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Multimode Gaussian quantum light, which includes multimode squeezed and multipartite quadrature entangled light, is a very general and powerful quantum resource with promising applications in quantum information processing and metrology. In this paper, we determine the ultimate sensitivity in the estimation of any parameter when the information about this parameter is encoded in such light, irrespective of the information extraction protocol used in the estimation and of the quantity measured. In addition we show that an appropriate homodyne detection scheme allows us to reach the Quantum Cramér-Rao bound. We show that, for a given set of available quantum resources, the most economical way to maximize the sensitivity is to put the most squeezed state available in a well-defined light mode. This implies that it is not relevant to take advantage of the existence of squeezed fluctuations in other modes, nor of quantum correlations and entanglement between different modes. We finally apply these considerations to the problem of optimal phase shift estimation.

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Optical techniques are widely used in many areas of science and technology to make accurate measurements and diagnostics, from microscopy, spectrography, chemical analysis, to gravitational wave detection and ranging. There are many reasons for this: light allows us to extract information in a remote and non destructive way, it carries information in a massively parallel way, and perhaps more importantly, optical measurements can reach very high precision and sensitivity levels.

It is therefore important to know what is the ultimate limit of sensitivity that can be possibly achieved in the estimation of a parameter θ that is encoded by one way or another in a light beam, given some constraints such as a fixed mean photon number N. This limit is imposed by the unavoidable quantum fluctuations of light and depends on the quantum state of light which conveys the information about θ. When the light is in a coherent state, this limit is called 'standard quantum limit' and scales as $1/N^{1/2}$.

Many studies have been devoted to finding ways to enhance the sensitivity of parameter estimation beyond the standard quantum limit using quantum resources. It has been shown that enhanced sensitivity can be achieved by using squeezed light [1] or entangled light [2]. This has been first experimentally demonstrated for measurements in which the information about the parameter θ is carried by the total intensity [3] or by the phase [4] of a light beam. Later situations were considered where the parameter θ does not change the total intensity of the light but modifies the details of the distribution of light in the transverse plane [5] (for example to estimate a very small lateral displacement of a beam [6]). As the energy of the squeezed state increases with the squeezing factor, the ultimate limit with squeezed state for a fixed total energy scales as $1/N^{3/4}$.

If one uses instead entangled states such as NOON states [7] one reaches the so-called Heisenberg-limit (HL) which scales as $1/N$. However, in the present state of technology real measurement schemes using these states do not lead to very high sensitivities, because of the small values of N experimentally reachable (so far, the highest achievable NOON state has $N \sim 100$ [8]), and decoherence tends to rapidly destroy these states, therefore limiting the performance of the measurement to a $1/N^{1/2}$ scaling for large N [9–11]. In [12] a scheme was proposed that reaches the HL without the use of an entangled state.

In this paper, we focus on multimode Gaussian states, which include quantum resources widely used in quantum optics like multimode squeezing and multipartite entanglement. These states are already generated experimentally with impressive amounts of squeezing [13] and entanglement [14] shared by many modes [15], and for large values of the mean photon number N, which can easily be as large as 10^{16} [16].

The development of gravitational wave antennae provides a good example of the practical utility of studying the ultimate limits of sensitivity: these interferometers are now very close to the standard quantum limit, and

the enhancement of their sensitivity by increasing the power of the laser light begins to reach practical limits due to material constraints or light-pressure fluctuations. The possibility which is now considered to improve their sensitivity is the use of non-classical Gaussian light, and optimizing the use of such resources is obviously an important issue [17].

Expression of QCR bound for pure states - For any quantum state depending on a parameter θ and described by a density matrix $\hat{\rho}_\theta$, the error in the estimation of θ based on Q repeated measurements of an observable $\hat{A}$ is given by [18]

$$\delta\theta = \frac{\left\langle \delta A_{\rm est}^2 \right\rangle_\theta^{1/2}}{\sqrt{Q}\left|\frac{\partial}{\partial\theta}\left\langle A_{\rm est}\right\rangle_\theta\right|}, \tag{1}$$

where $A_{\rm est}$ is an unbiased estimator of θ that depends on the results of the measurements of $\hat{A}$. By optimizing over all estimators $A_{\rm est}$ and all measurements, Braunstein and Caves [18] showed that the best achievable sensitivity for measuring θ is bounded by the so-called quantum Cramér-Rao (QCR) bound

$$\delta\theta \geq \delta\theta_{\rm min} \equiv \left(2\sqrt{Q}\frac{s(\hat{\rho}_\theta,\hat{\rho}_{\theta+d\theta})}{d\theta}\right)^{-1}, \tag{2}$$

where $s(\hat{\rho}_\theta,\hat{\rho}_{\theta+d\theta})$ is the Bures distance between $\hat{\rho}_\theta$ and $\hat{\rho}_{\theta+d\theta}$, which, in the case of pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ is equal to $\sqrt{2(1-|\langle\psi_1|\psi_2\rangle|)}$.

Let us now consider a pure quantum state of light $|\psi_\theta\rangle$ spanning over M different spatial or temporal modes $\{v_i(\mathbf{r},t)\}$ $(i=1,...,M)$. For mixed states with parameter independent mixing probabilities, the sensitivity can at most be as good as for the pure states from it is mixed [19]. We call $\hat{a}_i$ the annihilation operator in the mode v_i, and introduce the quadrature operators $\hat{x}_i = \hat{a}_i + \hat{a}_i^\dagger$ and $\hat{p}_i = \mathrm{i}(\hat{a}_i^\dagger - \hat{a}_i)$. We define the column vectors $\hat{\mathbf{x}} = (\hat{x}_1,\ldots,\hat{x}_M)^\top$, $\hat{\mathbf{p}} = (\hat{p}_1,\ldots,\hat{p}_M)^\top$, and $\hat{\mathbf{X}} = (\hat{\mathbf{x}},\hat{\mathbf{p}})^\top$.

The overlap between the states $|\psi_\theta\rangle$ and $|\psi_{\theta+d\theta}\rangle$ reads

$$|\langle\psi_\theta|\psi_{\theta+d\theta}\rangle|^2 = (4\pi)^M \int W_\theta(\mathbf{X}) W_{\theta+d\theta}(\mathbf{X})\ d^{2M}\mathbf{X}, \tag{3}$$

where W_θ is the Wigner function of $|\psi_\theta\rangle$,

$$W_\theta(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^M}\int e^{i\boldsymbol{\xi}\cdot\mathbf{p}}\langle\mathbf{x}-\boldsymbol{\xi}\,|\psi_\theta\rangle\,\langle\psi_\theta|\,\mathbf{x}+\boldsymbol{\xi}\rangle\ d^M\boldsymbol{\xi}\,. \tag{4}$$

At second order in $d\theta$, it is equal to

$$|\langle\psi_\theta|\psi_{\theta+d\theta}\rangle|^2 \simeq 1 - \frac{d\theta^2}{2}\left((4\pi)^M\int\left(W'_\theta(\mathbf{X})\right)^2\ d^{2M}\mathbf{X}\right). \tag{5}$$

The first order vanishes because the states are pure. Throughout this letter, for any function depending on the parameter θ, we use the convention $f'_\theta \equiv \left.\frac{\partial f}{\partial\theta}\right|_\theta$, regardless of what other explicit variables f might depend on.

This leads to the QCR bound for pure states

$$\delta\theta_{\rm min} = \left(2Q\,(4\pi)^M\int\left(W'_\theta(\mathbf{X})\right)^2\ d^{2M}\mathbf{X}\right)^{-1/2}. \tag{6}$$

This intermediate result is very interesting as it gives a simple expression of the QCR bound in terms of the Wigner function for any pure quantum state. In the remainder of this paper, we will apply this formula to Gaussian states.

QCR bound for pure Gaussian states - For a Gaussian state $|\psi_\theta\rangle$, the Wigner function takes the form

$$W_\theta(\mathbf{X}) = \frac{1}{(2\pi)^M}\exp\left(-\frac{1}{2}(\mathbf{X}-\overline{\mathbf{X}}_\theta)^\top\boldsymbol{\Gamma}_\theta^{-1}(\mathbf{X}-\overline{\mathbf{X}}_\theta)\right) \tag{7}$$

where $\overline{\mathbf{X}}_\theta$ is the column vector of the expectation values of the quadratures for the different modes, and $\boldsymbol{\Gamma}_\theta$ the symmetrized covariance matrix. Both possibly depend on θ. One finds from (6)

$$\delta\theta_{\rm min} = Q^{-1/2}\left(\overline{\mathbf{X}}_\theta'^\top\boldsymbol{\Gamma}_\theta^{-1}\overline{\mathbf{X}}'_\theta + \frac{\mathrm{tr}\left(\left(\boldsymbol{\Gamma}'_\theta\boldsymbol{\Gamma}_\theta^{-1}\right)^2\right)}{4}\right)^{-1/2}. \tag{8}$$

The expression in the big bracket of Eq. (8) corresponds to the quantum Fisher information $I_{\rm Fisher}$ for a pure Gaussian state. It is made of two terms which represent the information about θ that can be extracted respectively from the mean field and from the noise. In the limit of very large values of N, the second term often turns out to be negligible compared to the first, and we will neglect it from now on.

$I_{\rm Fisher}$ can be expressed in more physical terms if one introduces a mode basis $\{\widetilde{v}_i(\mathbf{r},t)\}$ specific to our problem. We first define the *normalized mean photon field mode* as

$$u_\theta(\mathbf{r},t) = \frac{\overline{a}_\theta(\mathbf{r},t)}{\|\overline{a}_\theta\|}, \tag{9}$$

where $\hat{a}(\mathbf{r},t) = \sum_i \hat{a}_i v_i(\mathbf{r},t)$ is the local annihilation operator, $\overline{a}_\theta(\mathbf{r},t) = \langle\psi_\theta|\,\hat{a}(\mathbf{r},t)\,|\psi_\theta\rangle$ the mean photon field, and $\|\overline{a}_\theta\|$ its norm,

$$\|\overline{a}_\theta\| = \left(\int|\overline{a}_\theta(\mathbf{r},t)|^2\ d^2\mathbf{r}dt\right)^{1/2}, \tag{10}$$

with spatial integration over a surface perpendicular to the light beam propagation, and time integration over the detection time. In the case of a monochromatic field, the mean photon field mode u_θ is proportional to the mean value of the electric field in the θ-dependent quantum state.

We can now define the *detection mode* by

$$\widetilde{v}_1(\mathbf{r},t) = \frac{\overline{a}'_\theta(\mathbf{r},t)}{\|\overline{a}'_\theta\|}. \tag{11}$$

One then completes the basis starting with mode $\widetilde{v}_1$ by other orthonormal modes $\widetilde{v}_{n>1}$.

The expression of the Fisher information in the $\{\widetilde{v}_i(\mathbf{r},t)\}$ mode basis is very simple as it involves only one matrix element of $\mathbf{\Gamma}_\theta^{-1}$:

$$I_{\text{Fisher}} = 4\mathbf{\Gamma}^{-1}_{\theta,[1,1]} \left\|\overline{a}'_\theta\right\|^2 \tag{12}$$

where $\mathbf{\Gamma}^{-1}_{\theta,[1,1]}$ is the first left, top element of the matrix $\mathbf{\Gamma}_\theta^{-1}$ in the basis $\{\widetilde{v}_i(\mathbf{r},t)\}$.

The Fisher information for a single measurement involving a coherent state ($\mathbf{\Gamma}_\theta = \mathbf{1}$), that we will call I_0, is found to be

$$I_0 = 4\left\|\overline{a}'_\theta\right\|^2 = N_\theta\left(4\left\|u'_\theta\right\|^2 + \left(\frac{N'_\theta}{N_\theta}\right)^2\right), \tag{13}$$

where $N_\theta = \left\|\overline{a}_\theta(\mathbf{r},t)\right\|^2$ is a quantity that tends to the mean photon number in the high N limit. We obtain finally the expression

$$\delta\theta_{\min} = \left[QN_\theta\left(4\left\|u'_\theta\right\|^2 + \left(\frac{N'_\theta}{N_\theta}\right)^2\right)\mathbf{\Gamma}^{-1}_{\theta,[1,1]}\right]^{-1/2} \tag{14}$$

for the QCR bound for parameter estimation using quantum Gaussian states. It depends on 3 factors: the first one is as usual the mean total number of photons measured QN_θ. The second one is related to the variation as a function of θ of the displacement of the mean field and the mean photon number. The more the light properties are affected by the variation of θ, the better the sensitivity one can expect for its estimation. While the general argument is obvious, the explicit formula (14) is not. The last factor is the influence on the measurement of the quantum fluctuations of the state, which is remarkably contained in a single element of the inverse covariance matrix in our specific mode basis.

Optimized multimode Gaussian state for parameter estimation - Let us now discuss under which conditions nonclassical multimode Gaussian states can be put to best use in the estimation of θ. We assume that these states are produced by linearly mixing the single mode squeezed beams produced by independent 'squeezers', such as degenerate parametric amplifiers, which is the most widely used technique to produce multimode squeezed and multipartite quadrature entangled states [20]. We will call $\sigma^2_{\min}$ the smallest quadrature noise among all the generated squeezed modes. $\sigma^{-2}_{\min}$ is the largest eigenvalue of the inverse covariance matrix in the initial basis of the independent squeezed modes. With the help of linear couplers i.e. of unitary transformations of the mode basis, the multimode squeezing can be transformed partially or totally into multipartite entanglement in a different mode basis. One can show that, under such unitary transformations, the diagonal matrix elements of the inverse of the covariance matrix are bound by the spectral radius of $\mathbf{\Gamma}_\theta^{-1}$, which is equal to $1/\sigma^2_{\min}$. Equality is reached only if the detection mode 1 is an eigenmode of the covariance matrix with the eigenvalue $\sigma^2_{\min}$, and thus *when the detection mode is the most squeezed mode and is not correlated with any other mode.* The QCR bound corresponding to the quantum resources that we have just described is thus

$$\delta\theta_{\min} = \frac{\sigma_{\min}}{\sqrt{QN_\theta}}\left(4\left\|u'_\theta\right\|^2 + \left(\frac{N'_\theta}{N_\theta}\right)^2\right)^{-1/2}. \tag{15}$$

We have shown here an important result: the only way to saturate the Cramér-Rao bound in the configuration that we have just described is to put the most squeezed state available into the detection mode and not to have correlations with the other modes. The presence of other squeezed modes, or of any kind of entanglement, will not help improve the sensitivity: one cannot take advantage of squeezed fluctuations or quantum correlations coming from different modes to improve the estimation of a single parameter [21].

A possible experimental implementation that reaches the QCR bound - The determination of the Quantum Cramér-Rao bound is very general and does not tell us which kind of detection, and which kind of measurement strategy are to be used in order to reach it. We show in this paragraph that a homodyne detection scheme in which the local oscillator is precisely taken in the detection mode allows us to reach the QCR bound.

If one uses an intense local oscillator in mode $\widetilde{v}_1$, the balanced homodyne detection operator, for a null relative phase between the local oscillator and the measured beam, is given by $\hat{D} = \hat{\tilde{x}}_1\sqrt{N_{\text{LO}}}$, where N_{LO} is the mean photon number of the local oscillator and $\hat{\tilde{x}}_1$ the real quadrature operator of the mode $\widetilde{v}_1$. A balanced detection set-up therefore allows us to measure the projection of a multimode field on the oscillator mode, even in presence of many other modes.

For a small variation $\delta\theta$ of the parameter θ the mean value of the homodyne signal is given by

$$\begin{aligned}\left\langle\hat{D}\right\rangle_{\theta+\delta\theta} &= \sqrt{N_{LO}}\left\langle\hat{\tilde{x}}_1\right\rangle_{\theta+\delta\theta} \\ &= 2\sqrt{N_{LO}}\,\mathcal{R}e\left(\int \widetilde{v}_1^*\overline{a}_{\theta+\delta\theta}\,d^2\mathbf{r}dt\right).\end{aligned} \tag{16}$$

As

$$\overline{a}_{\theta+\delta\theta} \approx \overline{a}_\theta + \delta\theta\,\overline{a}'_\theta, \tag{17}$$

one finally gets by using the orthonormality properties of the mode basis $\{\widetilde{v}_i(\mathbf{r},t)\}$, and the fact that $\int u^*_\theta u'_\theta\,d^2\mathbf{r}dt$ is a purely imaginary number,

$$\left\langle\hat{D}\right\rangle_{\theta+\delta\theta} = \sqrt{N_{LO}}\left(\sqrt{I_0}\delta\theta + 2\frac{N'_\theta}{\sqrt{I_0}}\right). \tag{18}$$

The homodyne signal, suitably calibrated, is therefore an estimator of $\delta\theta$. Because of the additional term in (18),

the estimation is biased. We then introduce the unbiased estimator $\widetilde{\delta\theta}$ of $\delta\theta$,

$$\widetilde{\delta\theta} = \frac{\left\langle \hat{D} \right\rangle_\theta - D_0}{\sqrt{N_{\rm LO} I_0}}. \tag{19}$$

where D_0 is the mean value of $\hat{D}$ for a zero value of $\delta\theta$. Considering the case when the light state is squeezed in the detection mode by a factor $\sigma^2_{\rm min}$ and assuming a unity signal to noise ratio, the sensitivity of the homodyne measurement can be shown to be

$$\delta\theta_{\rm homodyne} = \frac{\sigma_{\rm min}}{\sqrt{N_\theta \left(4 \left\| u'_\theta \right\|^2 + \left(N'_\theta / N_\theta \right)^2 \right)}}, \tag{20}$$

which is indeed equal to the QCR bound (15) for a single measurement.

Application to a phase shift estimation - Let us finally illustrate the interest of our approach by revisiting a well-known and important problem of quantum optics, namely the interferometric estimation of a phase shift ϕ [8, 22–24]. As Michelson or Mach-Zehnder interferometers have two orthogonal and normalized output modes v_1 and v_2 that can be detected separately, they are two-mode devices. In presence of a phase-shift ϕ between the two arms of the interferometer, the two-mode mean field $\overline{a}_\phi$ at the output of the interferometer is

$$\overline{a}_\phi = A \cos(F(\phi/2))\, v_1 + A \sin(F(\phi/2))\, v_2. \tag{21}$$

When the arms of the Michelson interferometer are empty, $F(x) = x$, but optical devices such as Fabry-Perot cavities can be inserted in the two arms in order to increase the phase sensitivity of the interferometer. Note that the total mean photon number $N_\phi = |A|^2$ does not change with ϕ. The normalized mean photon field mode defined in (9) is in the present case the mode:

$$u_\phi = v_1 \cos(F(\phi/2)) + v_2 \sin(F(\phi/2)) \tag{22}$$

whereas the detection mode defined in (11) is:

$$\widetilde{v}_1 = -v_1 \sin(F(\phi/2)) + v_2 \cos(F(\phi/2)). \tag{23}$$

According to equation(14), the QCR bound is

$$\delta\phi_{\rm min} = \frac{1}{|F'(\phi/2)| \sqrt{N_\phi \mathbf{\Gamma}^{-1}_{\phi,[1,1]}}}. \tag{24}$$

It is minimum when $|F'|$ is maximum, as expected. Its minimum value is obtained by using a quantum state consisting of a vacuum squeezed state in the detection mode $\widetilde{v}_1$ and any Gaussian state of mean photon number N_ϕ uncorrelated with the first one in the mode $\widetilde{v}_2$ orthogonal to $\widetilde{v}_1$. These modes are defined in the detection plane, i.e. at the output of the interferometer. It is easy to see that the output state that we have just described is obtained by sending a field with mean value A (for example a coherent state) at one input port of the beamsplitter and a vacuum squeezed state at the other input port: one thus finds that the well-known scheme introduced by C. Caves [1] is optimal when one uses Gaussian resources. This implies in particular that using Gaussian entangled states of the two input modes will not help improve the sensitivity of the interferometer [21], nor setting up more complicated detection schemes based on the measurement of other observables than field intensities.

In conclusion, we have derived the expression of the quantum Cramér-Rao bound for parameter estimation using a pure Gaussian multimode state. We have shown that this bound can be reached with the help of a balanced homodyne detection scheme. We have also shown that multimode squeezing and multipartite entanglement are of no help. These results are good news for the experimentalists because single mode highly squeezed Gaussian states can be readily generated experimentally and because a simple homodyne detection scheme, easily achievable in a laboratory, is sufficient for reaching the best possible sensitivity.

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