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# DYNAMICAL PROPERTIES OF MONOTONE DENDRITE MAPS

ISSAM NAGHMOUCHI

ABSTRACT. We show that for a monotone dendrite map  $f : D \rightarrow D$ , any  $\omega$ -limit set is either finite or a minimal Cantor set. We also prove that  $UR(f) = R(f) = \Lambda(f) = \overline{P(f)}$  where  $P(f)$ ,  $UR(f)$ ,  $R(f)$  and  $\Lambda(f)$  denote the sets of periodic points, uniformly recurrent points, recurrent points and the union of all  $\omega$ -limit sets respectively. Moreover, we prove that the following properties are equivalent: (i)  $R(f) = D$ , (ii)  $\overline{R(f)} = D$  and (iii)  $D \setminus \text{End}(D) \subset P(f)$ .

## 1. INTRODUCTION

This paper is devoted to monotone dendrite maps. The structure of  $\omega$ -limit set for monotone dendrite maps is studied here. Acosta and Eslami [1] proved that any infinite  $\omega$ -limit set of a dendrite homeomorphism is a Cantor minimal set. Efremova and Makhrova [4] constructed a homeomorphism of the Gehman dendrite having an infinite  $\omega$ -limit set which is a minimal Cantor set. In [12], we proved that for a monotone dendrite map, any infinite  $\omega$ -limit set does not contain periodic points. This paper can be viewed as a continuation of the paper [12]. We prove that any infinite  $\omega$ -limit set of monotone dendrite map is a minimal Cantor set (Corollary 1.3), this generalizes Acosta and Eslami result in [1]. For a graph map  $f$ , Hawete [5] proved that  $\overline{UR(f)} = \overline{R(f)}$ . For a monotone dendrite map  $f$ , we prove that  $UR(f) = R(f) = \Lambda(f) = \overline{P(f)}$ .

Oversteegen and Tymchatyn [13] showed that recurrent homeomorphisms of the plane are periodic. Kolev and Pérouème [6] proved that recurrent homeomorphisms of a compact surface with negative Euler characteristic are still periodic. In this direction, we prove that every relatively recurrent monotone dendrite map is a homeomorphism where every cut point is periodic (Theorem 1.6 and Corollary 1.7). Before stating our main results, we recall some basic properties of dendrites and monotone maps.

A continuum is a compact connected metric space. A topological space is arcwise connected if any two of its points can be joined by an arc. We use the terminologies from Nadler [11]. An arc is any space homeomorphic to the compact interval  $[0, 1]$ . A *dendrite*  $D$  is a locally connected continuum which contains no simple closed curve. Recall that any two distinct points

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$x, y$  of a dendrite  $D$  can be joined by a unique arc with endpoints  $x$  and  $y$ , denote this arc by  $[x, y]$  and let  $(x, y) = [x, y] \setminus \{y\}$  (resp.  $(x, y) = [x, y] \setminus \{x\}$  and  $(x, y) = [x, y] \setminus \{x, y\}$ ). A point  $e \in D$  is called an *endpoint* if  $D \setminus \{e\}$  is connected. The set of endpoints of  $D$  is denoted by  $End(D)$ . Any point  $x \in D \setminus End(D)$  is called a *cut point*. The set of cut points of  $D$  is dense in  $D$ . A continuous map from a dendrite into itself is called a dendrite map. Every dendrite has the fixed point property (see [11]); that is every dendrite map has a fixed point.

Let  $\mathbb{Z}_+$  and  $\mathbb{N}$  be the sets of non-negative integers and positive integers respectively. Let  $X$  be a compact metric space with metric  $d$ . For a subset  $A$  of  $X$ , denote by  $\overline{A}$  the closure of  $A$  and by  $diam(A)$  the diameter of  $A$ . For  $\delta > 0$  and  $x \in X$ , denote by  $B(x, \delta) := \{y \in X : d(x, y) < \delta\}$ . Let  $f : X \rightarrow X$  be a continuous map. Denote by  $f^n$  the  $n$ -th iterate of  $f$ ; that is,  $f^0 = \text{Identity}$  and  $f^n = f \circ f^{n-1}$  if  $n \geq 1$ . For any  $x \in X$  the subset  $O_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$  is called the  $f$ -orbit of  $x$ . A point  $x \in X$  is called periodic of prime period  $n \in \mathbb{N}$  if  $f^n(x) = x$  and  $f^i(x) \neq x$  for  $1 \leq i \leq n-1$ . We define the  $\omega$ -limit set of a point  $x$  to be the set

$$\begin{aligned} \omega_f(x) &= \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow \infty, \lim_{i \rightarrow \infty} d(f^{n_i}(x), y) = 0\} \\ &= \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}. \end{aligned}$$

A point  $x \in X$  is said to be recurrent for  $f$  if  $x \in \omega_f(x)$ . The set  $\omega_f(x)$  is a non-empty, closed and strongly invariant set, i.e.  $f(\omega_f(x)) = \omega_f(x)$ . If  $\omega_f(x)$  is finite then it is a periodic orbit. If  $\omega_{f^m}(x)$  is finite for some  $m \in \mathbb{N}$  then  $\omega_f(x)$  is also finite (see [3] for more details). A subset  $A \subset X$  is called  $f$ -invariant if  $f(A) \subset A$ . It is called a *minimal set of  $f$*  if it is nonempty, closed,  $f$ -invariant and minimal (in the sense of inclusion) for these properties. If  $X$  is a minimal set of  $f$ , we say that  $f$  is a minimal map; in this case, every  $f$ -orbit is dense in  $X$ . A point  $x \in X$  is called *uniformly recurrent* of  $f$  if for any neighborhood  $U$  of  $x$  there exists  $N \in \mathbb{N}$  such that  $\{f^{n+i}(x) : i = 0, 1, \dots, N\} \cap U \neq \emptyset$  for all  $n \in \mathbb{N}$ . Note that  $x$  is uniformly recurrent if and only if  $\overline{O_f(x)} = \omega_f(x)$  is a minimal set (see [3], Proposition 5, Chapter V). Let  $Fix(f)$ ,  $P(f)$ ,  $UR(f)$ ,  $R(f)$  and  $\Lambda(f)$  denote the sets of fixed points, periodic points, uniformly recurrent points, recurrent points and the union of all  $\omega$ -limit sets respectively. Then we have the inclusion relation  $Fix(f) \subset P(f) \subset UR(f) \subset R(f) \subset \Lambda(f)$ .

We say that  $f$  is *pointwise recurrent* (resp. *relatively recurrent*) if  $R(f) = X$  (resp.  $\overline{R(f)} = X$ ). We say that  $f$  is *transitive* if for any two nonempty open sets  $U$  and  $V$  in  $X$ , there exists  $n \in \mathbb{N}$  such that  $f^n(U) \cap V \neq \emptyset$ ; or equivalently if there is a point  $x \in X$  for which  $\omega_f(x) = X$  since  $X$  here is a compact metric space (see [7], Theorem 2.2.2).

A pair  $(x, y) \in X \times X$  is called *proximal* if  $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$ , it is called *distal* if  $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0$ . If  $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$ ,  $(x, y)$  is called *asymptotic*. A pair  $(x, y)$  is called a *Li-Yorke pair* (of  $f$ ) if it is proximal but not asymptotic. We say that  $f$  is distal if for any  $x, y \in X$  with  $x \neq y$ , the pair  $(x, y)$  is distal.

**Definition 1.1.** ([8]) *Let  $X, Y$  be two topological spaces. A map  $f : X \rightarrow Y$  is said to be monotone if for any connected subset  $C$  of  $Y$ ,  $f^{-1}(C)$  is connected.*

Notice that  $f^n$  is monotone for every  $n \in \mathbb{N}$  when  $f$  itself is monotone.

Our main results can be stated as follows:

**Theorem 1.2.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Then for any  $x \in D$ , we have:*

- (i)  $\omega_f(x)$  is a minimal set.
- (ii)  $\omega_f(x) \subset \overline{P(f)}$ .

**Corollary 1.3.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Then for any  $x \in D$ ,  $\omega_f(x)$  is either a finite set or a minimal Cantor set. In particular,  $f$  is not transitive.*

**Theorem 1.4.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Then  $UR(f) = R(f) = \Lambda(f) = \overline{P(f)}$ .*

**Corollary 1.5.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Then the restriction map  $f|_{R(f)}$  is a distal homeomorphism.*

**Theorem 1.6.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Then the following statements are equivalent:*

- (i)  $f$  is pointwise recurrent.
- (ii)  $f$  is relatively recurrent.
- (iii) every cut point is a periodic point.

So from Corollary 1.5, we have the following:

**Corollary 1.7.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. If  $f$  is relatively recurrent, then  $f$  is a distal homeomorphism.*

This paper is organized as follows: In section 2, we give some preliminaries which are useful for the rest of the paper. In section 3, we give some preparatory results concerning monotone dendrite maps and in section 4, we prove the main results of this paper.

## 2. PRELIMINARIES

We need the following results:

**Theorem 2.1.** ([2], Theorem 3.10) *Let  $(X, d)$  be compact metric space and let  $f : X \rightarrow X$  be a continuous map without Li-Yorke pairs. Then  $f$  is minimal when it is transitive.*

**Theorem 2.2.** ([12], Corollary 1.2) Let  $f : D \rightarrow D$  be a monotone dendrite map. Then  $f$  has no Li-Yorke pairs.

**Lemma 2.3.** *If  $J$  is a compact interval and  $f : J \rightarrow J$  is a continuous monotone map, then for any  $x \in J$ ,  $\omega_f(x)$  is either a fixed point or a periodic orbit of period 2. In particular,  $f$  has no Li-Yorke pair.*

*Proof.* The proof is trivial.  $\square$

**Lemma 2.4.** ([10], Lemma 2.1) Let  $(D, d)$  be a dendrite. Then for every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that, for any  $x, y \in D$  with  $d(x, y) \leq \delta$ , we have  $\text{diam}([x, y]) < \varepsilon$ .

**Lemma 2.5.** ([10], Lemma 2.2) Let  $[a, b]$  be an arc in a dendrite  $(D, d)$  and  $w \in [a, b]$ . There is  $\delta > 0$  such that if  $v \in D$  with  $d(v, b) \leq \delta$  then  $[v, a] \supset [w, a]$ .

**Lemma 2.6.** *Let  $[a, b]$  be a non degenerate arc in a dendrite  $(D, d)$ . Then there is  $\delta > 0$  such that  $[u, v] \cap [a, b] \neq \emptyset$  for any  $u, v \in D$  satisfying  $d(a, u) < \delta$  and  $d(b, v) < \delta$ .*

*Proof.* As  $a \neq b$ , there exist  $y, z \in (a, b)$  such that  $z \in (y, b)$ . By Lemma 2.5, there is  $\delta > 0$  such that if  $u \in B(a, \delta)$  and  $v \in B(b, \delta)$  then  $[u, z] \supset [y, z]$  and  $[v, y] \supset [z, y]$ , so  $[u, v] = [u, y] \cup [y, z] \cup [z, v] \supset [y, z]$  and hence  $[u, v] \cap [a, b] \neq \emptyset$ . This completes the proof.  $\square$

**Lemma 2.7.** ([10], Lemma 2.3) Let  $(C_i)_{i \in \mathbb{N}}$  be a sequence of connected subsets of a dendrite  $(D, d)$ . If  $C_i \cap C_j = \emptyset$  for all  $i \neq j$ , then

$$\lim_{n \rightarrow +\infty} \text{diam}(C_n) = 0.$$

**Lemma 2.8.** *Let  $(D, d)$  be a dendrite and  $f : D \rightarrow D$  a monotone dendrite map. Then for any  $x, y \in D$ ,  $f([x, y]) = [f(x), f(y)]$ .*

*Proof.* Since  $f$  is continuous and monotone, we have  $f([x, y]) \supset [f(x), f(y)]$  and  $f^{-1}([f(x), f(y)]) \supset [x, y]$  respectively. Hence,  $[f(x), f(y)] \supset f([x, y])$  and therefore,  $f([x, y]) = [f(x), f(y)]$ .  $\square$

**Lemma 2.9.** *Let  $(D, d)$  be a dendrite and  $f : D \rightarrow D$  a monotone dendrite map. Suppose that  $a \in \text{Fix}(f)$  and let  $x \in D$ . If for some  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$ ,  $(a, f^n(x)) \cap (a, f^{m+n}(x)) \neq \emptyset$ , then  $(a, x) \cap (a, f^m(x)) \neq \emptyset$ .*

*Proof.* Take  $z \in (a, f^n(x)) \cap (a, f^{m+n}(x))$ . Since  $f^n([a, x]) = [a, f^n(x)]$  and  $f^n([a, f^m(x)]) = [a, f^{m+n}(x)]$  (Lemma 2.8), there exist  $y_1 \in (a, x)$  and  $y_2 \in (a, f^m(x))$  such that  $f^n(y_1) = f^n(y_2) = z$ . By Lemma 2.8,  $f^n([y_1, y_2]) = \{z\}$  so  $a \notin [y_1, y_2]$  since  $z \neq a$ . Then necessarily,  $(a, x) \cap (a, f^m(x)) \neq \emptyset$ .  $\square$

**Lemma 2.10.** *Let  $f : D \rightarrow D$  be a monotone dendrite map,  $a \in \text{Fix}(f)$  and  $x \in D$ . If  $[a, x] \subset [a, f(x)]$  then there exists  $b \in \text{Fix}(f)$  such that  $\lim_{n \rightarrow +\infty} f^n(x) = b$  and  $[a, f^n(x)] \subset [a, b]$  for all  $n \in \mathbb{Z}_+$ .*

*Proof.* We prove by induction on  $n$  that  $[a, f^n(x)] \subset [a, f^{n+1}(x)]$  for every  $n \in \mathbb{Z}_+$ : For  $n = 0$ , we have  $[a, x] \subset [a, f(x)]$ . Suppose that for some  $n \in \mathbb{Z}_+$ ,  $[a, f^n(x)] \subset [a, f^{n+1}(x)]$  then by Lemma 2.8,  $[a, f^{n+1}(x)] \subset [a, f^{n+2}(x)]$ . Thus the closure  $\bar{I}$  of the set  $I = \cup_{n \in \mathbb{Z}_+} [a, f^n(x)]$  is an  $f$ -invariant arc and the sequence  $(f^n(x))_{n \in \mathbb{Z}_+}$  is monotone in this arc, so it converges to a fixed point  $b \in \bar{I}$ , and we get  $[a, f^n(x)] \subset [a, b] = \bar{I}$ , for all  $n \in \mathbb{Z}_+$ .  $\square$

### 3. SOME RESULTS

**Lemma 3.1.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Let  $a \in \text{Fix}(f)$  and  $x \in D$  be such that  $[a, x] \cap \text{Fix}(f) = \{a\}$  and  $[a, x] \cap [a, f(x)] = [a, u_1]$  where  $u_1 \in (a, x)$ . Then the following statements hold:*

- (i) *if  $f(u_1) \in [a, u_1)$  then  $\omega_f(x) = \{a\}$ .*
- (ii) *if  $f(u_1) \in (u_1, f(x))$  then there exists  $b \in \text{Fix}(f)$  such that  $\omega_f(x) = \{b\}$ .*

*Proof.* By Lemma 2.8,  $f([a, x]) = [a, f(x)]$ , then as  $u_1 \in (a, f(x))$ , there is  $u_0 \in (a, x)$  such that  $f(u_0) = u_1$ . Denote for all  $n \in \mathbb{N}$ ,  $u_n = f^n(u_0)$ .

*Proof of (i).* In this case,  $u_0 \in (u_1, x]$  and as  $[a, x] \cap \text{Fix}(f) = \{a\}$ , then for every  $y \in [a, u_0]$ ,  $\omega_f(x) = \{a\}$ . If for some  $k \in \mathbb{Z}_+$ ,  $f^k(x) \in [a, u_0]$ , then it is clear that  $\omega_f(x) = \{a\}$ . Suppose that for all  $k \in \mathbb{Z}_+$ ,  $f^k(x) \notin [a, u_0]$ . In this case, we will see that for each  $k \in \mathbb{N}$ ,

$$(3.1) \quad [u_k, f^k(x)] \cap [a, x] = \{u_k\}.$$

We proceed by induction on  $k$ : For  $k = 1$ ,  $[u_1, f(x)] \cap [a, x] = \{u_1\}$  (see Fig. 1).

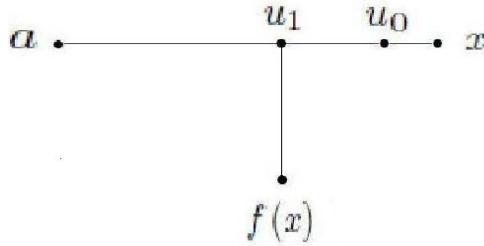


FIGURE 1

Now suppose that for some  $k \in \mathbb{N}$ ,  $[u_k, f^k(x)] \cap [a, x] = \{u_k\}$ . If  $[u_{k+1}, f^{k+1}(x)] \cap [a, x] \supsetneq \{u_{k+1}\}$ , then there is  $y \in (u_{k+1}, f^{k+1}(x)) \cap [a, u_1]$ . Since  $f([a, u_0]) = [a, u_1]$  and  $f([u_k, f^k(x)]) = [u_{k+1}, f^{k+1}(x)]$ , there exist  $w \in [a, u_0]$  and  $v \in [u_k, f^k(x)]$

such that  $f(w) = f(v) = y$ . By Lemma 2.8,  $f([w, v]) = \{y\}$ . Since  $[u_k, f^k(x)] \cap [a, x] = \{u_k\}$ , we get  $u_k \in [w, v]$  and so  $f(u_k) = y \neq u_{k+1}$ , a contradiction. Then  $[u_{k+1}, f^{k+1}(x)] \cap [a, x] = \{u_{k+1}\}$ .

We will prove now that the sets  $(u_k, f^k(x))$ ,  $k \in \mathbb{N}$ , are pairwise disjoint. Suppose that there exist  $i, j \in \mathbb{N}$  and  $z \in D$  such that  $z \in (u_i, f^i(x)) \cap (u_{i+j}, f^{i+j}(x))$ . As  $f^i([u_0, x]) = [u_i, f^i(x)]$  and  $f^i([u_j, f^j(x)]) = [u_{i+j}, f^{i+j}(x)]$ , there exist  $y_1 \in [u_0, x]$  and  $y_2 \in [u_j, f^j(x)]$  such that  $f^i(y_1) = f^i(y_2) = z$ , so by Lemma 2.8,  $f^i([y_1, y_2]) = \{z\}$ . By (3.1),  $[u_j, f^j(x)] \cap [a, x] = \{u_j\}$ , and since  $y_1 \in [u_0, x] \subset [a, x]$ , we have  $u_j \in [y_1, y_2]$ , thus  $u_{i+j} = f^i(u_j) = z$ , a contradiction. We conclude that the sets  $(u_k, f^k(x))$ ,  $k \in \mathbb{N}$  are pairwise disjoint, so by Lemma 2.7, we have  $\lim_{k \rightarrow +\infty} \text{diam}([u_k, f^k(x)]) = 0$  and therefore the pair  $(x, u_0)$  is asymptotic. Hence,  $\omega_f(x) = \omega_f(u_0) = \{a\}$ .

*Proof of (ii).* By Lemma 2.10, there exists  $b \in \text{Fix}(f)$  such that  $\lim_{n \rightarrow +\infty} f^n(u_1) = b$  and  $[a, u_n] \subset [a, b]$  for all  $n \in \mathbb{N}$ . Clearly,  $[u_1, b] \cap \text{Fix}(f) = \{b\}$  (Lemma 2.10). We distinguish three cases:

- Case 1.  $b \in [u_1, f(x)]$ : In this case  $b = u_2$ , indeed, we have  $u_1 \in [x, b]$ , so by Lemma 2.8,  $u_2 = f(u_1) \in [f(x), b]$ . Therefore,  $u_2 \in [f(x), b] \cap [a, b] = \{b\}$ , thus  $u_2 = b$  and  $f([u_1, b]) = \{b\}$ . Now, if the sets  $(b, f^n(x))$ , for  $n \in \mathbb{Z}_+$ , are pairwise disjoint then by Lemma 2.7,  $\lim_{n \rightarrow \infty} \text{diam}([b, f^n(x)]) = 0$  and so  $\omega_f(x) = \{b\}$ . Otherwise, there exist  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$  such that  $(b, f^n(x)) \cap (b, f^{m+n}(x)) \neq \emptyset$ . So by Lemma 2.9,  $[b, x] \cap [b, f^m(x)] = [b, v]$  for some  $v \in (b, x)$ . Let us show that  $v \in [b, u_1]$ : We have  $b \in [a, f(x)]$ , and since  $b \in \text{Fix}(f)$ , we have  $b \in [a, f^m(x)]$  (Lemma 2.8). So  $\{a, b, v\} \subset [a, f^m(x)]$ , then  $v$  does not belong to  $(u_1, x]$  (since otherwise the set  $\{a, b, v\}$  cannot be included in an arc). We have  $v \in (b, u_1]$  and  $f([u_1, b]) = \{b\}$  then  $f^m(v) = b$ . In result,  $[b, x] \cap [b, f^m(x)] = [b, v]$  where  $v \in (b, u_1] \subset (b, x)$  and  $f^m(v) = b$ . Applying Lemma 3.1, (i) to the map  $f^m$  by considering the fixed point  $b$  of  $f^m$  instead of  $a$  and the point  $v$  instead of  $u_1$ , we get  $\omega_{f^m}(x) = \{b\}$  and as  $b \in \text{Fix}(f)$ ,  $\omega_f(x) = \{b\}$ .

- Case 2.  $b \notin [u_1, f(x)]$  and  $f(x) \in [u_1, b]$ : In this case, we have, by Lemma 2.10,  $\omega_f(x) = \omega_f(f(x)) = \{b\}$ .

- Case 3.  $b \notin [u_1, f(x)]$  and  $f(x) \notin [u_1, b]$ : In this case,  $[b, x] \cap [b, f(x)] = [b, v]$  where  $v \in (u_1, b]$ . So  $f(v) \in (v, b]$  (Lemma 2.10). Applying Lemma 3.1, (i) to the map  $f$  by considering  $b$  instead of  $a$  and  $v$  instead of  $u$ , we get  $\omega_f(x) = \{b\}$ . The proof is complete.  $\square$

**Lemma 3.2.** *Let  $f : D \rightarrow D$  be a monotone dendrite map. Let  $a \in \text{Fix}(f)$  and  $x \in D$ . If  $\omega_f(x)$  is infinite then for every  $n \in \mathbb{N}$ ,  $[a, x] \cap [a, f^n(x)] = [a, u_n]$  where  $u_n \in \text{Fix}(f^n)$ .*

*Proof.* One can suppose that  $[a, x] \cap \text{Fix}(f) = \{a\}$ . Let  $n \in \mathbb{N}$  and  $[a, x] \cap [a, f^n(x)] = [a, u_n]$ . Then,  $u_n \in [a, x]$ , indeed, if  $u_n = x$ , then by Lemma 2.10 applied to  $f^n$ , we get  $\omega_{f^n}(x) = \{b\} \subset \text{Fix}(f^n)$  and so  $\omega_f(x)$

is finite, a contradiction. It follows that  $u_n \in \text{Fix}(f^n)$  if  $u_n = a$  and, by Lemma 3.1, if  $u_n \in (a, x)$ .  $\square$

#### 4. PROOF OF MAIN RESULTS

*Proof of Theorem 1.2:* If  $\omega_f(x)$  is finite, then Theorem 1.2 obviously holds. In the following we may assume that  $\omega_f(x)$  is infinite.

*Proof of (i).* Take  $a \in \text{Fix}(f)$  such that  $[a, x] \cap \text{Fix}(f) = \{a\}$ . By Lemma 2.7, the sets  $(a, f^{2n}(x)]$ , for  $n \in \mathbb{N}$ , cannot be pairwise disjoint (since otherwise,  $\omega_{f^2}(x) = \{a\}$  and so  $\omega_f(x)$  is finite). So by Lemma 2.9, there exist  $n_0 \in \mathbb{N}$  with  $n_0 > 1$  and  $u_0 \in D$  such that  $[a, x] \cap [a, f^{n_0}(x)] = [a, u_0]$  where  $u_0 \in (a, x]$ . By Lemma 2.10,  $u_0 \in (a, x)$  and by Lemma 3.2,  $u_0 \in \text{Fix}(f^{n_0})$ . If we consider now the map  $f^{n_0}$  then in the same way, we can prove that there exist an integer  $n_1 \in \mathbb{N}$  with  $n_1 > 1$  and a fixed point  $u_1$  of  $f^{n_0 n_1}$  in the arc  $(u_0, x)$  such that  $[a, x] \cap [a, f^{n_0 n_1}(x)] = [a, u_1]$ . By induction, we find a sequence of integers  $(n_i)_{i \in \mathbb{Z}_+}$  and a sequence of points  $(u_i)_{i \in \mathbb{Z}_+}$  in  $D$  such that for every  $i \in \mathbb{Z}_+$ , we have:

$$(4.1) \quad n_i > 1,$$

$$(4.2) \quad u_i \in \text{Fix}(f^{N_i}),$$

$$(4.3) \quad u_{i+1} \in (u_i, x),$$

$$(4.4) \quad [a, x] \cap [a, f^{N_i}(x)] = [a, u_i],$$

where  $N_i = \prod_{0 \leq j \leq i} n_j$ . Then by (4.3), the sequence of points  $(u_i)_{i \in \mathbb{N}}$  is monotone in the arc  $[a, x]$  so it converges to a point  $u_\infty \in [a, x]$  (see Figure 2). It is possible that  $u_\infty = x$ .

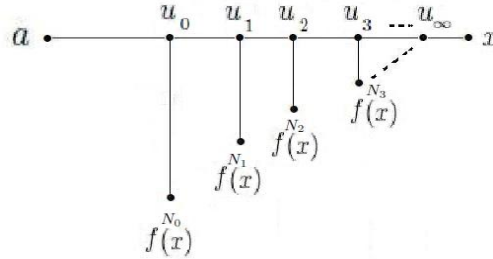


FIGURE 2

From (4.4), the sets  $[u_i, f^{N_i}(x)]$ , for  $i \in \mathbb{N}$  are pairwise disjoint, hence by Lemma 2.7,  $\lim_{i \rightarrow +\infty} \text{diam}([u_i, f^{N_i}(x)]) = 0$ . Then  $\lim_{i \rightarrow +\infty} f^{N_i}(x) = u_\infty$  and by (4.1),  $\lim_{i \rightarrow +\infty} N_i = +\infty$  so  $u_\infty \in \omega_f(x)$ . If  $u_\infty = x$ , then the map



$f|_{\omega_f(x)}$  is transitive. If  $u_\infty \neq x$ , then for every  $i \in \mathbb{N}$ , we have  $[u_\infty, x] \subset [u_i, x]$  and since  $f^{N_i}([u_i, x]) = [u_i, f^{N_i}(x)]$  (Lemma 2.8),

$$[f^{N_i}(u_\infty), f^{N_i}(x)] = f^{N_i}([u_\infty, x]) \subset [u_i, f^{N_i}(x)].$$

So  $\lim_{i \rightarrow +\infty} \text{diam}([f^{N_i}(u_\infty), f^{N_i}(x)]) = 0$ . Then the pair  $(u_\infty, x)$  is proximal and by Theorem 2.2, it is an asymptotic pair, so  $\omega_f(u_\infty) = \omega_f(x)$ . Therefore, regardless of whether  $u_\infty = x$  or  $u_\infty \neq x$ ,  $f|_{\omega_f(x)}$  is a transitive map without Li-Yorke pairs, hence, by Theorem 2.1, this map is minimal and so  $\omega_f(x)$  is a minimal set.

*Proof of (ii):* By (4.2),  $u_i \in P(f)$  for all  $i \in \mathbb{N}$ . Since  $\lim_{i \rightarrow +\infty} u_i = u_\infty$ , we have  $u_\infty \in \overline{P(f)}$ . As  $f(\overline{P(f)}) = \overline{P(f)}$ , then  $\omega_f(x) = \omega_f(u_\infty) \subset \overline{P(f)}$ .  $\square$

*Proof of Corollary 1.3:* Let  $x \in D$  be such that  $\omega_f(x)$  is infinite. By Theorem 1.2,  $\omega_f(x)$  is minimal so it has no isolated point. To prove that  $\omega_f(x)$  is a Cantor set, it suffices to prove that it is totally disconnected: Otherwise,  $\omega_f(x)$  contains a non-degenerate arc  $[a, b]$ . By Theorem 1.2,  $a, b \in \overline{P(f)}$ , so by Lemma 2.6, there are  $p, q \in P(f)$  such that  $[p, q] \cap [a, b] \neq \emptyset$ . Take  $y \in [p, q] \cap [a, b]$ . Since  $p$  and  $q$  are periodic, there is  $n \in \mathbb{N}$  such that  $f^n(p) = p$  and  $f^n(q) = q$ , so  $f^n([p, q]) = [p, q]$  (Lemma 2.8). By Lemma 2.3,  $\omega_{f^n}(y)$  is finite and so is  $\omega_f(y)$ . But this contradicts that  $\omega_f(y) = \omega_f(x)$  is an infinite minimal set. Thus  $\omega_f(x)$  must be totally disconnected.  $\square$

*Proof of Theorem 1.4:* By Theorem 1.2, we have  $UR(f) = R(f) = \Lambda(f) \subset \overline{P(f)}$ . So it suffices to prove that  $\overline{P(f)} \subset R(f)$ . Let  $x \in \overline{P(f)}$ . We distinguish two cases:

- Case 1.  $\omega_f(x)$  is a periodic orbit: Without loss of generality, one can assume that  $\omega_f(x) = \{a\} \subset \text{Fix}(f)$ . We will prove that  $x = a$  and so  $x \in R(f)$ . Suppose that  $x \neq a$ . Take  $w \in (x, a)$ , by Lemma 2.5, there is  $\delta > 0$  such that if  $p \in B(x, \delta)$  then  $[p, a] \supset [w, a]$ . As  $x \in \overline{P(f)}$ , one can choose  $p \in P(f)$ . We will show that  $[p, a] \supset [x, a]$ : otherwise,  $[p, a] \cap [x, a] = [v, a]$  where  $v \in (x, a)$ , so take  $v' \in (x, v)$ , then by Lemma 2.5 applying for  $v'$  instead of  $w$ , there is  $q \in P(f)$  such that  $[q, a] \supset [v', a]$ , hence  $[q, a] \cap [p, a] = [v, a]$ . Thus,  $a \notin [p, q]$  (since otherwise,  $[q, a] \cap [p, a] = \{a\}$ , but  $[q, a] \cap [p, a] = [v, a] \neq \{a\}$ ) and  $v \in [p, q]$ . As  $p$  and  $q$  are periodic points, there is  $n \in \mathbb{N}$  such that  $f^n(p) = p$  and  $f^n(q) = q$ , so by Lemma 2.8,  $f^n([p, q]) = [p, q]$  and hence  $\omega_{f^n}(v) \subset [p, q]$ . As  $\omega_f(x) = \{a\}$ , the pair  $(x, a)$  is asymptotic and then, by Lemma 2.4,  $\lim_{n \rightarrow +\infty} \text{diam}([f^n(x), a]) = 0$ . Hence, for any point  $y \in [x, a]$ ,  $\omega_f(y) = \{a\}$ . In particular,  $\omega_f(v) = \{a\}$  and then  $\omega_{f^n}(v) = \{a\}$ , a contradiction since  $a \notin [p, q]$ . Thus,  $[p, a] \supset [x, a]$ . The arc  $[p, a]$  is  $f^n$ -invariant, by Lemma 2.8, so  $f^n(x) \in [p, a]$ . In fact, since  $\omega_f(x) = \{a\}$ ,  $f^n(x) \in (x, a]$  and there is  $x_{-n} \in (p, x)$  such that  $f^n(x_{-n}) = x$ . So for each  $m \in \mathbb{N}$ ,  $\omega_{f^m}(x_{-n}) = \{a\}$ . Again since  $x \in \overline{P(f)}$  and  $x \neq a$ , one can find  $q \in P(f)$  such that  $[q, p] \supset [x_{-n}, p]$  and  $a \notin [q, p]$ . Take  $m \in \mathbb{N}$  such

that  $f^m(q) = q$  and  $f^m(p) = p$ , then, by Lemma 2.8,  $f^m([q, p]) = [q, p]$ . So  $\omega_{f^m}(x_{-n}) \subset [q, p]$ , a contradiction. Therefore,  $x = a$ .

• Case 2.  $\omega_f(x)$  is infinite: Let  $u_\infty$  and  $u_0 \in P(f)$  given in the proof of Theorem 1.2. We will prove that  $x = u_\infty$ , and so  $x \in \overline{\omega_f(x)}$  which implies that  $x \in R(f)$ : Assume that  $x \neq u_\infty$ . As  $x \in \overline{P(f)}$ , then by Lemma 2.5, we can find  $p \in P(f)$  such that  $[p, u_0] \supset [u_\infty, u_0]$ , so  $u_\infty \in [p, u_0]$ . Take  $n \in \mathbb{N}$  such that  $f^n(p) = p$  and  $f^n(u_0) = u_0$ , hence by Lemma 2.8,  $f^n([p, u_0]) = [p, u_0]$ , so, by Lemma 2.3,  $\omega_{f^n}(u_\infty)$  is finite and so is  $\omega_f(u_\infty) = \omega_f(x)$ , a contradiction.

For both cases 1 or 2, we proved that  $x \in R(f)$ . So  $\overline{P(f)} \subset R(f)$  and therefore  $UR(f) = R(f) = \Lambda(f) = \overline{P(f)}$ .  $\square$

*Proof of Corollary 1.5:* Take  $x, y \in R(f)$  with  $x \neq y$ . By Theorem 2.2, the pair  $(x, y)$  is either asymptotic or distal. Let us prove that the pair  $(x, y)$  is distal. Indeed, suppose that  $(x, y)$  is an asymptotic pair, then by Lemma 2.4,  $\lim_{n \rightarrow +\infty} \text{diam}([f^n(x), f^n(y)]) = 0$  and thus,  $\omega_f(x) = \omega_f(y) = \omega_f(z)$  for any  $z \in [x, y]$ . As  $x, y \in \overline{P(f)}$  (Theorem 1.4), then, by Lemma 2.6, there exist  $p, q \in P(f)$  such that  $[p, q] \cap [x, y] \neq \emptyset$ . Take  $z \in [p, q] \cap [x, y]$ . Let  $n \in \mathbb{N}$  be such that  $f^n(p) = p$  and  $f^n(q) = q$ , hence by Lemma 2.8, the arc  $[p, q]$  is  $f^n$  invariant. So by Lemma 2.3,  $\omega_{f^n}(z)$  is finite, hence it is a periodic orbit for  $f^n$  and so  $\omega_f(z) = \omega_f(x) = \omega_f(y)$  is a periodic orbit for  $f$ . As  $x, y \in R(f)$  we have  $x, y \in P(f)$  which is impossible. Therefore, the map  $f|_{R(f)}$  is one to one. Moreover, as  $f(R(f)) = R(f)$  and  $R(f) = \overline{P(f)}$  is compact, the map  $f|_{R(f)}$  is a homeomorphism.  $\square$

*Proof of Theorem 1.6:* (i)  $\Rightarrow$  (ii) is clear. (iii)  $\Rightarrow$  (i) follows from Theorem 1.4. It remains to prove (ii)  $\Rightarrow$  (iii): By Theorem 1.4,  $\overline{P(f)} = D$ . Take  $x$  a cut point of  $D$ , then  $D \setminus \{x\}$  has more than one connected component and let  $A$  and  $B$  be two disjoint connected components of  $D \setminus \{x\}$ . Since the set  $P(f)$  is dense, one can find two periodic points  $a$  and  $b$  with  $a \in A$  and  $b \in B$ , so  $x \in [a, b]$ . Without loss of generality, one can assume that  $a, b \in \text{Fix}(f)$ . Suppose that  $x$  is not a fixed point. If  $f(x) \in (x, b]$  then  $f^n(x) \in [f(x), b]$  for all  $n \in \mathbb{N}$ . Take  $w \in (x, f(x))$ . As  $w \neq f(x)$  and  $\overline{P(f)} = D$ , there is, by Lemma 2.5, a periodic point  $p$  such that  $[f(x), b] \cap [p, a] = \emptyset$  and  $[p, a] \supset [w, x]$ , so  $x \in [p, a]$ . Let  $m \in \mathbb{N}$  be the period of  $p$ , then  $f^m([p, a]) = [p, a]$  (Lemma 2.8). Therefore,  $f^{km}(x) \in [p, a]$  for all  $k \in \mathbb{Z}_+$  and we have  $f^n(x) \in [f(x), b]$  for all  $n \in \mathbb{N}$ , this is a contradiction since  $[f(x), b] \cap [p, a] = \emptyset$ . By a similar way, the case  $f(x) \in [a, x)$  leads to a contradiction. Therefore,  $x \in P(f)$ , this completes the proof.  $\square$

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ISSAM NAGHMOUCHI, UNIVERSITY OF CARTHAGE, FACULTY OF SCIENCE OF BIZERTE,  
DEPARTMENT OF MATHEMATICS, JARZOUNA, 7021, TUNISIA.

*E-mail address:* `issam.nagh@gmail.com`