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A Laplace-Variational Iteration Method for solving the homogeneous Smoluchowski coagulation equation

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Abstract. In this paper, we introduce a modified variational iteration method to solve the homogeneous Smoluchowski's coagulation equation. The approximate solutions are compared with the exact ones. The obtained results derived from the proposed method are shown graphically and discussed in detail. We conclude that the Laplace-variational iteration method (LVIM) is a powerful and easy-to-use technique for handling partial differential equations.

Keywords: Laplace-variational iteration method, Homogeneous Smoluchowski coagulation equation.

1 Introduction

Nonlinear ordinary or partial differential equations arise in various fields of science, physics and engineering. The wide applicability of these equations is the main reason why they have gained so much attention from many mathematicians and scientists. Unfortunately they are sometimes very difficult to solve, either numerically or theoretically. There are many methods to obtain approximate solutions of these kinds of equations. The variational iteration method (VIM) is powerful in investigating the approximate or analytical solutions of the nonlinear ordinary and partial differential equations. This method is proposed by the Chinese mathematician He in 1997, as a modification of a general Lagrange multiplier method proposed by Inokuti et al. in 1978. It has been shown that this procedure is a powerful tool for solving various kinds of problems [3][4]. The physical process of coagulation or coalescence of particles is often modeled by Smoluchowski's coagulation equation. This equation is a mean-field model for the growth of clusters (particles, droplets, etc.) by binary coalescence; that is, the driving growth mechanism is the merger of two particles into a single one. Analytical solutions for this equation are known only for a very limited number of kernels (constant, additive or multiplicative)[2]. Therefore, numerical methods have to be used to describe the time evolution of the cluster-size distribution. In this work we present an analytical-numerical technique (LVIM) for the solution of the homogeneous Smoluchowski's coagulation equation with both of constant and multiplicative

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kernels. The computations associated with the examples in this work were performed by using the software Maple 13.

2 Preliminaries

In this section we recall briefly the basic concept of (VIM). We begin with considering a differential equation in the general form,

$$Lf(x, t) + Nf(x, t) = g(x, t), \quad (2.1)$$

where L is a linear operator, N a nonlinear operator and $g(x, t)$ is the source term. According to the variational iteration method[3], one can construct a correction functional as follow

$$f_{k+1}(x, t) = f_k(x, t) + \int_0^t \lambda(\tau)[Lf_k(x, \tau) + N\tilde{f}_k(x, \tau) - g(x, \tau)]d\tau, \quad (2.2)$$

where λ is a general Lagrangian multiplier, the subscript k denotes the k^{th} order approximation, $f_0(x, t)$ is an initial approximation which can be known according to the initial conditions or the boundary conditions, and the function \tilde{f}_k is a restricted variation which means $\delta\tilde{f}_k = 0$. It is clear that the successive approximation f_k , $k \geq 1$ can be established by determining a general Lagrangian multiplier λ , which can be identified optimally via the variational theory. The successive approximations f_{k+1} , $k \geq 0$ of the solution $f(x, t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $f_0(x, t)$. When λ is known, then other several approximations f_k , $k \geq 1$, follow immediately. Consequently, the exact solution can be obtained by using

$$f(x, t) = \lim_{n \rightarrow \infty} f_n(x, t) \quad (2.3)$$

3 Applications and discussion.

The Smoluchowski's coagulation equation without diffusion part [1, 5, 7, 9] is defined as follows

$$\begin{cases} \frac{\partial f}{\partial t} = C^+(f) - C^-(f), & (x, t) \in \mathbb{R}_+^2, \\ f(x, 0) = f_0, & x \in \mathbb{R}_+, \end{cases} \quad (3.4)$$

where

$$\begin{aligned} C^+(f) &= \frac{1}{2} \int_0^x K(x-y, y)f(x-y, t)f(y, t)dy \\ C^-(f) &= \int_0^\infty K(x, y)f(x, t)f(y, t)dy, \end{aligned} \quad (3.5)$$

and f_0 is a known initial data, $f(x, t)$ is the density of cluster of mass x per unit volume at time t .

3.1 Example 1.

Consider the following Smoluchowski coagulation equation [1, 5, 9] with constant kernel $K(x, y) = 1$

$$\frac{\partial f}{\partial t} = \frac{1}{2} \int_0^x f(x-y, t) f(y, t) dy - \int_0^\infty f(x, t) f(y, t) dy, \quad (3.6)$$

$$f_0 := f(x, 0) = \exp(-x). \quad (3.7)$$

According to [5], eq.(3.6),(3.7) has an exact solution under the form

$$f(x, t) = N^2(t) \exp(-N(t)x), \quad (3.8)$$

with

$$N(t) = \frac{2M_0}{2 + M_0 t} \quad \text{and} \quad M_0 = 1, \quad (3.9)$$

for $(x, t) \in \mathbb{R}_+^2$, where $M(t)$ and $N(t)$ are the total mass of clusters and the total number of particles in time $t \geq 0$ respectively, defined by:

$$N(t) = \int_0^\infty f(x, t) dx \quad \text{and} \quad M(t) = \int_0^\infty x f(x, t) dx. \quad (3.10)$$

Applying the Laplace transform to eq.(3.6),(3.7) yields

$$\frac{\partial F(\xi, t)}{\partial t} = \frac{1}{2} F(\xi, t)^2 - F(\xi, t) \int_0^\infty f(y, t) dy \quad (3.11)$$

where F denotes the Laplace transform of the function f .

By the variational iteration method described in the previous section, we find that the Lagrange multiplier $\lambda = -1$, then the iteration formula for equation (3.11) reads

$$F_{k+1}(\xi, t) = F_k(\xi, t) - \int_0^t \left[\frac{\partial F}{\partial s}(\xi, s) - \frac{1}{2} F_k(\xi, s)^2 + F_k(\xi, s) \int_0^\infty f_k(y, s) dy \right] ds, \quad (3.12)$$

Accordingly, we obtain the following successive approximations

$$\begin{aligned} F_0(\xi, t) &= \frac{1}{\xi + 1}, \\ F_1(\xi, t) &= \frac{1}{\xi + 1} - \frac{t}{\xi + 1} + \frac{t}{2(\xi + 1)^2}, \\ F_2(\xi, t) &= \frac{1}{1 + \xi} - \frac{t}{1 + \xi} + \frac{1}{2} \frac{t}{(1 + \xi)^2} - \frac{1}{24} \frac{(1 + 4\xi + 6\xi^2 + 4\xi^3)t^3}{(1 + \xi)^4} - \frac{1}{16} \frac{(-4 - 16\xi - 24\xi^2 - 12\xi^3)t^2}{(1 + \xi)^4}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \quad (3.13)$$

By applying the inverse Laplace we get from (3.13)

$$\begin{aligned}
f_0(x, t) &= \exp(-x), \\
f_1(\xi, t) &= \exp(-x) \left(1 + \frac{t(x-2)}{2} \right), \\
f_2(\xi, t) &= \exp(-x) \left(2 + \frac{t(x-2)}{2} + \frac{t^2}{8}(6 - 6x + x^2) + \frac{t^3}{12} \left(\frac{1}{12}x^3 - x^2 + 3x - 2 \right) \right), \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned} \tag{3.14}$$

Where the f_k are the terms of the decomposition of $f(x, t)$, then the VIM solution for (3.6) reads

$$f(x, t) = \exp(-x) \left[2 + \frac{t(x-2)}{2} + \frac{t^2}{8}(6 - 6x + x^2) + \frac{t^3}{12} \left(\frac{1}{12}x^3 - x^2 + 3x - 2 \right) + \dots \right] \tag{3.15}$$

3.2 Example 2.

we now consider equation 3.4 with a multiplicative kernel as $K(x, y) = xy$ as the following

$$\frac{\partial f}{\partial t} = \frac{1}{2} \int_0^x (x-y)yf(x-y, t)f(y, t)dy - \int_0^\infty xyf(x, t)f(y, t)dy, \tag{3.16}$$

and

$$f_0 := f(x, 0) = \frac{\exp(-x)}{x} \tag{3.17}$$

Following [5], eq.(3.16) has the following exact solution

$$f(x, t) = \exp(-Tx) \frac{I_1(2xt^{1/2})}{x^2t^{1/2}} \tag{3.18}$$

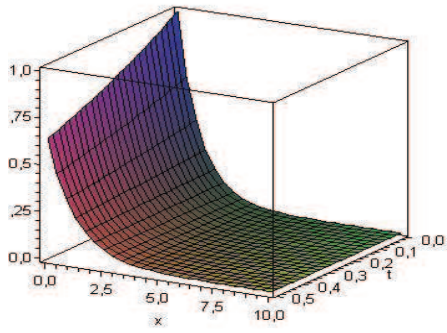
where $T = \begin{cases} 1+t, & \text{for } t \leq 1, \\ 2\sqrt{t}, & \text{otherwise} \end{cases}$

and I_1 is the modified Bessel function of the first kind

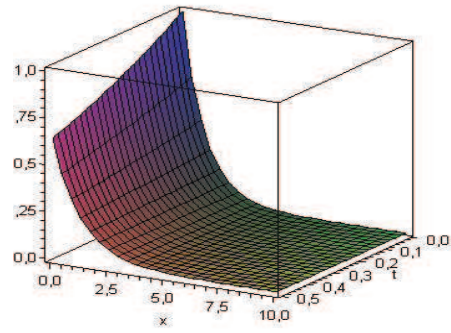
$$I_1(x) = \frac{1}{\pi} \int_0^\pi \exp(x \cos \theta) \cos \theta d\theta. \tag{3.19}$$

In the present example, the total volume $M_1(t)$ defined by (3.10) satisfies

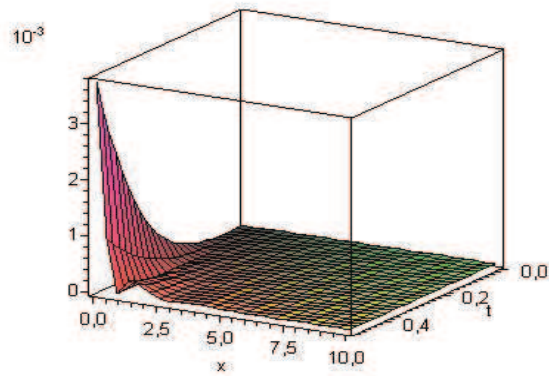
$$M_1(t) = \begin{cases} 1 & \text{for } t \in [0, 1] \\ t^{-1/2}, & \text{if } t \geq 1, \end{cases} \tag{3.20}$$



(a)



(b)



(c)

Figure 1: (a): The surface plot of $f(x,t)$ solution of (3.6): (a) exact solution (b) the approximate solution and (c) the absolute error $|f(x,t) - f_{app}(x,t)|$.

and the gelation phenomenon takes place at $t = 1$.

In similar way, Applying the inverse Laplace transform to eq.(3.16),(3.17) gives f_1 that will give f_2 , the determination of f_2 leads to find f_3 , and so on. This in turn will allows to complete determination of the components of f_k , $k \geq 0$, until convergence of the obtained series. The approximate solution is given by

$$f(x, t) = \frac{1}{261534873600} \exp(-x)(726485760t^3x^2(2x^3 - 25x^2 - 60 + 80x) + 4324320t^4x^4(798x - 1050 - 161x^2 + 9x^3) + \dots)$$

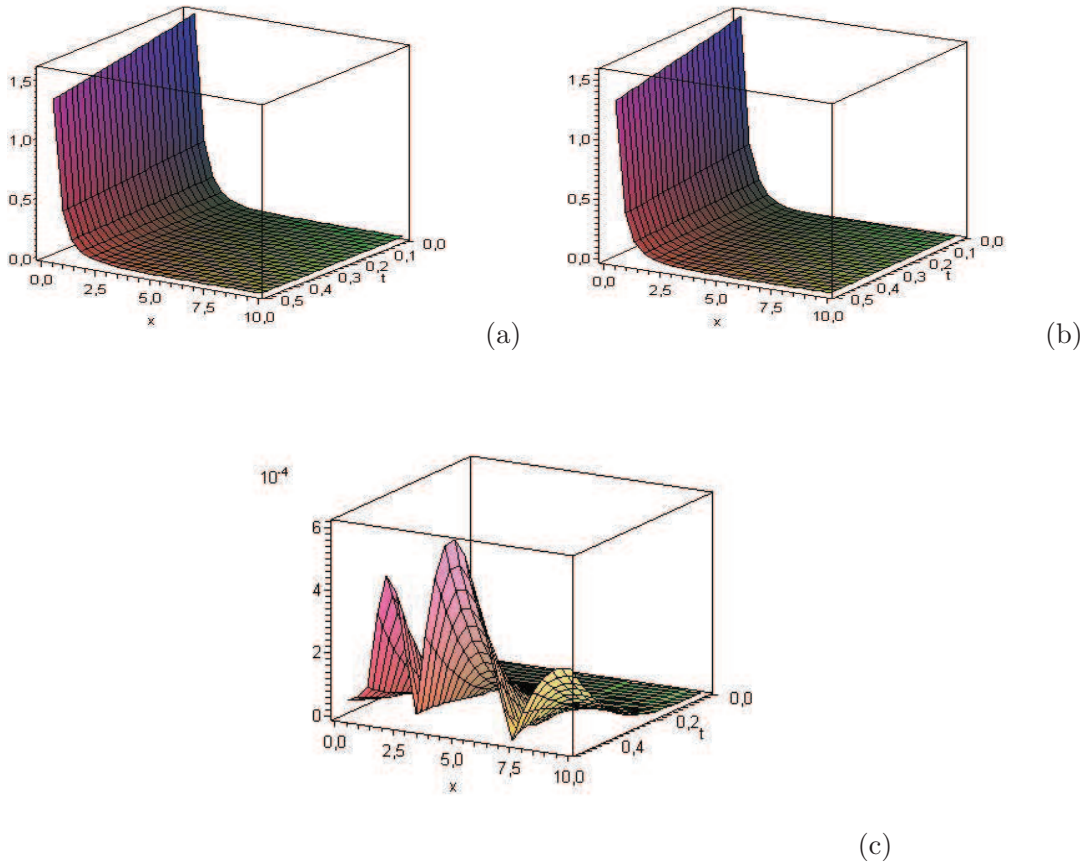


Figure 2: The surface plot of $f(x, t)$ solution of (3.16): (a) exact solution (b) the approximate solution and (c) the absolute error $|f(x, t) - f_{app}(x, t)|$.

From figures.1 and .2 We notice that the method presented by Yildirim [9] (homotopy perturbation method), the maximum absolute error for example.1 is 14.10^{-3} , while the maximum absolute error can be less than 4.10^{-3} using the present method. For example.2 the maximum absolute error found in [9] is 14.10^{-3} , while the maximum absolute error

is less than 6.10^{-4} using the present method. We stress that only three terms of the decomposition series were used for the approximate solutions.

4 Conclusion

In this paper, a combined form of the Laplace transform method with the variational iteration method is presented, and used to find solutions to the homogeneous Smoluchowski coagulation equation. The obtained solutions are compared to the exact solutions and show the high accuracy of the proposed method. Consequently the (LVIM) is promising and can be applied for other nonlinear equations that appearing in various scientific fields.

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