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Martin David, Jean-Jacques Marigo, C. Pideri

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M. David[1,2], J.-J. Marigo[1], C. Pideri[3]

[1] *LMS, École Polytechnique, France, {david,marigo}@lms.polytechnique.fr*
[2] *LaMSID, EDF R&D, France*
[3] *ANLA, Université de Toulon et du Var, France, pideri@univ-tln.fr*

Résumé — We undertake the homogenization of a three-dimensional elastic medium, which contains heterogeneities localized on a surface. These can be either reinforcements, like steel reinforcements in concrete, or defects, like microcracks periodically distributed. We propose a simple model describing the effective behaviour of these heterogeneities. This model, formulated in an energetic framework, combines an elastic interface behaviour with that of a membrane. Its interest is discussed through several applications.
Mots clés — Homogenization, Separation of scales, Matched asymptotic developments.

1 Introduction

Let us consider a three-dimensional elastic medium, which contains heterogeneities periodically distributed on a surface. The size of these heterogeneities is much smaller than the size of the overall structure, at least by a factor 5 or 10. These heterogeneities can be for example steel reinforcements in concrete, or microcracks or cavities localized on a surface. In the context of finite element simulations, this kind of problem is very difficult to treat. Indeed, modeling each one of the small heterogeneities requires a very fine mesh, which gives a huge numerical cost. On the other hand, these heterogeneities have an influence on the behaviour of the overall structure. It is therefore interesting to construct a simplified model, which reproduces the effective behaviour of the heterogeneities, while being much easier to compute.

For this purpose, we will combine the methods of *separation of scales* and *matched asymptotic expansions*. The former was used by Sanchez-Palencia, Léné and Suquet in the eighties to homogenize the behaviour of three-dimensional periodic materials [11,14]. By separating the so-called micro- and macro-scales, this method enables to identify an *effective behaviour*. The latter was introduced in this context by Nguetseng and Sanchez-Palencia [15]. Other simplified approaches have been made on this problem, by studying a thin homogeneous layer, whose behaviour is different from the surrounding volume. This problem has been first analyzed by Huy and Sanchez-Palencia [16], followed by Caillerie, Licht and Michaille, Geymonat, Benveniste and Bessoud [3-7,12]. These authors made the assumption that the layer is very soft (i.e. scales with the thickness of the layer), or very stiff (i.e. scales with its inverse). These two hypothesis lead respectively to an elastic interface and a membrane model.

Here we propose a more general approach, where the layer is heterogeneous, while its stiffness does not depend *a priori* on the thinness of the layer. This work is the sequel of previous papers made on this subject [1,2,13]. The peculiarity of this work is that we propose an energetic formulation for the behaviour of the homogenized interface. This leads to an interesting interpretation of the behaviour of the interface, and will help us analyze the stability of our model. The paper is organized in the following way. In the first section of the paper, we detail the setting of the problem, and apply the method of separation of scales to our problem. The second one is devoted to constructing an energetic formulation describing the homogenized behaviour of the interface. Finally, the third part illustrates the application of our method to steel reinforcements in concrete.

2 Separation of scales

2.1 Setting of the problem

The problem we want to study is the following. Consider a deformable three-dimensional body, with natural reference configuration Ω, a regular connected open subset of $\mathbb{R}^3$. The behaviour of the body is linear elastic, characterized by the stiffness tensor $\mathbf{A}$. We also assume that the body is homogeneous, except near a plane Γ. In the vicinity of this surface, the elastic tensor takes the form of a two-dimensional periodic network, with periodic vectors denoted by $\mathbf{t}_2$ and $\mathbf{t}_3$. These vectors are small compared to the size of the overall structure. For the sake of simplicity, the array is assumed orthogonal, i.e. $\mathbf{t}_2.\mathbf{t}_3 = 0$. The projected surface of a periodic pattern $\mathbf{t}_2 \otimes \mathbf{t}_3$ on Γ is denoted by Y.

The boundary $\partial\Omega$ of the solid is separated into two parts $\partial_F\Omega$ and $\partial_u\Omega$. Forces are imposed on $\partial_F\Omega$, while the displacement of the body is prescribed on $\partial_u\Omega$. The body is also subjected to volume forces. In this context, the behaviour of the body is governed by the following equations.

$$\begin{aligned}
&\sigma = \mathbf{A} : \varepsilon(\mathbf{u}) \\
&\text{div}(\sigma) + \mathbf{f} = 0 \\
&\sigma.\mathbf{n} = \mathbf{F}_d \quad \text{on} \quad \partial_F\Omega \\
&\mathbf{u} = \mathbf{u}_d \quad \text{on} \quad \partial_u\Omega
\end{aligned}$$

which are respectively the behaviour equation, the equation of equilibrium and the boundary conditions.

To simplify the following developments, we set the characteristic size of the structure to 1, which corresponds to a simple change of scale. We also introduce a small parameter η, which is defined by

$$\eta = \sqrt{\|\mathbf{t}_1\| \, \|\mathbf{t}_2\|}$$

and which characterizes the size of a periodic pattern.

2.2 Coordinates systems and variables

We now need to define the coordinate systems for our problem. At the global scale, we build an cartesian coordinate system (x_1, x_2, x_3). x_1 gives the (signed) distance to the plane Γ, while x_2 and x_3 parameterize the tangent plane, and are parallel to the periodic vectors $\mathbf{t}_2$ and $\mathbf{t}_3$. We also introduce microscopic coordinates (y_1, y_2, y_3) to describe the heterogeneities. They are simply derived from the previous ones near the interface by using the relation $\mathbf{y} = \mathbf{x}/\eta$.

The complete material can be decomposed into several parts. The heterogeneous part is restricted to the vicinity of the surface Γ, while the homogeneous region is the volume $\Omega \backslash \Gamma$. In the external region, the displacement and stress fields are denoted by $\mathbf{u}(\mathbf{x})$ and $\sigma(\mathbf{x})$. They do not depend on $\mathbf{y}$ since the volume is homogeneous. Similarly, we denote the displacement and stress fields near the interface by $\mathbf{v}(\mathbf{y}, x_2, x_3)$ and $\tau(\mathbf{y}, x_2, x_3)$.

2.3 Equations

Following the method of separation of scales, the fields are decomposed in powers of η

$$\mathbf{u} = \mathbf{u}_0 + \eta\mathbf{u}_1 + \eta^2\mathbf{u}_2 + \ldots \tag{1}$$

$$\sigma = \sigma_0 + \eta\sigma_1 + \eta^2\sigma_2 + \ldots \tag{2}$$

$$\mathbf{v} = \mathbf{v}_0 + \eta\mathbf{v}_1 + \eta^2\mathbf{v}_2 + \ldots \tag{3}$$

$$\tau = \tau_0 + \eta\tau_1 + \eta^2\tau_2 + \ldots \tag{4}$$

The equilibrium and constitutive equations near the interface are also modified in the following way

$$\begin{aligned}
&\text{div}_x(\tau) + \frac{1}{\eta}\text{div}_y(\tau) + \mathbf{f} = 0 \\
&\tau = \mathbf{A} : \left(\varepsilon_x(\mathbf{v}) + \frac{1}{\eta}\varepsilon_y(\mathbf{v}) \right)
\end{aligned}$$

By introducing the expansions (1-4) in the previous equations, we obtain a serie of equations which will be solved in the following section.

We finally need to take into account matching conditions between the homogeneous volume and the heterogeneous interface. These matching conditions for the displacement read

$$\mathbf{u}_0|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \mathbf{v}_0$$
$$\mathbf{u}_1|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \left(\mathbf{v}_1 - y_1 \left.\frac{\partial \mathbf{u}_0}{\partial x_1}\right|_{\Gamma^\pm} \right)$$

with $\mathbf{u}|_{\Gamma^\pm}$ respectively the limit of $\mathbf{u}$ on the positive and negative side of the interface. We also obtain similar matching conditions for the stresses.

2.4 Calculations

2.4.1 Micro problem of order 0

Using the previous equations, we can solve the whole problem in an iterative way, starting by the order 0, and studying alternatively the local problem near the interface, and the global problem in the external volume. Near the interface, the equation of behaviour of order -1 can be written

$$\mathbf{A} : \varepsilon_y(\mathbf{v}_0) = 0$$

$\mathbf{v}_0$ is therefore a rigid body motion. Since $\mathbf{v}_0$ is also periodic in y_2 and y_3, $\mathbf{v}_0$ can only be a translation, and does not depend upon $\mathbf{y}$.

2.4.2 Macro problem of order 0

Inside the homogeneous volume, we have the constitutive and equilibrium relations of order 0

$$\sigma_0 = \mathbf{A} : \varepsilon_x(\mathbf{u}_0)$$
$$\mathrm{div}_x(\sigma_0) + \mathbf{f} = 0$$

with the usual limit conditions

$$\sigma_0.\mathbf{n} = \mathbf{F}_d \quad \text{on} \quad \partial_F \Omega$$
$$\mathbf{u}_0 = \mathbf{u}_d \quad \text{on} \quad \partial_u \Omega$$

The matching conditions on each side of the interface show that the jumps of displacement and stress are zero

$$[\![\mathbf{u}_0]\!] = 0 \quad \text{on} \quad \Gamma$$
$$[\![\sigma_0.\mathbf{e}_1]\!] = 0 \quad \text{on} \quad \Gamma$$

In conclusion, the defects do not have any influence on the global structure at order 0. This result is well known from previous studies. It may be understood with simple arguments : when the thickness of the heterogeneous layer goes to zero, its own rigidity becomes negligible. We therefore have to push our analysis until order 1 to identify the effective behaviour of the interface.

2.4.3 Micro problem of order 1

The constitutive and equilibrium equations read

$$\tau_0 = \mathbf{A} : (\varepsilon_x(\mathbf{v}_0) + \varepsilon_y(\mathbf{v}_1))$$
$$\mathrm{div}_y(\tau_0) = 0$$

$\mathbf{v}_1$ and τ_0 are related to the macroscopic fields through the matching conditions

$$\mathbf{u}_1|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \left(\mathbf{v}_1 - y_1 \frac{\partial \mathbf{u}_0}{\partial x_1} \right)$$
$$\sigma_0|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \tau_0$$

These relations suggest to introduce the correctors $\hat{\mathbf{v}}_1$ and $\hat{\tau}_0$, defined by

$$\mathbf{v}_1 = y_1 \frac{\partial \mathbf{u}_0}{\partial x_1} + \hat{\mathbf{v}}_1$$
$$\tau_0 = \sigma_0 + \hat{\tau}_0$$

The constitutive equation then reads

$$\sigma_0 + \hat{\tau}_0 = \mathbf{A} : \varepsilon_y(\hat{\mathbf{v}}_1) + \mathbf{A} : \varepsilon_y \left(y_1 \frac{\partial \mathbf{u}_0}{\partial x_1} \right) + \mathbf{A} : \varepsilon_x(\mathbf{v}_0)$$

and since $\sigma_0 = \mathbf{A}_0 : \varepsilon_x(\mathbf{u}_0)$

$$\hat{\tau}_0 = \mathbf{A} : \varepsilon_y(\hat{\mathbf{v}}_1) + (\mathbf{A} - \mathbf{A}_0) : \varepsilon_x(\mathbf{u}_0)$$

with $\mathbf{A}_0$ the homogeneous behaviour of the volume. Since σ_0 is homogeneous at the scale of a pattern, the equation of equilibrium reads

$$\mathrm{div}(\hat{\tau}_0) = 0$$

while the matching conditions for the stress of order 0 become

$$\lim_{y_1 \to \pm\infty} \hat{\tau}_0 = 0$$

From the previous results, one can write the complete set of equations at the scale of a pattern

$$\hat{\tau}_0 = \mathbf{A} : \varepsilon_y(\hat{\mathbf{v}}_1) + (\mathbf{A} - \mathbf{A}_0) : \varepsilon_x(\mathbf{u}_0)$$
$$\mathrm{div}(\hat{\tau}_0) = 0$$
$$\lim_{y_1 \to \pm\infty} \hat{\tau}_0 = 0$$
$$\hat{\mathbf{v}}_1 \quad \text{and} \quad \hat{\tau}_1 \quad \text{are} \quad \text{periodic}$$

This problem is an elastic problem with prestress in infinite medium. It is well posed, even though rigid translations must be restricted. The loading parameter is $\varepsilon_x(\mathbf{u}_0)$, which is obtained from the global problem of order 0. Since this problem is linear, the superposition principle may apply. There are six different modes of sollicitations corresponding to the six independant components of $\varepsilon_x(\mathbf{u}_0)$. It is thus sufficient to solve six elementary problems at the scale of a pattern to characterize the behaviour of the interface.

2.4.4 Macro problem of order 1

The next step is to analyze the consequences of the response of the interface at the global scale. At order 1, the behaviour equation reads

$$\sigma_1 = \mathbf{A} : \varepsilon_x(\mathbf{u}_1)$$

and the equilibrium equation

$$\mathrm{div}_x(\sigma_1) = 0$$

with the homogeneous boundary conditions

$$\sigma_1 . \mathbf{n} = 0 \quad \text{on} \quad \partial_F \Omega$$
$$\mathbf{u}_1 = 0 \quad \text{on} \quad \partial_u \Omega$$

To complete the set of equations, we need to identify the jump conditions across the plane Γ. As before, we use the matching conditions between the volume and the interface

$$\mathbf{u}_1|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \hat{\mathbf{v}}_1$$
$$\sigma_1|_{\Gamma^\pm} = \lim_{y_1 \to \pm\infty} \left(\tau_1 - y_1 \frac{\partial \sigma_0}{\partial x_1} \right)$$

The jump of displacement is therefore given by

$$[\![u_1]\!] = \lim_{y_1 \to \infty} (\hat{v}_1(y_1, y_2, y_3) - \hat{v}_1(-y_1, y_2, y_3)) = \mathbf{d}(\varepsilon_0)$$

where $\mathbf{d}(\varepsilon_0)$ is the jump of displacement extracted from the microscopic problem. The jump of stress is similarly given by

$$[\![\sigma_1]\!] = \lim_{y_1 \to \infty} \left(\hat{\tau}_1(y_1, y_2, y_3) - \hat{\tau}_1(-y_1, y_2, y_3) - 2y_1 \frac{\partial \sigma_0}{\partial x_1} \right)$$

which may be put in the following form

$$[\![\sigma_1]\!] .\mathbf{e}_1 + \mathrm{div}_x (\sigma_I) = 0$$

with

$$\sigma_I(\varepsilon_0) = \int_{Y \times \mathbb{R}} \hat{\tau}_0 \, \mathrm{d}\Omega$$

σ_I may be seen as the stress going through the interface. One can show that it only contains membrane components, that is $\sigma_I.\mathbf{e}_1 = 0$.

We have shown that the main influence of the heterogeneities on the global structure is given by the jump of displacement and membrane stress $\mathbf{d}$ and σ_I. Since these values are extracted from the local problem of order 1, they are linear functions of the deformation $\varepsilon_x(\mathbf{u}_0)$. The complete set of equations of order 1 for the global problem is therefore

$$\begin{aligned}
&\sigma_1 = \mathbf{A} : \varepsilon_x(\mathbf{u}_1) \\
&\mathrm{div}_x(\sigma_1) = 0 \\
&\sigma_1.\mathbf{n} = 0 \quad \text{on} \quad \partial_F \Omega \\
&\mathbf{u}_1 = 0 \quad \text{on} \quad \partial_u \Omega \\
&[\![\mathbf{u}_1]\!] = \mathbf{d} \\
&[\![\sigma_1]\!] .\mathbf{e}_1 + \mathrm{div}_x(\sigma_I) = 0
\end{aligned}$$

3 Energetic formulation

3.1 Reformulation of the problem

Solving independantly the order 0 and 1 for the macroscopic problem is not very relevant in practice. It is much more efficient to solve the whole problem in one step. The purpose of this section is to construct an energetic formulation, whose solutions are accurate *until order 1 in* η. By adding order 0 and 1, we obtain the following problem

$$\begin{aligned}
&\sigma = \mathbf{A} : \varepsilon_x(\mathbf{u}) \\
&\mathrm{div}_x(\sigma) + \mathbf{f} = 0 \\
&\sigma.\mathbf{n} = \mathbf{F}_d \quad \text{on} \quad \partial_F \Omega \\
&\mathbf{u} = \mathbf{u}_d \quad \text{on} \quad \partial_u \Omega \\
&[\![\mathbf{u}]\!] = \eta \mathbf{d}(\overline{\varepsilon}_x(\mathbf{u})) \\
&[\![\sigma]\!] .n = -\eta \mathrm{div}_x(\sigma_I(\overline{\varepsilon}_x(\mathbf{u})))
\end{aligned}$$

where we made approximations which only impact on order 2. Note also that $\overline{\varepsilon}$ is the average of the (possibly discontinuous) deformation on each side of the interface.

3.2 Formulation in an energetic framework

To recast the set of equations in an energetic framework, it is necessary to define an interface energy, describing the energy which is carried by the interface Γ. The behaviour of the interface is currently defined by the linear functions $\mathbf{d}(\varepsilon_x(\mathbf{u}))$ and $\sigma_I(\varepsilon_x(\mathbf{u}))$, which are related to the jump of displacement $[\![\mathbf{u}]\!]$ and membrane stress σ_m through

$$\begin{pmatrix} [\![\mathbf{u}]\!] \\ \sigma_m \end{pmatrix} = \eta \begin{pmatrix} \mathbf{d} \\ \sigma_I \end{pmatrix}$$

The behaviour of the interface is obtained through the resolution of 6 elementary problems at the scale of a pattern. These elementary problems enable to identify the following linear functions, which completely define the behaviour of the inhomogeneities at first order.

$$\begin{pmatrix} [\![\mathbf{u}]\!] \\ \sigma_m \end{pmatrix} = \eta \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{pmatrix} \overline{\varepsilon}.\mathbf{e}_1 \\ \overline{\varepsilon}_m \end{pmatrix}$$

where A, B, C and D are 3×3 matrices. This formulation is not satisfactory, because it mixes primal and dual components. We therefore construct a better formulation by performing several matrices operations and obtain the following relation

$$\begin{pmatrix} \overline{\sigma}.\mathbf{e}_1 \\ \sigma_m \end{pmatrix} = \begin{bmatrix} \eta^{-1} K_{dd} & K_{d\varepsilon} \\ K_{\varepsilon d} & \eta K_{\varepsilon\varepsilon} \end{bmatrix} \cdot \begin{pmatrix} [\![\mathbf{u}]\!] \\ \overline{\varepsilon}_m \end{pmatrix}$$

This formulation of the behaviour of the interface is much more interesting, because it allows us to define the following mechanical energy

$$E_I = \frac{1}{2} \begin{pmatrix} [\![\mathbf{u}]\!] \\ \overline{\varepsilon}_m \end{pmatrix}^T \cdot \begin{bmatrix} \eta^{-1} K_{dd} & K_{d\varepsilon} \\ K_{\varepsilon d} & \eta K_{\varepsilon\varepsilon} \end{bmatrix} \cdot \begin{pmatrix} [\![\mathbf{u}]\!] \\ \overline{\varepsilon}_m \end{pmatrix}$$

This energetic formulation has several advantages, which will be drawn in the following.

3.3 Analysis of the formulation

In this formulation, one can recognize very common behaviours for an interface. K_{dd} corresponds to a jump of displacement $[\![u]\!]$ that is proportional to the stress applied on the interface $\overline{\sigma}.e_1$, which is typical of an elastic interface. This behaviour is coherent with the so-called *soft model* described in the literature. The prefactor η^{-1} indicates that the rigidity of the interface goes to infinity as the size of the heterogeneities goes to zero : in other words, when the heterogeneities become smaller, the jump of displacement vanishes. On the other hand, $K_{\varepsilon\varepsilon}$ corresponds to a membrane stress σ_m that is proportional to the membrane deformation ε_m, which is typical of a simple membrane. This behaviour is in turn coherent with the so-called *stiff model*. For this behaviour, the prefactor η shows that the rigidity of the membrane goes to zero when the size of the inhomogeneities goes to zero. In general, the behaviour of the interface combines membrane and elastic interface behaviour, with a coupling between the two given by the non-diagonal term $K_{\varepsilon d}$.

Notice however that the rigidity matrix is not positive definite in general. This will appear clearly in the illustrative example in the following. For now, let us just emphasize that the membrane behaviour $K_{\varepsilon\varepsilon}$ is generally negative when the inhomogeneities are softer than the surrounding volume, while the interface energy K_{dd} is negative when the inhomogeneities are stiffer. The global problem is therefore not convex anymore, which can lead to spurious oscillations and instabilities, depending on the algorithm used to solve the problem. This problem is however not detrimental, because it can be solved by considering a finite thickness homogenized behaviour. Furthermore, as will be underlined in the following section, the complex general behaviour can be reduced to simpler models depending on the applications, which can help restore the convexity of the problem.

4 Application of the model to reinforcements in concrete

We can finally apply our homogenenized model to steel reinforcements in concrete. The concrete used in civil engineering has a major drawback : it is not very resistant to traction. To overcome this limitation, the concrete is generally reinforced by steel bars, which have a strong resistance to traction. These bars are much smaller than the size of the overall structure, with a diameter of several centimeters. This large difference of scale results in a huge numerical cost if one tries to model each one of the reinforcement bars in three dimensions. It is therefore interesting to construct a simplified model, which reproduces the influence of these reinforcements on the whole structure.

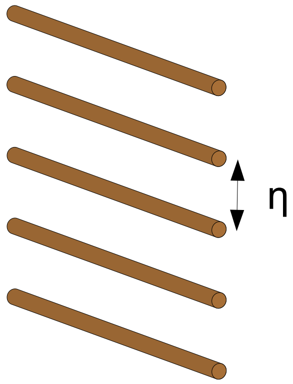

FIG. 1 – Unidirectional steel reinforcements.

We consider a set of unidirectional steel bars, whose diameter is 2 cm, and separated by 20 cm. The steel has a Young modulus of 200 GPa, and a Poisson ratio of 0.3. The reinforcements are surrounded by concrete, whose Young modulus and Poisson ratio are 31 GPa and 0.22. The elementary calculations at the scale of a pattern give the following homogenenized behaviour

$$K = \begin{pmatrix} -1.85e+13 & & & -4.61e+09 & -1.93e+09 & \\ & -5.75e+12 & & & & \\ & & -7.22e+12 & & & \\ -4.61e+09 & & & 2.66e+08 & 1.88e+07 & \\ -1.93e+09 & & & 1.88e+07 & 6.07e+07 & \\ & & & & & 5.64e+07 \end{pmatrix}$$

This homogenized behaviour shows a membrane and an elastic interface behaviour. The latter is unstable, because the diagonal terms are negative. However, we see that the energy terms involved in the elastic interface behaviour are very large, resulting in a very small jump of displacement. This justifies that we restrict the behaviour of the reinforcements to the following membrane behaviour

$$K_m = \begin{pmatrix} 26.6 & 1.9 & \\ 1.9 & 6.1 & \\ & & 5.6 \end{pmatrix} 10^7 \ \mathrm{J/m}$$

This membrane behaviour can be seen as the effective behaviour of the reinforcements at the global scale. It is very satisfactory, because it is easy to implement in a finite element code by using membrane elements. We can also analyze the behaviour of the reinforcements by comparing the different terms in the stiffness matrix. As expected, the rigidity of the reinforcements is especially large in the direction of the bars, but the rigidity in the transverse direction is not negligible. It is about a fifth of the main rigidity. Interestingly, since the membrane behaviour is positive definite, we get rid of the instability of the general model. This ensures that the global problem is convex, thus avoiding spurious oscillations of the stress field.

This membrane behaviour can be compared with simple models used by engineers to take into account the reinforcements in large simulations. These simple models are generally made of bars and shell elements. Since our model was rigorously derived by the method of separation of scales, it can be used as a reference model for simpler approaches.

5 Conclusion and perspectives

We have presented a new approach to identify the effective behaviour of periodic heterogeneities localized on a surface. This approach is very general, and is not restricted to stiff or soft interfaces, nor to homogeneous layers. We described in particular an energetic formulation which enables to analyze the major influence of these defects on the whole structure. This extensive model couples elastic interface and membrane behaviour, and is able to represent all possible kinds of behaviours of the interface. This model can obviously be reduced to simpler models when it is relevant. We have also emphasized the fact that this model may be unstable, in particular for elastodynamics applications. This particular point will be the subject of future works.

This approach can be easily extended to all kinds of linear differential equations of second order. This includes thermal diffusion, electrostatic, magnetism and so on. Other subsequent works may include generalisations of this study. We have made two major hypothesis : the array of defects is orthogonal, and the materials on each side of the interface are the same. These hypothesis are not always relevant, and this study may be modified to take into account such cases.

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