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# Scaling limits of a heavy tailed Markov renewal process.

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## Abstract

We first show the convergence of a rescaled heavy tailed Markov renewal process  $(\tau, J)$  towards the  $\alpha$ -stable regenerative set. As the state space of the governing Markov chain  $J$  is continuous, one has to resort to the integrated behavior of the Markov mass renewal function. In a second part we apply this result to the critical regime of the strip wetting model. We compute the scaling limits of the contact points of the polymer with the strip by giving the explicit form of its Radon Nykodym derivative with respect to the law of the zero level set of a standard brownian motion.

**Keywords:** heavy tailed Markov renewals, scaling limits, random walks conditioned to stay positive, regenerative sets.

**Mathematics subject classification (2000):** 60F77, 60K15, 60K20, 60K05, 82B27.

## 1 Introduction and main results.

### 1.1 The Markov renewal setup.

This work mainly focuses on the scaling limits of a Markov renewal process with heavy tailed renewal times. By a Markov renewal process, we mean a point process  $\tau$  on  $\mathbb{N}$  whose increments  $\tau_1, \dots, \tau_n - \tau_{n-1}, \dots$  are not necessarily i.i.d., but governed by a Markov chain  $J$  with measurable state space  $\mathcal{E}$ .

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Throughout this paper, we will assume that  $\mathcal{E} = [0, a]$  where  $a$  is strictly positive and that it is equipped with the standard Borel  $\sigma$ -field.

The Markov dependence of  $J$  on  $\tau$  may be described in various equivalent ways, one of them being the following:

1. First consider a kernel of the form  $K_{x,dy}(n)$  with  $x, y \in \mathbb{R}, n \in \mathbb{Z}^+$ , that is a function  $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \times \mathbb{Z}^+ \mapsto \mathbb{R}^+$  where  $\mathcal{B}(\mathbb{R})$  denotes the Borel  $\sigma$ -field of  $\mathbb{R}$  and such that for fixed  $n \in \mathbb{Z}^+$ ,  $K_{\cdot,\cdot}(n)$  verifies the following:
  - for each  $x \in \mathbb{R}$ ,  $K_{x,\cdot}(n)$  is a  $\sigma$ -finite measure on  $\mathbb{R}$ .
  - for every  $F \in \mathcal{B}(\mathbb{R})$ ,  $K_{\cdot,F}(n)$  is a Borel function.

We will always assume that the kernels  $K$  that we consider have a density with respect to the Lebesgue measure that we denote by  $k_{x,y}(n)$ .

2. Then sample  $J$  as a Markov chain on  $\mathcal{E}$  with  $J_0 = 0$  and with transition kernel

$$\mathbf{P}[J_{n+1} \in dy | J_n = x] = \sum_{m \geq 1} K_{x,dy}(m). \quad (1)$$

3. Finally sample the increments  $(\tau_i - \tau_{i-1})_{i \geq 1}$  as a sequence of independent, but not identically distributed random variables according to the conditional law:

$$\mathbf{P}[\tau_{n+1} - \tau_n = m | \{J_i\}_{i \geq 0}] = \frac{k_{J_n, J_{n+1}}(m)}{\sum_{j \geq 1} k_{J_n, J_{n+1}}(j)}, \quad n, m \geq 0. \quad (2)$$

Markov renewal processes have been introduced independently by Lévy [Lev56] and Smith [Smi55], and their basic properties have been studied by Pyke [Pyk61], Cinlar [Çin68] and Pyke and Schaufele [PS64] among others. They have been widely used recently as a convenient tool to describe phenomena arising in models issued from statistical mechanics, and more particularly in models related to pinning models, such as the periodic wetting model ([CGZ07] and the monograph [Gia07][Chapter 3]) or the 1 + 1 dimensional field with Laplacian interaction ([CD08] and [CD09]).

We will show our results in the case where the kernel  $K$  satisfies the following assumptions.

**Assumption 1.1.** *We make the following assumptions on the kernel  $K$ :*

- *the transition function of the Markov chain  $J$  is absolutely continuous with respect to Lebesgue measure, and its density  $k(\cdot, \cdot)$  is continuous and strictly positive on the whole square  $[0, a]^2$ .*

- There exist  $\Phi(\cdot)$  a strictly positive continuous bounded function on  $[0, a]^2$  and  $L(\cdot)$  a slowly varying function such that the equivalence

$$1/dy\mathbf{P}[\tau_1 = n, J_1 \in dy | J_0 = x] = k_{x,y}(n) \sim \Phi(x, y) \frac{L(n)}{n^{1+\alpha}} \quad (3)$$

holds uniformly for  $(x, y) \in [0, a]^2$ ; moreover, we assume that the Markovian renewal is non defective, that is for every  $x \in [0, a]$ :

$$\int_{y \in [0, a]} \sum_{n \geq 1} K_{x,dy}(n) = 1. \quad (4)$$

A few remarks about these assumptions are in order:

- We will use the following notation which is valid for every measurable  $A \in \sigma(\tau)$ :

$$\frac{1}{dy}\mathbf{P}[A, J_1 \in dy] =: \mathbf{P}[A, J_1 = y]. \quad (5)$$

The fact that  $k$  is continuous implies that this definition makes sense everywhere, and not only almost everywhere. Note also that one has the equality:

$$k(x, y) = \frac{1}{dy} \sum_{n \geq 1} K_{x,dy}(n) = \sum_{n \geq 1} k_{x,y}(n). \quad (6)$$

- The first part of Assumption 1.1 implies in particular the fact that  $J$  is a *regular* Markov chain (see [Fel71][VIII,7]). Regular Markov chains with strictly positive density are arguably the simplest example of Markov chains with continuous state space which still satisfy the basic ergodic theorem. More precisely, it is well known that every strictly positive regular kernel on a closed interval is ergodic, and also that the ergodicity of such a kernel is equivalent to the existence of a strictly positive stationary distribution which we will denote by  $\Pi$  (see [Fel71][VIII,7, Theorems 1 and 2]).
- The second part of Assumption 1.1 expresses some kind of long range independence between  $\tau$  and  $J$ ; we stress that this property is verified at least for the Markov renewal process we consider in the second part of this paper.

A consequence of equation (3) is the fact that the Markov renewal process  $\tau$  has steps distribution with *infinite mean*. We point out that even for ordinary renewal processes, the exact asymptotic behavior of the Green function

in the case of infinite mean has been a longstanding problem which has been solved in great generality only recently (see [Don97]). The extension of this deep result to the framework of Markov renewal theory has been achieved in the case of finite state modulating chain in [CGZ06], but it turns out that their techniques do not extend to the continuous state case we are considering here. As a matter of fact, we are not able to show precise estimates on the rate of decay of the Green function, but it turns out that we do not need them to show our main results.

The aim of this paper is to give a description of the large scale behavior of the set  $\tau$ ; we first need some terminology. We consider the class  $\mathcal{C}_\infty$  of closed subsets of  $[0, \infty)$  which we endow with the Mathéron topology which we describe next. For  $t \geq 0, F \in \mathcal{C}_\infty$ , we set  $d_t(F) := \inf(F \cap (t, \infty))$ . The function  $t \mapsto d_t(F)$  is right continuous and  $F$  may be identified to  $d_t(F)$  as  $F = \{t \in \mathbb{R}^+, d_{t-}(F) = 0\}$ . The space of such functions may be endowed with the standard Skorohod topology, and this topology gives the Mathéron topology *via* the previous identification. Under this topology,  $\mathcal{C}_\infty$  is compact, separable and compact, hence it is a Polish space.

Finally, we introduce the classical notion of  $\alpha$  stable regenerative set; recall that a subordinator is a non decreasing Lévy process. It is said to be  $\alpha$ -stable ( $\alpha \in (0, 1)$ ) if its Lévy-Khintchine exponent  $\phi(\lambda)$  is equal to  $\lambda^\alpha$ . The  $\alpha$  stable regenerative set is then defined as being the closure of the range of the  $\alpha$  stable subordinator.

If the Markov renewal process  $(\tau, J)$  verifies Assumption 1.1, we show that the following result holds, it is actually the main result of this paper.

**Theorem 1.2.** *The rescaled sets  $(\tau_{(N)})_N$  converge in law towards the set  $\mathcal{A}_\alpha \cap [0, 1]$  where  $\mathcal{A}_\alpha$  denotes the  $\alpha$  stable-regenerative set.*

Let us note that such a result has been shown in the classical renewal framework in [Gia07][Theorem 2.7] by making use of the pointwise convergence of the Lévy exponent of a Poisson process whose range is equal to  $\tau_{(N)}$  towards the one of the  $\alpha$  stable-regenerative set, which directly implies the result (see [FFM85]); the same idea is not available in our setup since the increments of  $\tau$  are no longer i.i.d. Another idea which does not work in our setup has been used in [CGZ07] to prove a very similar result in the context of weakly inhomogeneous wetting models; the authors use in an essential way the finiteness of the state space of the governing Markov chain  $J$ .

## 1.2 Application to the strip wetting model.

### 1.2.1 Definition of the model.

The main motivation for proving Theorem 1.2 has been provided by a model which originates from statistical mechanics and which we describe now.

We consider  $(S_n)_{n \geq 0}$  a random walk such that  $S_0 := 0$  and  $S_n := \sum_{i=1}^n X_i$  where the  $X_i$ 's are i.i.d. and  $X_1$  has a density  $h(\cdot)$  with respect to the Lebesgue measure. We denote by  $\mathbf{P}$  the law of  $S$ , and by  $\mathbf{P}_x$  the law of the same process conditioned on  $\{S_0 = x\}$ . We will assume that  $h(\cdot)$  is continuous and bounded on  $\mathbb{R}$ , that  $h(\cdot)$  is positive in a neighborhood of the origin, that  $\mathbf{E}[X] = 0$  and that  $\mathbf{E}[X^2] =: \sigma^2 \in (0, \infty)$ .

The fact that  $h$  is continuous and positive in the neighborhood of the origin entails that

$$n_0 := \inf_{n \in \mathbb{N}} \{(\mathbf{P}[S_n > a], \mathbf{P}[-S_n > a]) \in (0, 1)^2\} < \infty. \quad (7)$$

In the sequel, we will assume that  $n_0 = 1$  (and thus that  $(\mathbf{P}[-S_1 > a], \mathbf{P}[S_1 > a]) \in (0, 1)^2$ ). We stress that our work could be easily extended to the generic  $n_0 \geq 2$  case, although this should lead to some specific technical difficulties (for example one should extend Theorem 1.2 to the case where the transition function of  $J$  may vanish on  $[0, a]$ ).

We consider the event  $\mathcal{C}_N := \{S_1 \geq 0, \dots, S_N \geq 0\}$ . We define the probability law  $\mathbf{P}_{N,a,\beta}$  on  $\mathbb{R}^N$  by

$$\frac{d\mathbf{P}_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^N \mathbf{1}_{S_k \in [0,a]}\right) \mathbf{1}_{\mathcal{C}_N} \quad (8)$$

where  $N \in \mathbb{N}$ ,  $\beta \in \mathbb{R}$  and  $Z_{N,a,\beta}$  is the normalisation constant usually called the partition function of the system.

$\mathbf{P}_{N,a,\beta}$  is a  $(1+1)$ -dimensional model for a linear chain of length  $N$  which is attracted or repelled to a defect *strip* of width  $a$ . By  $(1+1)$ -dimensional, we mean that the configurations of the linear chain are described by the trajectories  $(i, \phi_i)_{i \leq N}$  of the field, so that we are dealing with directed models. The strength of this interaction with the strip is tuned by the parameter  $\beta$ . Regarding the terminology, note that the use of the term *wetting* has become customary to describe the positivity constraint  $\mathcal{C}_N$  and refers to the interpretation of the field as an effective model for the interface of separation between a liquid above a wall and a gas, see [DGZ05].

The purpose of this part is to investigate the behavior of  $\mathbf{P}_{N,a,\beta}$  in the large  $N$  limit. In particular, we would like to understand when the reward  $\beta$  is strong enough to pin the chain near the defect strip, a phenomenon that

we will call *localization*, and what are the macroscopic effects of the reward on the system. We point out that this kind of questions have been answered in depth in the case of the standard wetting model, that is formally in the  $a = 0$  case, and that the problem of extending these results to the case where the defects are in a strip has been posed in [Gia07, Chapter 2].

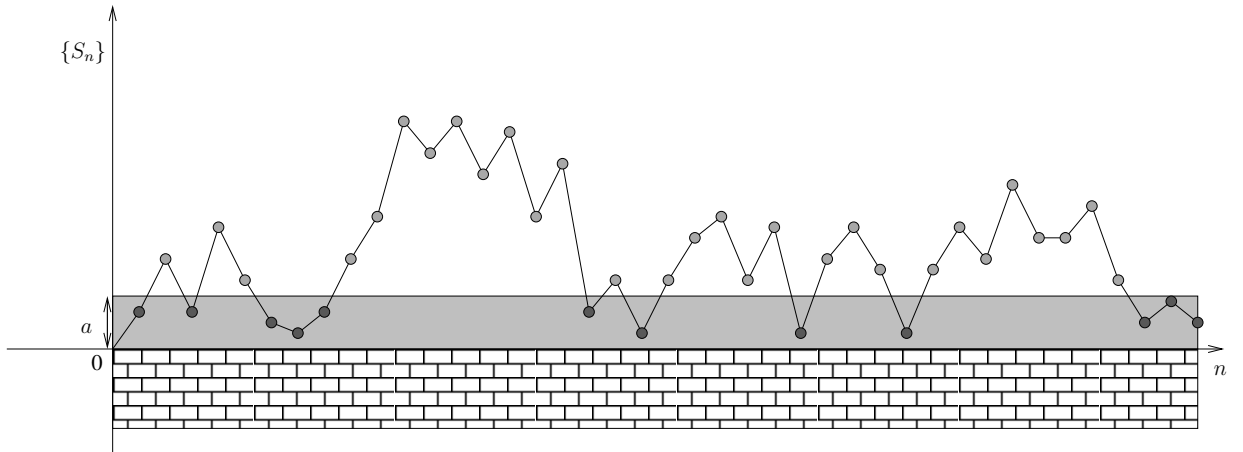


Figure 1: A trajectory of  $\mathbf{P}_{41,a,\beta}^c$  with 12 pinned sites. The graph  $(n, S_n)$  is an interface separating a dry zone from a wet zone. The interface is conditioned to stay above a wall which attracts or repels it when it comes *close to the wall*

### 1.2.2 The free energy.

A standard way to define localization for our model is by looking at the Laplace asymptotic behavior of the partition function  $Z_{N,a,\beta}$  as  $N \rightarrow \infty$ . More precisely, one may define the free energy  $F(\beta)$  by

$$F(\beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log(Z_{N,a,\beta}). \quad (9)$$

The basic observation is that the free energy is non-negative. One has the inequality:

$$\begin{aligned} Z_{N,a,\beta} &\geq \mathbf{E} \left[ \exp \left( \beta \sum_{k=1}^N \mathbf{1}_{S_k \in [0,a]} \right) \mathbf{1}_{S_k > a, k=12, \dots, N} \right] \\ &\geq \mathbf{P} [S_j > a, j = 1 \dots, N]. \end{aligned} \quad (10)$$

Choose some  $M > a$  such that  $\mathbf{P}[S_1 \in [a, M]] > 0$ . Integrating over  $S_1$ , one gets:

$$\mathbf{P}[S_j > a, j = 1, \dots, N] \geq \int_{[a, M]} h(t) \mathbf{P}_t[S_1 > a, \dots, S_{N-1} > a] dt. \quad (11)$$

A consequence of fluctuation theory estimates for killed random walk (see [Soh][Lemma 3.1] for the discrete case) is that for fixed  $M$ , the quantity  $N^{3/2} \mathbf{P}_t[S_1 > a, \dots, S_{N-1} > a] \in [c, c']$  for every  $N$  and every  $t \in [a, M]$  where  $c, c'$  are positive constants. Thus:

$$Z_{N, a, \beta} \geq \frac{c}{N^{3/2}} \int_{[a, M]} h(t) dt. \quad (12)$$

Therefore  $F(\beta) \geq 0$  for every  $\beta$ . Since the lower bound has been obtained by ignoring the contribution of the paths that touch the strip, one is led to the following:

**Definition 1.3.** *The model  $\{\mathbf{P}_{N, a, \beta}\}$  is said to be localized if  $F(\beta) > 0$ .*

It is easy to show that  $F(\cdot)$  is a convex function, in particular it is a continuous function as long as it is finite. Obviously,  $F(\cdot)$  is increasing since for fixed  $N$ ,  $Z_{N, a, \beta}$  is increasing in  $\beta$ . Therefore, there exists a critical value  $\beta_c^a \in \mathbb{R}$  such that the strip wetting model is localized for  $\beta > \beta_c^a$ .

### 1.2.3 Scaling limits of the system.

A common feature shared by the strip wetting model and the classical homogeneous one is that the measure  $\mathbf{P}_{N, a, \beta}$  exhibits a remarkable decoupling between the contact level set  $\mathcal{I}_N := \{i \leq N, S_i \in [0, a]\}$  and the excursions of  $S$  between two consecutive contact points (see [DGZ05] for more details in the standard homogeneous pinning case). In fact, conditionally on  $I_N = \{t_1, \dots, t_k\}$  and on  $(S_{t_1}, \dots, S_{t_k})$ , the *bulk* excursions  $e_i = \{e_i(n)\}_n := \{\{S_{t_i+n}\}_{0 \leq n \leq t_{i+1}-t_i}\}$  are independent under  $\mathbf{P}_{N, a, \beta}$  and are distributed like the random walk  $(S', \mathbf{P}_{S_{t_i}})$  conditioned on the event  $\left\{S'_{t_{i+1}-t_i} = S_{t_{i+1}}, S'_{t_i+j} > a, j \in \{1, \dots, t_{i+1} - t_i - 1\}\right\}$ . It is therefore clear that to extract scaling limits on  $\mathbf{P}_{N, a, \beta}$ , one has to combine good control over the law of the contact set  $\mathcal{I}_N$  and suitable asymptotics properties of the excursions.

This decoupling is the basis to the resolution of the large scale limits of the laws  $\{\mathbf{P}_{N, a, \beta}\}$ ; one might show that for  $\beta > \beta_c^a$ , the diffusive rescaling limit of the laws  $\{\mathbf{P}_{N, a, \beta}\}$  is simply the null function, whereas in the delocalized



phase the limiting law is the brownian motion conditioned to stay positive, that is the brownian meander. We stress that the proofs are quite similar although technically more involved than in the classical homogeneous wetting model, see [Soh10] for more details.

What is left by these considerations is the critical case. It turns out that it is much more convenient to describe the scaling limit of the critical case in terms of the convergence of the set of contact points with the strip, mainly because additional tightness conditions should be made on the free process  $S$  in order to ensure the convergence in law of the entire trajectories towards the reflected brownian motion (see [CGZ] to get some insight into this issue). It turns out that the scaling limits of the set of contact times with the strip is almost the  $1/2$ -stable regenerative set; more precisely, we compute its density with respect to the law of  $\mathcal{A}_{1/2}$ .

**Theorem 1.4.** *Consider the set  $\tau := \{j \in \mathbb{N}, \phi_j \in [0, a]\}$  where  $\phi_j$  is distributed according to the law  $\mathbf{P}_{N, a, \beta_0^a}$ . One has the convergence in law:*

$$\tau_{(N)} \Rightarrow \mathcal{B}_{1/2} \cap [0, 1] \quad (13)$$

where  $\mathcal{B}_{1/2}$  is a random closed set whose law is absolutely continuous with respect to the one of  $\mathcal{A}_{1/2}$  with Radon Nykodym derivative given by

$$\frac{\pi}{2} (1 - \max(\mathcal{A}_{1/2} \cap [0, 1]))^{1/2}. \quad (14)$$

We stress that the density term that appears in the above result is pushing the rightmost point of the process away from 1 and closer to the origin: it is a reminiscence of the transient nature of the original process.

### 1.3 Outline of the paper.

The remaining of this paper is divided into two main parts, the first one dealing with the generic Markovian renewal setup, the second one being devoted to the application of the main result of the first part to the strip wetting model. More specifically, the exposition of the paper will be organized as follows:

1. in section 2, we explicit the asymptotic behavior of the Markov mass renewal function which forms the cornerstone to the proof of our main result.
2. in section 3, we prove Theorem 1.2.
3. in section 4, we describe the underlying Markov renewal structure inherent to the strip wetting model.

4. in section 4.1, we give some results issued from fluctuation theory for random walks which will be the basis of our approach for the proof of Theorem 1.4.
5. in section 4.2, we give a suitable representation for the free energy of the strip wetting model and apply it to the proof of Theorem 1.4.

## 2 Asymptotic equivalence of the mass Markov renewal function.

### 2.1 Notations.

Given two kernels  $A_{x,dy}(n), B_{x,dy}(n)$  we define their convolution

$$(A * B)_{x,dy}(n) := \sum_{m=0}^n \int_{\mathbb{R}} A_{x,dz}(m) B_{z,dy}(n-m) \quad (15)$$

and the  $k$ -fold convolution of the kernel  $A$  with itself will be denoted by  $A_{x,dy}^{*k}$  where by definition  $A_{x,dy}^{*0} := \delta_x(dy) \mathbf{1}_{n=0}$ .

We denote by  $U(n, x, dy)$  the Markov renewal mass distribution function defined by  $U(n, x, dy) := \sum_{i=0}^n \sum_{k \geq 1} \mathbf{P}_x[\tau_k = i, J_k \in dy] =: \sum_{i=0}^n u(i, x, dy)$ . Note that  $U(n, x, dy) = \sum_{k \geq 0} \mathbf{P}_x[\tau_k \leq n, J_k \in dy]$ . The purpose of this part is to give asymptotics on  $U$  for large  $n$ , and to do that, we generalize the well known Tauberian theorems which hold for the classical renewal. More specifically, we show the following:

**Theorem 2.1.** *As  $n \rightarrow \infty$ , for every  $x \in [0, a]$ , the following weak convergence holds :*

$$\frac{U(n, x, dy)L(n)}{n^\alpha} \rightarrow \frac{\alpha}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \frac{\Pi[dy]}{\mathbf{E}_{\Pi^{(2)}}[\Phi/k]} \quad (16)$$

where  $\Pi^{(2)}$  is the invariant measure associated to the Markov chain  $(J_i, J_{i+1})_{i \geq 0}$ .

By this we mean that for every continuous bounded function  $f$  on  $[0, a]$  and every  $x \in [0, a]$ , as  $n \rightarrow \infty$ , the following convergence holds :

$$\frac{L(n)}{n^\alpha} \int_{y \in [0, a]} U(n, x, dy) f(y) \rightarrow \frac{\alpha}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \frac{\int_{y \in [0, a]} f(y) \Pi[dy]}{\mathbf{E}_{\Pi^{(2)}}[\Phi/k]} \quad (17)$$

Note that it is quite clear that  $\Pi^{(2)}[du, dv] = \Pi(du)k(u, v)dv$  where  $\Pi(\cdot)$  is the invariant measure associated to the Markov chain  $J$ , and thus in particular one has the equality

$$\mathbf{E}_{\Pi^{(2)}}[\Phi/k] = \int_{[0, a]^2} \Phi(u, v)\Pi(du)dv. \quad (18)$$

For  $(x, y) \in [0, a]^2, \lambda > 0$ , we consider the Laplace transform measure  $U_{x, dy}(\lambda)$  associated to  $(u(n, x, dy))_n$  defined by

$$U_{x, dy}(\lambda) := \sum_{j \geq 0} e^{-\lambda j} u(j, x, dy). \quad (19)$$

Our main technical step will be to prove the following proposition:

**Proposition 2.2.** *For every  $x \in [0, a]$ , as  $\lambda \searrow 0$ , one has the following weak convergence:*

$$\frac{\Gamma(1 - \alpha)\lambda^\alpha L(1/\lambda)\mathbf{E}_{\Pi^{(2)}}[\Phi/k]}{\alpha} U_{x, dy}(\lambda) \rightarrow \Pi[dy]. \quad (20)$$

To prove Proposition 2.2, we will need some technical results which we summarize in the next section.

## 2.2 Proof of Proposition 2.2

The Markov renewal equation writes:

$$u(n, x, dy) = \sum_{k \geq 0} K_{x, dy}^{*k}(n) \quad (21)$$

which implies

$$U(n, x, dy) = \sum_{k \geq 0} \sum_{j=0}^n K_{x, dy}^{*k}(j). \quad (22)$$

For conciseness, we will note  $\mathcal{S} := [0, a]^2$  and  $z := (x, y)$  is an element of  $\mathcal{S}$ . Note that the assumption made on  $\Phi(\cdot)$  implies that  $0 < \min_{\mathcal{S}} \Phi \leq \max_{\mathcal{S}} \Phi < \infty$ .

For  $z \in \mathcal{S}, \lambda > 0$ , define

$$\phi_z(\lambda) := \frac{1}{dy} \mathbf{E}_x[e^{-\lambda \tau_1} | J_1 = y]. \quad (23)$$

We are able to control the behavior of  $\phi_z(\cdot)$  close to the origin, and this uniformly on  $\mathcal{S}$ .

**Proposition 2.3.** *As  $\lambda \searrow 0$ , one has:*

$$\sup_{z:=(x,y) \in \mathcal{S}} \left| k(x,y) \frac{(1 - \phi_z(\lambda))}{\lambda^\alpha L(1/\lambda)} - \frac{\Phi(z)\Gamma(1 - \alpha)}{\alpha} \right| \rightarrow 0. \quad (24)$$

To prove Proposition 2.3, we will need the following result:

**Lemma 2.4.** *Given  $\rho > 0$  and a family of measures  $V_z^{(n)}$  on  $\mathbb{R}^+$  which are without atoms indexed by  $z \in \mathcal{S}, n \in \mathbb{N}$ . Denote by  $\Psi_z^{(n)}(\lambda)$  their Laplace transforms. Assume that they verify that for every  $x > 0$ , one has*

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathcal{S}} |V_z^{(n)}([0, x]) - x^\rho| = 0. \quad (25)$$

and assume moreover that there exists  $d > 0$  such that  $\sup_{(n,z) \in \mathbb{N} \times \mathcal{S}} \Psi_z^{(n)}(d) < \infty$ . Then for every  $\lambda > d$ :

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathcal{S}} \left| \Psi_z^{(n)}(\lambda) - \frac{\Gamma(\rho + 1)}{\lambda^\rho} \right| = 0. \quad (26)$$

Note that in our case, the sequence of measures  $V^{(n)}$  will have a density with respect with the Lebesgue measure, and thus will be without atoms.

*Proof of Lemma 2.4.* Let  $t > 0$  be fixed; for any  $\lambda > a$ , uniformly on  $z \in \mathcal{S}$ , as  $n \rightarrow \infty$ , we have:

$$\int_{[0,t]} e^{-\lambda x} V_z^{(n)}(dx) \rightarrow \int_{[0,t]} e^{-\lambda x} \rho x^{\rho-1} dx. \quad (27)$$

Indeed, integrating by part in the left hand side above and making use of the fact that the sequence  $V^{(n)}$  has no atoms, one obtains:

$$\int_{[0,t]} e^{-\lambda x} V_z^{(n)}(dx) = e^{-\lambda t} V_z^{(n)}(t) + \lambda \int_{[0,t]} e^{-\lambda x} V_z^{(n)}(x) dx \quad (28)$$

and making use of the uniformity statement of equation (25) (which allows us in particular to use dominated convergence) and reintegrating by part, equation (27) is proved.

On the other hand, still for  $\lambda > d$ , it is clear that

$$\int_{[t,\infty)} e^{-\lambda x} V_z^{(n)}(dx) \leq e^{-(\lambda-d)t} \sup_{(n,z) \in \mathbb{N} \times \mathcal{S}} \Psi_z^{(n)}(d) \quad (29)$$

which can be made arbitrarily small (independently of  $z, n$ ) as soon as  $t$  is large enough. Noting that

$$\int_{[0,\infty)} e^{-\lambda x} \rho x^{\rho-1} dx = \frac{\Gamma(\rho + 1)}{\lambda^\rho}, \quad (30)$$

we get the result. □

Now we prove Proposition 2.3.

*Proof of Proposition 2.3.* For  $z \in \mathcal{S}$ , we define the (infinite mass) measure on  $\mathbb{R}^+$   $U_z([0, n]) := \sum_{j \leq n} \mathbf{P}_x[\tau_1 > j, J_1 = y]$ . The following equivalence holds uniformly on  $\mathcal{S}$ :

$$U_z([0, n]) \sim \sum_{j \leq n} \frac{L(j)\Phi(z)}{\alpha j^\alpha} \sim \frac{n^{1-\alpha}L(n)\Phi(z)}{\alpha(1-\alpha)}. \quad (31)$$

Indeed, we use the general fact that if  $u_n(z)$  is a positive sequence depending on  $z \in \mathcal{S}$  verifying  $\sup_{z \in \mathcal{S}} |u_n(z)| < \infty$  and, as  $n \rightarrow \infty$ ,

$$\sup_{z \in \mathcal{S}} \left| \frac{n^\alpha u_n(z)}{L(n)} - \Phi(z) \right| \rightarrow 0, \quad (32)$$

then

$$\sup_{z \in \mathcal{S}} \left| \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n} u_j(z) - \Phi(z) \right| \rightarrow 0. \quad (33)$$

Let us prove this quickly. First we recall the standard equivalence

$$\sum_{j=1}^n \frac{L(j)}{j^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha} L(n). \quad (34)$$

For every  $z \in \mathcal{S}$ , using the triangle inequality, we get:

$$\begin{aligned} & \left| \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n} u_j(z) - \Phi(z) \right| \\ & \leq \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n} \frac{L(j)}{j^\alpha} \left| \frac{j^\alpha u_j(z)}{L(j)} - \Phi(z) \right| \\ & + \left( \sup_{z \in \mathcal{S}} |\Phi(z)| \right) \left| \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n} \frac{L(j)}{j^\alpha} - 1 \right|. \end{aligned} \quad (35)$$

The second term in the right hand side above vanishes when  $n \rightarrow \infty$ ; for the first one, we fix  $\varepsilon > 0$ , an integer  $n_0 > 0$  and we write:

$$\frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n} \frac{L(j)}{j^\alpha} \left| \frac{j^\alpha u_j(z)}{L(j)} - \Phi(z) \right| \leq \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{j \leq n_0} \dots + \frac{1-\alpha}{n^{1-\alpha}L(n)} \sum_{n_0 \leq j \leq n} \dots \quad (36)$$

and the first term above is easily seen to be smaller than  $\varepsilon$  for  $n_0$  large enough uniform for  $z \in \mathcal{S}$  (since the sequence  $\left(\sup_{z \in \mathcal{S}} \left| \frac{n^\alpha u_n(z)}{L(n)} - \Phi(z) \right| \right)_n$  is bounded). As for the second one, just fix  $n_0$  such that

$$\sup_{z \in \mathcal{S}} \left| \frac{j^\alpha u_j(z)}{L(j)} - \Phi(z) \right| \leq \varepsilon \quad (37)$$

as soon as  $j \geq n_0$  to get that it is smaller than  $\frac{(1-\alpha)\varepsilon}{n^{1-\alpha}L(n)} \sum_{n \geq n_0} \frac{L(j)}{j^\alpha}$  to conclude.

We denote by  $\Psi_z(\lambda)$  the Laplace transform associated to the measure  $U_z(\cdot)$ . Integrating by part, we then have the equality which is valid for  $z \in \mathcal{S}, \lambda \geq 0$ :

$$\begin{aligned} 1 - \phi_z(\lambda) &= (1 - e^{-\lambda}) \sum_{n \geq 0} e^{-\lambda n} \mathbf{P}_x[\tau_1 > n | J_1 = y] \\ &= (1 - e^{-\lambda}) \frac{\Psi_z(\lambda)}{k(x, y)}. \end{aligned} \quad (38)$$

Making use of equation (31), we give the asymptotic behavior of  $\Psi_z(\cdot)$  close to the origin; more precisely, uniformly on  $z \in \mathcal{S}$ , the convergence

$$\Psi_z(\lambda) \sim \frac{L(1/\lambda)\Phi(z)\Gamma(1-\alpha)}{\alpha\lambda^{1-\alpha}} \quad (39)$$

holds as  $\lambda \searrow 0$ . Clearly, if this is true, making use of equation (38), Proposition 2.3 will be proved.

We define a sequence of measures  $V_z^{(n)}$  on  $\mathbb{R}^+$  by  $V_z^{(n)}([0, u]) := \frac{U_z([0, nu])}{U_z([0, n])}$ . Equation (31) implies that the hypothesis of Lemma 2.4 is verified by the sequence  $V_z^{(n)}$  with  $\rho = 1 - \alpha$  with an arbitrary  $d > 0$ . It is plain that, for every fixed  $\lambda > 0$ ,

$$\Psi_z^{(n)}(\lambda) = \frac{\Psi_z(\lambda/n)}{U_z([0, n])}. \quad (40)$$

Making use of Lemma 2.4, as  $n \rightarrow \infty$ , the convergence

$$\frac{\Psi_z(\lambda/n)}{U_z([0, n])} \rightarrow \frac{\Gamma(2-\alpha)}{\lambda^{1-\alpha}} \quad (41)$$

holds uniformly on  $\mathcal{S}$ . Hence we get:

$$\Psi_z(\lambda/n) \sim \frac{\Phi(z)L(n)\Gamma(2-\alpha)}{\alpha(1-\alpha)(\lambda/n)^{1-\alpha}} \quad (42)$$

and this still holds uniformly on  $\mathcal{S}$ . Extending this convergence to any sequence which decreases to zero (which is straightforward), making use of standard properties of slowly varying functions and of the Gamma function, one gets the equivalence (39).  $\square$

To finally prove Proposition 2.2, we will need the following easy lemma.

**Lemma 2.5.** *Let  $(c_n)_n$  be a positive sequence which converges towards  $c > 0$ ,  $u_k$  a positive sequence which converges towards  $u > 0$  and  $g(\cdot)$  a function  $\mathbb{R}^+ \rightarrow \mathbb{R}$  which is  $o(\lambda)$  as  $\lambda \searrow 0$ . Then, as  $\lambda \searrow 0$ , one has:*

$$\sum_{k \geq 0} c_k e^{-k(\lambda u_k + g(\lambda))} \sim \frac{c}{u\lambda}. \quad (43)$$

*Proof of Lemma 2.5.* Note that one can ignore the first  $k_1$  terms in the sum appearing in equation (43), where  $k_1 \geq 0$  is an arbitrarily large fixed integer. Let  $\varepsilon \in (0, \min(u, c))$  be fixed. For  $k_1$  large enough and  $\lambda$  small enough (independently from  $k_1$ ), one has :

$$(c - \varepsilon) \sum_{k \geq k_1} e^{-k(\lambda(u+\varepsilon))} \leq \sum_{k \geq k_1} c_k e^{-k(\lambda u_k + g(\lambda))} \leq (c + \varepsilon) \sum_{k \geq k_1} e^{-k(\lambda(u-\varepsilon))}. \quad (44)$$

Therefore

$$\frac{c - \varepsilon}{c} \frac{u}{u + \varepsilon} \leq \liminf_{\lambda \searrow 0} \frac{\lambda u}{c} \sum_{k \geq 0} c_k e^{-k(\lambda u_k + g(\lambda))} \leq \limsup_{\lambda \searrow 0} \frac{\lambda u}{c} \sum_{k \geq 0} c_k e^{-k(\lambda u_k + g(\lambda))} \leq \frac{c + \varepsilon}{c} \frac{u}{u - \varepsilon} \quad (45)$$

and considering  $\varepsilon$  arbitrarily close to zero yields the statement.  $\square$

Consider now a function  $f$  which is continuous and positive on  $[0, a]$ , and let  $x \in [0, a]$  be fixed. Making use of the Markov renewal equation and of the fact that conditionally on the  $J_i$ 's, the  $(\tau_i - \tau_{i-1})_{i \geq 1}$  are independent, we get:

$$\int_{[0, a]} f(y) U_{x, dy}(\lambda) = \sum_{n \geq 0} \int_{[0, a]} f(y) \int_{[0, a]^{n-1}} \prod_{j=0}^{n-1} \phi_{x_j, x_{j+1}}(\lambda) \mathbf{P}_{x_j} [J_1 = x_{j+1}] \quad (46)$$

where  $x_0 := x$  and  $x_n := y$ . For  $(u, v) \in \mathcal{S}$ , we define

$$g_{u, v}(\lambda) := 1 - \phi_{u, v}(\lambda) - \frac{\Gamma(1 - \alpha) \lambda^\alpha L(1/\lambda) \Phi(u, v)}{\alpha k(u, v)}. \quad (47)$$

Recall that Proposition 2.3 asserts that  $\sup_{(u, v) \in \mathcal{S}} |g_{u, v}(\lambda)| = o(\lambda^\alpha L(1/\lambda))$  as  $\lambda \searrow 0$ .

For fixed  $n$  and fixed  $z \in \mathcal{S}$ , one has:

$$\begin{aligned} & \int_{[0,a]^{n-1}} \prod_{j=0}^{n-1} \phi_{x_j, x_{j+1}}(\lambda) \mathbf{P}_{x_j} [J_1 = x_{j+1}] \\ &= \int_{[0,a]^{n-1}} \exp \left( \sum_{j=0}^{n-1} \log (1 - (1 - \phi_{x_j, x_{j+1}}(\lambda))) \right) \mathbf{P}_x [J_1 = x_1, \dots, J_n = x_n] \end{aligned} \quad (48)$$

Given  $\varepsilon \in (0, 1/2)$ , there exists  $\lambda_0$  such that as soon as  $\lambda \in (0, \lambda_0)$ , for every  $(u, v) \in \mathcal{S}$ , there exists  $c_{u,v}(= c_{u,v}(\lambda)) \in [1/2 - \varepsilon, 1/2 + \varepsilon]$  such that the last term above is equal to:

$$\begin{aligned} & \int_{[0,a]^{n-1}} \exp \left( \sum_{j=0}^{n-1} (1 - \phi_{x_j, x_{j+1}}(\lambda)) + c_{x_j, x_{j+1}} (1 - \phi_{x_j, x_{j+1}}(\lambda))^2 \right) \\ & \quad \mathbf{P} [J_1 = x_1, \dots, J_n = x_n] \\ &= \mathbf{E} \left[ \exp \left( \sum_{j=0}^{n-1} -\frac{\Gamma(1-\alpha)\lambda^\alpha L(1/\lambda)}{\alpha} \frac{\Phi(J_j, J_{j+1})}{k(J_j, J_{j+1})} - g_{J_j, J_{j+1}}(\lambda) + c_{J_j, J_{j+1}} (1 - \phi_{J_j, J_{j+1}}(\lambda))^2 \right) \mathbf{1}_{J_n=y} \right] \end{aligned} \quad (49)$$

Making use of the ergodic theorem (which holds in our case, since  $J$  is a regular Markov chain (see [Fel71][VIII,7 Theorem 1]), we get that

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{\Phi(J_j, J_{j+1})}{k(J_j, J_{j+1})} \rightarrow \mathbf{E}_{\Pi^{(2)}} \left[ \frac{\Phi}{k} \right] = \int_{[0,a]} \Pi(dx) \Phi(x, y) dy \quad (50)$$

$\mathbf{P}_x$  almost surely. When  $n$  becomes large, the generic term appearing in the sum of the right hand side of (46) finally becomes equivalent to

$$\begin{aligned} & \int_{y \in [0,a]} f(y) \mathbf{P}_x [J_n = y] \\ & \times \exp \left( -n \left( \frac{\Gamma(1-\alpha)\lambda^\alpha L(1/\lambda)}{\alpha} \mathbf{E}_{\Pi^{(2)}} [\Phi/k] + \mathbf{E}_{\Pi^{(2)}} [g_{J_0, J_1}(\lambda) + c_{J_0, J_1} (1 - \phi_{J_0, J_1}(\lambda))^2] \right) \right). \end{aligned} \quad (51)$$

The function  $\lambda \mapsto \mathbf{E}_{\Pi^{(2)}} [g_{J_0, J_1}(\lambda) + c_{J_0, J_1} (1 - \phi_{J_0, J_1}(\lambda))^2]$  is then  $o(\lambda^\alpha L(1/\lambda))$ .

On the other hand, making use of the classical ergodic theorem, as  $n \rightarrow \infty$ , the following convergence holds:

$$\int_{y \in [0,a]} f(y) \mathbf{P}_x [J_n = y] \rightarrow \int_{[0,a]} f(y) \Pi[dy]. \quad (52)$$

Finally, we can apply Lemma 2.5 to prove Proposition 2.2.



### 2.3 Proof of Theorem 2.1.

To prove Theorem 2.1, we will use the following result, which is known as the *extended continuity theorem* (see [Fel71][XIII Theorem 2a]):

**Theorem 2.6.** *Let  $U_n$  be a sequence of measures defined on  $\mathbb{R}^+$ , and let  $\phi_n$  denote their Laplace transforms. For a fixed  $d > 0$ , if the sequence of functions  $\phi_n$  converges pointwise towards a function  $\phi$  on  $(d, \infty)$ , then the sequence of measures  $U_n$  converges weakly towards a measure  $U$  with Laplace transform  $\phi$ .*

We consider a continuous function  $f$  on  $[0, a]$  and a fixed  $x \in [0, a]$ . Assume first that  $f(\cdot)$  is positive; making use of Proposition 2.2, for  $t \searrow 0$  and fixed  $\lambda > 0$ , we get the convergence:

$$\frac{\int_{[0,a]} U_{x,dy}(\lambda t) f(y)}{\int_{[0,a]} U_{x,dy}(t) f(y)} \rightarrow \frac{1}{\lambda^\alpha}. \quad (53)$$

Consider now the measure  $m_f(\cdot)$  on  $\mathbb{R}^+$  which is defined by

$$m_f([0, n]) := \int_{[0,a]} f(y) U(n, x, dy), \quad (54)$$

and denote by  $\Psi_f(\cdot)$  its Laplace transform. It is then quite clear that one may rewrite equation (53) as

$$\frac{\Psi_f(\lambda t)}{\Psi_f(t)} \rightarrow \frac{1}{\lambda^\alpha} \quad (55)$$

when  $t \searrow 0$ . On the other hand,  $\Psi_f(\lambda t)$  is the Laplace transform associated to the measure  $m_f(u/t)$ , and thus by making use of the extended continuity theorem, this implies the following convergence (for  $t \searrow 0$ ):

$$\frac{m_f(u/t)}{\Psi_f(t)} \rightarrow \frac{u^\alpha}{\Gamma(1 + \alpha)}. \quad (56)$$

Finally, considering the convergence (56) with  $u = 1$  along the subsequence  $t_n = 1/n$ , as  $n \rightarrow \infty$ , one gets:

$$\int_{[0,a]} f(y) U(n, x, dy) \sim \frac{\int_{[0,a]} U_{x,dy}(n) f(y)}{\Gamma(1 + \alpha)} \quad (57)$$

and making use of Proposition 2.2, this is exactly the statement of Theorem 2.1. Making use of the linearity of the equivalence (57) and writing  $f = f_+ - (-f_-)$  where  $f_+$  (respectively  $f_-$ ) is the positive (respectively the negative) part of  $f$ , we are done.

### 3 Proof of Theorem 1.2.

In this section, we finally prove Theorem 1.2.

The compactness of  $\mathcal{C}_\infty$  implies that every sequence of probability measures on  $\mathcal{C}_\infty$  is tight, so that in order to check the convergence in law of a sequence  $(\mathbf{P}_n)$  towards  $\mathbf{P}$ , one just has to show finite dimensional convergence, namely that for every  $t_1 < t_2 < \dots < t_n$ , the following weak convergence holds:

$$\mathbf{P}_N G_t^{-1} \Rightarrow \mathbf{P}_N G_t^{-1} \quad (58)$$

where  $G_t : \mathcal{C}_\infty \rightarrow [0, \infty]^n$  is defined by  $G_t(F) := (d_{t_1}(F), \dots, d_{t_n}(F))$ .

Let us recall that for  $t > 0$ , the law of  $d_t$  under  $\mathbf{P}_\alpha$  is given by (see [Ber99][Proposition 3.1])

$$\mathbf{P}_\alpha[d_t \in dy] = \frac{\sin(\alpha\pi)}{\pi} \frac{t^\alpha}{y(y-t)^\alpha} \mathbf{1}_{y>t} dy. \quad (59)$$

For conciseness, we show (58) in the  $n = 1$  case when  $J$  starts from an initial state  $x \in [0, a]$ . Considering an arbitrary  $y > t$ , we have:

$$\begin{aligned} \mathbf{P}_N[J_0 = x, d_t > y] &= \sum_{k \geq 1} \int_{[0, a]} \mathbf{P}_x[\tau_k \leq Nt, J_k \in dv, \tau_{k+1} > Ny] \\ &= \sum_{j \leq [Nt]} \int_{[0, a]} \left( \sum_{k \geq 1} \mathbf{P}_x[\tau_k = j, J_k \in dv] \right) \mathbf{P}_v[\tau_1 > Ny - j] \\ &= \sum_{j \leq [Nt]} \int_{[0, a]} u(j, x, dv) \mathbf{P}_v[\tau_1 > Ny - j]. \end{aligned} \quad (60)$$

Using Abel's summation, we get:

$$\begin{aligned} \mathbf{P}_N[J_0 = x, d_t > y] &= \sum_{j \leq [Nt]} \int_{[0, a]} (U(j, x, dv) - U(j-1, x, dv)) \mathbf{P}_v[\tau_1 > Ny - j] = \\ &\int_{[0, a]} U(Nt, x, dv) \mathbf{P}_v[\tau_1 > N(t-y) - 1] \\ &- \sum_{1 \leq j \leq Nt} \int_{[0, a]} U(j, x, dv) \mathbf{P}_v[\tau_1 = Ny - j - 1] - \mathbf{P}_x[\tau_1 > Ny] \end{aligned} \quad (61)$$

Of course,  $\mathbf{P}_x[\tau_1 > Ny] \rightarrow 0$  as  $N \rightarrow \infty$ .

As for the first term in the right hand side of equation (61), we get :

$$\begin{aligned}
& \int_{[0,a]} U(Nt, x, dv) \mathbf{P}_v[\tau_1 > N(t-y) - 1] \\
& \rightarrow \int_{[0,a]} \frac{\alpha t^\alpha}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \frac{\Pi(dv)}{\mathbf{E}_{\Pi^{(2)}}[\Phi/k]} \frac{\int_{[0,a]} \Phi(v, u) du}{\alpha(y-t)^\alpha} \\
& = \frac{\sin(\alpha\pi)}{\pi} \left( \frac{t}{y-t} \right)^\alpha.
\end{aligned} \tag{62}$$

For the second term in the right hand side of equation (61), first note that for  $\varepsilon \in (0, y)$ , making use of the uniform convergence property for slowly varying functions, one has :

$$\begin{aligned}
& \sum_{\varepsilon N \leq j \leq Nt} \int_{[0,a]} U(j, x, dv) \mathbf{P}_v[\tau_1 = Ny - j - 1] \\
& \sim \int_{[0,a]} \sum_{\varepsilon N \leq j \leq Nt} \frac{\alpha j^\alpha \Pi[dv]}{L(j) \Gamma(1+\alpha) \Gamma(1-\alpha) \mathbf{E}_{\Pi^{(2)}}[\Phi/k]} \times \frac{L(Ny - j - 1) \int_{[0,a]} \Phi(v, u) du}{(Ny - j - 1)^{\alpha+1}} \\
& \sim \frac{\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_\varepsilon^t \frac{u^\alpha}{(y-u)^{\alpha+1}} du.
\end{aligned} \tag{63}$$

On the other hand, we get easily that

$$\lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \sum_{j \leq \varepsilon N} \int_{[0,a]} U(j, x, dv) \mathbf{P}_v[\tau_1 = Ny - j - 1] = 0. \tag{64}$$

Thus we are left with checking the equality:

$$\frac{\sin(\alpha\pi)}{\pi} \int_y^\infty \frac{t^\alpha}{u(u-t)^\alpha} du = \frac{\sin(\alpha\pi)}{\pi} \left( \frac{t}{y-t} \right)^\alpha - \frac{\alpha \sin(\alpha\pi)}{\pi} \int_0^t \frac{u^\alpha}{(y-u)^{\alpha+1}} du \tag{65}$$

which is easy using integration by parts; just note that

$$\int_0^t \frac{u^\alpha}{(y-u)^{\alpha+1}} du = \frac{1}{\alpha} \left[ \left( \frac{u}{y-u} \right)^\alpha \right]_0^t - \int_0^t \frac{u^{\alpha-1}}{(y-u)^\alpha} du \tag{66}$$

and finally the equality

$$\int_0^t \frac{u^{\alpha-1}}{(y-u)^\alpha} du = \int_y^\infty \frac{t^\alpha}{u(u-t)^\alpha} du \tag{67}$$

is obvious.

## 4 Application to the strip wetting model

### 4.1 Asymptotics about the kernel of the free process.

The following transition kernel will be of basic importance for the proof of Theorem 1.4:

$$\begin{aligned} F_{x,dy}(n) &:= \mathbf{P}_x[S_1 > a, S_2 > a, \dots, S_{n-1} > a, S_n \in dy] \mathbf{1}_{x,y \in [0,a]} \text{ if } n \geq 2, \\ F_{x,dy}(1) &:= h(y-x) \mathbf{1}_{x,y \in [0,a]} dy \end{aligned} \quad (68)$$

and consider the process  $(J_n)_{n \geq 0}$  where  $J_n := S_{\tau_n}$ ; the process  $\tau$  is then a Markov renewal process whose modulating chain is the Markov chain  $J$ .

Note that the kernel  $F$  is defective, in the sense that  $\int_{[0,a]} \sum_{k \geq 1} F_{x,dy}(k) = \mathbf{P}_x[S_1 \geq 0] \leq \mathbf{P}[S_1 \geq -a] < 1$  for every  $x \in [0, a]$ . In particular, for every  $x \in [0, a]$ , the quantity  $\mathbf{P}_x[\tau_1 = \infty] := 1 - \int_{[0,a]} \sum_{k \geq 1} F_{x,dy}(k)$  is strictly positive.

We denote by  $l_N$  the cardinality of  $\{k \leq N | S_k \in [0, a]\}$ , and we define  $\bar{F}_x(k) := \int_{[0,a]} \sum_{j > k} F_{x,dy}(j)$ . With these notations, we can write the joint law of  $(l_N, (\tau_n)_{n \leq l_N}, (J_n)_{n \leq l_N})$  under  $\mathbf{P}_{N,a,\beta}$  under the following form:

$$\begin{aligned} \mathbf{P}_{N,a,\beta}[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1, \dots, k] \\ = \frac{e^{\beta k}}{Z_{N,a,\beta}} F_{0,dy_1}(t_1) F_{y_1,dy_2}(t_2 - t_1) \dots F_{y_{k-1},dy_k}(t_k - t_{k-1}) (\bar{F}_{y_k}(N - t_k) + \mathbf{P}_{y_k}[\tau_1 = \infty]) \end{aligned} \quad (69)$$

where  $k \in \mathbb{N}$ ,  $0 < t_1 < \dots < t_k \leq N$  and  $(y_i)_{i=1,\dots,k} \in \mathbb{R}^k$ .

To describe the asymptotic behavior of the kernel  $F$ , we will need some results issued from fluctuation theory for random walks. Let us collect some basic facts about this topic.

For  $n$  an integer, we denote by  $T_n$  the  $n$ th ladder epoch; that is  $T_0 := 0$  and, for  $n \geq 1$ ,  $T_n := \inf\{k \geq T_{n-1}, S_k > S_{T_{n-1}}\}$ . We also introduce the so-called ascending ladder height  $(H_n)_{n \geq 0}$ , which, for  $k \geq 1$ , is given by  $H_k := S_{T_k}$ . Note that the process  $(T, H)$  is a bivariate renewal process on  $(\mathbb{R}^+)^2$ . In a similar way, one may define the strict descending ladder variables process  $(T^-, H^-)$  as  $(T_0^-, H_0^-) := (0, 0)$  and

$$T_n^- := \inf\{k \geq T_{n-1}^-, S_k < S_{T_{n-1}^-}\} \quad \text{and} \quad H_k^- := -S_{T_k^-}. \quad (70)$$

It is well known (see [Fel71]) that the following relation holds

$$\sigma^2 = 2\mathbf{E}[H_1] \mathbf{E}[H_1^-]. \quad (71)$$

In particular, it follows that both  $\mathbf{E}[H_1]$  and  $\mathbf{E}[H_1^-]$  are finite. It is also a well known fact that the continuity of  $h(\cdot)$  implies the continuity of the distribution function of  $H_1$  as one can deduce from the identity  $\mathbf{P}[H_1^- \in I] = \sum_{k \geq 1} \mathbf{P}[T_1^- = k, S_k \in I]$  which is valid for every interval  $I$ .

Let us consider the renewal function  $U(\cdot)$  associated to the ascending ladder heights process:

$$U(x) := \sum_{k=0}^{\infty} \mathbf{P}[H_k \leq x] = \mathbf{E}[\mathcal{N}_x] \quad (72)$$

where  $\mathcal{N}_x$  is the cardinality of  $\{k \geq 0, H_k \leq x\}$ . It follows in particular from this definition that  $U(\cdot)$  is a subadditive increasing function, and in our context it is also continuous. Note also that  $U(0) = 1$ . We denote by  $V(x)$  the analogous quantity for the process  $H^-$ . Finally, for a positive integer  $n$ , for  $(u, x) \in (\mathbb{R}^+)^2$ , we will make use of the following transition density:

$$g_{n,u}(x) := \frac{1}{dx} \mathbf{P}_{-u}[S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0, S_n \in dx]. \quad (73)$$

With these notations, we show the following result in the chapter 3 of the thesis [Soh10]:

**Theorem 4.1.** *Assume that  $X$  is given by a bounded density, with zero mean and variance  $\sigma^2$ . Then:*

(i) *for any fixed  $u \geq 0$  the following convergence holds uniformly for  $x \geq 0$ :*

$$\frac{1}{dx} \mathbf{P}_{-u}[S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0, S_n \in dx] \sim U(u) \frac{\mathbf{P}[H_1 \geq x]}{\sigma \sqrt{2\pi n^{3/2}}}. \quad (74)$$

(ii) *moreover, the sequence of functions  $\sigma \sqrt{2\pi n^{3/2}} g_{n,u}(x)$  is dominated by a multiple of its limit:  $n^{3/2} g_{n,u}(x) \leq \mathcal{C} U(u) \mathbf{P}[H_1 \geq x]$  for every  $n, u, x$ ; consequently, one has also that*

$$\mathbf{P}_{-u}[S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0] \sim U(u) \frac{\mathbf{E}[H_1]}{\sigma \sqrt{2\pi n^{3/2}}}. \quad (75)$$

**Remark 4.2.** *Of course, considering the random walk  $\tilde{S}$  whose transitions are given by  $\mathbf{P}_x[\tilde{S}_1 \in dy] := h(x - y)$ , Theorem 4.1 implies immediately that, for fixed  $u \geq 0$ , the following equivalence holds uniformly for  $x \geq 0$ :*

$$\tilde{g}_{n,u}(x) := \mathbf{P}_u[S_1 \geq 0, \dots, S_{n-1} \geq 0, -S_n \in dx] \sim V(u) \frac{\mathbf{P}[H_1^- \geq x]}{\sigma \sqrt{2\pi n^{3/2}}} \quad (76)$$

*and obviously a similar statement as (ii) of the above theorem also holds for the quantity  $\mathbf{P}_u[S_1 \geq 0, \dots, S_{n-1} \geq 0, S_n < 0]$ .*

Note that this result has been shown in the case where  $S$  is lattice by Alili and Doney ([AD99]).

We define the following function:

$$\Phi_a(x, y) := \frac{\mathbf{P}[H_1^- \geq a - y]}{\sigma\sqrt{2\pi}} \int_a^{+\infty} h(t-x)V(t-a)dt \mathbf{1}_{x,y \in [0,a]}. \quad (77)$$

The integral defining  $\Phi_a(x, y)$  is finite for every  $(x, y) \in [0, a]^2$ . Indeed, making use of the classical renewal theorem and of the finiteness of  $\mathbf{E}[H_1^-]$ , one has  $V(t) \sim t/\mathbf{E}[H_1^-]$  as  $t \rightarrow \infty$ , and then we use that  $X_1 \in L^1(\mathbb{R})$ . Making use of the continuity of  $h(\cdot)$ , by an argument involving dominated convergence, it is also quite easy to see that  $\Phi_a(\cdot, \cdot)$  is a continuous function on  $[0, a]^2$ . The (strict) positivity of  $V(\cdot)$  on  $\mathbb{R}^+$  coupled with the fact that  $\mathbf{P}[S_1 > a] > 0$  entails that the integral in the definition of  $\Phi_a$  is strictly positive for  $x \in [0, a]$ ; on the other hand, as for all  $y \in [0, a]$ ,  $\mathbf{P}[H_1^- > a - y] \geq \mathbf{P}[H_1^- > a] \geq \mathbf{P}[-S_1 > a] > 0$ , we get that  $\Phi_a$  is strictly positive on  $[0, a]^2$ .

The next result provides useful estimates on  $F(\cdot, \cdot)(n)$  :

**Lemma 4.3.** *The following equivalence holds uniformly on  $(x, y) \in [0, a]^2$ :*

$$n^{3/2} \frac{1}{dy} F_{x,dy}(n) \sim \Phi_a(x, y). \quad (78)$$

*Proof.* For  $n \geq 2$ , integrating over the first step of  $S$  which is not in  $[0, a]$ , one gets:

$$F_{x,dy}(n) = \int_a^{+\infty} h(t-x) \mathbf{P}_t[S_1 > a, S_2 > a, \dots, S_{n-2} > a, S_{n-1} \in dy] dt \mathbf{1}_{x,y \in [0,a]}. \quad (79)$$

Let  $\varepsilon > 0$  be fixed; for every  $n$ :

$$\begin{aligned} & \sup_{(x,y) \in [0,a]^2} |n^{3/2} F_{x,dy}(n) - \Phi_a(x, y)| \\ & \leq \sup_{x \in [0,a]} \int_a^{+\infty} dt h(t-x) \sup_{y \in [0,a]} \left| n^{3/2} \tilde{g}_{n,t-a}(y-a) - V(t-a) \frac{\mathbf{P}[H_1^- \geq a-y]}{\sigma\sqrt{2\pi}} \right|. \end{aligned} \quad (80)$$

The integrand appearing in the right hand side of the inequality above vanishes for large  $n$  for fixed  $x$ ; defining

$$f_n(t) := \sup_{y \in [0,a]} \left| n^{3/2} \tilde{g}_{n,t-a}(y-a) - V(t-a) \frac{\mathbf{P}[H_1^- \geq a-y]}{\sigma\sqrt{2\pi}} \right|, \quad (81)$$

we get that  $f_n$  converges simply towards the null function on  $(a, \infty)$ , and making use of the domination part of Theorem 4.1 and of Remark 4.2, we get that it is dominated by a multiple of  $V(t - a)$  which verifies  $\int_{(a, \infty)} V(t - a)h(t - x)dt < \infty$ .

Consider then  $\varepsilon > 0$ ; there exists  $M_1 > 0$  such that for every  $n \in \mathbb{N}$ ,

$$\sup_{x \in [0, a]} \int_{M_1}^{\infty} h(t - x)f_n(t)dt \leq \varepsilon. \quad (82)$$

Getting the convergence towards zero of the quantity  $\sup_{x \in [0, a]} \int_a^{M_1} h(t - x)f_n(t)dt$  is easy (just make use of the boundedness of  $h(\cdot)$  and of the dominated convergence theorem). Thus the uniform convergence which is stated in Lemma 4.3. □

## 4.2 Defining the free energy through an infinite dimensional problem.

In this section, we define the free energy in a way that allows us to make use of the underlying Markov renewal structure inherent to this model. For  $\lambda \geq 0$ , we introduce the following kernel:

$$B_{x, dy}^\lambda := \sum_{n=1}^{\infty} e^{-\lambda n} F_{x, dy}(n) \quad (83)$$

and the associated integral operator

$$(B^\lambda h)(x) := \int_{[0, a]} B_{x, dy}^\lambda h(y). \quad (84)$$

We then have the

**Lemma 4.4.** *For  $\lambda \geq 0$ ,  $B_{x, dy}^\lambda$  is a compact operator on the Hilbert space  $L^2([0, a])$ .*

*Proof.* Note that for all  $n \in \mathbb{N}$ ,  $F_{x, dy}(n)$  has a density  $f_{x, y}(n)$  with respect to the Lebesgue measure restricted to  $[0, a]$ , and this implies that  $B_{x, dy}^\lambda$  also has one (we denote it by  $b^\lambda(x, y)$ ). To show Lemma 4.4, it is sufficient to show that  $B_{x, dy}^\lambda$  is actually Hilbert-Schmidt, that is that

$$\int b^\lambda(x, y)^2 \mathbf{1}_{x, y \in [0, a]} dx dy < \infty. \quad (85)$$

Note that we already used in the proof of Lemma 4.3 that there exists  $C' > 0$  such that, for all  $n \in \mathbb{N}$  and  $x, y \in [0, a]$ ,  $f_{x,y}(n) \leq C' \frac{\tilde{g}(y-a)}{n^{3/2}}$ .

It is then straightforward to show (85). Actually, we have the inequalities:

$$\begin{aligned} & \int b^\lambda(x, y)^2 \mathbf{1}_{x,y \in [0,a]} dx dy \\ &= \int \left( \sum_{n=1}^{\infty} e^{-\lambda n} f_{x,y}(n) \right)^2 \mathbf{1}_{x,y \in [0,a]} dx dy \\ &\leq a \int \sum_{n,m \geq 1} e^{-\lambda(n+m)} C'^2 g(y-a)^2 n^{-3/2} m^{-3/2} \mathbf{1}_{y \in [0,a]} dy < \infty. \end{aligned} \quad (86)$$

□

Lemma 4.4 enables us to introduce  $\delta^a(\lambda)$ , the spectral radius of the operator  $B^\lambda$ . It is easy to check that  $\delta^a(\lambda) \in (0, \infty)$  for  $\lambda \geq 0$ ;  $\delta^a(\lambda)$  is an isolated and simple eigenvalue of  $B_{x,dy}^\lambda$  (see Theorem 1 in [Zer87]). The function  $\delta^a(\cdot)$  is non-increasing, continuous on  $[0, \infty)$  and analytic on  $(0, \infty)$  because the operator  $B_{x,dy}^\lambda$  has these properties. The analyticity and the fact that  $\delta^a(\cdot)$  is not constant (as  $\delta^a(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ ) force  $\delta^a(\cdot)$  to be strictly decreasing.

We denote by  $(\delta^a)^{-1}(\cdot)$  its inverse function, defined on  $(0, \delta^a(0)]$ . We now define  $\beta_c^a$  and  $F^a(\beta)$  by:

$$\beta_c^a := -\log(\delta^a(0)), \quad F^a(\beta) := (\delta^a)^{-1}(\exp(-\beta)) \text{ if } \beta \geq \beta_c^a \text{ and } 0 \text{ otherwise.} \quad (87)$$

Note that this definition entails in particular the analyticity of  $F(\cdot)$  on  $\mathbb{R} \setminus \{\beta_c^a\}$ . We discuss below the relevance of this definition in accordance with the definition given in equation (9).

The fact that  $b^{F^a(\beta)}(x, y) > 0$  for every  $(x, y) \in [0, a]$  implies the uniqueness (up to a multiplication by a positive constant) and the positivity almost everywhere of the so-called right and left Perron Frobenius eigenfunctions of  $B_{x,dy}^{F^a(\beta)}$ ; more precisely, Theorem 1 in [Zer87] implies that there exist two functions  $v_\beta(\cdot)$  and  $w_\beta(\cdot)$  in  $L^2([0, a])$  such that  $v_\beta(x) > 0$  and  $w_\beta(x) > 0$  for almost every  $x \in [0, a]$ , and moreover:

$$\int_{y \in [0,a]} B_{x,dy}^{F^a(\beta)} v_\beta(y) = \left( \frac{1}{e^\beta} \wedge \frac{1}{e^{\beta_c^a}} \right) v_\beta(x) \quad (88)$$

$$\int_{x \in [0,a]} w_\beta(x) B_{x,dy}^{F^a(\beta)} dx = \left( \frac{1}{e^\beta} \wedge \frac{1}{e^{\beta_c^a}} \right) w_\beta(y) dy. \quad (89)$$

Spelling out these equalities, we get that

$$v_\beta(x) = \frac{1}{\frac{1}{e^\beta} \wedge \frac{1}{e^{\beta_c^a}}} \sum_{n \geq 0} e^{-F^a(\beta)n} \int_{y \in [0,a]} f_{x,y}(n) v_\beta(y) dy, \quad (90)$$



which implies in particular the fact that  $v_\beta(\cdot)$  is positive and continuous on the whole  $[0, a]$  and not only almost everywhere. Similarly, the function  $w_\beta(\cdot)$  is everywhere positive and continuous. These considerations lead us to define the new kernel

$$K_{x,dy}^\beta(n) := e^\beta F_{x,dy}(n) e^{-F^a(\beta)n} \frac{v_\beta(y)}{v_\beta(x)}. \quad (91)$$

It is then straightforward to check that

$$\begin{aligned} \int_{y \in \mathbb{R}} \sum_{n \in \mathbb{N}} K_{x,dy}^\beta(n) &= \frac{e^\beta}{v_\beta(x)} \int_{y \in \mathbb{R}} \sum_{n \in \mathbb{N}} F_{x,dy}(n) e^{-F^a(\beta)n} v_\beta(y) \mathbf{1}_{y \in [0, a]} \\ &= \frac{e^\beta}{v_\beta(x)} \int_{y \in \mathbb{R}} B_{x,dy}^{F^a(\beta)} v_\beta(y) = 1 \wedge \frac{e^\beta}{e^{\beta a}}. \end{aligned} \quad (92)$$

In particular, when  $\beta = \beta_c^a$ ,  $K_{\cdot, \cdot}^{\beta_c^a}$  is a non defective Markov renewal kernel, which satisfies Assumption 1.1 with a trivial slowly varying function (that is converging at infinity),  $\alpha = 1/2$  and  $\tilde{\Phi}_\beta(x, y) := e^{\beta \frac{v_\beta(y)}{v_\beta(x)}} \Phi_a(x, y)$ . Note that it is not hard to see that for every  $\beta \in \mathbb{R}$ , the measure  $\Pi_\beta(dx) := v_\beta(x) w_\beta(x) dx$  is invariant for the Markov chain  $J$ ; moreover, both  $v_\beta(\cdot)$  and  $w_\beta(\cdot)$  being defined up to a multiplicative constant, one can use this degree of freedom to make  $\Pi_\beta$  a probability measure on  $[0, a]$ .

Let us point out that it is not clear *a priori* that the quantity we define in (87) actually coincides with the definition given in equation (9). We will not delve into this issue in this paper; however, we stress that this identification relies on applying the Markov renewal theorem to the kernel  $K^\beta$  in the localized phase (thus when the Markov renewal associated to  $K^\beta$  is recurrent positive), and on direct (polynomial) asymptotics on the partition functions in the delocalized one (for more details, see the thesis [Soh10][Chapter 2]). For all relevant purposes, we will use the definition given in equation (87) as a definition for the free energy.

### 4.3 Proof of Theorem 1.4.

We will need the asymptotic behavior of  $Z_{N,a,\beta_c^a}(x)$ , which is given in the next lemma:

**Lemma 4.5.** *As  $N \rightarrow \infty$  and for  $x \in [0, a]$ , the following equivalence holds:*

$$\begin{aligned} Z_{N,a,\beta_c^a}(x) &\sim N^{1/2} \frac{v_{\beta_c^a}(x)(1 - e^{-\beta_c^a})}{\pi e^{\beta_c^a} \int_{[0,a]^2} v_{\beta_c^a}(t) w_{\beta_c^a}(s) \Phi_a(s, t) ds dt} \int_{[0,a]} w_{\beta_c^a}(s) ds \\ &\sim N^{1/2} v_{\beta_c^a}(x) C_{a,\Phi}. \end{aligned} \quad (93)$$

*Proof of Lemma 4.5.* For  $(x, y, n) \in [0, a]^2 \times \mathbb{N}$ , we denote by  $\kappa$  the Markov renewal mass function associated to the kernel  $K_{\cdot, \cdot}^{\beta_c^a}$ , that is  $\kappa(n, x, dy) := \sum_{k \geq 1} (K^{\beta_c^a})_{x, dy}^{*k}(n)$ .

Conditioning on the last excursion before time  $N$ , we have:

$$\begin{aligned}
Z_{N, a, \beta_c^a}(x) &= v_{\beta_c^a}(x) \int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \frac{1}{v_{\beta_c^a}(y)} \mathbf{P}_y[\tau_1 > N - n] \\
&\quad + v_{\beta_c^a}(x) \int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \frac{1}{v_{\beta_c^a}(y)} \mathbf{P}_y[\tau_1 = \infty] \\
&= v_{\beta_c^a}(x) \int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \sum_{j > N-n} \int_{[0, a]} \frac{K_{y, du}^{\beta_c^a}(j)}{v_{\beta_c^a}(u)} \\
&\quad + v_{\beta_c^a}(x) \int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \frac{1}{v_{\beta_c^a}(y)} \mathbf{P}_y[\tau_1 = \infty].
\end{aligned} \tag{94}$$

Taking into account the fact that  $v_{\beta_c^a}(\cdot)$  is continuous and positive on  $[0, a]$ , and that  $K_{\cdot, \cdot}^{\beta_c^a}$  is non defective (and thus that  $\int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \sum_{j > N-n} \int_{[0, a]} K_{y, du}^{\beta_c^a}(j) = \mathbf{P}_x[S_1 > 0] \leq 1$ ), the first term in the right hand side of equation (94) is bounded as  $N$  becomes large. As for the second term, denoting by  $k_{\beta_c^a}(s, t) := \frac{1}{dt} \sum_{n \geq 1} K_{s, dt}^{\beta_c^a}(n)$ , we make use of Proposition 2.1 to get the equivalence:

$$\int_{[0, a]} \sum_{n=0}^N \kappa(n, x, dy) \frac{\mathbf{P}_y[\tau_1 = \infty]}{v_{\beta_c^a}(y)} \sim \frac{N^{1/2}}{\pi} \int_{[0, a]} \frac{\Pi_{\beta_c^a}(dy)}{\mathbf{E}_{\Pi^{(2)}}[\tilde{\Phi}_{\beta_c^a}/k_{\beta_c^a}]} \frac{\mathbf{P}_y[\tau_1 = \infty]}{v_{\beta_c^a}(y)}. \tag{95}$$

Then we recall that  $\Pi_{\beta_c^a}(du) = v_{\beta_c^a}(u)w_{\beta_c^a}(u)du$  and we make use of the equality

$$\int_{[0, a]} w_{\beta_c^a}(s) \left( 1 - \sum_{j \geq 1} \int_{[0, a]} F_{s, dt}(j) \right) ds = (1 - e^{-\beta_c^a}) \int_{[0, a]} w_{\beta_c^a}(t) dt. \tag{96}$$

Finally making use of equality (18) we get Lemma 4.5.  $\square$

Now we go to the proof of Theorem 1.4.

*Proof of Theorem 1.4.* As in the proof of Theorem 1.2, we just have to show finite dimensional convergence.

We denote by  $\mathbf{P}_{\mathcal{B}_{1/2}}$  the law of the limiting random set appearing in Theorem 1.2, by  $R$  the length of the brownian excursion and for  $F \in \mathcal{F}$ ,  $g_t(F) = \sup(F \cap [0, t])$ .

Let us first compute the limiting quantity we are looking for. For  $0 < s < t < 1$ , we write:

$$\begin{aligned}
\mathbf{P}_{\mathcal{B}_{1/2}}(d_s > t) &= \mathbf{P}_{\mathcal{B}_{1/2}}(g_t < s, g_1 = g_s) + \mathbf{P}_{\mathcal{B}_{1/2}}(g_t < s, g_1 > t) \\
&= \mathbf{P}_{\mathcal{B}_{1/2}}(g_1 \leq s) \\
&+ \frac{\pi}{2} \int_{u \in [0, s]} \int_{v \in [t-u, 1-u]} \int_{w \in [u+v, 1]} \mathbf{P}(g_t \in du, d_t - g_t \in dv) \sqrt{1-w} \mathbf{P}(g_1 \in dw | d_t = u+v).
\end{aligned} \tag{97}$$

We first make use of the explicit law of  $g_1$  (see [Ber99][Proposition 3.1]), and we deduce the following equalities:

$$\begin{aligned}
\mathbf{P}_{\mathcal{B}_{1/2}}(d_s > 1) &= \frac{\pi}{2} \int_{[0, s]} \sqrt{1-u} \mathbf{P}_{1/2}(g_1 \in du) \\
&= \frac{\pi}{2} \int_{[0, s]} \sqrt{1-u} \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} du \\
&= \sqrt{s}.
\end{aligned} \tag{98}$$

Then we note that for every  $x \in (t, 1)$  and  $w \in (x, 1)$ , the following holds:

$$\mathbf{P}(g_1 \in dw | d_t = x) = \mathbf{P}(x + g_{1-x} \in dw) = \frac{1}{\pi} \frac{dw}{\sqrt{(w-x)(1-w)}}. \tag{99}$$

On the other hand, the following equality is well known (see [BS02]):

$$\mathbf{P}(g_t \in du, d_t - g_t \in dv) = \frac{dudv}{2\pi\sqrt{uv^3}}. \tag{100}$$

Finally one has:

$$\mathbf{P}_{\mathcal{B}_{1/2}}(d_s > t) = \sqrt{s} + \frac{1}{2\pi} \int_{(u,v) \in [0, s] \times [t, 1]} \sqrt{\frac{1-v}{u(v-u)^3}} dudv. \tag{101}$$

In the same spirit as for the proof of Theorem 1.2, we then establish the convergence of  $\mathbf{P}_{N, a, \beta_c^a}(d_s^N > t)$  towards  $\mathbf{P}_{\mathcal{B}_{1/2}}(d_s > t)$  as  $N \rightarrow \infty$ , thus proving Theorem 1.4. A computation similar to the one we made for the proof of Lemma 4.5 leads to the following convergence (for  $s \in (0, 1)$ ):

$$\mathbf{P}_{N, a, \beta_c^a}(d_s^N > 1) \rightarrow \sqrt{s}. \tag{102}$$

Thus we are left with computing the limit of the second term in the right hand side of the equality below:

$$\mathbf{P}_{N, a, \beta_c^a}(d_s^N > t) = \mathbf{P}_{N, a, \beta_c^a}(d_s^N \geq 1) + \mathbf{P}_{N, a, \beta_c^a}(d_s^N \in (t, 1)). \tag{103}$$

We write:

$$\begin{aligned} \mathbf{P}_{N,a,\beta_c^a}(d_s^N \in (t, 1)) &= \frac{v_{\beta_c^a}(0)}{Z_{N,a,\beta_c^a}} \sum_{n=0}^{sN} \int_{x \in [0,a]} \kappa(n, 0, x) \\ &\times \int_{y \in [0,a]} \sum_{m=tN}^N \frac{K_{x,dy}^{\beta_c^a}(m-n)}{v_{\beta_c^a}(y)} Z_{N-m,a,\beta_c^a}(y). \end{aligned} \quad (104)$$

Once again, we do not have access to the local behavior of the quantities  $\kappa(n, 0, x)$  for  $n$  large, thus we use integration by part as in the proof of Theorem 1.2 and we have:

$$\begin{aligned} \mathbf{P}_{N,a,\beta_c^a}(d_s^N \in (t, 1)) &= \frac{v_{\beta_c^a}(0)}{Z_{N,a,\beta_c^a}} \int_{x \in [0,a]} \kappa(sN, 0, dx) \sum_{m=tN}^N \int_{y \in [0,a]} \frac{K_{x,dy}^{\beta_c^a}(m-sN)}{v_{\beta_c^a}(y)} Z_{N-m,a,\beta_c^a}(y) \\ &+ \frac{v_{\beta_c^a}(0)}{Z_{N,a,\beta_c^a}} \sum_{n=0}^{sN-1} \int_{(x,y) \in [0,a]^2} \sum_{j=0}^n \kappa(j, 0, dx) \sum_{m=tN}^N \frac{K_{x,dy}^{\beta_c^a}(m-n) - K_{x,dy}^{\beta_c^a}(m-n-1)}{v_{\beta_c^a}(y)} Z_{N-m,a,\beta_c^a}(y) \\ &+ \frac{v_{\beta_c^a}(0)}{Z_{N,a,\beta_c^a}} \int_{(x,y) \in [0,a]^2} \sum_{m=tN}^N \frac{K_{0,dy}^{\beta_c^a}(m)}{v_{\beta_c^a}(y)} Z_{N-m,a,\beta_c^a}(y) \\ &=: (\star) + (\star\star) + (\star\star\star). \end{aligned} \quad (105)$$

Making use of Lemma 4.5, it is straightforward to check that  $(\star\star\star)$  is  $\mathcal{O}(N^{-1/2})$  as  $N \rightarrow \infty$ . As  $N \rightarrow \infty$ , the following holds:

$$\begin{aligned} (\star) &\sim \frac{1}{N^{1/2} C_{a,\Phi}} \int_{x \in [0,a]} \frac{\sqrt{sN}}{\pi} \frac{\Pi_{\beta_c^a}(dx)}{\mathbf{E}_{\Pi^{(2)}}[\tilde{\Phi}_{\beta_c^a}/k_{\beta_c^a}]} \sum_{m=tN}^N \int_{y \in [0,a]} \sqrt{N-m} \frac{C_{a,\Phi} \tilde{\Phi}_{\beta_c^a}(x,y)}{(m-sN)^{3/2}} dy \\ &\rightarrow \frac{\sqrt{s}}{\pi} \int_t^1 \frac{\sqrt{1-u}}{(u-s)^{3/2}} du. \end{aligned} \quad (106)$$

On the other hand, one gets easily that, as  $N \rightarrow \infty$ , the following convergence holds uniformly on  $[0, a]^2$ :

$$\frac{1}{dy} (K_{x,dy}^{\beta_c^a}(N) - K_{x,dy}^{\beta_c^a}(N-1)) \sim -\frac{3\tilde{\Phi}_{\beta_c^a}(x,y)}{2N^{5/2}}. \quad (107)$$

Thus we get the equivalence:

$$\begin{aligned}
(\star\star) &\sim -\frac{3}{2N^{1/2}C_{a,\Phi}} \sum_{n=0}^{sN-1} \int_{x \in [0,a]} \frac{\sqrt{n}}{\pi} \frac{\Pi_{\beta_c^a}(dx)}{\mathbf{E}_{\Pi(2)}[\tilde{\Phi}_{\beta_c^a}/k_{\beta_c^a}]} \sum_{m=tN}^N \int_{y \in [0,a]} \sqrt{N-m} \frac{C_{a,\Phi} \tilde{\Phi}_{\beta_c^a}(x,y)}{(m-n)^{5/2}} \\
&\sim -\frac{3}{2\pi} \int_{(u,v) \in [0,s] \times [t,1]} \frac{\sqrt{u(1-v)}}{(v-u)^{5/2}} dudv.
\end{aligned} \tag{108}$$

Putting everything together, we finally have the following convergence:

$$\mathbf{P}_{N,a,\beta_c^a}(d_s^N \in (t, 1)) \rightarrow \frac{\sqrt{s}}{\pi} \int_t^1 \frac{\sqrt{1-u}}{(u-s)^{3/2}} du - \frac{3}{2\pi} \int_{(u,v) \in [0,s] \times [t,1]} \frac{\sqrt{u(1-v)}}{(v-u)^{5/2}} dudv. \tag{109}$$

On the other hand, integrating by part, we get the equality:

$$\begin{aligned}
\int_{(u,v) \in [0,s] \times [t,1]} \sqrt{\frac{1-v}{u(v-u)^3}} &= \left[ \int_{v \in [t,1]} 2 \frac{\sqrt{u(1-v)}}{(v-u)^{3/2}} dv \right]_{u=0}^s - \frac{3}{2} \int_{(u,v) \in [0,s] \times [t,1]} \frac{2\sqrt{u(1-v)}}{(v-u)^{5/2}} dudv \\
&= 2\sqrt{s} \int_t^1 \frac{\sqrt{1-v}}{(v-s)^{3/2}} dv - \frac{3}{2} \int_{(u,v) \in [0,s] \times [t,1]} \frac{2\sqrt{u(1-v)}}{(v-u)^{5/2}} dudv.
\end{aligned} \tag{110}$$

With the help of the above, comparing the equality (101) and the convergence appearing in equation (109), we are done.  $\square$

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