  $\alpha_q > 0$  and the coordinates of  $r = \pi\alpha$  are distinct. Then, SBM is identifiable.

The assumption on  $r$  is not strongly restrictive since the set of vectors violating this assumption is Lebesgue measure 0. Therefore, Theorem 2.1 actually

asserts the generic identifiability of SBM (see Allman *et al.*, 2009). However the conditions of Theorem 2.1 are not always true. For example in the case of the affiliation model with equal class-frequencies, defined by  $\pi_{q,l} = a + b\delta_{q,l}$  and  $\alpha_q = \frac{1}{Q}$ . Therefore it is interesting to see if such models are also identifiable. Theorem 2.2 gives the complete identifiability for the particular case  $Q = 2$ .

Note that the Theorem 2.1 also holds with  $r' = \pi^t \alpha$  (instead of  $r = \pi^t \alpha$ ), and also with vectors  $r''$  given by  $r''_q = \sum_l \pi_{q,l} \pi_{l,q} \alpha_l$  for every  $1 \leq q \leq Q$ .

*Proof of Theorem 2.1.* First, let  $P_{[n]}$  denote the probability distribution function of the adjacency matrix  $X_{[n]}$  of SBM. Let us show that there exists a unique  $(\alpha, \pi)$  corresponding to  $P_{[n]}$ .

Up to reordering, let  $r_1 < r_2 < \dots < r_Q$  denote the coordinates of the vector  $r$  in the increasing order:  $r_q$  is equal to the probability of an edge from a given vertex in the class  $C_q$ .

Let  $R$  denote the Van der Monde matrix defined by  $R_{i,q} = r_q^i$ , for  $0 \leq i < Q$  and  $1 \leq q \leq Q$ .  $R$  is invertible since the coordinates of  $r$  are all different. For  $i \geq 1$ ,  $R_{i,q}$  is the probability that  $i$  given vertices in  $C_q$  have an edge.

Let us also define

$$u_i = \sum_{1 \leq k \leq Q} \alpha_k r_k^i, \quad i = 0, \dots, 2Q - 1 .$$

For  $i \geq 1$ ,  $u_i$  denotes the probability that the first  $i$  coefficients of the first row of  $X_{[n]}$  are equal to 1. Note that  $n \geq 2Q$  is a necessary requirement on  $n$  since  $X_{ii} = 0$  by assumption. Hence given  $P_{[n]}$ ,  $u_0 = 1$  and  $u_1, \dots, u_{2Q-1}$  are known.

Futhermore, set  $M$  the  $(Q+1) \times Q$  matrix given by  $M_{i,j} = u_{i+j}$  for every  $0 \leq i \leq Q$  and  $0 \leq j < Q$ , and let  $M_i$  denote the square matrix obtained by removing the row  $i$  from  $M$ . The coefficients of  $M_Q$  are

$$M_{i,j} = u_{i+j} = \sum_{1 \leq k \leq Q} r_k^i \alpha_k r_k^j, \quad \text{with } 0 \leq i, j < Q .$$

Defining the diagonal matrix  $A = \text{Diag}(\alpha)$ , it comes that  $M_Q = RAR^t$ , where  $R$  and  $A$  are invertible, but unknown at this stage. With  $D_k = \det(M_k)$  and the polynomial  $B(x) = \sum_{k=0}^Q (-1)^{k+Q} D_k x^k$ , it yields  $D_Q = \det(M_Q) \neq 0$  and the degree of  $B$  is equal to  $Q$ .

Set  $V_i = (1, r_i, \dots, r_i^Q)^t$  and let us notice that  $B(r_i)$  is the determinant of the square matrix produced when appending  $V_i$  as last column to  $M$ . The  $Q+1$  columns of this matrix are linearly dependent, since they are all linear combinations of the  $Q$  vectors  $V_1, V_2, \dots, V_Q$ . Hence  $B(r_i) = 0$  and  $r_i$  is a root of  $B$  for every  $1 \leq i \leq Q$ . This proves that  $B = D_Q \prod_{i=1}^Q (x - r_i)$ . Then, one knows  $r = (r_1, \dots, r_Q)$  (as the roots of  $B$  defined from  $M$ ) and  $R$ . It results that  $A = R^{-1} M_Q (R^t)^{-1}$ , which yields a unique  $(\alpha_1, \dots, \alpha_Q)$ .

It only remains to determine  $\pi$ . For  $0 \leq i, j < Q$ , let us introduce  $U_{i,j}$  the probability that the first row of  $X_{[n]}$  begins with  $i + 1$  occurrences of 1, and the second row of  $X$  ends up with  $j$  occurrences of 1 ( $i + 1 + j \leq n - 1$  implies  $n \geq 2Q$ ).

Then,  $U_{i,j} = \sum_{k,l} r_k^i \alpha_k \pi_{k,l} \alpha_l r_l^j$ , for  $0 \leq i, j < Q$ , and the  $Q \times Q$  matrix  $U = RA\pi AR^t$ . The conclusion results from  $\pi = A^{-1}R^{-1}U(R^t)^{-1}A^{-1}$ . □

The assumption of Theorem 2.1 on  $r$  ( $r'$  or  $r''$ ), leading to *generic identifiability*, can be further relaxed in the particular case where  $n = 4$  and  $Q = 2$ .

**Theorem 2.2.** *Set  $n = 4$ ,  $Q = 2$  and let us assume that  $\alpha_q > 0$  for every  $1 \leq q \leq Q$ , and the coefficients of  $\pi$  are not all equal.*

*Then, SBM is identifiable.*

The proof of this result is deferred to Appendix A.

Note that when  $Q = 2$ , (A2) implies that the coefficients of  $\pi$  are not all equal.



### 3. Maximum-likelihood estimation of SBM parameters

#### 3.1. Asymptotics of $\mathbb{P}(Z_{[n]} = \cdot \mid X_{[n]})$

In this section we study the *a posteriori* probability distribution function of  $Z_{[n]}$ ,  $\mathbb{P}(Z_{[n]} = \cdot \mid X_{[n]})$ , which is a random variable depending on  $X_{[n]}$ .

Let us consider a realization of the random graph generated with the sequence of true labels  $Z = z^*$ , where  $z^* = \{z_i^*\}_{i \in \mathbb{N}^*}$ . In the sequel,  $P^* := \mathbb{P}(\cdot \mid Z = z^*)$  denotes the conditional distribution given the whole label sequence.

The following result provides the convergence rate of  $\mathbb{P}(Z_{[n]} = z_{[n]}^* \mid X_{[n]})$  towards 1 with respect to  $P^*$ , that is given  $Z = z^*$ . Theorem 3.1 is an important result that will be repeatedly used all along the paper.

**Theorem 3.1.** *Let us assume that assumptions (A2)–(A4) hold. For every  $t > 0$ ,*

$$P^* \left[ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\mathbb{P}(Z_{[n]} = z_{[n]} \mid X_{[n]})}{\mathbb{P}(Z_{[n]} = z_{[n]}^* \mid X_{[n]})} > t \right] = \mathcal{O}(ne^{-\kappa n}) ,$$

where  $\kappa > 0$  is a constant depending on  $\pi^*$ , but not on  $z^*$ .

The proof of Theorem 3.1 is deferred to Appendix B.

A noticeable feature of this result is that the convergence rate does not depend on  $z^*$ . This point turns out to be crucial when deriving results for the MLE and the variational estimator (respectively Section 3.2 and Section 4.2). Besides, the exponential bound of Theorem 3.1 allows the use of Borel-Cantelli's lemma to get the  $P^*$ -almost sure convergence.

**Corollary 3.2.** *With the same notation as Theorem 3.1,*

$$\sum_{z_{[n]} \neq z_{[n]}^*} \frac{\mathbb{P}(Z_{[n]} = z_{[n]} \mid X_{[n]})}{\mathbb{P}(Z_{[n]} = z_{[n]}^* \mid X_{[n]})} \xrightarrow{n \rightarrow +\infty} 0 , \quad P^* - a.s. .$$

Moreover,

$$\mathbb{P}(Z_{[n]} = z_{[n]}^* \mid X_{[n]}) \xrightarrow{n \rightarrow +\infty} 1 , \quad P^* - a.s. ,$$

and for every  $z_{[n]} \neq z_{[n]}^*$ ,

$$\mathbb{P}(Z_{[n]} = z_{[n]} \mid X_{[n]}) \xrightarrow{n \rightarrow +\infty} 0 , \quad P^* - a.s. .$$

According to Corollary 3.2, observing the connections between the edges of the realization of the graph as  $n$  grows provides us enough information to be able to (asymptotically) recover, given  $Z = z^*$ , the sequence labels  $z^*$  used to generate this graph.

As a consequence of previous Corollary 3.2, one can also understand the above phenomenon in terms of the conditional distribution of  $Z_{[n]}$  given  $X_{[n]}$ .

**Corollary 3.3.**

$$\mathcal{D}(Z_{[n]} \mid X_{[n]}) \xrightarrow[n \rightarrow +\infty]{w} \delta_{z^*}, \quad P^* - a.s. ,$$

where  $\mathcal{D}(Z_{[n]} \mid X_{[n]})$  denotes the distribution of  $Z_{[n]}$  given  $X_{[n]}$ ,  $\xrightarrow[n \rightarrow +\infty]{w}$  refers to the weak convergence in  $\mathcal{M}_1(\mathcal{Z})$ , the set of probability measures on  $\mathcal{Z} = \{1, \dots, Q\}^{\mathbb{N}}$  and  $\delta_{z^*}$  is the Dirac function at the point  $z^*$ .

*Proof of Corollary 3.3.* For every  $n \in \mathbb{N}^*$ , let us define  $\mathcal{Z}_n = \{1, \dots, Q\}^n$ , and introduce a metric space  $(\mathcal{Z}_n, d_n)$ , where the distance  $d_n$  is given by

$$\forall z, z' \in \mathcal{Z}_n, \quad d_n(z, z') = \sum_{k=1}^n 2^{-k} \mathbf{1}_{(z_k \neq z'_k)} .$$

Similarly,  $(\mathcal{Z}, d)$  denotes a metric space with

$$\forall z, z' \in \mathcal{Z}, \quad d(z, z') = \sum_{k \geq 1} 2^{-k} \mathbf{1}_{(z_k \neq z'_k)} .$$

Then,  $\mathcal{Z}_n$  can be embedded into  $\mathcal{Z}$  thanks to the mapping  $i_n : z_{[n]} = (z_1, \dots, z_n) \in \mathcal{Z}_n \mapsto (z_1, \dots, z_n, 1, 1, \dots) \in \mathcal{Z}$ .  $\mathcal{Z}_n$  can be seen as a subset  $\tilde{\mathcal{Z}}_n$  of  $\mathcal{Z}$ . In the sequel,  $\mathcal{Z}_n$  and  $\tilde{\mathcal{Z}}_n$  will be identified.

Let us introduce  $\mathcal{B}$  the Borel  $\sigma$ -algebra on  $\mathcal{Z}$ , and  $\mathcal{B}_n$  the  $\sigma$ -algebra induced by  $\mathcal{B}$  on  $\mathcal{Z}_n$ . Let also  $\mathbb{P}^n = P[\cdot \mid X_{[n]}]$  denote a probability measure on  $\mathcal{B}$ , and  $\mathbb{E}_n[\cdot]$  is the expectation with respect to  $\mathbb{P}^n$ .

Set  $h \in C_b(\mathcal{Z})$  (continuous bounded functions on  $\mathcal{Z}$ ) such that  $\|h\|_\infty \leq M$  for  $M > 0$ . By continuity at point  $z^*$ , for every  $\epsilon > 0$ , there exists  $\eta > 0$  such that

$$d(z, z^*) \leq \eta \quad \Rightarrow \quad |h(z^*) - h(z)| \leq \epsilon .$$

Then,

$$\begin{aligned} |\mathbb{E}_n [ h (Z_{[n]}) ] - h(z^*)| &\leq \sum_{z_{[n]}} |h(z_{[n]}) - h(z^*)| \mathbb{P}^n (Z_{[n]} = z_{[n]}) \\ &\leq \epsilon + 2M \sum_{z_{[n]} \in (B_\eta^*)^c} \mathbb{P}^n (Z_{[n]} = z_{[n]}) \\ &\leq \epsilon + o_{\mathbb{P}}(1) \quad P^* - a.s. , \end{aligned}$$

where  $B_\eta^* = B(z^*, \eta)$  denotes the ball in  $\mathcal{Z}$  with radius  $\eta$  with respect to  $d$ . In the last inequality,  $o_{\mathbb{P}}(1)$  results from Corollary 3.2, which yields the result.  $\square$

### 3.2. MLE consistency

Let us start by recalling the SBM log-likelihood (1):

$$\mathcal{L}_2(X_{[n]}; \alpha, \pi) = \log \left( \sum_{z_{[n]}} e^{\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)} \mathbb{P} [ Z_{[n]} = z_{[n]} ] \right) ,$$

where  $\mathbb{P} [ Z_{[n]} = z_{[n]} ] = \prod_{i=1}^n \alpha_{z_i}$ , and  $(\alpha, \pi)$  are the SBM parameters. The main focus of this section is to settle the consistency of the  $(\alpha, \pi)$  MLE.  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$  is a complex expression for two reasons. First, the  $X_{ij}$  are not independent, so the log-likelihood is not a sum. Moreover, the sum inside the log-function is composed of  $Q^n$  terms, so the log-likelihood is not computable for graphs with more than 20 vertices. For these reasons, no result about the MLE estimators has been obtained till now. The key point in all the paper is given by Theorem 3.1, which sets that the sum inside the log-function contains one overriding term which crush the  $Q^n - 1$  other terms when  $n \rightarrow \infty$ . Therefore, the sum asymptotically reduces to only one term.

Note that another non standard setting is the number of random variables which is  $n(n-1)$  and not  $n$  as usual. More precisely we have  $n(n-1)$  variables which give some information about the connectivity matrix  $\pi$ , but we have only  $n$  vertices in the graph, so we have only  $n$  information about the classes frequencies  $\alpha$ . This specificity of the SBM log-likelihood implies a refined treatment of the normalizing constant  $n$  or  $n(n-1)$ . For this reason we separate the proof of consistency in two parts, one concerning  $\pi$  and one about  $\alpha$ .

The strategy can be decomposed into two steps. First, the consistency of the  $\pi$  estimator is addressed by use of an approach based on M-estimators. Second, a result similar to Theorem 3.1 is combined with a “deconditionning” argument to get the desired consistency of the  $\alpha$  estimator.

The consistency derivation for the MLE of  $\pi$  strongly relies on a general theorem which is inspired from that for M-estimators (van der Vaart and Wellner, 1996).

**Theorem 3.4.** *Let  $(\Theta, d)$  and  $(\Psi, d')$  denote metric spaces, and let  $M_n : \Theta \times \Psi \rightarrow \mathbb{R}$  be a random function and  $\mathbb{M} : \Theta \rightarrow \mathbb{R}$  a deterministic one such that for every  $\epsilon > 0$ ,*

$$\sup_{d(\theta, \theta_0) \geq \epsilon} \mathbb{M}(\theta) < \mathbb{M}(\theta_0) , \tag{3}$$

$$\sup_{(\theta, \psi) \in \Theta \times \Psi} |M_n(\theta, \psi) - \mathbb{M}(\theta)| := \|M_n - \mathbb{M}\|_{\Theta \times \Psi} \xrightarrow[n \rightarrow +\infty]{P} 0 . \tag{4}$$

Moreover, set  $(\hat{\theta}, \hat{\psi}) = \text{Argmax}_{\theta, \psi} M_n(\theta, \psi)$ . Then,

$$d(\hat{\theta}, \theta_0) \xrightarrow[n \rightarrow +\infty]{P} 0 .$$

One important difference between Theorem 3.4 and its usual counterpart for M-estimators (van der Vaart and Wellner, 1996) is that  $M_n$  and  $\mathbb{M}$  do not depend on the same number of arguments. Our consistency result for the MLE of  $\pi$  strongly relies on this point.

*Proof of Theorem 3.4.* For every  $\eta > 0$ , there exists  $\delta > 0$  such that

$$P \left[ d(\hat{\theta}, \theta_0) \geq \eta \right] \leq P \left[ \mathbb{M}(\hat{\theta}) \leq \mathbb{M}(\theta_0) - 3\delta \right] .$$

Since  $\|M_n - \mathbb{M}\|_{\Theta \times \Psi} \xrightarrow[n \rightarrow +\infty]{P} 0$ , it comes that for large enough values of  $n$ ,

$$\begin{aligned} P \left[ d(\hat{\theta}, \theta_0) \geq \eta \right] &\leq P \left[ M_n(\hat{\theta}, \hat{\psi}) \leq M_n(\theta_0, \psi_0) - \delta \right] + o(1) \\ &\leq o(1) . \end{aligned}$$

□

The leading idea in what follows is to check the assumptions of Theorem 3.4.

The main point of our approach consists in using  $P^*$  as a reference probability measure, that is, working as if  $Z_{[n]} = z_{[n]}^*$  were known. In this setting, a key quantity is

$$\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi) = \sum_{i \neq j} X_{i,j} \log \pi_{z_i, z_j} + (1 - X_{i,j}) \log(1 - \pi_{z_i, z_j}) ,$$

where  $(z_{[n]}, \pi)$  are interpreted as parameters. For any  $(z_{[n]}, \pi)$ , let us introduce

$$\begin{aligned} \phi_n(z_{[n]}, \pi) &:= \frac{1}{n(n-1)} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi) , \\ \Phi_n(z_{[n]}, \pi) &:= \mathbb{E} \left[ \phi_n(z_{[n]}, \pi) \mid Z_{[n]} = z_{[n]}^* \right] . \end{aligned}$$

Only a subset of the whole set of possible  $(z_{[n]}, \pi)$  will be considered. Therefore, let

$$\mathcal{P} = \{(z_{[n]}, \pi) \mid |\Phi_n(z_{[n]}, \pi)| < +\infty\}$$

denote the set of *admissible* parameters. Indeed, any  $(z_{[n]}, \pi) \in \mathcal{P}^c$  leads to  $|\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)| = +\infty$ , with large probability as  $n \rightarrow +\infty$ . In other words, this  $(z_{[n]}, \pi)$  does not matter in the sum in the expression (1). Note that any  $(z_{[n]}, \pi) \in \mathcal{P}$  satisfies for every  $(i, j)$ ,

$$\pi_{z_i, z_j} \in \{0, 1\} \Rightarrow \pi_{z_i, z_j} = \pi_{z_i^*, z_j^*}^* = \mathbb{E} \left[ X_{i,j} \mid Z_{[n]} = z_{[n]}^* \right] .$$

The following Proposition 3.5 settles a uniform convergence result for  $\phi_n - \Phi_n$ . Its proof, which is deferred to Appendix C, strongly relies on Talagrand's (Massart, 2007) concentration inequality. This is a consequence of the uniformity requirement, at least with respect to  $\pi$ .

**Proposition 3.5.** *With the above notation, let us assume that (A3) holds. Then,*

$$\sup_{\mathcal{P}} |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 .$$

The following theorem settles the desired properties for  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$ , that is (3) (uniform convergence) and (4) (well-identifiability). The proof of uniform convergence exploits the connection between  $\phi_n(z_{[n]}, \pi)$  and  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$  and subsequently Proposition 3.5.

**Theorem 3.6.** *Let us assume that (A3) and (A4) hold, and for every  $(\alpha, \pi)$ , set  $M_n(\alpha, \pi) = [n(n-1)]^{-1} \mathcal{L}_2(X_{[n]}; \alpha, \pi)$ , and*

$$\begin{aligned} & \mathbb{M}(\pi) \\ &= \max_{\{a_{i,j}\} \in \mathcal{A}} \left\{ \sum_{q,l} \alpha_q^* \alpha_l^* \sum_{q',l'} [a_{q,q'} a_{l,l'} \pi_{q,l}^* \log \pi_{q',l'} + (1 - \pi_{q,l}^*) \log(1 - \pi_{q',l'})] \right\} , \end{aligned}$$

where  $(\alpha^*, \pi^*)$  denotes the true parameter of SBM, and  $\mathcal{A} = \left\{ A = (a_{i,j})_{1 \leq i,j \leq Q} \mid a_{q,q'} \geq 0, \sum_{q'} a_{q,q'} = 1 \right\} \subset \mathcal{M}_Q(\mathbb{R})$ . Then for any  $\eta > 0$ ,

$$\begin{aligned} & \sup_{d(\pi, \pi^*) \geq \eta} \mathbb{M}(\pi) < \mathbb{M}(\pi^*) , \\ & \sup_{\alpha, \pi} |M_n(\alpha, \pi) - \mathbb{M}(\pi)| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 , \end{aligned}$$

where  $d$  denotes a distance.

The proof of Theorem 3.6 is given in Appendix D.

Let us now deduce the Corollary 3.7, which asserts the consistency of the MLE of  $\pi$ .

**Corollary 3.7.** *Under the same assumptions as Theorem 3.6, let us define the MLE of  $(\alpha, \pi)$*

$$(\hat{\alpha}, \hat{\pi}) := \text{Argmax}_{(\alpha, \pi)} \mathcal{L}_2(X_{[n]}; \alpha, \pi) .$$

Then for any distance  $d(\cdot, \cdot)$  on the set of parameters  $\pi$ ,

$$d(\hat{\pi}, \pi^*) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 .$$

*Proof of Corollary 3.7.* This is a straightforward consequence of Theorem 3.4 and Theorem 3.6.  $\square$

A quick inspection of the proof of uniform convergence in Theorem 3.6 shows that the asymptotic behavior of log-likelihood  $\mathcal{L}_2$  does not depend on  $\alpha$ . Roughly speaking, this is a consequence of the expression of  $\mathcal{L}_2$  in which the number of

terms involving  $\pi$  is of order  $n^2$ , whereas about only  $n$  terms involve  $\alpha$ . This difference of scaling with respect to  $n$  between  $\pi$  and  $\alpha$  justifies to some extent a different approach for the MLE of  $\alpha$ .

The proposed strategy heavily relies on an analogous result to Theorem 3.1, where the true value  $(\alpha^*, \pi^*)$  of SBM parameters is replaced by an estimator  $(\hat{\alpha}, \hat{\pi})$ . Let us first state this result in a general framework. It will be also used in Section 4.

**Proposition 3.8.** *Let us assume that assumptions (A2)–(A4) hold, and that there exists an estimator  $\hat{\pi} = \pi^* + o_{\mathbb{P}}(v_n)$ , with  $nv_n = \mathcal{O}(1)$ . Let also  $\hat{\alpha}$  denote any estimator of  $\alpha^*$ . Then for any  $t > 0$ ,*

$$P^* \left[ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\hat{P}(Z_{[n]} = z_{[n]} | X_{[n]})}{\hat{P}(Z_{[n]} = z_{[n]}^* | X_{[n]})} > t \right] = \mathcal{O} \left( ne^{-\kappa n} \vee \mathbb{P} [\|\hat{\pi} - \pi^*\|_{\infty} > v_n] \right) ,$$

where  $\kappa > 0$  is a constant depending on  $\pi^*$ , and

$$\begin{aligned} & \log \left( \frac{\hat{P}(Z_{[n]} = z_{[n]} | X_{[n]})}{\hat{P}(Z_{[n]} = z_{[n]}^* | X_{[n]})} \right) \\ &= \sum_{i \neq j} \left\{ X_{ij} \log \left( \frac{\hat{\pi}_{z_i, z_j}}{\hat{\pi}_{z_i^*, z_j^*}} \right) + (1 - X_{ij}) \log \left( \frac{1 - \hat{\pi}_{z_i, z_j}}{1 - \hat{\pi}_{z_i^*, z_j^*}} \right) \right\} + \sum_i \log \frac{\hat{\alpha}_{z_i}}{\hat{\alpha}_{z_i^*}} . \end{aligned}$$

The proof of Proposition 3.8 is given in Appendix E.

Note that the novelty of this result, with respect to Theorem 3.1, lies in the convergence rate which depends on the behaviour of  $\hat{\pi}$ . This is the reliable price for estimating rather than knowing  $\pi^*$ . The requirement on the rate of convergence  $v_n$  of  $\hat{\pi}$  arises from the proof as a necessary requirement to get the convergence in probability toward 0 as  $n$  tends to  $+\infty$ . However, we do not know whether this artificially results from the strategy of proof or whether this is essentially a necessary condition.

There is empirical evidence (see Gazal *et al.*, 2011) that the rate of convergence of  $\hat{\pi}$  is of order  $\frac{1}{n}$ , but this property is assumed and not proven in this paper. Besides, in the same way as in Theorem 3.1, one crucial point in Proposition 3.8 is the independence of the convergence rate with respect to  $z_{[n]}^*$ . The strategy of proof presented in the sequel strongly relies on this property.

Let us now settle the consistency of the MLE of  $\alpha^*$  on the basis of previous Proposition 3.8.

**Theorem 3.9.** *Let us assume the MLE  $\hat{\pi} = \pi^* + o_{\mathbb{P}}(1/n)$ . With the same assumptions as Theorem 3.6, and the notation of Corollary 3.7, then*

$$d(\hat{\alpha}, \alpha^*) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 ,$$

where  $d$  denotes any distance between vectors in  $\mathbb{R}^Q$ .

*Proof of Theorem 3.9.* As usually for mixture models, it is easy to see that the MLE of  $\alpha$  is given for any  $q$  by

$$\hat{\alpha}_q = \frac{1}{n} \sum_{i=1}^n \hat{P}(Z_i = q \mid X_{[n]}) .$$

First, let us work with respect to  $P^*$ , that is, as if  $Z_{[n]} = z_{[n]}^*$  were known. Setting  $N_q(z_{[n]}) = \sum_{i=1}^n \mathbb{1}_{(z_i=q)}$ , it comes

$$\begin{aligned} \left| \hat{\alpha}_q - N_q(z_{[n]}^*)/n \right| &\leq \left| \frac{1}{n} \sum_{i=1}^n \hat{P}(Z_i = z_i^* \mid X_{[n]}) \mathbb{1}_{(z_i^*=q)} - N_q(z_{[n]}^*)/n \right| \\ &\quad + \hat{P}(Z_{[n]} \neq z_{[n]}^* \mid X_{[n]}) \\ &\leq \frac{1}{n} \sum_{i=1}^n \left( 1 - \hat{P}(Z_i = z_i^* \mid X_{[n]}) \right) \mathbb{1}_{(z_i^*=q)} \\ &\quad + \hat{P}(Z_{[n]} \neq z_{[n]}^* \mid X_{[n]}) \\ &\leq \frac{1}{n} \sum_{i=1}^n \left( \hat{P}(Z_i \neq z_i^* \mid X_{[n]}) \right) \mathbb{1}_{(z_i^*=q)} + \hat{P}(Z_{[n]} \neq z_{[n]}^* \mid X_{[n]}) \\ &\leq 2\hat{P}(Z_{[n]} \neq z_{[n]}^* \mid X_{[n]}) . \end{aligned}$$

Second, let us now use a deconditionning argument replacing  $P^*$  by  $\mathbb{P}$ . Let  $N_q = N_q(Z_{[n]})$  denote a binomial random  $\mathcal{B}(n, \alpha_q^*)$  for every  $q$ . Then for every  $\epsilon > 0$ ,

$$\begin{aligned} &\mathbb{P} \left[ |\hat{\alpha}_q - \alpha_q^*| > \epsilon \right] \\ &\leq \mathbb{P} \left[ |\hat{\alpha}_q - N_q/n| > \epsilon/2 \right] + \mathbb{P} \left[ |N_q/n - \alpha_q^*| > \epsilon/2 \right] \\ &\leq \mathbb{P} \left[ |\hat{\alpha}_q - N_q/n| > \epsilon/2 \right] + o(1) , \end{aligned}$$

by use of LLG. Finally, a straightforward use of Proposition 3.8 leads to

$$\begin{aligned} &\mathbb{P} \left[ |\hat{\alpha}_q - N_q/n| > \epsilon/2 \right] \\ &= \mathbb{E}_{Z_{[n]}} \left[ P \left( |\hat{\alpha}_q - N_q/n| > \epsilon/2 \mid Z_{[n]} \right) \right] \\ &\leq \sum_{z_{[n]}} P \left[ \hat{P}(Z_{[n]} \neq z_{[n]} \mid X_{[n]}) > \epsilon/4 \mid Z_{[n]} = z_{[n]} \right] \mathbb{P} \left[ Z_{[n]} = z_{[n]} \right] \\ &= o(1) . \end{aligned}$$

□

#### 4. Variational estimators of SBM parameters

In Section 3, the consistency has been proved for the maximum likelihood estimators. In a sense this result is a purely theoretical one, for the MLE can be computed in practice only for very small graphs with less than 20 vertices. However the MLE results of the previous section are necessary to obtain the consistency of the variational estimators, which in turn, are the practically useful estimators.

Indeed, the log-likelihood  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$  involves a sum over  $Q^n$  terms, which is intractable in practice except for very small and unrealistic values of  $n$ :

$$\mathcal{L}_2(X_{[n]}; \alpha, \pi) = \log \left\{ \sum_{z_{[n]} \in \mathcal{Z}_n} e^{\sum_{i \neq j} b_{ij}(z_i, z_j)} P_{Z_{[n]}}(z_{[n]}) \right\},$$

with  $b_{ij}(z_i, z_j) = X_{ij} \log \pi_{z_i, z_j} + (1 - X_{ij}) \log(1 - \pi_{z_i, z_j})$ . To circumvent this problem, alternatives are Markov chain Monte Carlo (MCMC) algorithms (Andrieu and Atchadé, 2007) and variational approximation (Jordan et al., 1999). However, MCMC algorithms suffer a high computational cost, which makes them unattractive compared to variational approximation. Actually the variational method is the only one which can deal with thousands of vertices in a reasonable computing time, because its complexity is only  $\mathcal{O}(n^2)$ . For example, the Mixnet (2009) package (based on variational approximation) works with up to 2000 vertices, whereas the STOCNET package (see Boer et al., 2006) (Gibbs sampling) only deals with 200 vertices.

The purpose of the present section is to prove that the variational approximation yields consistent estimators of the SBM parameters. The resulting estimators will be called *variational estimators* (VE).

##### 4.1. Variational approximation

To the best of our knowledge, the first use of variational approximation for SBM has been made by Daudin et al. (2008). The variational method consists in approximating  $P^{X_{[n]}} = \mathbb{P}(Z_{[n]} = \cdot | X_{[n]})$  by a product of  $n$  multinomial distributions. This leads to approximate  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$  by a sum of  $n^2$  terms. The computational virtue of this trick is that a sum of  $Q^n$  terms is replaced by a sum of  $n^2$  terms.

Let us define  $\mathcal{D}_n$  as a set of product multinomial distributions

$$\mathcal{D}_n = \left\{ D_{\tau_{[n]}} = \prod_{i=1}^n \mathcal{M}(1, \tau_{i,1}, \dots, \tau_{i,Q}) \mid \tau_{[n]} \in \mathcal{S}_n \right\}, \quad (5)$$

where

$$\mathcal{S}_n = \left\{ \tau_{[n]} = (\tau_1, \dots, \tau_n) \in ([0, 1]^Q)^n \mid \forall i, \tau_i = (\tau_{i,1}, \dots, \tau_{i,Q}), \sum_{q=1}^Q \tau_{i,q} = 1 \right\}.$$



For any  $D_{\tau_{[n]}} \in \mathcal{D}_n$ , the variational log-likelihood,  $\mathcal{J}(\cdot; \cdot, \cdot, \cdot)$  is defined by

$$\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi) = \mathcal{L}_2(X_{[n]}; \alpha, \pi) - K(D_{\tau_{[n]}}, P^{X_{[n]}}) \quad , \quad (6)$$

where  $K(\cdot, \cdot)$  denotes the Kullback-Leibler divergence, and  $P^{X_{[n]}} = \mathbb{P}(Z_{[n]} = \cdot \mid X_{[n]})$ . With this choice of  $\mathcal{D}_n$ ,  $\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi)$  has the following expression (see [Daudin et al., 2008](#)):

$$\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi) = \sum_{i \neq j} \sum_{q, l} b_{ij}(q, l) \tau_{i,q} \tau_{j,l} - \sum_{iq} \tau_{i,q} (\log \tau_{i,q} - \log \alpha_q) \quad . \quad (7)$$

Then, the variational approximation  $R_{X_{[n]}}$  to  $P^{X_{[n]}}$  is given by solving the minimization problem over  $\mathcal{D}_n$ :

$$R_{X_{[n]}} \in \text{Argmin}_{D_{\tau} \in \mathcal{D}_n} K(D_{\tau}, P^{X_{[n]}}) \quad ,$$

as long as such a minimizer exists.

Thus, minimizing  $K(D_{\tau_{[n]}}, P^{X_{[n]}})$  with respect to  $\tau_{[n]}$  is equivalent to maximizing  $\mathcal{J}(X_{[n]}; \tau_{[n]}, \pi, \alpha)$ , which leads to

$$\hat{\tau}_{[n]} = \hat{\tau}_{[n]}(\pi, \alpha) := \text{Argmax}_{\tau_{[n]}} \mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi) \quad .$$

The variational estimators (VE) of  $(\alpha, \pi)$  are

$$(\hat{\alpha}, \hat{\pi}) = \text{Argmax}_{\alpha, \pi} \mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) \quad . \quad (8)$$

Note that in practice, the variational algorithm maximizes  $\mathcal{J}(X_{[n]}; \tau, \alpha, \pi)$  alternatively with respect to  $\tau$  and  $(\alpha, \pi)$  (see [Daudin et al., 2008](#)).

In the sequel, the same notation as in Section 3 is used. In particular, it is often assumed that a realization of SBM is observed, which has been generated from the sequence of true labels  $Z = z^*$ . In this setting,  $P^*$  denotes the conditional distribution  $\mathbb{P}(\cdot \mid Z = z^*)$  given the whole label sequence. The first result provides some assurance about the reliability of the variational approximation to  $P^{X_{[n]}}$ .

**Proposition 4.1.** *For every  $n$ , let  $\mathcal{D}_n$  denote the set defined by (5), and  $P^{X_{[n]}}(\cdot)$  be the distribution of  $Z_{[n]}$  given  $X_{[n]}$ . Then, assuming (A2) – (A4) hold,*

$$K(R_{X_{[n]}}, P^{X_{[n]}}) := \inf_{D \in \mathcal{D}_n} K(D, P^{X_{[n]}}) \xrightarrow{n \rightarrow \infty} 0 \quad , \quad P^* - a.s. \quad .$$

Note that this convergence result is given with respect to  $P^*$  (and not to  $\mathbb{P}$ ). Stronger results can be obtained (see Section 4.1) thanks to fast convergence rates. Proposition 4.1 yields some confidence in the reliability of the variational approximation, which gets closer to  $P^{X_{[n]}}$  as  $n$  tends to  $+\infty$ . However, it does not provide any warranty about the good behavior of variational estimators, which is precisely the goal of following Section 4.2.

*Proof of Proposition 4.1.*

By definition of the variational approximation,

$$K(R_{X_{[n]}}, P^{X_{[n]}}) \leq K(\delta_{z_{[n]}^*}, P^{X_{[n]}}) ,$$

where  $\delta_{z_{[n]}^*} = \prod_{1 \leq i \leq n} \delta_{z_i^*} \in \mathcal{D}_n$ . Then,

$$K(R_{X_{[n]}}, P^{X_{[n]}}) \leq \sum_{1 \leq i \leq n} -\log(P(Z_i = z_i^* | X_{[n]})) = -\log \left[ P \left( Z_{[n]} = z_{[n]}^* | X_{[n]} \right) \right] .$$

The conclusion results from Theorem 3.1, and Corollary 3.2 since  $P \left( Z_{[n]} = z_{[n]}^* | X_{[n]} \right) \xrightarrow[n \rightarrow \infty]{} 1$   $P^* - a.s.$   $\square$

#### 4.2. Consistency of the variational estimators

Since the variational log-likelihood  $\mathcal{J}(\cdot; \cdot, \cdot, \cdot)$  (6) is defined from the log-likelihood  $\mathcal{L}_2(\cdot; \cdot, \cdot)$ , the properties of  $\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi)$  are strongly connected to those of  $\mathcal{L}_2(X_{[n]}; \alpha, \pi)$ . Therefore, the strategy followed in the present section is very similar to that of Section 3. In particular, the consistency of  $\tilde{\pi}$  (VE of  $\pi$ , see (8)) is addressed first. Then, the consistency of the VE of  $\alpha$  ( $\tilde{\alpha}$ , see (8)) exploits the convergence of the estimator of  $\pi$ .

The first step consists in applying Theorem 3.4 to settle the  $\tilde{\pi}$  consistency. Following results aim at justifying the use of this theorem by checking its assumptions.

Theorem 4.2 states that  $\mathcal{L}_2$  and  $\mathcal{J}$  are asymptotically equivalent uniformly with respect to  $\alpha$  and  $\pi$ .

**Theorem 4.2.** *With the same notation as Theorem 3.6 and Section 4.1, let us define*

$$J_n(\alpha, \pi) := \frac{1}{n(n-1)} \mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) .$$

Then, (A3) and (A4) yield

$$\sup_{\alpha, \pi} \{ |J_n(\alpha, \pi) - M_n(\alpha, \pi)| \} = o(1), \quad \mathbb{P} - a.s. ,$$

where the supremum is computed over sets given in (A3) and (A4).

This statement is stronger than Proposition 4.1 in several respects. On the one hand, convergence applies almost surely with respect to  $\mathbb{P}$ , and not with respect to  $P^*$ . On the other hand, Theorem 4.2 exhibits the convergence rate toward 0, which is not faster than  $n(n-1)$ .

*Proof of Theorem 4.2.*

Recall the definitions of  $\mathcal{L}_1$  (2),  $\mathcal{L}_2$  (1),  $\mathcal{J}$  (6) and let  $\hat{z}_{[n]} = \hat{z}_{[n]}(\pi) = \text{Argmax}_{z_{[n]}} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)$ . Lemma F.1 yields

$$\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi) \leq \mathcal{L}_2(X_{[n]}; \alpha, \pi) \leq \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi) .$$

Then, applying Assumption (A4) and Lemma F.2, there exists  $0 < \gamma < 1$  independent of  $(\alpha, \pi)$  such that

$$|\mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)| \leq n \log(1/\gamma) .$$

The conclusion results straightforwardly.  $\square$

The consistency of  $\tilde{\pi}$  is provided by the following result, which is simple a consequence of Theorem 4.2, Proposition 3.5, and Theorem 3.4.

**Corollary 4.3.** *With the notation of Theorem 4.2 and assuming (A3) and (A4) hold, let us define the VE of  $(\alpha, \pi)$*

$$(\tilde{\alpha}, \tilde{\pi}) = \text{Argmax}_{\alpha, \pi} J_n(\alpha, \pi) .$$

*Then for any distance  $d(\cdot, \cdot)$  on the set of  $\pi$  parameters,*

$$d(\tilde{\pi}, \pi^*) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 .$$

The proof is completely similar to that of Corollary 3.7 and is therefore not reproduced here.

The consistency of the VE of  $\alpha^*$  results from the same deconditioning argument as the MLE of  $\alpha^*$  (Section 3.2). There is some empirical evidence (see Gazal *et al.*, 2011) about the rate  $\frac{1}{n}$  of the convergence of  $\tilde{\pi}$ . This rate is assumed in the following theorem.

**Theorem 4.4.** *Let us assume the VE  $\tilde{\pi}$  converges at rate  $1/n$  to  $\pi^*$ . With the same assumptions as Theorem 4.2 and assuming (A3) and (A4) hold, then*

$$d(\hat{\alpha}, \alpha^*) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 ,$$

*where  $d$  denotes any distance between vectors in  $\mathbb{R}^Q$ .*

The crux of the proof is an other use of Proposition 3.8.

*Proof of Theorem 4.4.*

First, let us show that given  $Z_{[n]} = z_{[n]}^*$ ,

$$\left| \tilde{\alpha}_q - N_q(z_{[n]}^*)/n \right| \xrightarrow[n \rightarrow \infty]{P^*} 0 .$$

For every  $q$ ,

$$\tilde{\alpha}_q = \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_{i,q},$$

where  $\tilde{\tau}_{i,q} = \hat{\tau}_{i,q}(\tilde{\alpha}, \tilde{\pi})$  (see (8)). Introducing  $z_i^*$ , it comes that

$$\tilde{\alpha}_q = \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_{i,z_i^*} \mathbf{1}_{(z_i^*=q)} + \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_{i,q} \mathbf{1}_{(z_i^*\neq q)} .$$

From (5), let us consider the *a posteriori* distribution of  $\tilde{Z}_{[n]} = (\tilde{Z}_1, \dots, \tilde{Z}_n)$  denoted by

$$D_{\tilde{\tau}_{[n]}}(z_{[n]}) = \mathbb{P} \left[ \tilde{Z}_{[n]} = z_{[n]} \mid X_{[n]} \right] = \prod_{i=1}^n \tilde{\tau}_{i,z_i} .$$

Then,

$$\begin{aligned} \left| \tilde{\alpha}_q - N_q(z_{[n]}^*)/n \right| &= \left| \frac{1}{n} \sum_{i=1}^n (\tilde{\tau}_{i,z_i^*} - 1) \mathbf{1}_{(z_i^*=q)} + \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_{i,q} \mathbf{1}_{(z_i^*\neq q)} \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n (1 - \tilde{\tau}_{i,z_i^*}) \mathbf{1}_{(z_i^*=q)} + \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_{i,q} \mathbf{1}_{(z_i^*\neq q)} \\ &\leq \frac{1}{n} \sum_{i=1}^n (1 - \tilde{\tau}_{i,z_i^*}) , \end{aligned}$$

using that when  $z_i^* \neq q$ ,  $\tilde{\tau}_{i,q} \leq \sum_{q \neq z_i^*} \tilde{\tau}_{i,q} = 1 - \tilde{\tau}_{i,z_i^*}$ . Hence,

$$\left| \tilde{\alpha}_q - N_q(z_{[n]}^*)/n \right| \leq \frac{1}{n} \sum_{i=1}^n \mathbb{P} \left[ \tilde{Z}_{[n]} \neq z_{[n]}^* \mid X_{[n]} \right] = 1 - D_{\tilde{\tau}_{[n]}}(z_{[n]}^*) .$$

It remains to show that  $D_{\tilde{\tau}_{[n]}}(z_{[n]}^*) \xrightarrow[n \rightarrow \infty]{P^*} 1$ , at a rate which does not depend of  $z_{[n]}^*$ . Let  $\tilde{P}(z_{[n]}^*)$  denote the *a posteriori* distribution of  $Z_{[n]}$  with parameters  $(\tilde{\alpha}, \tilde{\pi})$ . According to Lemma 4.5, the asymptotic behavior of  $D_{\tilde{\tau}_{[n]}}(z_{[n]}^*)$  is closely related to that of  $\tilde{P}(z_{[n]}^*)$ . Then, another use of Proposition 3.8 applied to  $\tilde{P}(z_{[n]}^*)$  and  $\tilde{\pi}$  yields

$$\tilde{P}(z_{[n]}^*) = 1 - \mathcal{O} \left( n e^{-\kappa n} \vee \mathbb{P} [ \|\tilde{\pi} - \pi^*\|_\infty > v_n ] \right) ,$$

where  $\kappa > 0$  is a constant depending on  $\pi^*$ . Therefore, Lemma 4.5 implies

$$D_{\tilde{\tau}_{[n]}}(z_{[n]}^*) = 1 - \sqrt{\mathcal{O} \left( n e^{-\kappa n} \vee \mathbb{P} [ \|\tilde{\pi} - \pi^*\|_\infty > v_n ] \right)} .$$

Result follows from the same reasoning as in the end of the proof of Proposition 3.8. □

**Lemma 4.5.**

$$\left| D_{\tilde{\tau}_{[n]}}(z_{[n]}^*) - \tilde{P}(z_{[n]}^*) \right| \leq \sqrt{-\frac{1}{2} \log \left[ \tilde{P}(z_{[n]}^*) \right]} .$$

*Proof of Lemma 4.5.*

$$\begin{aligned} \left| D_{\tilde{\tau}_{[n]}}(z_{[n]}^*) - \tilde{P}(z_{[n]}^*) \right| &\leq \left\| D_{\tilde{\tau}_{[n]}} - \tilde{P} \right\|_{TV} \\ &\leq \sqrt{\frac{1}{2} K \left( D_{\tilde{\tau}_{[n]}} , \tilde{P} \right)} \\ &\leq \sqrt{\frac{1}{2} K \left( \delta_{z_{[n]}^*} , \tilde{P} \right)} = \sqrt{-\frac{1}{2} \log \left[ \tilde{P}(z_{[n]}^*) \right]} . \end{aligned}$$

□

## 5. Conclusion

This paper provides theoretical (asymptotic) results about the stochastic block model (SBM) inference, especially applying to directed graphs, unlike most of existing results. This is typically the setting of true graphs such as biological networks. In particular, asymptotic equivalence between maximum-likelihood and variational estimators is proved, as well as the consistency of resulting estimators. To the best of our knowledge, these are the first results of this type for variational estimators of the SBM parameters. Such theoretical properties are essential since they validate the empirical practice which uses variational approaches as a reliable means to deal with up to several thousands of nodes.

Besides, this work can be seen as a preliminary step towards a deeper analysis of maximum-likelihood and variational estimators of SBM parameters. In particular, a further interesting question is the choice of the number  $Q$  of classes in the mixture model. Indeed, it is important to develop a *data-driven* strategy to choose  $Q$  in order to make the variational approach fully applicable in practice, and validate the empirical practice.

## Appendix A: Proof of Theorem 2.2

*Proof of Theorem 2.2.*

Let us just assume  $Q = 2$ ,  $n = 4$ , and that no element of  $\alpha$  is zero.

If the coordinates of  $r = \pi\alpha$  are distinct, then Theorem 2.1 applies and the desired result follows.

Otherwise, the two coordinates are  $r$ ,  $r'$  and  $r''$  are not distinct. Set  $r_1 = r_2 = a$  and  $u_i = \alpha_1 r_1^i + \alpha_2 r_2^i$ , for  $i \geq 0$ . Let us also define  $b = r'_1 = r'_2$ , and  $c = r''_1 = r''_2$ . Then, the following equalities hold:

$$\begin{aligned} a &= \pi_{11}\alpha_1 + \pi_{12}\alpha_2 = \pi_{21}\alpha_1 + \pi_{22}\alpha_2 \quad , \\ b &= \pi_{11}\alpha_1 + \pi_{21}\alpha_2 = \pi_{12}\alpha_1 + \pi_{22}\alpha_2 \quad , \\ c &= \pi_{11}^2\alpha_1 + \pi_{21}\pi_{12}\alpha_2 = \pi_{12}\pi_{21}\alpha_1 + \pi_{22}^2\alpha_2 \quad . \end{aligned}$$

From  $a - b = (\pi_{12} - \pi_{21})\alpha_2 = -(\pi_{12} - \pi_{21})\alpha_1$  we deduce  $\pi_{12} = \pi_{21}$  and  $a = b$ . Then,

$$\begin{aligned} \alpha_1\alpha_2(\pi_{11} - \pi_{12})^2 &= (\alpha_1 + \alpha_2)(\alpha_1\pi_{11}^2 + \alpha_2\pi_{12}^2) - (\alpha_1\pi_{11} + \alpha_2\pi_{12})^2 \\ &= c - a^2 \\ &= c - b^2 \\ &= \alpha_1\alpha_2(\pi_{22} - \pi_{12})^2 \quad . \end{aligned}$$

If  $c = a^2$ , then  $\pi_{11} = \pi_{12} = \pi_{21} = \pi_{22} = a$  and  $\alpha$  cannot be found.

If  $c \neq a^2$ , then  $|\pi_{11} - \pi_{12}| = |\pi_{22} - \pi_{12}| \neq 0$ . But  $\alpha_1(\pi_{11} - \pi_{12}) = a - \pi_{12} = b - \pi_{12} = \alpha_2(\pi_{22} - \pi_{12})$  leads to  $|\alpha_1| = |\alpha_2|$  and  $\alpha_1 = \alpha_2 = 1/2$ . Hence  $\pi_{11} = \pi_{22}$ . Then,  $\pi_{11}$  and  $\pi_{12}$  are the roots of the polynomial  $x^2 - 2ax + 2a^2 - c$ .

At this stage, we need to distinguish between  $\pi_{11}$  and  $\pi_{12}$ . Let us introduce the probability  $d$  that  $X_{[n]}$  fits the pattern

$$\begin{array}{cccc} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} .$$

Then,  $d = (\pi_{11}^3 + 3\pi_{11}\pi_{12}^2)/4$  and one can compute  $e = \sqrt[3]{d - a^3} = (\pi_{11} - \pi_{12})/2$ . This leads to  $\pi_{11} = \pi_{22} = a + e$  and  $\pi_{12} = \pi_{21} = a - e$ , which yields the conclusion.  $\square$

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## Appendix B: Proof of Theorem 3.1 and Proposition 3.8

Let us first give the proof under Assumption (A1) and (A3). The proof in the more general setting of Assumptions (A2), (A3), and (A6) is given in Appendix B.5.

$$\mathbf{B.1. Upper bounding} \quad \mathbb{P} \left[ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\mathbb{P}[Z_{[n]} = z_{[n]} | X_{[n]}]}{\mathbb{P}[Z_{[n]} = z_{[n]}^* | X_{[n]}]} > t \mid Z = z^* \right]$$

Let  $P^{X_{[n]}}(z_{[n]})$  denote  $\mathbb{P}[Z_{[n]} = z_{[n]} \mid X_{[n]}]$  for every  $z_{[n]}$ . The sum on  $z_{[n]}$  is partitioned according to the number  $r$  of differences between  $z_{[n]}$  and  $z_{[n]}^*$ .

$$\sum_{z_{[n]} \neq z_{[n]}^*} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} = \sum_{r=1}^n \sum_{\|z - z^*\|_0 = r} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)},$$

where  $\|z\|_0$  designates the number of non-zero components of the vector  $z$ .

$$\begin{aligned} & \mathbb{P} \left[ \sum_{z \neq z^*} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > t \mid Z = z^* \right] \\ &= \mathbb{P} \left[ \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > t \mid Z = z^* \right] \\ &\leq \mathbb{P} \left[ \bigcup_{r=1}^n \left\{ \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n} \mid Z = z^* \right\} \right] \\ &\leq \sum_{r=1}^n \mathbb{P} \left[ \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n} \mid Z = z^* \right] \\ &\leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \mathbb{P} \left[ \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n \binom{n}{r} (Q-1)^r} \mid Z = z^* \right]. \end{aligned}$$

Note that the number of  $z_{[n]} \neq z_{[n]}^*$  such that  $\|z_{[n]} - z_{[n]}^*\|_0 = r$  is equal to  $\binom{n}{r} (Q-1)^r$  since  $z_{[n]} \in \{1, \dots, Q\}^n$ . This leads us to

$$\begin{aligned} & \mathbb{P} \left[ \sum_{z \neq z^*} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > t \mid Z = z^* \right] \\ &\leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \mathbb{P} \left[ \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n^{r+1} (Q-1)^r} \mid Z = z^* \right]. \end{aligned}$$



**B.2. Upper bounding**  $\mathbb{P} \left[ \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n^{r+1}(Q-1)^r} \mid Z = z^* \right]$

Let us first notice that

$$\begin{aligned} & \mathbb{P} \left[ \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \frac{t}{n^{r+1}(Q-1)^r} \mid Z = z^* \right] \\ = & \mathbb{P} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > \log \left( \frac{t}{n^{r+1}(Q-1)^r} \right) \mid Z = z^* \right] \\ = & \mathbb{P} \left\{ \frac{1}{r(2n-r-1)} \left( \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} - \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \right) > \right. \\ & \left. \frac{1}{r(2n-r-1)} \left( \log \frac{t}{n^{r+1}(Q-1)^r} - \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \right) \mid Z = z^* \right\}. \end{aligned}$$

Moreover,

$$\begin{aligned} & \log \left( \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right) - \mathbb{E}^{Z=z^*} \left[ \log \left( \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right) \right] \\ = & \sum_{i \neq j} \left\{ \left( X_{ij} - \pi_{z_i^*, z_j^*}^* \right) \log \left[ \frac{\pi_{z_i, z_j}^* (1 - \pi_{z_i^*, z_j^*}^*)}{\pi_{z_i^*, z_j^*}^* (1 - \pi_{z_i, z_j}^*)} \right] \right\}. \end{aligned}$$

Under condition (A1), Lemma B.4 implies that there are  $r(2n-r-1)$  non-zero term in the sum.

Let us then apply Proposition B.1 (Hoeffding's inequality), with  $a_{ij} = -b_{ij} = \log \left[ (1-\zeta)^2 \zeta^{-2} \right]$  (see Lemma B.2),  $L = 2(b_{i,j} - a_{i,j})$ . For any  $s > 0$ , it comes

$$\begin{aligned} & \mathbb{P} \left[ \frac{1}{r(2n-r-1)} \left( \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} - \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \right) > s \mid Z = z^* \right] \\ & \leq \exp \left( \frac{-r(2n-r-1)s^2}{L^2} \right). \end{aligned}$$

### B.3. Conclusion

One then apply the previous result with a particular choice of  $s$ . Thus,

$$s = \frac{1}{r(2n-r-1)} \left( \log \frac{t}{n^{r+1}(Q-1)^r} - \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \right)$$

leads to

$$s = \frac{\log t - (r+1)\log(n) - r\log(Q-1)}{r(2n-r-1)} - \frac{1}{r(2n-r-1)} \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right].$$

With Lemma B.3, it is not difficult to show that for large enough values of  $n$ ,

$$s^2 \geq \frac{3}{4} \left( \frac{1}{r(2n-r-1)} \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \right)^2 \geq \frac{3}{4} (c^*)^2.$$

It results that

$$\exp\left(\frac{-r(2n-r-1)s^2}{L^2}\right) \leq \exp\left(\frac{-3r(2n-r-1)(c^*)^2}{4L^2}\right),$$

which implies

$$\begin{aligned} \mathbb{P} \left[ \sum_{z \neq z^*} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > t \mid Z = z^* \right] &\leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} \exp\left(\frac{-3r(2n-r-1)(c^*)^2}{4L^2}\right) \\ &\leq \sum_{r=1}^n \binom{n}{r} (Q-1)^r \exp\left(\frac{-3r(2n-r-1)(c^*)^2}{4L^2}\right) \\ &\leq \sum_{r=1}^n \binom{n}{r} [(Q-1)u_n]^r, \end{aligned}$$

where  $u_n = \exp\left[\frac{-3(n-1)(c^*)^2}{4L^2}\right]$ .

Using the inequality  $e^x - 1 \leq xe^x$ , one obtains

$$\begin{aligned} \mathbb{P} \left[ \sum_{z \neq z^*} \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} > t \mid Z = z^* \right] &\leq (1 + (Q-1)u_n)^n - 1 \\ &\leq e^{(Q-1)nu_n} - 1 \\ &\leq (Q-1)u_n n e^{(Q-1)nu_n} \xrightarrow[n \rightarrow +\infty]{} 0, \end{aligned}$$

which concludes the proof.

#### B.4. Hoeffding's inequality and related lemmas

**Proposition B.1** (Hoeffding's inequality). *Let  $\{Y_{i,j}\}_{1 \leq i \neq j \leq n}$  independent random variables such that for every  $i \neq j$ ,  $Y_{i,j} \in [a_{i,j}, b_{i,j}]$  almost surely. Then, for any  $t > 0$ ,*

$$\mathbb{P} \left[ \sum_{i \neq j}^n (Y_{i,j} - \mathbb{E}[Y_{i,j}]) > t \right] \leq \exp\left(\frac{-t^2}{\sum_{i \neq j} (b_{i,j} - a_{i,j})^2}\right).$$

**Lemma B.2** (Values of  $a_{i,j}$  and  $b_{i,j}$ ). *Assuming (A3) holds for  $\pi^*$  with  $\zeta > 0$ , it comes for every  $1 \leq i \neq j \leq n$ ,*

$$\left| X_{ij} \log \left[ \frac{\pi_{z_i, z_j}^* (1 - \pi_{z_i^*, z_j^*}^*)}{\pi_{z_i^*, z_j^*}^* (1 - \pi_{z_i, z_j}^*)} \right] \right| \leq 2 \log \left[ \left( \frac{1 - \zeta}{\zeta} \right) \right] .$$

**Lemma B.3** (Bounds for the exponent of the exponential bound). *Assuming (A1) and (A3) hold, there exist positive constants  $c^* = c(\pi^*)$  and  $C^* = C(\pi^*)$  such that*

$$0 < c^* \leq -\frac{1}{r(2n - r - 1)} \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \leq C^* .$$

*Proof of Lemma B.3.*

$$\begin{aligned} & \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \\ = & \mathbb{E}^{Z=z^*} \left[ \sum_{i \neq j} \left\{ X_{ij} \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - X_{ij}) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \right\} + \sum_i \log \frac{\alpha_{z_i}^*}{\alpha_{z_i^*}^*} \right] \\ = & \sum_{i \neq j} \mathbb{E}^{Z=z^*} \left[ X_{ij} \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - X_{ij}) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \right] + \sum_i \log \frac{\alpha_{z_i}^*}{\alpha_{z_i^*}^*} \\ = & \sum_{i \neq j} - \left[ \pi_{z_i^*, z_j^*}^* \log \left( \frac{\pi_{z_i^*, z_j^*}^*}{\pi_{z_i, z_j}^*} \right) + (1 - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{1 - \pi_{z_i^*, z_j^*}^*}{1 - \pi_{z_i, z_j}^*} \right) \right] + \sum_i \log \frac{\alpha_{z_i}^*}{\alpha_{z_i^*}^*} . \end{aligned}$$

Since  $\|z_{[n]} - z_{[n]}^*\|_0 = r$ , Lemma B.4 implies that there are  $r(2n - r - 1)$  couples  $(i, j)$  such that  $\pi_{z_i^*, z_j^*}^* \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \neq 0$ . Let us set

$$\begin{aligned} C^* & := \max \left\{ \pi_{q,l}^* \log \left( \frac{\pi_{q,l}^*}{\pi_{q',l'}^*} \right) + (1 - \pi_{q,l}^*) \log \left( \frac{1 - \pi_{q,l}^*}{1 - \pi_{q',l'}^*} \right) \right\} , \\ c^* & := \min \left\{ \pi_{q,l}^* \log \left( \frac{\pi_{q,l}^*}{\pi_{q',l'}^*} \right) + (1 - \pi_{q,l}^*) \log \left( \frac{1 - \pi_{q,l}^*}{1 - \pi_{q',l'}^*} \right) \right\} , \end{aligned}$$

where *maximum* and *minimum* are taken over the  $\{(q, l), (q', l') \mid \pi_{q,l}^* \neq \pi_{q',l'}^*\}$ . Then for each  $(i, j) \in D(z_{[n]}^*, z_{[n]})$  (see Lemma B.4),

$$0 < c^* \leq \pi_{z_i^*, z_j^*}^* \log \left( \frac{\pi_{z_i^*, z_j^*}^*}{\pi_{z_i, z_j}^*} \right) + (1 - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{1 - \pi_{z_i^*, z_j^*}^*}{1 - \pi_{z_i, z_j}^*} \right) \leq C^* .$$

Condition C3 implies that  $\left| \log \frac{\alpha_{z_i}^*}{\alpha_{z_i^*}^*} \right| \leq \log \frac{1-\gamma}{\gamma}$ .

Therefore  $\frac{1}{r(2n-r-1)} \sum_i \log \frac{\alpha_{z_i}^*}{\alpha_{z_i^*}^*} \leq \frac{r}{r(2n-r-1)} \log \frac{1-\gamma}{\gamma} \xrightarrow{n \rightarrow +\infty} 0$ .

Hence for  $n$  sufficiently high,

$$0 < c^* \leq -\frac{1}{r(2n-r-1)} \mathbb{E}^{Z=z^*} \left[ \log \frac{P^{X_{[n]}}(z_{[n]})}{P^{X_{[n]}}(z_{[n]}^*)} \right] \leq C^* .$$

□

**Lemma B.4.** For every  $z_{[n]}^*$  and  $z_{[n]}$ , let us define

$$D^* \left( z_{[n]}^*, z_{[n]} \right) = \left\{ (i, j) \mid i \neq j, \pi_{z_i^*, z_j^*}^* \neq \pi_{z_i, z_j}^* \right\} ,$$

and let  $\left| D^* \left( z_{[n]}^*, z_{[n]} \right) \right|$  denote its cardinality. Let us further assume that there are  $r > 0$  differences between  $z_{[n]}^*$  and  $z_{[n]}$ , that is  $\left\| z_{[n]}^* - z_{[n]} \right\|_0 = r$ , and that assumptions (A1) and (A3) hold. Then,

$$\left| D^* \left( z_{[n]}^*, z_{[n]} \right) \right| = r(2n - r - 1) .$$

*Proof of Lemma B.4.*

Let  $T_n$  denote the  $n \times n$  matrix defined by  $T_n = \{t_{i,j}\}_{1 \leq i,j \leq n}$ , where  $t_{i,j} = \pi_{z_i^*, z_j^*}^* - \pi_{z_i, z_j}^*$ . Let us assume that the  $r$  differences occur at the first  $r$  coordinates of  $z_{[n]}$ . Then,  $T_n$  is organized as four-blocks matrix, where diagonal coefficients are 0 as well as the bottom-right block, which is of size  $n-r \times n-r$ . Thus,  $\left| D^*(z_{[n]}^*, z_{[n]}) \right| = n^2 - [(n-r)^2 + r]$ , which concludes the proof. □

**B.5. Condition (A1) may be replaced by (A2) and (A4)**

For  $\left\| z_{[n]} - z_{[n]}^* \right\|_0 = r$ , there are at most  $r(2n-r-1)$  couples  $(i, j)$  such that  $\pi_{z_i, z_j}^* \neq \pi_{z_i^*, z_j^*}^*$ . When replacing (A2) by (A1), two situations arise. Either, the number of couples  $(i, j)$  such that  $\pi_{z_i, z_j}^* \neq \pi_{z_i^*, z_j^*}^*$  is  $o(rn)$ , or it is of the order  $rn$ . With Assumption (A4), the first setting cannot occur. Then, there exists  $C > 0$  such that  $Crn$  couples  $(i, j)$  satisfy  $\pi_{z_i, z_j}^* \neq \pi_{z_i^*, z_j^*}^*$ . One gets the same convergence rate (up to a constant) as under Assumption (A1).

### Appendix C: Proof of Proposition 3.5

*Proof of Proposition 3.5.* Let us first recall that

$$\begin{aligned}\phi_n(z_{[n]}, \pi) &:= \frac{1}{n(n-1)} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi) \ , \\ \Phi_n(z_{[n]}, \pi) &:= \mathbb{E} \left[ M_n(z_{[n]}, \pi) \mid z_{[n]} = z_{[n]}^* \right] \ .\end{aligned}$$

Then,

$$\begin{aligned}|\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| &= \rho_n \left| \sum_{i \neq j} \left( X_{ij} - \pi_{z_i^*, z_j^*} \right) \log \left[ \pi_{z_i, z_j} / (1 - \pi_{z_i, z_j}) \right] \right| \ , \\ &= \rho_n \left| \sum_{i \neq j} \xi_{ij} g(\pi_{z_i, z_j}) \right| \ ,\end{aligned}$$

where  $\rho_n = n(n-1)^{-1}$ ,  $\xi_{ij} = X_{ij} - \pi_{z_i^*, z_j^*}$ , and  $g(t) = \log(t/(1-t))$ ,  $t \in ]0, 1[$ .

With  $g_{i,j} = g(\pi_{z_i, z_j})$ , let us introduce

$$Z(g) = \left| \sum_{i \neq j} \xi_{ij} g_{ij} \right| \ ,$$

where  $g = \{g_{i,j}\}_{1 \leq i \neq j \leq n}$ .

Let  $\mathcal{P}_1$  denote the set of couples  $(z_{[n]}, \pi)$  such that the number of couples  $(i, j)$  such that  $g_{i,j} \neq 0$  is negligible with respect to  $n(n-1)$ , and  $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$ . Then,  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$  and for any  $\eta > 0$ ,

$$\begin{aligned}\mathbb{P} \left[ \sup_{\mathcal{P}} \rho_n Z(g) > \eta \mid Z = z^* \right] &= P^* \left[ \sup_{\mathcal{P}} \rho_n Z(g) > \eta \right] \\ &\leq P^* \left[ \sup_{\mathcal{P}_1} \rho_n Z(g) > \eta \right] + P^* \left[ \sup_{\mathcal{P}_2} \rho_n Z(g) > \eta \right] \ .\end{aligned}\tag{9}$$

On the one hand, the first probability in the right-hand side of (9) converges to 0  $\mathbb{P} - a.s.$ , since Lemma C.2 implies that

$$\sup_{\mathcal{P}_1} \rho_n Z(g) \xrightarrow[n \rightarrow \infty]{} 0, \quad \mathbb{P} - a.s. \ .$$

On the other hand, the second probability in the right-hand side of (9) can be dealt with thanks to Talagrand's inequality. For every  $z_{[n]}$  and  $\epsilon > 0$ , let us define

$$\Omega_n(\epsilon; z_{[n]}) = \left\{ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n Z(g) \leq (1 + \epsilon) \sqrt{\rho_n} \Lambda + \sqrt{\rho_n \Gamma^2 x_n} + (1/\epsilon + 1/3) \rho_n \Gamma x_n \right\} \ ,$$

where  $\Gamma$  and  $\Lambda$  are constants respectively defined in Lemma C.2 and Lemma C.3,  $\mathcal{P}_2(z_{[n]}) = \{\pi \mid (z_{[n]}, \pi) \in \mathcal{P}_2\}$ , and  $\{x_n\}_n$  is a sequence of positive real numbers to be chosen later. Talagrand's inequality (Theorem C.1) implies for any  $z_{[n]}$

$$P^* \left[ \Omega_n(\epsilon; z_{[n]})^c \right] \leq e^{-x_n} .$$

Combining the previous result with

$$P^* \left[ \sup_{\mathcal{P}_2} |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| > \eta \right] \leq \sum_{z_{[n]}} P^* \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n Z(g) > \eta \right] ,$$

it comes

$$\begin{aligned} & P^* \left[ \sup_{\mathcal{P}_2} |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| > \eta \right] \\ & \leq \sum_{z_{[n]}} P^* \left[ \left\{ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n Z(g) > \eta \right\} \cap \Omega_n(\epsilon; z_{[n]}) \right] + \sum_{z_{[n]}} e^{-x_n} \\ & \leq \sum_{z_{[n]}} P^* \left[ (1 + \epsilon) \sqrt{\rho_n} \Lambda + \sqrt{\rho_n \Gamma^2 x_n} + (1/\epsilon + 1/3) \rho_n \Gamma x_n > \eta \right] + \sum_{z_{[n]}} e^{-x_n} . \end{aligned}$$

Since  $z_{[n]}$  belongs to a set of cardinality at most  $Q^n$ , choosing  $x_n = n \log(n)$  entails the first sum is equal to 0 for large enough values of  $n$ , while the second sum converges to 0.

Finally, a quick inspection of the proof shows this convergence is uniform with respect to  $z_{[n]}^*$ , which provides the desired result.  $\square$

**Theorem C.1** (Talagrand). *Let  $\{Y_{ij}\}_{1 \leq i \neq j \leq n}$  denote independent centered random variables, and define*

$$\forall g \in \mathcal{G}, \quad Z(g) = \sum_{i \neq j} Y_{ij} g_{ij} ,$$

where  $\mathcal{G} \subset \mathbb{R}^{n^2}$ . Let us further assume that there exist  $b > 0$  and  $\sigma^2 > 0$  such that  $|Y_{ij} g_{ij}| \leq b$  for every  $(i, j)$ , and  $\sup_{g \in \mathcal{G}} \sum_{i \neq j} \text{Var}(Y_{ij} g_{ij}) \leq \sigma^2$ . Then, for every  $\epsilon > 0$ , and  $x > 0$ ,

$$\mathbb{P} \left[ \sup_g Z(g) \geq \mathbb{E} \left[ \sup_g Z(g) \right] + (1 + \epsilon) \sqrt{2\sigma^2 x} + b(1/\epsilon + 1/3)x \right] \leq e^{-x} .$$

*Proof.* A proof can be found in Massart (2007) (p.95, (4.50)).  $\square$

**Lemma C.2.** *With the same notation as Theorem C.1, Assumption (A3) entails that there exists  $\Gamma(\zeta) > 0$  only depending on  $\zeta$  such that*

$$\sup_{\mathcal{P}} \max_{i \neq j} |\xi_{ij} g_{ij}| \leq \Gamma, \quad \text{and} \quad \sup_{\mathcal{P}} \max_{i \neq j} \text{Var}(\xi_{ij} g_{ij}) \leq \frac{\Gamma^2}{4} .$$

*Proof.* If  $(z_{[n]}, \pi) \in \mathcal{P}$ , then

$$\left( \pi_{z_i, z_j} \in \{0, 1\} \Rightarrow \pi_{z_i^*, z_j^*}^* = \pi_{z_i, z_j} \right) \Rightarrow (g_{i,j} = 0) .$$

Then for every  $(z_{[n]}, \pi) \in \mathcal{P}$ , there exists  $\Gamma = \Gamma(\zeta) > 0$  (Assumption (A3)) such that

$$\forall i \neq j, \quad |\xi_{ij} g_{ij}| \leq \Gamma ,$$

for every  $(z_{[n]}, \pi) \in \mathcal{P}$ . This also leads to

$$\forall i \neq j, \quad \text{Var}(\xi_{ij} g_{ij}) \leq \Gamma^2/4 .$$

□

**Lemma C.3.** *With the same notation as Proposition 3.5, for every  $z_{[n]}$  such that  $(z_{[n]}, \pi) \in \mathcal{P}_2$ , there exists a constant  $\Lambda = \Lambda(\zeta) > 0$  (Assumption (A3)) such that*

$$\mathbb{E} \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \left| \sum_{i \neq j} (X_{ij} - \pi_{z_i^*, z_j^*}^*) g_{ij} \right| \middle| Z = z^* \right] \leq \Lambda \sqrt{n(n-1)}^{-1} .$$

*Proof of Lemma C.3.* Let  $\mathbb{E}^*[\cdot]$  denote the expectation given  $Z = z^*$ . Then,

$$\begin{aligned} & \mathbb{E}^* \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \left| \sum_{i \neq j} (X_{ij} - \pi_{z_i^*, z_j^*}^*) g_{ij} \right| \\ & \leq \mathbb{E}_{X, X'}^* \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \left| \sum_{i \neq j} (X_{ij} - X'_{ij}) g_{ij} \right| \right] , \end{aligned}$$

where the  $X'_{i,j}$ s are independent random variables with the same distribution as the  $X_{i,j}$ s. A symetrization argument based on Rademacher variables

$\{\epsilon_{i,j}\}_{1 \leq i \neq j \leq n}$  leads to

$$\begin{aligned} & \mathbb{E}^* \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \left| \sum_{i \neq j} \left( X_{ij} - \pi_{z_i^*, z_j^*}^* \right) g_{ij} \right| \\ & \leq 2 \mathbb{E}^* \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \mathbb{E}_\epsilon \left[ \left| \sum_{i \neq j} \epsilon_{ij} X_{ij} g_{ij} \right| \right] \right] , \end{aligned}$$

where  $\mathbb{E}_\epsilon[\cdot]$  denotes the expectation with respect to  $\epsilon_{i,j}$ s. Then, Jensen's inequality yields

$$\begin{aligned} & \mathbb{E}^* \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \left| \sum_{i \neq j} \left( X_{ij} - \pi_{z_i^*, z_j^*}^* \right) g_{ij} \right| \\ & \leq 2 \mathbb{E}^* \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \sqrt{\text{Var}_\epsilon \left[ \sum_{i \neq j} \epsilon_{ij} X_{ij} g_{ij} \right]} \right] \\ & \leq 2 \mathbb{E}^* \left[ \sup_{\mathcal{P}_2(z_{[n]})} \rho_n \sqrt{n(n-1)g_{ij}^2} \right] \\ & \leq \Lambda(\zeta) \sqrt{\rho_n} . \end{aligned}$$

□

*Justification of Talagrand in the proof of Proposition 3.5.* Let us first recall that

$$\begin{aligned} \phi_n(z_{[n]}, \pi) & := \frac{1}{n(n-1)} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi) , \\ \Phi_n(z_{[n]}, \pi) & := \mathbb{E} \left[ M_n(z_{[n]}, \pi) \mid z_{[n]} = z_{[n]}^* \right] . \end{aligned}$$

Let  $\Theta$  be the set of parameters  $(z_{[n]}, \pi)$  defined by assumptions of Section 2.2. In particular, any  $z_{[n]}$  is allowed whereas only values of  $\pi$  satisfying (A3) are allowed.

$$\begin{aligned} \phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi) & = \rho_n \sum_{i \neq j} \left( X_{ij} - \pi_{z_i^*, z_j^*}^* \right) \log \left[ \pi_{z_i, z_j} / (1 - \pi_{z_i, z_j}) \right] \\ & = \rho_n \sum_{q,l} \sum_{i \neq j} \left( X_{ij} - \pi_{q,l}^* \right) \mathbb{1}_{(z_i^*=q, z_j^*=l)} a_{i,j} , \end{aligned}$$

where  $a_{i,j} = \log \left[ \pi_{z_i, z_j} / (1 - \pi_{z_i, z_j}) \right]$ . For every  $(q, l)$ , Hoeffding's inequality and (A3) yields

$$P^* \left[ \left| \sum_{i \neq j} \left( X_{ij} - \pi_{q,l}^* \right) \mathbb{1}_{(z_i^*=q, z_j^*=l)} a_{i,j} \right| \geq t \right] \leq 2 \exp \left[ - \frac{t^2}{4N_{q,l}(z^*)\Gamma^2} \right] ,$$



with  $N_{q,l}(z_{[n]}^*) = \sum_{i \neq j} \mathbb{1}_{(z_i^*=q, z_j^*=l)}$ . Then,

$$\begin{aligned} & P^* [ |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| \geq t\rho_n ] \\ & \leq \sum_{q,l} P^* \left[ \left| \sum_{i \neq j} (X_{ij} - \pi_{q,l}^*) \mathbb{1}_{(z_i^*=q, z_j^*=l)} a_{i,j} \right| \geq t/Q^2 \right] \\ & \leq 2 \sum_{q,l} \exp \left[ -\frac{t^2}{4N_{q,l}(z_{[n]}^*)Q^4\Gamma^2} \right]. \end{aligned}$$

Thus for every  $x > 0$ ,

$$P^* [ |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| \geq x ] \leq 2 \sum_{q,l} \exp \left[ -\frac{x^2}{4\rho_n^2 N_{q,l}(z_{[n]}^*)Q^4\Gamma^2} \right].$$

Furthermore,

$$\begin{aligned} & \mathbb{P} [ |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| \geq x ] \\ & = \sum_{z_{[n]}} \mathbb{P} [ |\phi_n(z_{[n]}, \pi) - \Phi_n(z_{[n]}, \pi)| \geq x \mid Z_{[n]} = z_{[n]} ] \mathbb{P} [ Z_{[n]} = z_{[n]} ] \\ & \leq \sum_{z_{[n]}} 2 \sum_{q,l} \exp \left[ -\frac{x^2}{4\rho_n^2 N_{q,l}(z_{[n]})Q^4\Gamma^2} \right] \mathbb{P} [ Z_{[n]} = z_{[n]} ] \\ & = 2 \sum_{q,l} \mathbb{E} \left[ \exp \left[ -\frac{x^2}{4\rho_n^2 N_{q,l}(Z_{[n]})Q^4\Gamma^2} \right] \right]. \end{aligned} \tag{10}$$

Then, random variables  $N_{q,l}(Z_{[n]})$  follow a binomial distribution  $\mathcal{B}(n, \pi_{q,l}^*)$ . It entails that

$$\mathbb{P} [ |N_{q,l}(Z_{[n]}) - \alpha_q^* \alpha_l^* \rho_n^{-1}| \geq t ] \leq 2 \exp \left[ -\frac{t^2}{\rho_n^{-1}} \right],$$

hence for every  $u > 0$

$$\mathbb{P} [ |\rho_n N_{q,l}(Z_{[n]}) - \alpha_q^* \alpha_l^*| \geq u ] \leq 2 \exp \left[ -\frac{u^2}{\rho_n} \right].$$

Plugging this in (10), it comes for every  $u > 0$

$$\begin{aligned} & \mathbb{E} \left[ \exp \left[ -\frac{x^2}{4\rho_n^2 N_{q,l}(Z_{[n]})Q^4\Gamma^2} \right] \right] \\ & \leq 2 \exp \left[ -\frac{u^2}{\rho_n} \right] + \mathbb{E} \left[ \exp \left[ -\frac{x^2}{4\rho_n (u + \alpha_q^* \alpha_l^*) Q^4\Gamma^2} \right] \right] \\ & \leq 2 \exp \left[ -\frac{u^2}{\rho_n} \right] + \exp \left[ -\frac{x^2}{4\rho_n (u + 1) Q^4\Gamma^2} \right]. \end{aligned}$$

It enables to conclude that for every  $x > 0$ ,

$$\begin{aligned} & \mathbb{P} \left[ \left| \phi_n (z_{[n]}, \pi) - \Phi_n (z_{[n]}, \pi) \right| \geq x \right] \\ & \leq 2Q^2 \exp \left[ -\frac{x^2}{4\rho_n (u+1) Q^4 \Gamma^2} \right] + 4Q^2 \exp \left[ -\frac{u^2}{\rho_n} \right] . \end{aligned}$$

□

**Appendix D: Theorem 3.6**

**Notation** Let  $\alpha^*$  and  $\pi^*$  be the true values of  $\alpha$  and  $\pi$ ,  $\mathcal{A}$  be the set of stochastic matrices of size  $Q$  given by  $\mathcal{A} = \{A = (a_{k,l})_{1 \leq k,l \leq Q} \mid a_{k,l} \geq 0, \sum_{l=1}^Q a_{k,l} = 1\}$ .

Furthermore, let us introduce the following quantities

$$\begin{aligned} \phi_n(\pi, z_{[n]}) &= \frac{1}{n(n-1)} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi), \quad \widehat{z}_{[n]}(\pi) = \operatorname{Argmax}_z \phi_n(z_{[n]}, \pi), \\ \Phi_n(\pi, z_{[n]}) &= \frac{1}{n(n-1)} \sum_{i,j \neq i} \pi_{z_i^* z_j^*}^* \log \pi_{z_i, z_j} + (1 - \pi_{z_i^* z_j^*}^*) \log(1 - \pi_{z_i, z_j}), \\ \widetilde{z}_{[n]}(\pi) &= \operatorname{Argmax}_z \Phi_n(z_{[n]}, \pi), \\ M_n(\alpha, \pi) &= \frac{1}{n(n-1)} \mathcal{L}_2(X_{[n]}; \alpha, \pi), \\ \mathbb{M}(\pi, A) &= \sum_{q,l} \alpha_q^* \alpha_l^* \sum_{q'l'} a_{q,q'} a_{l,l'} [\pi_{q,l}^* \log \pi_{q'l'} + (1 - \pi_{q,l}^*) \log(1 - \pi_{q'l'})], \\ \bar{A}_\pi &= \operatorname{Argmax}_{A \in \mathcal{A}} \mathbb{M}(\pi, A), \quad \mathbb{M}(\pi) = \mathbb{M}(\pi, \bar{A}_\pi). \end{aligned}$$

Note that  $\bar{A}_{\pi^*} = I_Q$  and  $\mathbb{M}(\pi^*) = \sum_{q,l} \alpha_q^* \alpha_l^* \mathbb{H}_{q,l}^*$ , where  $\mathbb{H}_{q,l}^* = \pi_{q,l}^* \log \pi_{q,l}^* + (1 - \pi_{q,l}^*) \log(1 - \pi_{q,l}^*)$ .

**Proof**

First, let us prove that:  $\forall \eta > 0, \sup_{d(\pi, \pi^*) \geq \eta} \mathbb{M}(\pi) < \mathbb{M}(\pi^*)$ . In the sequel, let  $(\bar{a}_{q,l})_{1 \leq q,l \leq Q}$  denote coefficient of  $\bar{A}_\pi$ . Without any further indication,  $\bar{a}_{q,l}$  refers to the matrix  $\pi$ . One has

$$\begin{aligned} &\mathbb{M}(\pi) - \mathbb{M}(\pi^*) \\ &= \sum_{q,l} \alpha_q^* \alpha_l^* \sum_{q'l'} \bar{a}_{q,q'} \bar{a}_{l,l'} [\pi_{q,l}^* \log \frac{\pi_{q'l'}}{\pi_{q,l}^*} + (1 - \pi_{q,l}^*) \log \frac{1 - \pi_{q'l'}}{1 - \pi_{q,l}^*}] \\ &= - \sum_{q,l} \alpha_q^* \alpha_l^* \sum_{q'l'} \bar{a}_{q,q'} \bar{a}_{l,l'} K(\pi_{q,l}^*, \pi_{q'l'}) . \end{aligned}$$

Since  $\{\pi \mid d(\pi, \pi^*) \geq \eta, (A2)\}$  is a compact set, there exists  $\pi^0 \neq \pi^*$  satisfying (A2) such that

$$\sup_{d(\pi, \pi^*) \geq \eta} \mathbb{M}(\pi) - \mathbb{M}(\pi^*) = \mathbb{M}(\pi^0) - \mathbb{M}(\pi^*) < 0 .$$

Otherwise for every  $(q, l)$ ,  $\sum_{q'l'} \bar{a}_{q,q'} \bar{a}_{l,l'} K(\pi_{q,l}^*, \pi_{q'l'}^0) = 0$  would entail that for every  $(q', l')$ ,  $\bar{a}_{q,q'} \bar{a}_{l,l'} K(\pi_{q,l}^*, \pi_{q'l'}^0) = 0$ . It implies that there exists  $f : \{1, \dots, Q\} \rightarrow \{1, \dots, Q\}$ ,  $\pi_{q,l}^* = \pi_{f(q), f(l)}^0$ , which is excluded since  $\pi^0 \neq \pi^*$  up to label switching.

Second, let us prove that  $\|M_n - \mathbb{M}\|_{\Theta_n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0$ . Set

$$|M_n(\alpha, \pi) - \mathbb{M}(\pi)| \leq |M_n(\alpha, \pi) - \phi_n(\pi, \widehat{z}_{[n]})| \quad (11)$$

$$+ |\phi_n(\pi, \widehat{z}_{[n]}) - \Phi_n(\pi, \widetilde{z}_{[n]})| \quad (12)$$

$$+ |\Phi_n(\pi, \widetilde{z}_{[n]}) - \mathbb{M}(\pi)| . \quad (13)$$

The three terms of the preceding expression are controlled as follows:

- upper bound of (11): Lemma F.2 implies that  $\mathbb{P} - a.s.$ ,

$$\begin{aligned} \sup_{\alpha, \pi} |M_n(\alpha, \pi) - \phi_n(\pi, \widehat{z}_{[n]})| &= \sup_{\alpha, \pi} \frac{|\mathcal{L}_2(X_{[n]}; \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \pi, \widehat{z}_{[n]})|}{n(n-1)} \\ &\leq \frac{\log(1/\gamma)}{n-1} \xrightarrow[n \rightarrow \infty]{} 0 , \end{aligned}$$

- upper bound of (12): the definitions of  $\widehat{z}_{[n]}$  and  $\widetilde{z}_{[n]}$  imply that  $\phi_n(\pi, \widehat{z}_{[n]}) \geq \phi_n(\pi, \widetilde{z}_{[n]})$  and  $\Phi_n(\pi, \widehat{z}_{[n]}) \leq \Phi_n(\pi, \widetilde{z}_{[n]})$ , hence

$$|\phi_n(\pi, \widehat{z}_{[n]}) - \Phi_n(\pi, \widetilde{z}_{[n]})| \leq \sup_{\pi, z_{[n]}} |\phi_n(\pi, z_{[n]}) - \Phi_n(\pi, z_{[n]})| .$$

Theorem 3.5 implies that for every  $z_{[n]}^*$  and  $\eta > 0$ ,

$$\mathbb{P} \left( \sup_{\pi, z_{[n]}} |\phi_n(\pi, z_{[n]}) - \Phi_n(\pi, z_{[n]})| > \eta \mid Z_{[n]} = z_{[n]}^* \right) \xrightarrow[n \rightarrow +\infty]{} 0 ,$$

where the rate of convergence does not depend on  $z_{[n]}^*$ . Therefore

$$\mathbb{P} \left( \sup_{\pi} |\phi_n(\pi, \widehat{z}_{[n]}) - \Phi_n(\pi, \widetilde{z}_{[n]})| > \eta \right) \xrightarrow[n \rightarrow +\infty]{} 0.$$

- upper bound of (13):  $\Phi_n(\pi, z_{[n]})$  can be expressed as:

$$\begin{aligned} &\Phi_n(\pi, z_{[n]}) \\ &= \sum_{qq'l'} \frac{N_{qq'}(z_{[n]})N_{l'l}(z_{[n]})}{n(n-1)} [\pi_{q,l}^* \log \pi_{q'l'} + (1 - \pi_{q,l}^*) \log(1 - \pi_{q'l'})] , \quad (14) \end{aligned}$$

where  $N_{qq'}(z_{[n]}) = \#\{i \mid z_i^* = q, \text{ and } z_i = q'\}$ .

Let  $\widetilde{N}_{qq'}(\pi) = N_{qq'}(\widetilde{z}_{[n]}(\pi))$ ,  $N_q^* = \#\{i \mid z_i^* = q\}$ ,  $\widetilde{a}_{qq'}(\pi) = \frac{\widetilde{N}_{qq'}(\pi)}{N_q^*}$ , and  $\widetilde{A}_\pi$  the stochastic matrix of  $\widetilde{a}_{qq'}(\pi)$ . Coefficient  $\widetilde{a}_{qq'}(\pi)$  yield the proportion of vertices from class  $q$  attributed to class  $q'$  by  $z_{[n]}$ . Note that (14) shows that  $\Phi_n(\pi, z_{[n]})$  only depends on  $z_{[n]}$  through the matrix  $\widetilde{A}_\pi$ . Therefore, one uses the notation  $\Phi_n(\pi, \widetilde{A}_\pi)$  in place of  $\Phi_n(\pi, z_{[n]})$ .

Definitions of  $\widetilde{A}_\pi$  and  $\bar{A}_\pi$  imply that  $\Phi_n(\pi, \widetilde{A}_\pi) \geq \Phi_n(\pi, \bar{A}_\pi)$  and  $\mathbb{M}(\pi) = \mathbb{M}(\pi, \bar{A}_\pi) \geq \mathbb{M}(\pi, \widetilde{A}_\pi)$ . Therefore,

1.  $\Phi_n(\pi, \tilde{A}_\pi) \geq \mathbb{M}(\pi)$   
 $\Rightarrow 0 \leq \Phi_n(\pi, \tilde{A}_\pi) - \mathbb{M}(\pi) \leq \Phi_n(\pi, \tilde{A}_\pi) - \mathbb{M}(\pi, \tilde{A}_\pi),$
2.  $\Phi_n(\pi, \tilde{A}_\pi) \leq \mathbb{M}(\pi)$   
 $\Rightarrow 0 \leq \mathbb{M}(\pi) - \Phi_n(\pi, \tilde{A}_\pi) \leq \mathbb{M}(\pi, \bar{A}_\pi) - \Phi_n(\pi, \bar{A}_\pi).$

Then,

$$\left| \Phi_n(\pi, \tilde{A}_\pi) - \mathbb{M}(\pi) \right| \leq \max_{A \in \mathcal{A}} |\Phi_n(\pi, A) - \mathbb{M}(\pi, A)| .$$

Moreover for every  $A \in \mathcal{A}$ ,

$$\begin{aligned} \Phi_n(\pi, A) - \mathbb{M}(\pi, A) = \\ \sum_{qq'l'l'} \left[ \frac{N_q^* N_l^*}{n(n-1)} - \alpha_q^* \alpha_l^* \right] a_{qq'} a_{ll'} \left[ \pi_{q,l}^* \log \pi_{q'l'} + (1 - \pi_{q,l}^*) \log(1 - \pi_{q'l'}) \right] . \end{aligned}$$

Since any  $\pi_{q'l'} \in \{0, 1\}$  such that  $\pi_{q,l}^* \neq \pi_{q'l'}$  is excluded, (A3) provides

$$\left| \pi_{q,l}^* \log \pi_{q'l'} + (1 - \pi_{q,l}^*) \log(1 - \pi_{q'l'}) \right| \leq \Delta(\zeta) < +\infty ,$$

where  $\Delta(\zeta) > 0$  is independent of  $\pi$  and  $q$ , and only depends on  $\zeta$  from Assumption (A3).

Then, the strong law of large numbers applied to  $N_q^*$  entails that  $\sup_{\pi} \left\{ |\Phi_n(\pi, \tilde{z}_{[n]}(\pi)) - \mathbb{M}(\pi)| \right\} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \mathbb{P} - a.s. .$

**Appendix E: Proof of Proposition 3.8**

*Proof of Proposition 3.8.*

$$\begin{aligned} & \log \frac{\widehat{P}^{X[n]}(z_{[n]})}{\widehat{P}^{X[n]}(z_{[n]}^*)} \\ &= \sum_{i \neq j} \left\{ X_{ij} \log \left( \frac{\widehat{\pi}_{z_i, z_j}}{\widehat{\pi}_{z_i^*, z_j^*}} \right) + (1 - X_{ij}) \log \left( \frac{1 - \widehat{\pi}_{z_i, z_j}}{1 - \widehat{\pi}_{z_i^*, z_j^*}} \right) \right\} + \sum_i \log \frac{\widehat{\alpha}_{z_i}}{\widehat{\alpha}_{z_i^*}}. \end{aligned}$$

Notice that

$$\log \widehat{\pi}_{z_i, z_j} = \log \pi_{z_i, z_j}^* + \log \left[ 1 + \frac{\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*}{\pi_{z_i, z_j}^*} \right]$$

and

$$\log(1 - \widehat{\pi}_{z_i, z_j}) = \log(1 - \pi_{z_i, z_j}^*) + \log \left[ 1 - \frac{\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*}{1 - \pi_{z_i, z_j}^*} \right].$$

Therefore,

$$\begin{aligned} & X_{ij} \log \widehat{\pi}_{z_i, z_j} + (1 - X_{ij}) \log(1 - \widehat{\pi}_{z_i, z_j}) \\ &= X_{ij} \log \pi_{z_i, z_j}^* + (1 - X_{ij}) \log(1 - \pi_{z_i, z_j}^*) \\ &+ \log \left[ 1 + \frac{(\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*)(X_{ij} - \pi_{z_i, z_j}^*)}{\pi_{z_i, z_j}^*(1 - \pi_{z_i, z_j}^*)} \right], \end{aligned}$$

and thus

$$\begin{aligned} & \log \frac{\widehat{P}^{X[n]}(z_{[n]})}{\widehat{P}^{X[n]}(z_{[n]}^*)} \\ &= \sum_{i \neq j} \left\{ X_{ij} \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - X_{ij}) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \right\} + \sum_i \log \frac{\widehat{\alpha}_{z_i}}{\widehat{\alpha}_{z_i^*}} \\ &+ \sum_{i \neq j} \log \left[ 1 + \frac{(\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*)(X_{ij} - \pi_{z_i, z_j}^*)}{\pi_{z_i, z_j}^*(1 - \pi_{z_i, z_j}^*)} \right] \\ &- \sum_{i \neq j} \log \left[ 1 + \frac{(\widehat{\pi}_{z_i^*, z_j^*} - \pi_{z_i^*, z_j^*}^*)(X_{ij} - \pi_{z_i^*, z_j^*}^*)}{\pi_{z_i^*, z_j^*}^*(1 - \pi_{z_i^*, z_j^*}^*)} \right] \\ &= T_1 + T_2 - T_3. \end{aligned}$$

In the following, one successively upper bound  $T_1$ ,  $T_2$ , and  $T_3$ .

**Upper bounding  $T_1$** 

The magnitude of  $T_1$  is given by a similar argument to that in the proof of Theorem 3.1. Let us consider

$$\begin{aligned}
 T_1 &= \sum_{i \neq j} \left\{ X_{i,j} \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - X_{i,j}) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \right\} + \sum_i \log \frac{\widehat{\alpha}_{z_i}}{\widehat{\alpha}_{z_i^*}} \\
 &= \sum_{i \neq j} \left\{ (X_{i,j} - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \frac{1 - \pi_{z_i^*, z_j^*}^*}{1 - \pi_{z_i, z_j}^*} \right) + \sum_i \log \frac{\widehat{\alpha}_{z_i}}{\widehat{\alpha}_{z_i^*}} \right\} \\
 &\quad + \sum_{i \neq j} \left\{ \pi_{z_i^*, z_j^*}^* \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \right) + (1 - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{1 - \pi_{z_i, z_j}^*}{1 - \pi_{z_i^*, z_j^*}^*} \right) \right\} \\
 &= T_{1,1} + T_{1,2} .
 \end{aligned}$$

Then,

$$P^* [T_1 > t] = P^* [T_{1,1} + T_{1,2} > t] .$$

Since (with the assumption that every coefficients of  $\pi^*$  are different)

$$\begin{aligned}
 &T_{1,2} [r(2n - r - 1)]^{-1} \\
 &\leq \max_{(q,l) \neq (q',l'), \pi_{q,l}^* \notin \{0,1\}} \pi_{q,l}^* \log \left( \frac{\pi_{q',l'}^*}{\pi_{q,l}^*} \right) + (1 - \pi_{q,l}^*) \log \left( \frac{1 - \pi_{q',l'}^*}{1 - \pi_{q,l}^*} \right) \\
 &= K(\pi^*) = K^* < 0 ,
 \end{aligned}$$

it comes

$$P^* [T_1 > t] \leq P^* [T_{1,1} + K^* [r(2n - r - 1)] > t] .$$

Similarly, Assumption (A4) yields a constant  $C(\gamma) > 0$  such that

$$\sum_i \log \frac{\widehat{\alpha}_{z_i}}{\widehat{\alpha}_{z_i^*}} \leq nC(\gamma) ,$$

which entails

$$\begin{aligned}
 &P^* [T_1 > t] \\
 &\leq P^* \left[ \sum_{i \neq j} (X_{i,j} - \pi_{z_i^*, z_j^*}^*) \log \left( \frac{\pi_{z_i, z_j}^*}{\pi_{z_i^*, z_j^*}^*} \frac{1 - \pi_{z_i^*, z_j^*}^*}{1 - \pi_{z_i, z_j}^*} \right) > t - nC(\gamma) - r(2n - r - 1)K^* \right] .
 \end{aligned}$$

Another use of Hoeffding's inequality associated with (A3) provides a constant  $C_\zeta > 0$  such that

$$P^* [T_1 > t] \leq \exp \left[ - \frac{(t - nC(\gamma) - r(2n - r - 1)K^*)^2}{r(2n - r - 1)C_\zeta} \right] .$$

**Upper bounding  $T_2$**

Since  $\widehat{\pi} = \pi^* + o_{\mathbb{P}}(1/n)$

$$\begin{aligned} T_2 &= \sum_{i \neq j} \log \left[ 1 + \frac{(\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*)(X_{i,j} - \pi_{z_i, z_j}^*)}{\pi_{z_i, z_j}^*(1 - \pi_{z_i, z_j}^*)} \right] \\ &= \sum_{i \neq j} \frac{(\widehat{\pi}_{z_i, z_j} - \pi_{z_i, z_j}^*)(X_{i,j} - \pi_{z_i, z_j}^*)}{\pi_{z_i, z_j}^*(1 - \pi_{z_i, z_j}^*)} (1 + o_{\mathbb{P}}(1)) . \end{aligned}$$

The terms of this sum have to be gathered in another way to yield the magnitude of  $T_2$ :

$$\begin{aligned} |T_2| &\leq \|\widehat{\pi} - \pi^*\|_{\infty} \sum_{q,l} K_{q,l} (1 + o_{\mathbb{P}}(1)) \left\{ \left| \sum_{q',l'} \sum_{\substack{i \neq j \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} (X_{i,j} - \pi_{q',l'}^*) \right| \right. \\ &\quad \left. + \left| \sum_{q',l'} N_{q',l'}^{q,l} (\pi_{q',l'}^* - \pi_{q,l}^*) \right| \right\} , \end{aligned}$$

where  $N_{q',l'}^{q,l} = \sum_{i \neq j} \mathbb{1}_{(z_i^* = q', z_j^* = l')} \mathbb{1}_{(z_i = q, z_j = l)}$ . Let us now introduce an event of large probability:

$$\Omega_n = \{ \|\widehat{\pi} - \pi^*\|_{\infty} \leq 1/n \} . \quad (15)$$

Note that by assumption,  $\mathbb{P}[\Omega_n^c] \xrightarrow[n \rightarrow \infty]{} 0$ . Hence,

$$\begin{aligned} &P^* [\Omega_n \cap \{|T_2| > t\}] \\ &\leq P^* \left[ \sum_{q,l} K_{q,l} (1 + o_{\mathbb{P}}(1)) \left\{ \left| \sum_{q',l'} \sum_{\substack{i \neq j \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} (X_{i,j} - \pi_{q',l'}^*) \right| \right\} > nt/2 \right] \\ &+ P^* \left[ 1/n \sum_{q,l} K_{q,l} (1 + o_{\mathbb{P}}(1)) \left\{ \left| \sum_{q',l'} N_{q',l'}^{q,l} (\pi_{q',l'}^* - \pi_{q,l}^*) \right| \right\} > t/2 \right] . \end{aligned}$$



Let us deal with the first term.

$$\begin{aligned}
 & P^* \left[ \sum_{q,l} K_{q,l} (1 + o_{\mathbb{P}}(1)) \left\{ \sum_{\substack{q',l' \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} \sum_{i \neq j} (X_{i,j} - \pi_{q',l'}^*) \right\} > nt/2 \right] \\
 & \leq \sum_{q,l} P^* \left[ (1 + o_{\mathbb{P}}(1)) \left| \sum_{\substack{q',l' \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} \sum_{i \neq j} (X_{i,j} - \pi_{q',l'}^*) \right| > nt/(2K_{q,l}Q^2) \right] \\
 & \leq Q^2 \max_{q,l} P^* \left[ \left| \sum_{\substack{q',l' \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} \sum_{i \neq j} (X_{i,j} - \pi_{q',l'}^*) \right| > nt/(4K_{q,l}Q^2) \right].
 \end{aligned}$$

A straightforward use of Hoeffding inequality yields

$$\begin{aligned}
 & P^* \left[ \left| \sum_{\substack{q',l' \\ (z_i^*, z_j^*) = (q', l') \\ (z_i, z_j) = (q, l)}} \sum_{i \neq j} (X_{i,j} - \pi_{q',l'}^*) \right| > nt/(4K_{q,l}Q^2) \right] \\
 & \leq \exp \left[ -\frac{1}{(4K_{q,l}Q^2)^2} \frac{n^2 t^2}{N_{q,l}} \right].
 \end{aligned}$$

For the second term, it comes that

$$\begin{aligned}
 & P^* \left[ 1/n \sum_{q,l} K_{q,l} (1 + o_{\mathbb{P}}(1)) \left\{ \left| \sum_{q',l'} N_{q',l'}^{q,l} (\pi_{q',l'}^* - \pi_{q,l}^*) \right| \right\} > t/2 \right] \\
 & \leq Q^2 \max_{q,l} P^* \left[ \left| \sum_{q',l'} N_{q',l'}^{q,l} (\pi_{q',l'}^* - \pi_{q,l}^*) \right| > nt/(4K_{q,l}Q^2) \right] \\
 & \leq Q^2 \max_{q,l} P^* \left[ \max_{(q',l')} |\pi_{q,l}^* - \pi_{q',l'}^*| N_{q,l} > nt/(4K_{q,l}Q^2) \right].
 \end{aligned}$$

Hence,

$$\begin{aligned}
 P^* [\Omega_n \cap \{|T_2| > t\}] & \leq Q^2 \max_{q,l} \exp \left[ -\frac{1}{(4K_{q,l}Q^2)^2} \frac{n^2 t^2}{N_{q,l}} \right] \\
 & \quad + Q^2 \max_{q,l} P^* \left[ \max_{(q',l')} |\pi_{q,l}^* - \pi_{q',l'}^*| > nt/(4N_{q,l}K_{q,l}Q^2) \right].
 \end{aligned}$$

### Upper bounding $T_3$

Since  $\hat{\pi} = \pi^* + o_{\mathbb{P}}(1/n)$ , it comes

$$\begin{aligned}
 & \log \left[ 1 + \frac{(\hat{\pi}_{z_i^*, z_j^*} - \pi_{z_i^*, z_j^*}^*)(X_{i,j} - \pi_{z_i^*, z_j^*}^*)}{\pi_{z_i^*, z_j^*}^*(1 - \pi_{z_i^*, z_j^*}^*)} \right] \\
 & = \frac{(\hat{\pi}_{z_i^*, z_j^*} - \pi_{z_i^*, z_j^*}^*)(X_{i,j} - \pi_{z_i^*, z_j^*}^*)}{\pi_{z_i^*, z_j^*}^*(1 - \pi_{z_i^*, z_j^*}^*)} (1 + o_{\mathbb{P}}(1)).
 \end{aligned}$$

Let us recall that since the likelihood is finite, the denominator is not null (and even bounded away from 0 and 1). Then,  $T_3$  can be written as

$$T_3 = \sum_{q,l} \frac{(Y_{q,l} - \pi_{q,l}^* N_{q,l})(\hat{\pi}_{q,l} - \pi_{q,l}^*)}{\pi_{q,l}^*(1 - \pi_{q,l}^*)} (1 + o_{\mathbb{P}}(1)).$$

Using the event  $\Omega_n$  defined by (15), one has

$$\begin{aligned}
 P^* [\Omega_n \cap \{|T_3| > t\}] &= P^* \left[ \left| \sum_{q,l} \frac{(Y_{q,l} - \pi_{q,l}^* N_{q,l})1/n}{\pi_{q,l}^*(1 - \pi_{q,l}^*)} (1 + o_{P^*}(1)) \right| > t \right] \\
 &\leq \sum_{q,l} P^* \left[ \left| \frac{(Y_{q,l} - \pi_{q,l}^* N_{q,l})}{n\pi_{q,l}^*(1 - \pi_{q,l}^*)} (1 + o_{P^*}(1)) \right| > t/Q^2 \right] \\
 &\leq Q^2 \max_{q,l} P^* \left[ \left| \frac{(Y_{q,l} - \pi_{q,l}^* N_{q,l})}{n\pi_{q,l}^*(1 - \pi_{q,l}^*)} (1 + o_{P^*}(1)) \right| > t/Q^2 \right] \\
 &\leq Q^2 \max_{q,l} P^* \left[ \frac{|Y_{q,l} - \pi_{q,l}^* N_{q,l}|}{\pi_{q,l}^*(1 - \pi_{q,l}^*)} > nt/(2Q^2) \right] \\
 &\leq Q^2 \max_{q,l} P^* \left[ \frac{1}{N_{q,l}} \frac{|Y_{q,l} - \pi_{q,l}^* N_{q,l}|}{\pi_{q,l}^*(1 - \pi_{q,l}^*)} > \frac{nt}{2Q^2 N_{q,l}} \right] \\
 &\leq 2Q^2 \max_{q,l} \left\{ \exp \left[ -N_{q,l} K^* \left( \frac{nt}{N_{q,l}} \right)^2 \right] \right\} \\
 &\leq 2Q^2 \max_{q,l} \left\{ \exp \left[ -K^* \frac{n^2 t^2}{N_{q,l}} \right] \right\} .
 \end{aligned}$$

### Gathering $T_1$ -, $T_2$ -, and $T_3$ -upper bounds

Then, for any  $\epsilon > 0$

$$\begin{aligned}
 &P^* \left[ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > \epsilon \right] \\
 &\leq P^* \left[ \left\{ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > \epsilon \right\} \cap \Omega_n \right] + P^* [\Omega_n^c] .
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 & P^* \left[ \left\{ \sum_{z_{[n]} \neq z_{[n]}^*} \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > \epsilon \right\} \cap \Omega_n \right] \\
 & \leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} P^* \left[ \left\{ \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > \epsilon / (n^{r+1} Q^r) \right\} \cap \Omega_n \right] \\
 & \leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} P^* \left[ \left\{ \log \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > -(r+1) \log n - r \log Q + \log \epsilon \right\} \cap \Omega_n \right] \\
 & \leq \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} P^* \left[ \left\{ \log \frac{\widehat{P}^{X_{[n]}}(z_{[n]})}{\widehat{P}^{X_{[n]}}(z_{[n]}^*)} > -5r \log n \right\} \cap \Omega_n \right] \quad (n \gg 1) \\
 & = \sum_{r=1}^n \sum_{\|z_{[n]} - z_{[n]}^*\|_0 = r} P^* [ \{T_1 + T_2 - T_3 > -5r \log n\} \cap \Omega_n ] .
 \end{aligned}$$

It remains to deal with  $P^* [ \{T_1 + T_2 - T_3 > -5r \log n\} \cap \Omega_n ]$ :

$$\begin{aligned}
 & P^* [ \{T_1 + T_2 - T_3 > -5r \log n\} \cap \Omega_n ] \\
 & \leq P^* [ \{T_1 + T_2 - T_3 > -5r \log n\} \cap \Omega_n \cap \{|T_3| \leq r \log n\} ] \\
 & \quad + P^* [ \{|T_3| > r \log n\} \cap \Omega_n ] \\
 & \leq P^* [ \{T_1 + T_2 > -6r \log n\} \cap \Omega_n ] + P^* [ \{|T_3| > r \log n\} \cap \Omega_n ] \\
 & \leq P^* [ T_1 > -7r \log n ] + P^* [ \{|T_2| > r \log n\} \cap \Omega_n ] \\
 & \quad + P^* [ \{|T_3| > r \log n\} \cap \Omega_n ] .
 \end{aligned}$$

Combining the previous bounds of  $T_1$ ,  $T_2$ , and  $T_3$  with the above inequality, it

comes

$$\begin{aligned}
 & P^* [\{T_1 + T_2 - T_3 > -r \log n\} \cap \Omega_n] \\
 \leq & \exp \left[ -\frac{[nC(\gamma) + 7r \log n + r(2n - r - 1)K^*]^2}{r(2n - r - 1)C_\zeta} \right] \\
 & + 2Q^2 \max_{q,l} \exp \left[ -\frac{1}{(4K_{q,l}Q^2)^2} \frac{n^2 (r \log n)^2}{N_{q,l}} \right] \\
 & + 2Q^2 \max_{q,l} P^* \left[ \zeta > \frac{1}{4K_{q,l}Q^2} \frac{nr \log n}{N_{q,l}} \right] \\
 & + 2Q^2 \max_{q,l} \exp \left[ -K^* \frac{n^2 (r \log n)^2}{N_{q,l}} \right] \\
 \leq & \exp \left[ -r(2n - r - 1) \frac{K^{*2}}{C_\zeta} \right] \\
 & + 2Q^2 \max_{q,l} \exp \left[ -\frac{1}{(4K_{q,l}Q^2)^2} \frac{n^2 r (\log n)^2}{(2n - r - 1)} \right] \\
 & + 2Q^2 \max_{q,l} P^* \left[ \zeta > \frac{1}{4K_{q,l}Q^2} \frac{n \log n}{(2n - r - 1)} \right] \\
 & + 2Q^2 \max_{q,l} \exp \left[ -K^* \frac{n^2 r (\log n)^2}{(2n - r - 1)} \right] .
 \end{aligned}$$

One gets that for  $n$  large enough,

$$\begin{aligned}
 & P^* [\{T_1 + T_2 - T_3 > -r \log n\} \cap \Omega_n] \\
 \leq & \exp \left[ -r(2n - r - 1) \frac{K^{*2}}{C_\zeta} \right] + 4Q^2 \exp \left[ -\bar{K}r(\log n)^2 \right] \\
 \leq & \bar{K}_2 \exp \left[ -\bar{K}r(\log n)^2 \right] .
 \end{aligned}$$

The end of the proof follows from the same argument as in the proof of Theorem 3.1.  $\square$

**Appendix F: Proof of Theorem 4.2**

**Lemma F.1.** *Let  $\hat{z}_{[n]} = \hat{z}_{[n]}(\pi) = \text{Argmax}_{z_{[n]}} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)$ . For every  $X_{[n]} \in \mathcal{X}_n$ ,  $(\alpha, \pi) \in \Theta$ , and  $\tau_{[n]} \in \mathcal{S}_n$ , it comes that*

$$\mathcal{J}(X_{[n]}; \tau_{[n]}, \alpha, \pi) \leq \mathcal{L}_2(X_{[n]}; \alpha, \pi) \leq \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi) .$$

*Proof of Lemma F.1.* The first inequality comes from the definition of  $\mathcal{J}$ . The second one results from  $\hat{z}_{[n]}(\pi) = \text{Argmax}_{z_{[n]}} \mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)$ .

For any  $(\alpha, \pi)$ ,

$$\begin{aligned} \mathcal{L}_2(X_{[n]}; \alpha, \pi) &= \log \left\{ \sum_{z_{[n]} \in \mathcal{Z}_n} e^{\mathcal{L}_1(X_{[n]}; z_{[n]}, \pi)} P_{Z_{[n]}}(z_{[n]}) \right\} \\ &\leq \log \left\{ e^{\mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)} \sum_{z_{[n]} \in \mathcal{Z}_n} P_{Z_{[n]}}(z_{[n]}) \right\} \\ &\leq \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi) . \end{aligned}$$

□

**Lemma F.2.** *Lemma F.1 and Assumption (A4) entail that there exists  $0 < \gamma < 1$  such that for every  $(\alpha, \pi)$ ,*

$$\begin{aligned} |\mathcal{L}_2(X_{[n]}; \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)| &\leq n \log(1/\gamma) , \\ |\mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)| &\leq n \log(1/\gamma) . \end{aligned}$$

*Proof of Lemma F.2.* From Lemma F.1 and definition of  $\hat{\tau}_{[n]}$  it comes for every  $(\alpha, \pi)$ :

$$\mathcal{J}(X_{[n]}; \hat{z}_{[n]}, \alpha, \pi) \leq \mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) \leq \mathcal{L}_2(X_{[n]}; \alpha, \pi) \leq \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi) .$$

Combined with  $\mathcal{J}(X_{[n]}; \hat{z}_{[n]}, \alpha, \pi) = \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi) + \sum_{i=1}^n \log \alpha_{\hat{z}_i}$ , it leads to both

$$\begin{aligned} |\mathcal{L}_2(X_{[n]}; \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)| &\leq - \sum_{i=1}^n \log \alpha_{\hat{z}_i} , \\ |\mathcal{J}(X_{[n]}; \hat{\tau}_{[n]}, \alpha, \pi) - \mathcal{L}_1(X_{[n]}; \hat{z}_{[n]}, \pi)| &\leq - \sum_{i=1}^n \log \alpha_{\hat{z}_i} . \end{aligned}$$

Assumption (A4) yields the conclusion.

□

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