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## ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

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### 1. Introduction

Let  $(\Omega, \mathcal{F}_t, \mathbf{P})$  be a probability space with an increasing right continuous family of  $(\mathcal{F}_\infty, \mathbf{P})$ -complete  $\sigma$ -algebras  $(\mathcal{F}_t)$ , and let  $\mathcal{P}$  be the predictable  $\sigma$ -algebra induced on  $\Omega \times \mathbf{R}_+$  by the family  $(\mathcal{F}_t)$ .

For  $H \in \mathcal{P}$ , we write  $H_s$  for the random variable  $\omega \rightarrow 1_H(s, \omega)$ . If  $Z = N + B$  is a semi-martingale such that  $N$  is a square integrable martingale and  $B$  an adapted process with square integrable variation, the mapping

$$(1) \quad H \rightarrow \int_0^\infty H_s dZ_s$$

defines a  $\sigma$ -additive vector measure on  $(\Omega \times \mathbf{R}_+, \mathcal{P})$  with values in  $L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$ . It has been shown by several authors that conversely if  $\mu$  is a  $\sigma$ -additive measure from  $\mathcal{P}$  to  $L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$  given on the elementary predictable sets  $H$  of the form

$$H = h \times ]s, t] \quad 0 < s < t, \quad h \in \mathcal{F}_s$$

by

$$(2) \quad \mu(H) = 1_h(Z_t - Z_s)$$

for a mean square right-continuous adapted process  $Z$ , then there is a modification of  $Z$  which is a semi-martingale [2].

Nevertheless, if we consider an other probability space  $(W, \mathcal{W}, \mathbf{Q})$ , an adapted process  $(\omega, t) \rightarrow Z_t(\omega, w)$  depending on  $w \in W$ , and a measure  $\mu$  which satisfies (2) for elementary predictable sets, and if we replace  $\sigma$ -additivity in  $L^2(\mathbf{P})$  for each  $w \in W$  by  $\sigma$ -additivity in  $L^2(\mathbf{P} \times \mathbf{Q})$ , it becomes possible that  $Z_t$  fails to be a semi-martingale for fixed  $w$ .

In the example that we give,  $Z_t$  is, for fixed  $w$ , the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.

## 2. Random mixing of semi-martingales

Let  $(U_\alpha(w))_{\alpha \in \mathbf{R}}$  be a second order process on  $(W, \mathcal{W}, \mathcal{Q})$  which is right continuous in  $L^2$ , with orthogonal increments and  $\mathcal{B}(\mathbf{R}) \times \mathcal{W}$  measurable and let  $m$  be the positive Radon measure on  $\mathbf{R}$  associated to  $U_\alpha$  by

$$m([\alpha, \beta]) = \mathbf{E}_Q(U_\beta - U_\alpha)^2, \quad \alpha < \beta.$$

Let  $(M_t^\alpha(\omega))_{\alpha \in \mathbf{R}}$  be a family of right continuous and left limited martingales, and  $(A_t^\alpha(\omega))_{\alpha \in \mathbf{R}}$  a family of continuous increasing adapted processes on  $(\Omega, \mathcal{F}_t, \mathbf{P})$  such that the maps  $(\alpha, \omega, s) \rightarrow M_s^\alpha(\omega)$  and  $(\alpha, \omega, s) \rightarrow A_s^\alpha(\omega)$  are  $\mathcal{B}(\mathbf{R}) \times \mathcal{F}_t \times \mathcal{B}(\mathbf{R}_+)$  measurable on  $\mathbf{R} \times \Omega \times [0, t]$  and such that

$$(3) \quad \int_{\alpha \in \mathbf{R}} \mathbf{E}_P[(M_\infty^\alpha)^2 + (A_\infty^\alpha)^2] dm(\alpha) < +\infty.$$

Then we set  $Z_t^\alpha(\omega) = M_t^\alpha(\omega) + A_t^\alpha(\omega)$  and

$$(4) \quad Z_t(\omega, w) = \int_{\alpha \in \mathbf{R}} Z_t^\alpha(\omega) dU_\alpha(w)$$

where the stochastic integral is of Wiener's type and exists for  $\mathbf{P}$  almost all  $\omega$  since by (3)  $Z_t^\alpha(\omega)$  belongs to  $L^2(\mathbf{R}, \mathcal{B}(\mathbf{R}), dm(x))$  for  $\mathbf{P}$ -almost all  $\omega$ .

For  $\mathbf{P}$ -almost  $\omega$  the process  $Z_t(\omega, w)$  is right continuous and left limited in  $L^2(W, \mathcal{W}, \mathcal{Q})$ .

If  $G$  is an elementary predictable process on  $(\Omega, \mathcal{F}_t, \mathbf{P})$  given by:

$$G_s(\omega) = G_0(\omega) 1_{]0, t_1]}(s) + \dots + G_n(\omega) 1_{]t_n, t_{n+1}]}(s)$$

for  $0 < t_1 < \dots < t_{n+1}$ , where  $G_i$  is a  $\mathcal{F}_{t_i}$ -measurable bounded random variable, it follows immediately

$$G_0(Z_{t_1} - Z_0) + \dots + G_n(Z_{t_{n+1}} - Z_{t_n}) = \int_{\alpha \in \mathbf{R}} \left( \int_0^\infty G_s dZ_s^\alpha \right) dU_\alpha.$$

And we have:

**Proposition 1.** *The map  $H \in \mathcal{P} \rightarrow \int_0^t H_s dZ_s$  defined by*

$$\int_0^t H_s dZ_s = \int_{\alpha \in \mathbf{R}} \left( \int_0^t H_s dZ_s^\alpha \right) dU_\alpha$$

*is a  $\sigma$ -additive  $L^2(\mathbf{P} \times \mathcal{Q})$  valued measure on  $(\Omega \times \mathbf{R}_+, \mathcal{P})$ .*

*Proof.* Let  $H^{(n)}$  be a sequence of disjoint predictable subsets of  $\Omega \times \mathbf{R}_+$ , we have

$$\begin{aligned} & \mathbf{E}_P \mathbf{E}_Q \left[ \int_{\alpha \in \mathbf{R}} \left( \int_0^t \sum_{n=N}^\infty H_s^{(n)} dZ_s^\alpha \right) dU_\alpha \right]^2 \\ &= \int_{\alpha \in \mathbf{R}} \mathbf{E}_P \left( \int_0^t \sum_{n=N}^\infty H_s^{(n)} dZ_s^\alpha \right)^2 dm(\alpha) \end{aligned}$$

which can be made arbitrarily small for  $N$  large enough because

$$E_P \left( \int_0^t \sum_{n=N}^{\infty} H_s^{(n)} dZ_s^\alpha \right)^2$$

tends to zero and remains bounded by

$$2 E_P [(M_\infty^\alpha)^2 + (A_\infty^\alpha)^2] < +\infty. \quad \square$$

Set

$$Z_t^{(1)} = \int_{\alpha \in \mathbb{R}} M_t^\alpha dU_\alpha \quad \text{and} \quad Z_t^{(2)} = \int_{\alpha \in \mathbb{R}} A_t^\alpha dU_\alpha.$$

**Lemma 2.** *There is a  $P \times Q$ -modification  $\tilde{Z}_t^{(1)}$  of  $Z_t^{(1)}$  which is a  $(\Omega, \mathcal{F}_t, P)$  right continuous and left limited martingale for  $Q$ -almost all  $w$ .*

*Proof.* Let  $G \in \mathcal{F}_s$ , the following equalities hold in  $L^2(W, \mathcal{W}, Q)$  for  $s < t$ :

$$\begin{aligned} E_P[1_G Z_t^{(1)}] &= \int_{\alpha \in \mathbb{R}} E_P[1_G M_t^\alpha] dU_\alpha = \int_{\alpha \in \mathbb{R}} E_P[1_G M_s^\alpha] dU_\alpha \\ &= E_P[1_G Z_s^{(1)}], \end{aligned}$$

therefore, if we choose a  $\mathcal{F}_t \times \mathcal{W}$ -measurable element  $z_t^{(1)}(\omega, w)$  in the  $L^2(P \times Q)$  equivalence class of  $Z_t^{(1)}$ , for  $w$  outside a  $Q$ -negligible set  $\mathcal{N}$ ,  $z_s^{(1)}$  is a  $(\mathcal{F}_s, P)$ -martingale for rational  $s$ .

Then, if we put  $\tilde{Z}_t^{(1)} = \lim_{\substack{s \text{ rational} \\ s \downarrow t}} z_s^{(1)}$ , for  $w \notin \mathcal{N}$ ,  $\tilde{Z}_t^{(1)}$  is  $P$ -almost surely a right

continuous and left limited  $(\mathcal{F}_t)$ -martingale and

$$\tilde{Z}_t^{(1)} = Z_t^{(1)} \quad P \times Q\text{-a.e.}$$

because  $Z_t^{(1)}$  is right continuous in  $L^2(P \times Q)$ .  $\square$

As concerns  $Z_t^{(2)}$ , it is a zero energy process:

**Lemma 3.** *Let  $\tau_n$  be a sequence of partitions of  $[0, t]$  with diameter tending to zero, then*

$$E_Q E_P \left[ \sum_{t_i \in \tau_n} (Z_{t_{i+1}}^{(2)} - Z_{t_i}^{(2)})^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* The expression is equal to

$$E_P \int_{\alpha \in \mathbb{R}} \sum_{\tau_n} (A_{t_{i+1}}^\alpha - A_{t_i}^\alpha)^2 dm(\alpha),$$

and  $\sum_{\tau_n} (A_{t_{i+1}}^\alpha - A_{t_i}^\alpha)^2$  tends to zero, because  $A_t^\alpha$  is continuous, and remains majorized by  $(A_\infty^\alpha)^2$ , which gives the result by (3).

Nevertheless, in general  $Z_t^{(2)}$  has no modification with finite variation, as shown by the following example:

Let  $X$  be a continuous martingale on  $(\Omega, \mathcal{F}_t, \mathbf{P})$  such that

$$E_{\mathbf{P}} X_{\infty}^2 < +\infty.$$

Let

$$M_t^{\alpha} = \int_0^t 1_{\{X_s > \alpha\}} dX_s$$

and  $A_t^{\alpha} = \frac{1}{2} L_t^{\alpha}$

where  $L_t^{\alpha}$  is the local time of  $X$  at  $\alpha$ . Condition (3) is satisfied as soon as the measure  $m$  is finite. If we put

$$Z_t = \int_{\alpha \in \mathbf{R}} M_t^{\alpha} dU_{\alpha} + \int_{\alpha \in \mathbf{R}} A_t^{\alpha} dU_{\alpha}$$

we have, from Meyer-Tanaka's formula:

$$Z_t = \int_{\alpha \in \mathbf{R}} [(X_t - \alpha)^+ - (X_0 - \alpha)^+] dU_{\alpha} = \int_{x_0}^{x_t} U_{\lambda} d\lambda \quad \mathbf{P} \times \mathbf{Q} \text{ a.e.}$$

If  $Z_t$  had a  $\mathbf{P} \times \mathbf{Q}$ -modification such that, for fixed  $w \in W$ ,  $\tilde{Z}_t$  were a  $(\Omega, \mathcal{F}_t, \mathbf{P})$  semi-martingale, then, since  $\tilde{Z}_t$  and  $\int_{x_0}^{x_t} U_{\lambda} d\lambda$  are both right continuous,  $\int_{x_0}^{x_t} U_{\lambda} d\lambda$  would be a semi-martingale. So, from ([1], theorem 5, 6), if we took for  $X$  a real stopped brownian motion starting at 0, the map

$$x \mapsto \int_0^x U_{\lambda}(w) d\lambda$$

would be the difference of two convex functions. But, if for example,  $U$  itself is a stopped brownian motion, that can be true only on a  $\mathbf{Q}$ -negligible set because almost all brownian sample paths have not finite variation. So, in this case, the  $\mathbf{P} \times \mathbf{Q}$ -modifications of  $Z_t$  are  $\mathbf{Q}$  a.e. not semi-martingales.

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