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Fan Ai-Hua, Lingmin Liao, Ji-Hua Ma

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AI-HUA FAN, LINGMIN LIAO, AND JI-HUA MA

Abstract. We propose to study multiple ergodic averages from multifractal analysis point of view. In some special cases in the symbolic dynamics, Hausdorff dimensions of the level sets of multiple ergodic average limit are determined by using Riesz products.

1. Introduction

Let (X, T) be a topological dynamical system and let $\ell \geq 2$ be a positive integer. We consider the following multiple ergodic averages

$$\frac{1}{n}\sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x), \tag{1.1}$$

where $f_1, \cdots, f_\ell$ are ℓ given continuous functions. Such multiple ergodic averages were introduced and studied by Furstenberg [9] in his ergodic theoretic proof of Szemerédi's theorem on arithmetic progressions. Since then these averages have received extensive studies in various contexts. For example, the L^2-normal convergence of (1.1) is proved by Host and Kra [11] with respect to a given invariant measure, and the almost sure convergence is proved earlier by Bourgain [2] in the case of $\ell = 2$. In this note we propose to study these multiple ergodic averages from multifractal analysis point of view.

Multifractal analysis of ergodic averages concerns the Hausdorff dimension of the level sets of the ergodic average limit. It reflects the complex behavior of the underlying chaotic dynamical system. There was a wide study in the case of simple ergodic averages ($\ell = 1$) in the last decades ([6, 7, 8, 13, 14, 15, 16]). Our first investigation shows that the multifractal analysis of multiple ergodic averages ($\ell \geq 2$) is much more difficult. This note aims at a special case where X is the symbolic space $\mathbb{D} = \{+1, -1\}^{\mathbb{N}}$ ($\mathbb{N}$ denoting the set of positive integers) and the dynamics is defined by the shift transformation $T : (x_1, x_2, \cdots) \mapsto (x_2, x_3 \cdots)$. The metric on $\mathbb{D}$ is chosen to be

$$\rho(x, y) = 2^{-\min\{k \geq 1 : x_k \neq y_k\}} \quad \text{for } x, y \in \mathbb{D}.$$

The Hausdorff dimension of a set A will be denoted by $\dim_H A$. See [3] for notions of dimensions of a set and [4] for notions of dimensions of a measure. Let $\ell \geq 1$. We shall examine the averages (1.1) with the functions

$$f_1(x) = f_2(x) = \cdots = f_\ell(x) = x_1 \quad \text{for } x \in \mathbb{D}. \tag{1.2}$$

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Then for $\theta \in [-1,1]$, we consider the level set

$$B_\theta := \left\{ x \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

We prove the following result.

Theorem 1.1. *For any $\theta \in [-1,1]$, we have*

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\Big(\frac{1+\theta}{2}\Big),$$

where $H(t) = -t\log_2 t - (1-t)\log_2(1-t)$ is the entropy function.

This result was known to Besicovitch and Eggleston when $\ell = 1$. Remark that the Hausdorff dimension of B_θ is strictly positive for any $\theta \in [0,1]$ when $\ell \geq 2$. Actually,

$$\dim_H B_\theta \geq 1 - 1/\ell > 0 \quad \text{if} \quad \ell \geq 2.$$

The proof of the theorem is based on the fact that $\mathbb{D}$ has a group structure and the functions $x \mapsto x_k x_{2k} \cdots x_{\ell k}$ are group characters and even they constitute a dissociated set of characters in the sense of Hewitt-Zuckermann [10]. As we shall show, the set B_θ supports a Riesz product, a nice measure which has the same Hausdorff dimension as that of B_θ. The idea of using Riesz product is inspired by [5] where oriented walks were studied. Although the Riesz product works perfectly for the above case concerned by Theorem 1.1, it has its limit for the general case.

We point out that the situation seems very different when the functions in (1.2) are replaced by other functions. For example, when f_i are chosen as $(x_1 + 1)/2$ which takes 0 and 1 as values. The obtained set can be identified with

$$A_\theta := \left\{ x \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

The set A_θ is similar to B_θ, but the determination of its dimension is more difficult.

Actually, we are motivated by the study of A_θ. The Riesz product method is not adapted to it. Then we propose to looking at the following set

$$X_0 := \left\{ x \in \{0,1\}^{\mathbb{N}} : \ x_n x_{2n} = 0, \quad \text{for all } n \right\},$$

which is a subset of A_θ with $\ell = 2$ and $\theta = 0$. We obtain the box dimension (denoted by $\dim_B$) for X_0 by a combinatoric method.

Theorem 1.2. *Let $\{a_n\}$ be the Fibonacci sequence defined by*

$$a_0 = 1, \ a_1 = 2, \quad a_n = a_{n-1} + a_{n-2} \ (n \geq 2).$$

We have

$$\dim_B(X_0) = \frac{1}{2\log 2} \sum_{n=1}^{\infty} \frac{\log a_n}{2^n} = 0.8242936\cdots.$$

The problem of determining the Hausdorff dimension of X_0 is now solved by Kenyon, Peres and Solomyak [12], where a class of sets similar to X_0 is studied. The result in [12] together with Theorem 1.2 shows that $\dim_H X_0 < \dim_B X_0$.

2. Riesz products

Let us consider $\mathbb{D}$ as an infinite product group of the multiplicative group $\{+1,-1\}$. The dual group of $\mathbb{D}$ consists of the Walsh functions $\{w_n(x)\}_{n=0}^{\infty}$ defined as follows. Define $w_0=1$. For each $n\geq 1$, let

$$n=2^{n_1-1}+2^{n_2-1}+\cdots+2^{n_s-1},\quad 1\leq n_1<n_2<\cdots<n_s,$$

be the unique expansion of the integer n in base 2. Then we define

$$w_n(x)=x_{n_1}x_{n_2}\cdots x_{n_s}.$$

An important subset of Walsh functions is the set of the Rademacher functions $\{r_n(x)\}_{n=1}^{\infty}$ defined by $r_n(x)=x_n$. The Rademacher functions are mutually independent with expectation zero with respect to the Haar measure. The following immediate consequence of the independence will be frequently used in the sequel.

Lemma 2.1. *Let f and g be two Haar integrable functions on $\mathbb{D}$. Suppose that f depends only on the first n coordinates of x and g is independent of the first n coordinates. Then*

$$\int f(x)g(x)dx=\int f(x)dx\int g(x)dx$$

where dx stands for the Haar measure on $\mathbb{D}$.

The n-th Fourier coefficient of an integrable function f is defined by

$$\hat{f}(n)=\int f(x)w_n(x)dx.$$

In the follows, we shall denote

$$\xi_k(x)=x_kx_{2k}\cdots x_{\ell k}\quad \text{ for all } k\geq 1.$$

Consider the product

$$dP_\theta(x)=\prod_{k=1}^{\infty}\big(1+\theta\xi_k(x)\big)dx.$$

The following lemma shows that the above product defines a probability measure on $\mathbb{D}$, which will be called Riesz product.

Lemma 2.2. *The partial products of the above infinite product converge in the weak-$*$ topology to a probability measure P_θ. Furthermore, for any function f depending only on the first n coordinates of x, we have*

$$\mathbb{E}_\theta[f]=\int f(x)\prod_{k=1}^{\lfloor n/\ell\rfloor}\big(1+\theta\xi_k(x)\big)dx, \tag{2.1}$$

where $\mathbb{E}_\theta[\,\cdot\,]$ stands for the expectation with respect to P_θ and "$\lfloor\,\cdot\,\rfloor$" is the integer part function.

Proof For $N\geq 1$, let

$$P_N(x)=\prod_{k=1}^{N}\big(1+\theta\xi_k(x)\big).$$

Then

$$P_{N+1}(x)-P_N(x)=\theta P_N(x)\xi_{N+1}(x).$$

Observe that for the fixed Walsh function $w_n(x) = x_{n_1}x_{n_2}\cdots x_{n_s}$, by Lemma 2.1, one has

$$\int P_N(x)\xi_{N+1}(x)w_n(x)dx = 0$$

whenever $(N+1)\ell > n_s$. It follows that $\hat{P}_N(n) = \hat{P}_{N+1}(n)$ for large N, so the limit

$$\lim_{N\to\infty}\int P_N(x)w_n dx$$

exists. That is to say, the measures $P_N(x)dx$ converge weakly to a limit measure P_θ.

The formula (2.1) follows directly from Lemma 2.1 and the definition of the Riesz product P_θ as a weak limit. □

The functions ξ_n are not P_θ-independent, but they are orthogonal. Therefore, we can get the following law of large numbers.

Lemma 2.3. *Suppose that g is a function on the interval $[-1,1]$ such that*

$$g(t) = \sum_{n=0}^{\infty} g_n t^n \quad \textit{with} \quad \sum_{n=1}^{\infty}|g_n| < \infty.$$

Then for P_θ-almost all x,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} g(\xi_k(x)) = \mathbb{E}_\theta[g(\xi_1)].$$

Proof Notice that $\xi_k^{2n}(x) = 1$ and $\xi_k^{2n-1}(x) = \xi_k(x)$ for any integer $n \geq 1$. Then we get

$$g(\xi_k) = \sum_{n=0}^{\infty} g_{2n} + \xi_k \sum_{n=1}^{\infty} g_{2n-1}.$$

By the formula (2.1), we have

$$\mathbb{E}_\theta(\xi_k) = \theta, \quad \mathbb{E}_\theta(\xi_j\xi_k) = \theta^2, \quad (j \neq k).$$

It follows that

$$\mathbb{E}_\theta[g(\xi_k)] = \sum_{n=0}^{\infty} g_{2n} + \theta\sum_{n=1}^{\infty} g_{2n-1}, \qquad \mathrm{Cov}_\theta[g(\xi_j), g(\xi_k)] = 0 \quad (j \neq k).$$

Therefore, the system $g(\xi_k) - \mathbb{E}_\theta[g(\xi_k)]$ $(k = 1, 2, \cdots)$ is orthogonal in $L^2(P_\theta)$. By the Menchoff Theorem ([17]), the series

$$\sum_{k=0}^{\infty}\frac{1}{k}\Big(g(\xi_k) - \mathbb{E}_\theta[g(\xi_k)]\Big)$$

converges P_θ-almost surely. Now the desired result follows from Kronecker's theorem. □

3. Proof of Theorem 1.1

Applying Lemma 2.3 to $g(t) = t$, we get that for P_θ-almost all x,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \xi_k(x) = \mathbb{E}(\xi_1) = \theta.$$

This means that the Riesz product P_θ is supported by the set B_θ. Now we are going to compute the local dimension of the Riesz product P_θ and we will apply Billingsley's theorem to conclude Theorem 1.1.

For each $x \in \mathbb{D}$ and $n \geq 1$, let

$$I_n(x) = I(x_1, \cdots, x_n) = \{y \in \mathbb{D} : y_k = x_k \text{ for } 1 \leq k \leq n\}.$$

It is the n-cylinder containing x, a ball of diameter 2^{-n}. By the formula (2.1), for any $n \geq \ell$, we have

$$P_\theta(I_n(x)) = \frac{1}{2^n} \prod_{k=1}^{\lfloor n/\ell \rfloor} \big(1 + \theta\xi_k(x)\big).$$

Recalling that $\xi_k(x) = +1$ or -1 for all x, by Taylor formula, we have

$$\log(1 + \theta\xi_k(x)) = -\sum_{n=1}^{\infty} \frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{2n-1} \xi_k(x).$$

Then for all points $x \in B_\theta$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \log(1 + \theta\xi_k(x)) = -\sum_{n=1}^{\infty} \frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{2n-1} \theta.$$

The right hand side can be written as

$$\theta \log(1+\theta) - \frac{\theta - 1}{2} \log(1 - \theta^2) = \left[1 - H\left(\frac{1+\theta}{2}\right)\right] \log 2.$$

It then follows that for all points $x \in B_\theta$,

$$\lim_{n\to\infty} \frac{\log P_\theta(I_n(x))}{\log |I_n(x)|} = \lim_{n\to\infty} \frac{\sum_{k=1}^{\lfloor n/\ell \rfloor} \log\big(1 + \theta\xi_k(x)\big) - \log 2^n}{\log 2^{-n}} = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\Big(\frac{1+\theta}{2}\Big).$$

The proof is completed by applying Billingsley's theorem ([1]). □

4. Proof of Theorem 1.2

It is clear that

$$\dim_B X_0 = \lim_{n\to\infty} \frac{\log_2 N_n}{n}$$

if the limit exists, where N_n is the cardinality of the following set

$$\{(x_1 x_2 \cdots x_n) : x_\ell x_{2\ell} = 0 \ \text{ for } \ell \geq 1 \text{ such that } \ 2\ell \leq n\}.$$

Each equality $x_\ell x_{2\ell} = 0$ defines a condition on the sequence $(x_1 \cdots x_n)$ which determines the cylinder $I(x_1, \cdots, x_n)$. We observe that all these conditions can be

divided into "independent" groups of conditions. Let

$$\begin{aligned}
C_0 &:= \{1,\ 3,\ 5,\ \ldots,\ 2n_0-1\},\\
C_1 &:= \left\{2\cdot 1,\ 2\cdot 3,\ 2\cdot 5,\ \ldots,\ 2\cdot\big(2n_1-1\big)\right\},\\
&\cdots\\
C_k &:= \left\{2^k\cdot 1,\ 2^k\cdot 3,\ 2^k\cdot 5,\ \ldots,\ 2^k\cdot(2n_k-1)\right\},\\
&\cdots\\
C_m &:= \{2^m\cdot 1\},
\end{aligned}$$

where n_k is the biggest integer such that

$$2^k(2n_k-1)\le n,\quad \text{i.e.,}\quad n_k=\left\lfloor \frac{n}{2^{k+1}}+\frac{1}{2}\right\rfloor$$

and m is the biggest integer such that

$$2^m\le n,\quad \text{i.e.,}\quad m=\lfloor \log_2 n\rfloor.$$

We have the decomposition $\{1,\cdots,n\}=C_0\sqcup C_1\sqcup\cdots\sqcup C_m$ and

$$n_0>n_1>\cdots>n_{m-1}>n_m=1.$$

The conditions $x_\ell x_{2\ell}=0$ with ℓ in different columns in the table defining $C_0,\cdots,C_m$ are independent. We are going to use this independence to count the number of possible choices for $(x_1,\cdots,x_n)$.

We have $n_m(=1)$ columns each of which has $m+1$ elements. Then we have a_{m+1} choices for x_ℓ with ℓ in the first column since $(x_\ell,x_{2\ell})$ is conditioned to be different from $(1,1)$. Each of the next $n_{m-1}-n_m$ columns has m elements, then we have $a_m^{n_{m-1}-n_m}$ choices for the x_ℓ's with ℓ in these columns. By induction, we get

$$N_n=a_{m+1}^{n_m}a_m^{n_{m-1}-n_m}a_{m-1}^{n_{m-2}-n_{m-1}}\cdots a_1^{n_0-n_1}.$$

Now, the box dimension of the set X_0 equals to

$$\begin{aligned}
\lim_{n\to\infty}\frac{\log_2 N_n}{n}&=\lim_{n\to\infty}\frac{1}{n}\left(n_m\log_2 a_{m+1}+\sum_{k=0}^{m}(n_{k-1}-n_k)\log_2 a_k\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\log_2 a_{m+1}+\sum_{k=0}^{\lfloor\log_2 n\rfloor}\left(\left\lfloor\frac{n}{2^k}+\frac{1}{2}\right\rfloor-\left\lfloor\frac{n}{2^{k+1}}+\frac{1}{2}\right\rfloor\right)\log_2 a_k\right)\\
&=\sum_{k=1}^{\infty}\frac{\log_2 a_k}{2^{k+1}}.
\end{aligned}$$

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LAMFA, UMR 6140 CNRS, Université de Picardie, 33 rue Saint Leu, 80039 Amiens, France
E-mail address: ai-hua.fan@u-picardie.fr

LAMA, CNRS UMR 8050, Université Paris-Est, 61 Avenue du Général de Gaulle, 94010, Créteil Cedex, France
E-mail address: lingmin.liao@u-pec.fr

Department of Mathematics, Wuhan University, 430072 Wuhan, China
E-mail address: jhma@whu.edu.cn