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Stéphane Nonnenmacher

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SPECTRAL PROBLEMS IN OPEN QUANTUM CHAOS

STÉPHANE NONNENMACHER

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1. INTRODUCTION

This review article will present some recent results and methods in the study of 1-particle quantum or wave scattering systems, in the semiclassical/high frequency limit, in cases where the corresponding classical/ray dynamics is chaotic.

The study of such systems has a long history in physics and mathematics, ranging from mesoscopic semiconductor physics to number theory. We will focus on some mathematical aspects, adopting a *quantum chaos* point of view: one wants to understand how the classical dynamics influences the quantum one, both regarding time dependent and time independent (that is, spectral) quantities. Equivalently, one searches for traces of classical chaos in the quantum mechanical system. In this introduction, I will focus on a simple system we will

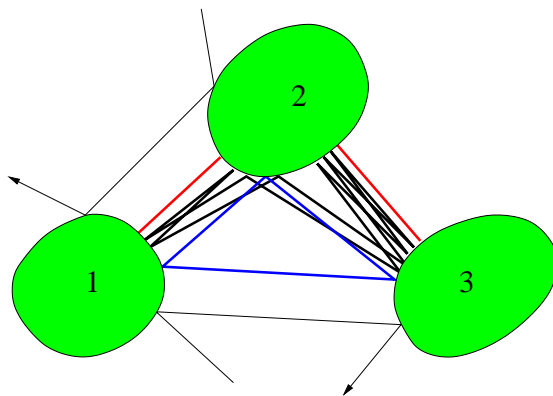


FIGURE 1. Configuration of 3 convex obstacles in the plane satisfying the no-eclipse condition, leading to a fractal hyperbolic trapped set. The numbering of the obstacles leads to the associated symbolic dynamics.

be dealing with: the scattering by three or more balls $(B_j)_{j=1,\dots,J}$ (more generally, strictly convex bodies) in \mathbb{R}^d , satisfying a no-eclipse condition [36]¹. The classical nature of the dynamics will be explained in §2.

At the quantum level, one wants to understand the wave propagation in this geometry, that is solve the (scalar) wave equation

$$(1) \quad \partial_t^2 u(x, t) - \Delta_\Omega u(x, t) = 0,$$

with given initial conditions $u(x, 0)$, $\partial_t u(x, 0)$. Here Δ_Ω is the Laplacian outside the disks $(\Omega = \mathbb{R}^d \setminus \sqcup_i D_i)$, with Dirichlet boundary conditions. Through a Fourier transform in time, we get the Helmholtz equation

$$(2) \quad \Delta_\Omega u(x) + k^2 u(x) = 0,$$

which describes stationary waves of energy k^2 (k is the wavevector, that is the inverse of the wavelength).

This systems is relevant physically, and has been studied in theoretical physics (see for instance the review paper by Wirzba [87] for the 2-dimensional scattering by J disks, and references therein) and mathematics literature [36]. It has also been implemented in various experimental realizations, most recently on microwave tables by the Marburg group [85].

1.1. Scattering vs. metastable states. For a given wavevector k , Eq. (2) admits an infinite dimensional space of solutions $u(x)$, which can be parametrized by decomposing $u(x)$, away from the obstacles (say, outside a ball $B(0, R_0)$) into a basis of incoming and outgoing waves:

$$(3) \quad u(x) = u_{in}(x) + u_{out}(x).$$

For instance, in dimension $d = 2$ the ingoing waves can be expanded in angular momentum eigenstates: using polar coordinates,

$$(4) \quad u_{in}(x) = \sum_{n \in \mathbb{Z}} a_n^{in} e^{in\theta} H_n^{in}(kr),$$

where H_n^{in} are the incoming Hankel functions, and similarly for $u_{out}(x)$. Any such solution $u(x)$ is called a scattering state. It is not L^2 -normalizable, reflecting the fact that the spectrum of $-\Delta_\Omega$ is absolutely continuous on \mathbb{R}_+ , without any embedded eigenvalue. We will briefly address the phase space structure of these scattering states in §9.2.

Beyond the a.c. spectrum, this system admits a discrete of quantum *resonances*, or complex generalized eigenvalues. They can be obtained as follows. The resolvent $(\Delta + k^2)^{-1}$ is a bounded operator on $L^2(\Omega)$ for $\text{Im } k > 0$; its norm diverges when $\text{Im } k \rightarrow 0$, reflecting the presence of the continuous spectrum. However, if we cut it off by a compactly supported (or fast decaying) function $\chi(x)$, the cutoff resolvent $\chi(\Delta + k^2)^{-1}\chi$ can be meromorphically continued from the upper half plane $\{\text{Im } k > 0\}$ to the lower half plane $\{\text{Im } k < 0\}$, where

¹Namely, the convex hull of any pair of obstacles B_i, B_j does not intersect any third obstacle.

it generally admits a discrete set of poles $\{k_j\}$ of finite multiplicities². These poles are called the resonances of $-\Delta_\Omega$. Each pole k_j (assuming it is simple) is associated with a generalized eigenfunction $u_j(x)$, which satisfies the equation

$$(\Delta_\Omega + k_j^2)u_j(x) = 0, \quad \text{with Dirichlet boundary conditions on } \partial\Omega,$$

and is *purely outgoing* (meaning that its decomposition (3) only contains outgoing components). This function grows exponentially when $|x| \rightarrow \infty$, an “unphysical” behaviour, so this function is only relevant inside a compact set (the *interaction region* formed by the ball $B(0, R_0)$). The time dependent function

$$(5) \quad \tilde{u}_j(x, t) \stackrel{\text{def}}{=} u_j(x)e^{-ik_j t}, \quad t \geq 0,$$

satisfies Eq. (1). The time decay in (5) explains why $u_j(x)$ is called a *metastable state*, with *lifetime*

$$\tau_j = \frac{2}{|\text{Im } k_j|}.$$

Similarly as in the case of a closed system, one can expand the time dependent wave $u(x, t)$ when $t \rightarrow \infty$ into a sum over (at least some of) the metastable states (5). Such an expansion is less straightforward than in the case of a closed system (it isn’t based on an L^2 orthogonal decomposition), but often gives a good description of the wave $u(x, t)$ for long times [82, 13, 31].

Another application of the study of resonances: the presence of a resonance free strip below the real axis (together with estimates of the resolvent in the strip) can be used to quantitatively estimate the dispersion and/or local energy decay for the wave $u(t)$, both in the case of the wave equation and the Schrödinger equation $u(t) = e^{it\Delta}u(0)$ [16, 17, 12].

1.2. Semiclassical distribution of resonances. We will not investigate these time dependent aspects any further, but will concentrate on the spectral one, namely the distribution of the resonances and the associated metastable states. The first mathematical works on the subject consisted in counting resonances in large disks $k_j \in D(0, k)$, $k \rightarrow \infty$. Melrose [46] obtained the general Weyl type upper bound $\mathcal{O}(k^d)$ for compact obstacles in odd dimension; this bound was generalized to obstacles in even dimension [84] as well as to scattering by a potential [89].

In the following we will focus on the *long living* resonances, that is those k_j sitting within a fixed distance from the real axis (equivalently, the resonances with lifetimes τ_j uniformly bounded from below). These resonances are obviously the most relevant for the long time behaviour of the waves. We will consider the high frequency limit $\text{Re } k \gg 1$, which is equivalent with the semiclassical limit in quantum mechanics (see §3), in order to establish a connection with the classical dynamics.

²In even dimension the continuation has a logarithmic singularity at $k = 0$, often represented by a cut along the negative imaginary axis.

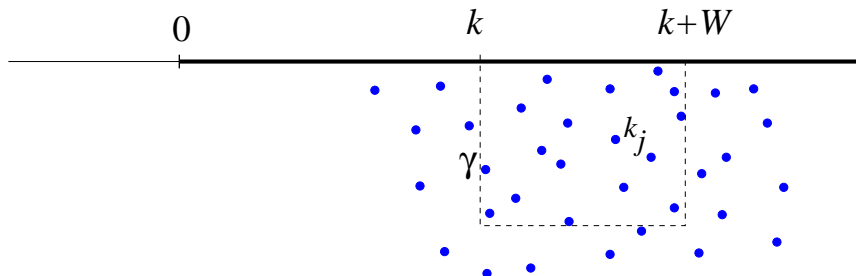


FIGURE 2. Absolutely continuous spectrum of Δ_Ω , together with the resonances below the real axis, near some value $k \gg 1$.

Questions:

- (1) For a given width $W > 0$ and depth $\gamma > 0$, what is the asymptotic number of resonances in the rectangle $[k, k + W] - i[0, \gamma]$ when $k \rightarrow \infty$?
- (2) In particular, is there some $\gamma > 0$ such that this rectangle is empty of resonances for k large enough? (such a γ is called a *resonance gap*).
- (3) Given an infinite sequence of long living resonances (k_{j_ℓ}) , what is the spatial, or phase space structure of the associated metastable states when $\text{Re } k_{j_\ell} \rightarrow \infty$?

In this high frequency limit, these spectral objects will be connected with properties of the long time classical dynamics of the system. The latter is the flow $\Phi^t : S^*\Omega \rightarrow S^*\Omega$, which consists in following straight rays at unit speed outside the obstacles, and reflecting specularly on the obstacles. With our conditions on the obstacles, this dynamics is chaotic in the following sense: the set of trapped trajectories

$$(6) \quad K = \{\rho \in S^*\Omega, \Phi^t(\rho) \text{ uniformly bounded when } t \rightarrow \pm\infty\}$$

is a fractal set, and the flow on it is uniformly hyperbolic (equivalently, one says that K is a hyperbolic set for Φ^t , see §2 for details). For future use we also define the outgoing (K^+) and incoming (K^-) tails of the trapped set,

$$(7) \quad K^+ = \{\rho \in S^*\Omega, \Phi^t(\rho) \text{ uniformly bounded when } t \rightarrow -\infty\},$$

with the obvious relation $K = K^- \cap K^+$.

The question (2) above has been addressed around the same time by Ikawa [36] and Gaspard and Rice [27] (see also [11]). Both these works establish the presence of a gap, provided the trapped set K is sufficiently “filamentary”. The precise criterium depends on a certain dynamical quantity associated with the flow, called the *topological pressure* (the pressure and the unstable Jacobian will be defined in §2).

Theorem 1. [36] *Consider the obstacle scattering problem in \mathbb{R}^d , with strictly convex obstacles satisfying the no-eclipse condition. If the topological pressure for the flow on the trapped set $K \subset S^*\Omega$ satisfies*

$$\mathcal{P} = \mathcal{P}(-1/2\varphi^+, \Phi^t \upharpoonright_K) < 0,$$

then for any small $\epsilon > 0$ there exists $k_\epsilon > 0$ such that there are no resonances of Δ_Ω in the strip

$$(8) \quad [k_\epsilon, \infty) - i[0, |\mathcal{P}| - \epsilon].$$

In dimension $d = 2$, the sign of the above topological pressure is determined by purely geometric data, namely the *Hausdorff dimension* of the trapped set:

$$(9) \quad \mathcal{P}(-1/2\varphi^+, \Phi^t \upharpoonright_K) < 0 \iff \dim_H(K) < 2,$$

which explains the notion of "filamentary" or "thin" trapped set.

In §8.1 we will sketch the proof of the above theorem (in a more general context), using the tool of quantum monodromy operators. The intuitive idea is the following: wavepackets propagating along K disperse exponentially fast due to the hyperbolicity of the trajectories; on the other hand, the wavepackets propagating on nearby trajectories could also interfere constructively in order to recombine themselves along K . The pressure criterium ensures that the dispersion is stronger than the possible constructive interference, leading to a global decay of the wave near K . The pressure $\mathcal{P}(-\varphi^+/2)$ will appear several times in the text, including in §9.2 devoted to the description of scattering states.

The value of this pressure somehow splits the systems we consider between "very open" systems with "thin" trapped sets ($\mathcal{P}(-\varphi^+/2) < 0$) vs. "almost closed" with "thick" trapped sets ($\mathcal{P}(-\varphi^+/2) \geq 0$). It will be relevant also in the description of the scattering states in §9.2.

The question (1) has first been addressed by Sjöstrand in the case of a real analytic Hamiltonian flow with a chaotic trapped set [72], leading to the first example of *fractal Weyl upper bound*. His result was generalized and sharpened in [76], see Thm 4 below. For the above obstacle scattering, a similar fractal upper bound had been conjectured in [72], but proved only recently [52].

To state the result, we recall the definition of the upper box (or Minkowski) dimension of a bounded set $K \subset \mathbb{R}^n$:

$$\overline{\dim}(K) \stackrel{\text{def}}{=} \limsup_{\epsilon \rightarrow 0} \left(n - \frac{\log \text{Vol}(K_\epsilon)}{\log \epsilon} \right),$$

where K_ϵ is the ϵ -neighbourhood of K . The dimension is said to be *pure* if $\frac{\text{Vol}(K_\epsilon)}{\epsilon^{n-\overline{\dim}(K)}}$ is uniformly bounded as $\epsilon \rightarrow 0$.

Theorem 2. [52] *Consider the obstacle scattering problem in \mathbb{R}^d , with strictly convex obstacles satisfying the no-eclipse condition.*

Let $2\nu + 1$ be the upper box dimension of the trapped set $K \subset S^\Omega$.*

Then, the resonances of Δ_Ω satisfy the following bound. For any $\gamma > 0$ and any $\epsilon > 0$, there exists $k_{\gamma,\epsilon}, C_{\gamma,\epsilon} > 0$ such that

$$\#\{k_j \in [k, k+1] - i[0, \gamma]\} \leq C_{\gamma,\epsilon} k^{\nu+\epsilon}, \quad \forall k > k_{\gamma,\epsilon}.$$

If K is of pure dimension $2\nu + 1$, one can take $\epsilon = 0$.

In dimension $d = 2$, the trapped set is always of pure dimension, and its box dimension is equal to its Hausdorff dimension. In that case, the dimension ν can be obtained through the topological pressure of the flow on Γ (see §2.2), namely ν is the (unique) real root s_0 of the equation

$$\mathcal{P}(-s\varphi^+, \Phi^t \upharpoonright K) = 0.$$

(because $\mathcal{P}(-s\varphi^+)$ is strictly decreasing with s , this equation directly leads to the equivalence (9)).

The question (3) has been studied only recently, in the case of a smooth Hamiltonian flow, or a discrete time dynamics (open quantum map). The main phenomenon is that, in the high frequency limit, the long living metastable states are microlocalized near the outgoing tail K^+ , and can be described in terms of semiclassical measures which are invariant through the flow, up to a global decay (see §9.1). Although no rigorous result on this question has been obtained in the case of obstacles, it is very likely that Thm. 10 can be adapted to the obstacles setting.

1.3. Outline of the paper. The rest of the paper is organized as follows. In the next section we describe the dynamical properties of the classical flows we wish to consider, namely Hamiltonian flows for which the trapped set is a compact hyperbolic repeller. We also define the relevant dynamical quantities associated with the flow, like the unstable Jacobian and the topological pressure.

In §3 we extend the above two theorems to more general systems, namely semiclassical Schrödinger operators $P(\hbar)$ involving a compactly supported potential, and Laplace-Beltrami operators on Riemannian manifolds, where the dynamics is only driven by the geometry. We state the analogues of Thms. 1 and 2 in these settings. The case of hyperbolic manifolds of infinite volume (obtained as quotients of the Poincaré half-space \mathbb{H}^d by certain discrete groups) is particularly interesting: the quantum resonances of the Laplacian can then be directly connected with the classical dynamics.

In §4 we interpret the quantum resonances as the eigenvalues of a (nonselfadjoint) operator obtained by “deforming” $P(\hbar)$ into the complex. We then (sketchily) explain how a further deformation, using microlocal weights, allows to prove a fractal Weyl upper bound for the number of resonances [76].

In §5 we introduce the model of open (chaotic) maps and their quantizations, which correspond to discrete time dynamics instead of flows. They have been used as a convenient toy model for the “true” scattering systems, being much more amenable to numerical studies. We then construct *quantum monodromy operators* associated with a scattering Hamiltonian $P(\hbar)$; they form a family of open quantum maps associated with the Poincaré map for the classical flow, and can be used as a tool to extract and study the resonances of $P(\hbar)$.

In §6 we formulate a weak and a strong form of *fractal Weyl law*, and discuss their validity for the various systems introduced above, mostly guided by numerical data. In §7 we give a heuristic explanation of the Weyl law for quantum maps, and provide a proof of its upper bound, eventually leading to Thm 2 and its analogues. The proof shows how the full quantum system can be reduced to an effective operator of minimal rank.

In §8 we show that the pressure criterium of Thm 1 applies to all the systems considered above. We sketch the proof of this gap in the case of open quantum maps and monodromy operators, leading to the general case of Schrödinger operators. In §8.2 we discuss the sharpness of this criterium, using both analytical and numerical results.

In §9 we briefly describe what is known about the phase space structure of metastable states associated with the long living resonances, in particular using the tool of semiclassical measures. We also consider the scattering states.

Finally, §10 presents a brief conclusion, and mentions possible extensions of the methods to similar nonselfadjoint quantum problems, like damped waves on a compact manifold of negative curvature.

Most of the above results have appeared elsewhere (or are bound to do so in a near future). The spectral radius estimate for open quantum maps, Thm 9, has not been formulated before, but it is a rather direct application of [55]. The numerics of §8.2.2 had not been published either.

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2. CHAOTIC DYNAMICS

We have already introduced the flow Φ^t on $S^*\Omega$ describing the classical scattering system outside the obstacles: it consists in the free motion outside the obstacles, plus specular reflection at the boundaries. This flow is the Hamiltonian flow generated by the Hamilton function

$$(10) \quad p(x, \xi) = \frac{|\xi|^2}{2} + V_{\mathcal{O}}(x), \quad \text{with the singular potential} \quad \begin{cases} V_{\mathcal{O}}(x) = \infty, & x \in \Omega, \\ V_{\mathcal{O}}(x) = 0, & \text{otherwise.} \end{cases}$$

The unit cotangent bundle $S^*\Omega$ is given by the energy shell $p^{-1}(1/2)$.

All the flows we will consider will be Hamiltonian. The discrete time models (open maps) introduced in §5 will be given by local diffeomorphisms on a symplectic manifold, which preserve the symplectic structure.

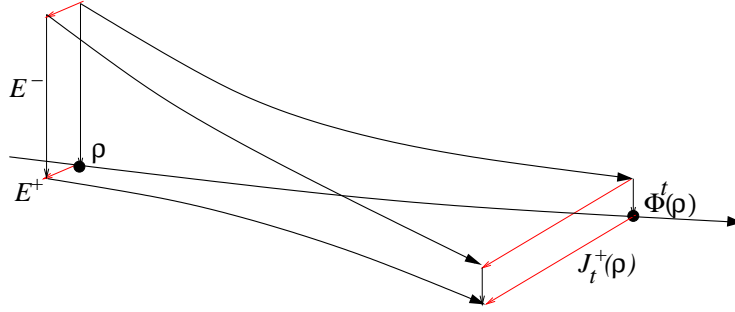


FIGURE 3. Hyperbolicity of the trajectory $\Phi^t(\rho)$: nearby orbits along the stable (resp. unstable) directions approach $\Phi^t(\rho)$ exponentially fast in the future (resp. past). The unstable Jacobian $J_t^+(\rho)$ measures this exponential divergence.

2.1. Hyperbolicity. Let us now describe the dynamical assumptions on these flows or maps. These properties generally refer to long time properties of the flow restricted to the trapped set K (we keep the notations of the obstacle problem).

The flow $\Phi^t \upharpoonright K$ is uniformly hyperbolic. This means that, for any $\rho \in K$, the tangent space $T_\rho(S^*\Omega)$ splits into the flow direction $\mathbb{R}H_p(\rho)$, a stable subspace $E^-(\rho)$, and an unstable subspace $E^+(\rho)$:

$$T_\rho(S^*\Omega) = \mathbb{R}H_p(\rho) \oplus E^-(\rho) \oplus E^+(\rho).$$

The (un)stable subspaces are characterized as follows: there exist $C, \lambda > 0$ such that, for each $\rho \in K$,

$$v \in E^\mp(\rho) \iff \forall t > 0, \quad \|d\Phi^{\pm t}v\| \leq C e^{-\lambda t} \|v\|.$$

The subspaces $E^\pm(\rho)$ depend continuously on $\rho \in K$, and are uniformly transverse to each other. An important quantity associated with this splitting is the *unstable Jacobian* of the flow,

$$J_t^+(\rho) \stackrel{\text{def}}{=} \det(d\Phi^t \upharpoonright_{E^+(\rho)}), \quad t > 0,$$

which measures the expansion of the flow along the unstable manifold³. This Jacobian grows exponentially when $t \rightarrow \infty$. The infinitesimal version of this Jacobian reads

$$(11) \quad \varphi^+(\rho) \stackrel{\text{def}}{=} \frac{dJ_t^+}{dt}(\rho) \upharpoonright_{t=0},$$

and it is possible to choose a metric near K such that φ^+ is always positive.

For the obstacle scattering system, the hyperbolicity of the flow is due to the strict convexity (or positive curvature) of the obstacles, which defocusses incoming parallel rays very fast.

³Although this Jacobian depends on the choice of coordinates and metric near ρ and $\Phi^t(\rho)$, its asymptotical behaviour for $t \rightarrow \infty$ does not.

The (un)stable subspaces have nonlinear counterparts, namely the (un)stable manifolds of the flow:

$$\forall \rho \in K, \quad W^\mp(\rho) = \{\rho' \in S^*\Omega, d(\Phi^t(\rho'), \Phi^t(\rho)) \xrightarrow{t \rightarrow \pm\infty} 0\}.$$

The unions of these manifolds makes up the incoming/outgoing tails K^\mp defined in (7).

A trapped set K hosting such dynamical properties is often called a hyperbolic set, or hyperbolic repeller.

2.2. Complexity. The second ingredient of a chaotic flow is *complexity*, in the information theoretic sense. It means that the trapped set K cannot be too simple, e.g it cannot just consist in finitely many periodic trajectories. Grossly speaking, complexity means that, if one groups the long segments of trajectories into “pencils” made of nearby segments, then the number of such pencils grows exponentially with the length of the segments.

Let make this notion more explicit with our obstacle problem (see Fig. 1). To any point $\rho \in K \setminus \mathcal{O}$, we can construct a bi-infinite sequence of symbols

$$\epsilon = \cdots \epsilon_{-2}, \epsilon_{-1} \cdot \epsilon_0 \epsilon_1 \epsilon_2 \cdots, \quad \epsilon_i \in \{1, \dots, J\}, \quad \epsilon_i \in \{1, 2, \dots, J\}$$

representing the obstacles successively hit by $\Phi^t(\rho)$ in the future or in the past. It automatically satisfies the condition $\epsilon_i \neq \epsilon_{i+1}$ for all i .

Conversely, the condition we put on the obstacles imply that, for any such sequence satisfying the above condition, one can construct a trajectory with the above properties, and this trajectory is (essentially) unique. In particular, this trajectory is periodic iff the sequence ϵ is so. This description of the trapped orbits in terms of sequences of “symbols” is called a *symbolic dynamics*. It is a simple way to classify the trajectories of the flow, and estimate its complexity. For instance, if one decides to group trajectories by specifying the obstacles they hit from $i = 0$ to $i = n$, then the number of such “pencils” is simply given by $J(J-1)^n$, which obviously grows exponentially with the “discrete time” n . The number of periodic orbits also grows exponentially with their period.

This complexity can be “measured” by the *topological pressures* $\mathcal{P}(f, \Phi^t \upharpoonright K)$ associated with the flow and an “observable” $f \in C(K)$. The pressure provides a statistical information on the “pencils” of long orbit segments. It is a very general concept, which can be defined in many ways.

In the present case of a hyperbolic repeller, this pressure can be defined in terms of long periodic orbits:

$$(12) \quad \mathcal{P}(f, \Phi^t \upharpoonright K) \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \frac{1}{T} \log \sum_{p: T-1 \leq T_p \leq T} \exp(f(p)), \quad f(p) = \int_0^{T_p} f(\Phi^t(\rho_p)) dt.$$

The sum runs over all the periodic orbits p of periods $T_p \in [T-1, T]$, and ρ_p is any point on p . In §9 we will give an alternative way to compute the topological pressure for the open baker’s map, in terms of its symbolic dynamics.

If one takes $f \equiv 0$, the above expression measures the exponential growth rate of the number of long periodic orbits, which defines the *topological entropy* of the flow:

$$(13) \quad \mathcal{P}(0, \Phi^t \upharpoonright_K) = H_{top}(\Phi^t \upharpoonright_K).$$

For this reason, complexity is often defined by a positive H_{top} .

If f is negative everywhere, the sum in (12) shows a competition between exponentially decreasing terms $e^{f(p)} \sim e^{-Tf}$, and the exponentially increasing number of terms. This is the case, for instance, if one uses the unstable Jacobian (11) and takes $f = -s\varphi^+$ for some parameter $s > 0$. In that case $e^{f(p)} = J_{T_p}^+(\rho_p)^{-s}$ measures the instability of the orbit p . When $s = 1$, the exponential damping exceeds the exponential proliferation, and the pressure is negative. Actually,

$$\gamma_{cl} \stackrel{\text{def}}{=} -\mathcal{P}(-\varphi^+, \Phi^t \upharpoonright_K) > 0$$

defines the *classical decay rate of the flow*, which can also be obtained as follows: consider an initial smooth probability distribution $\mu_0 = g_0(\rho)d\rho$ on $S^*\Omega$, with the density g_0 supported inside the interaction region $S^*B(0, R_0)$, with $g_0(\rho) > 0$ at some point $\rho \in K^-$. If we let this measure evolve through Φ^t , the mass of the interaction region will asymptotically decay as

$$(14) \quad \Phi^{t*}\mu(S^*B(0, R_0)) \sim C e^{-t\gamma_{cl}}, \quad t \rightarrow \infty.$$

In the text we will mostly be interested by the pressure with coefficient $s = 1/2$. It can be compared with the two quantities defined above. Indeed, using basic properties of the pressure, one shows the following bounds:

$$(15) \quad -\gamma_{cl}/2 \leq \mathcal{P}(-\varphi^+/2) \leq \frac{1}{2}(H_{top} - \gamma_{cl}).$$

The upper bound clearly demonstrates that $\mathcal{P}(-\varphi^+/2)$ is negative if the dynamics on K is “more unstable than complex”.

As we have seen in the discussion after Thm 2, in dimension $d = 2$ the Hausdorff dimension of K can be obtained in terms of the topological pressure $\mathcal{P}(-s\varphi^+)$. This is a sign that hyperbolicity and complexity directly influences the (fractal) geometry of the trapped set.

In §5 we introduce open maps κ , which are local diffeomorphisms defined on some open subset of a symplectic manifold. The definition of the trapped set, of uniform hyperbolicity, are very similar with the case of flows. Since the Jacobian J_t^+ only makes sense for integer times, we define in this case $\varphi^+ = \log J_1^+$. The topological pressure can be defined as in (12), with $f(p)$ given by the sum over the points in p .

3. SEMICLASSICAL FORMULATION AND MORE EXAMPLES OF CHAOTIC FLOWS

We have already explained that the ray dynamics associated with the obstacle scattering was generated by the Hamiltonian (10). Similarly, in the high frequency ($k \gg 1$) limit, the Helmholtz equation (2) can be rewritten using an small positive parameter, which we call \hbar by analogy with Planck's constant. This parameter scales as

$$\frac{1}{C} \leq \hbar k \leq C \quad (\text{in short, } \hbar \asymp k^{-1}),$$

so the high-frequency limit is equivalent with the semiclassical limit $\hbar \rightarrow 0$. The equation (2) now takes the form of a time-independent Schrödinger equation:

$$-\frac{\hbar^2 \Delta_\Omega}{2} u = E(\hbar) u, \quad \text{with energy } E(\hbar) \in \left[\frac{1}{2C^2}, \frac{C^2}{2} \right].$$

Here the operator (quantum Hamiltonian)

$$(16) \quad P(\hbar) = -\frac{\hbar^2 \Delta_\Omega}{2}$$

is the quantization of the classical Hamiltonian (10).

3.1. Potential scattering. The introduction of \hbar in the obstacle problem is merely a convenient bookkeeping parameter in the high frequency limit.

More importantly, it allows to naturally generalize the above study to more general scattering Hamiltonian flows, typically by replacing the obstacle potential $V_{\mathcal{O}}$ by a smooth potential $V \in C_c^\infty(\mathbb{R}^d)$, leading to the classical Hamiltonian

$$(17) \quad p(x, \xi) = \frac{|\xi|^2}{2} + V(x), \quad (x, \xi) \in T^*\mathbb{R}^d,$$

which generates a smooth flow Φ^t on $T^*\mathbb{R}^d$.

The \hbar -quantization of this Hamiltonian (see Appendix A) is the Schrödinger operator

$$(18) \quad P(\hbar) = -\frac{\hbar^2 \Delta}{2} + V(x), \quad ,$$

where Δ is the Laplacian on \mathbb{R}^d . We say that $p(x, \xi)$ is the *semiclassical symbol* of the operator $P(\hbar)$ (see the Appendix for a short reminder on \hbar -quantization). If $\text{supp } V$ is contained in a ball $B(0, R_0)$, we will call this ball the interaction region.

As opposed to the obstacle case, the flow on the energy shell $p^{-1}(E)$, $E > 0$, is not obtained through a simple rescaling of the one at energy 1/2: both dynamics can be drastically different. Similarly, at the quantum level, $P(\hbar)$ depends on \hbar in a nontrivial way. It is easy to produce a smooth potential $V(x)$ such that the flow on the energy shell $p^{-1}(E)$ is chaotic in some range $[E_1, E_2]$, $E_1 > 0$ in the sense that for any energy $E \in [E_1, E_2]$, the trapped set E ,

$$K_E = \{ \rho \in p^{-1}(E), \Phi^t(\rho) \text{ is uniformly bounded for } t \in \mathbb{R} \},$$

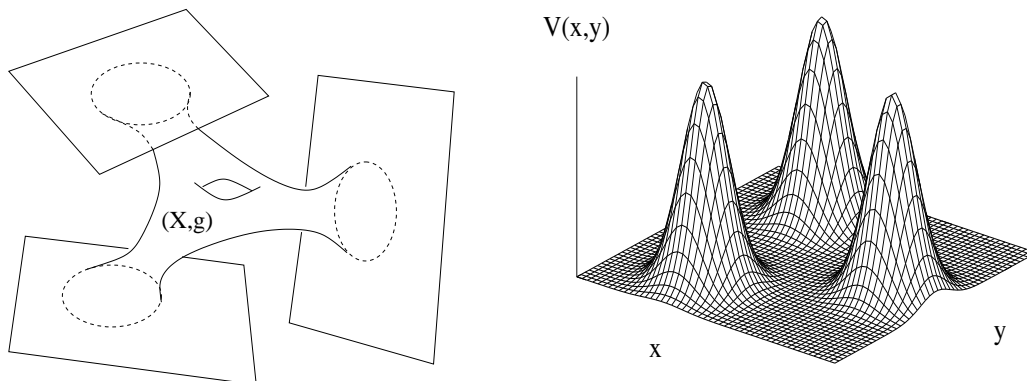


FIGURE 4. Left: a Riemannian surface with 3 Euclidean ends. Right: a potential $V \in C_c^\infty(\mathbb{R}^2)$, with Hamiltonian flow hyperbolic on the trapped set K_E in a range of energies.

is a hyperbolic repeller. Following Sjöstrand [72, Appendix c], one can for instance “smoothen” the hard body potential $V_{\mathcal{O}}$ associated with the above obstacle problem, and obtain a potential with J steep “bumps” (see Fig. 4), such that the flow is chaotic in some intermediate energy range.

The Schrödinger operator (18) admits a continuous spectrum on \mathbb{R}_+ , but like for the obstacle problem, its truncated resolvent $\chi(P(\hbar) - z)\chi$ can be meromorphically continued from the upper to the lower half-plane. The poles $\{z_j(\hbar)\}$ of this continued resolvent form a discrete set, which are the resonances of $P(\hbar)$. In general these resonances depend on \hbar in a nontrivial way.

We specifically consider the vicinity of positive energies E for which the trapped set K_E is a hyperbolic, and ask the same questions as in §1.2. The following results are direct analogues of Thm 1 and Thm 2 in the semiclassical setting.

Theorem 3. [55] *Consider the semiclassical Hamiltonian $P(\hbar)$ of (18), such that for some energy $E > 0$ the flow generated by the Hamiltonian (17) has a hyperbolic trapped set K_E .*

If the topological pressure

$$\mathcal{P} = \mathcal{P}\left(-\frac{1}{2}\varphi^+, \Phi^t \upharpoonright_{K_E}\right) \text{ is negative,}$$

then for any $\delta, \epsilon > 0$ small enough, there exists $\hbar_{\delta, \epsilon} > 0$ such that, for any $\hbar \in (0, \hbar_{\delta, \epsilon}]$ the operator $P(\hbar)$ does not have resonances in the strip $[E - \delta, E + \delta] - i[0, |\mathcal{P}| - \epsilon]$.

Theorem 4. [72, 76] *Let $P(\hbar)$ be a semiclassical Hamilton operator as in Thm 3, and let $2\nu_E + 1$ be the upper box dimension of K_E .*

Then, for any $C > 0$, the number of resonances of $P(\hbar)$ in the disk $D(E, C\hbar)$, counted with multiplicities, is bounded as follows. For any $\epsilon > 0$, there exists $C_{C,\epsilon}, \hbar_{C,\epsilon} > 0$ such that

$$\forall \hbar < \hbar_{C,\epsilon}, \quad \#\{\text{Res } P(\hbar) \cap D(E, C\hbar)\} \leq C_{C,\epsilon} \hbar^{-\nu_E - \epsilon}.$$

If K_E is of pure dimension, one can take $\epsilon = 0$ in the above estimate.

One can generalize the above scattering problems on \mathbb{R}^d by considering a Schrödinger operator $P(\hbar)$ of the form (18) on an unbounded Riemannian manifold (X, g) with a “nice enough” geometry near infinity. This geometry should allow to meromorphically extend the truncated resolvent in some strip. For instance, Thm. 3 applies if X is a union of Euclidean infinities outside a compact part [55]. It has been extended by Datchev [19] and Datchev-Vasy [20] to more complicated geometries near infinity, in particular asymptotically Euclidean or asymptotically hyperbolic manifolds. Their strategy is to “glue together” the resolvent estimates of two model manifolds, one with the true, trapping structure in the interaction region but a simple (say, Euclidean) structure near infinity, and the other one with the true infinity but a simple (nontrapping) interaction region. In parallel, Vasy [83] recently developed a new method to analyze the resolvent at high frequency, in a variety of asymptotically hyperbolic geometries.

The next section provides some examples of results specific to manifolds of constant negative curvature.

3.2. Geometric scattering. If the metric of X has negative sectional curvature, the geodesic flow on S^*X becomes hyperbolic. This is also the case if X is the union of a compact component X_0 of negative curvature (so that all the orbits staying in X_0 are unstable), glued to a certain number of Euclidean ends (see Fig. 4). The operator quantizing the geodesic flow is the (rescaled) Laplace-Beltrami operator on X , $P(\hbar) = -\hbar^2 \Delta_X / 2$.

One particularly appealing class of examples consists in manifolds (X, g) obtained by quotienting the Poincaré half-space \mathbb{H}^{n+1} (which has uniform curvature -1) by certain discrete subgroups Γ of the group of isometries of \mathbb{H}^{n+1} . Such a manifold $X = \Gamma \backslash \mathbb{H}^{n+1}$ inherits the uniform hyperbolic geometry of \mathbb{H}^{n+1} , so that all trajectories are hyperbolic. For a certain type of subgroups Γ (called convex co-compact), the manifold X has infinite volume and the trapped set is compact; this trapped set is then automatically a hyperbolic set. This definition of X through group theory leads to remarkable properties of the spectrum of Δ_X , which we now summarize (a recent review of the theory in dimension 2 can be found in Borthwick’s book [9]).

The absolutely continuous spectrum of $-\Delta_X$ consists in the half-line $[n^2/4, \infty)$, leaving the possibility of finitely many eigenvalues in the interval $(0, 1/4)$. It is common to write the energy as

$$k^2 = s(n - s), \quad \text{so that the essential spectrum corresponds to } s \in n/2 + i\mathbb{R}.$$

This parametrization has the following advantage. The resonances of Δ_X , parametrized by a discrete set $\{s_j\} \subset \{\operatorname{Re} s < n/2\}$, exactly correspond to the nontrivial zeros of the Selberg zeta function [59]

$$(19) \quad Z_\Gamma(s) = \exp \left(- \sum_p \sum_{m \geq 1} \frac{1}{m} \det(1 - P_p)^{-1/2} e^{-sm\ell(p)} \right).$$

Here p are the primitive closed geodesics of the flow, and P_p is the linearized Poincaré return map around p . This shows that the (quantum) resonances are solely determined by the classical dynamics on K .

One can show that the rightmost zero of $Z_X(s)$ is located at $s_0 = \delta$, where $\delta > 0$ is the Hausdorff dimension of the limit set⁴ $\Lambda(\Gamma)$ [58, 80], while all other zeros satisfy

$$(20) \quad \forall j \neq 0, \quad \operatorname{Re} s_j < \delta.$$

If $\delta \in [n/2, n]$ (“thick” trapped set), δ corresponds to the eigenvalue of the ground state of the Laplacian. There may be finitely many other eigenvalues ($s_j \in [n/2, \delta)$), while the resonances will be located in the half-space $\{\operatorname{Re} s < n/2\}$.

If $\delta \in (0, n/2)$ (“thin” trapped set), all the zeros of $Z_X(s)$ correspond to resonances of Δ_X , and the bound (20) shows the presence of a resonance gap. The topological pressure of $\Phi^t \upharpoonright_K$ is exactly given by

$$\mathcal{P}(-1/2\varphi^+, \Phi^t \upharpoonright_K) = \delta - n/2,$$

so the bound (20) is a (more precise) analogue of the semiclassical gaps in Thms 1 and 3.

In this geometric setting one can also obtain fractal upper bounds for the number of resonances near the essential spectrum.

Theorem 5. [90, 34] *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a hyperbolic manifold of infinite volume, with Γ a Schottky group⁵. Then the resonances of Δ_X (counted with multiplicities) satisfy*

$$\forall \gamma > 0, \exists C_\gamma > 0, \forall r > 1, \quad \#\{s_j \in [n/2 - \gamma, n/2] + i[r, r + 1]\} \leq C_\gamma r^\delta.$$

Notice that $2\delta + 1$ is the (Hausdorff or Minkowski) dimension of the trapped set $K \subset S^*X$. A slightly weaker result was first obtained in [90] in dimension 2, using microlocal methods similar with those of [72] (see §4.2). The above result was obtained in [34] by analyzing the zeta function in terms of a *classical* expanding map on $\partial\mathbb{H}^{n+1}$, and the corresponding transfer operator. This possibility to rely on purely classical dynamics is specific to the locally homogeneous space $\Gamma \backslash \mathbb{H}^{n+1}$.

⁴Fix one point $x_0 \in \mathbb{H}^{n+1}$. Then the limit set $\Lambda(\Gamma) \stackrel{\text{def}}{=} \overline{\{\gamma \cdot x_0, \gamma \in \Gamma\}} \cap \partial\mathbb{H}^{n+1}$ is actually independent of x_0 .

⁵Schottky groups form a certain subclass of convex co-compact groups of isometries.

4. “MASSAGING” $P(\hbar)$ INTO A PROOF OF THE FRACTAL WEYL UPPER BOUND

This section presents two consecutive methods, which were used in [76] to prove the fractal Weyl upper bound of Thm 4, that is in the case of a semiclassical Schrödinger operator (18). Both methods consist in “deformations” of the original operator $P(\hbar)$, which can be easily analyzed at the level of the *symbols* of the operators, so as to draw consequences on the spectrum of the deformed operators.

The first method, called “complex scaling”, or rather “complex deformation”, provide an alternative definition of the resonances of $P(\hbar)$ in terms of the eigenvalues of a nonselfadjoint operator. We remind that resonances were originally obtained as poles of the meromorphic continuation of the truncated resolvent $\chi(P(\hbar) - z)^{-1}\chi$. Each resonance $z_j(\hbar)$ is associated with a metastable state $u_j(\hbar)$, which is not in L^2 but satisfies the differential equation $P(\hbar)u_j(\hbar) = z_j(\hbar)u_j(\hbar)$.

4.1. Complex scaling: resonances as spectrum of a nonselfadjoint operator. The “complex scaling” strategy [1, 35] (below we follow the presentation of [74]) consists in deforming the configuration space \mathbb{R}^d into a complex contour $\Gamma_\theta \subset \mathbb{C}^d$, $\theta \in [0, \theta_0]$, of the form

$$\Gamma_\theta \cap \{|x| \leq R_0\} = \mathbb{R}^d \cap \{|x| \leq R_0\}, \quad \Gamma_\theta \cap \{|x| \geq 2R_0\} = \{e^{i\theta}x, x \in \mathbb{R}^d, |x| \geq 2R_0\}.$$

We recall that $B(0, R_0)$ is the interaction region, which contains the support of the potential. The differential operator $P(\hbar)$, when analytically extended on Γ_θ , is then equivalent with an operator $P_\theta(\hbar)$ acting on \mathbb{R}^d . Outside $B(0, 2R_0)$ this operator is simply given by $-e^{-2i\theta} \frac{\hbar^2 \Delta}{2}$: this shows that $P_\theta(\hbar)$ is not selfadjoint on $L^2(\mathbb{R}^d)$, and has essential spectrum on the half-line $e^{-2i\theta}\mathbb{R}_+$. More importantly, its spectrum in the cone $\{-2\theta < \arg(z) < 0\}$ is discrete, with eigenvalues $z_j(\hbar)$ equal to the resonances of the original operator $P(\hbar)$. The associated eigenstates $u_{j,\theta}(\hbar)$ are equal to the metastable states $u_j(\hbar)$ inside $B(0, R_0)$, but they are now globally square-integrable.

Through this deformation, the study of resonances has become a spectral problem for the nonselfadjoint differential operator $P_\theta(\hbar)$, which can make advantage of pseudodifferential calculus, that is study the spectrum of $P_\theta(\hbar)$ by analyzing its semiclassical symbol $p_\theta(x, \xi)$. For $\theta \ll 1$, this symbol takes the form

$$(21) \quad p_\theta(x, \xi) = p(x, \xi), \quad |x| \leq R_0$$

$$(22) \quad p_\theta(x, \xi) = e^{-2i\theta} \frac{|\xi|^2}{2}, \quad |x| \geq 2R_0.$$

In particular, for any positive energy $E > 0$ and $\delta > 0$ small, the phase space region

$$V_\theta(\delta) \stackrel{\text{def}}{=} \{(x, \xi), |p_\theta(x, \xi) - E| \leq \delta\}$$

is compact (it is contained inside $T^*B(0, 2R_0)$). This has for consequence that, for \hbar small enough, $P_\theta(\hbar)$ has discrete spectrum in $D(E, C\hbar)$. The compactness of $V_\theta(\delta)$ also provides

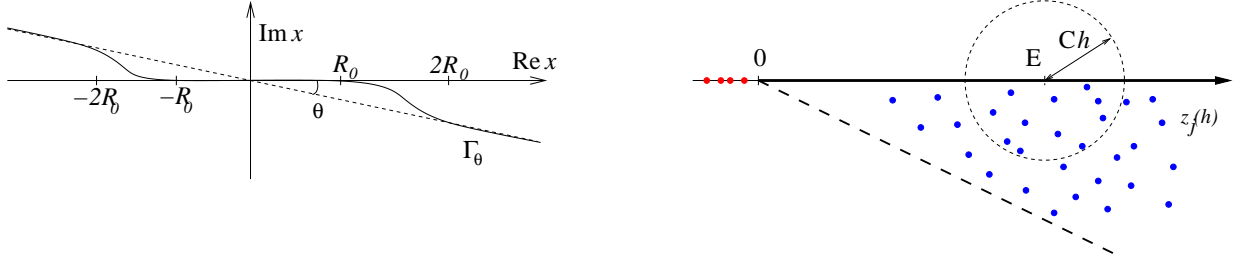


FIGURE 5. Left: deformation of the configuration space \mathbb{R}^d into a contour in $\Gamma_\theta \subset \mathbb{C}^d$. Right: the spectrum of the deformed operator $P_\theta(\hbar)$.

a rough upper bound on the number of eigenvalues near E . Heuristically, the number of eigenvalues of $P_\theta(\hbar)$ in $D(E, C\hbar)$ is bounded from above by the number of quantum states which can be squeezed in the region $V_\theta(C\hbar)$, each state occupying a volume $\sim \hbar^d$. This argument can be made rigorous [76, Thm 2], and produces the bound

$$(23) \quad \# \text{Spec } P_\theta(\hbar) \cap D(E, C\hbar) = \mathcal{O}(\hbar^{-d} \text{Vol } V_\theta(C\hbar)) = \mathcal{O}(\hbar^{-d+1}).$$

This estimate does not depend on the nature of the dynamics in the interaction region. In case the flow on K_E contains stable orbits surrounded by elliptic islands, one can show that this estimate is optimal, by explicitly constructing sufficiently many quasimodes with quasi-energies in $D(E, C\hbar)$ very close to the real axis, and showing that actual resonances must lie nearby (see [81] and references therein).

Remark 4.1. When solving the Schrödinger equation $i\hbar\partial_t u = P_\theta(\hbar)u$, the negative imaginary part of P_θ acts as an “absorbing” term. Indeed, a wavepacket u_0 microlocalized near a point $(x, \xi) \in p^{-1}(E)$, $|x| \geq 2R_0$, will be absorbed fast, in the sense that its norm will be reduced by a factor $\sim e^{t \text{Im } p_\theta(x, \xi)/\hbar} = e^{-t \sin(2\theta)E/\hbar}$. Hence, the complex scaling has the effect to absorb the waves propagating outside the interaction region.

The above complex deformation can be used for any type of potential $V(x)$. In order to refine the counting estimate (23), one strategy [72, 76] consists in a second deformation of the operator $P_\theta(\hbar)$, obtained by conjugating it with an appropriate *microlocal weight*, such as to shrink the region $V_\theta(C\hbar)$ up to the close vicinity of the trapped set. Eventhough we will present an alternative (yet related) proof of Thm 4 in §7.2, we chose to sketch this strategy below, which features the power of symbol calculus.

4.2. Conjugation by an escape function. One constructs by hand an *escape function* $G \in C_c^\infty(T^*X)$, which is adapted to the flow Φ^t in some energy layer $p^{-1}([E - \delta, E + \delta])$ in the following way. The function G is required to strictly grow along the flow outside an ε -neighbourhood \tilde{K}_E^ε of $\tilde{K}_E \stackrel{\text{def}}{=} \cup_{|E' - E| \leq \delta} K_{E'}$ (and for $|x| \leq 2R_0$). The microlocal weight is then obtained by quantizing this escape function into the operator $G^w = \text{Op}_\hbar(G)$, and

exponentiating: for some factor $t \gg 1$ one defines the deformed operator

$$P_{\theta,tG}(\hbar) \stackrel{\text{def}}{=} e^{-tG^w} P_{\theta}(\hbar) e^{tG^w} .$$

$P_{\theta,tG}$ and P_{θ} obviously have the same spectrum, but the pseudodifferential calculus shows that the former has a symbol of the form

$$(24) \quad p_{\theta,tG} = p_{\theta} - i\hbar t \{p, G\} + \mathcal{O}((\hbar t)^2),$$

where the Poisson bracket $\{p, G\} = H_p G$ is the derivative of G along the flow generated by p (see the Appendix). From the construction of G , this symbol has a (substantial) negative imaginary part outside $\tilde{K}_E^{\varepsilon}$, showing that $P_{\theta,tG}(\hbar)$ is absorbing there. The same volume argument as above then shows (with obvious notations) that

$$(25) \quad \# \text{Spec } P_{\theta,tG}(\hbar) \cap D(E, C\hbar) = \mathcal{O}(\hbar^{-d} \text{Vol } V_{\theta,tG}(C\hbar)).$$

If $\varepsilon > 0$ is very small, the above bound is sharper than (23), because the set $V_{\theta,tG}(C\hbar)$ has a much smaller volume than $V_{\theta}(C\hbar)$. Indeed, the former set is contained inside $\tilde{K}_E^{\varepsilon} \cap V_{\theta}(C\hbar)$, the volume of which scales as⁶

$$(26) \quad \text{Vol} \left(\tilde{K}_E^{\varepsilon} \cap \{|p(x, \xi) - E| \leq C\hbar\} \right) \asymp \hbar \varepsilon^{2(d-\nu)-2}, \quad \varepsilon, \hbar \ll 1,$$

where $2\nu + 1$ is the box dimension of K_E inside $p^{-1}(E)$. In order to gain a power of \hbar in the right hand side of (25), we need to take $\varepsilon \sim \hbar^{\alpha}$ for some $\alpha > 0$, which implies that the escape function $G(x, \xi)$ has to be \hbar -dependent. On the other hand, as explained in the Appendix, the pseudodifferential calculus leading to (24) is valid only if G belongs to a *bona fide* symbol class, implying that it cannot fluctuate too strongly. The limiting class corresponds to functions fluctuating on distances of order

$$(27) \quad \varepsilon = \varepsilon(\hbar) \sim \hbar^{1/2}.$$

Injecting (27) into (26) leads to the fractal Weyl upper bound of Thm 4.

The construction of an optimal escape function G is a bit tricky, it uses the hyperbolicity of the flow on \tilde{K}_E . To give a schematic idea, let us consider the simple example of the model Hamiltonian [15]

$$p = \xi_1 + \lambda x_2 \xi_2 \quad \text{on } T^*(S^1 \times \mathbb{R}),$$

for which K_E consists in a single hyperbolic periodic orbit $\{\xi_1 = E, x_2 = \xi_2 = 0\}$. An “optimal” escape function is then

$$(28) \quad G_1(x, \xi) = \left(\log(\varepsilon^2 + x_2^2) - \log(\varepsilon^2 + \xi_2^2) \right) \implies H_p G_1 = \lambda \left(\frac{x_2^2}{\varepsilon^2 + x_2^2} + \frac{\xi_2^2}{\varepsilon^2 + \xi_2^2} \right),$$

with the scaling (27). Indeed, the gradient $H_p G \geq 1$ for $|(x, \xi)| \geq C\varepsilon$, while the function remains in a reasonable symbol class.

⁶here we assume \tilde{K}_E is of “pure” Minkowski dimension. In the general case one needs to replace \dim by $\dim + \epsilon$ for any arbitrary $\epsilon > 0$.

In the case of a fractal trapped set K_E , the function G will locally have a structure similar with G_1 near K_E , except that the coordinates x_2, ξ_2 are replaced by functions more or less measuring the distance from the incoming/outgoing tails K_E^\mp .

5. OPEN QUANTUM MAPS AND QUANTUM MONODROMY OPERATORS

In this section we present *open quantum maps*, which are a toy model used to study the distribution of quantum resonances in chaotic situations. The main advantage of this toy model has been on the numerical side. Furthermore, the recent introduction of *quantum monodromy operators* in the context of chaotic scattering, established a direct link between the resonances of a Schrödinger operator, and a certain family of open quantum maps (quantizing a Poincaré return map for the classical flow).

5.1. Open quantum maps. In this section we introduce the concept of *open map*, and its quantization into an *open quantum map*.

At the classical level, an open map is a symplectic diffeomorphism $\kappa : V \mapsto \kappa(V)$, where V and $\kappa(V)$ are bounded precompact open subsets of a symplectic manifold Σ , which locally looks like $T^*\mathbb{R}^d$. The map κ is “open” because we assume that $\kappa(V) \neq V$, so there exist points $\rho \in V$ such that $\kappa(\rho)$ has no further image; we interpret it by saying that $\kappa(\rho)$ has “fallen in the hole”, or has “escaped to infinity” (this interpretation will become clearer when we specifically treat Poincaré maps). By time inversion, the map $\kappa^{-1} : \kappa(V) \rightarrow V$ is also an open map, and points in $V \setminus \kappa(V)$ have escaped to infinity “in the past”.

This escape phenomenon naturally leads to the notion of incoming/outgoing tails and trapped set: we define

$$\mathcal{K}^\mp \stackrel{\text{def}}{=} \{\rho \in V \cup \kappa(V), \kappa^{\pm n}(\rho) \in V, \forall n > 0\}, \quad \mathcal{K} = \mathcal{K}^- \cap \mathcal{K}^+.$$

We will assume that \mathcal{K} is compact and at finite distance from the boundary ∂V . As in the case of flows, we will say that the open map κ is *chaotic* iff the dynamics generated by κ on \mathcal{K} is hyperbolic and complex, in the sense of §2. We will see below that the Poincaré return map of for a chaotic scattering flow is a chaotic open map. Still, it is not difficult to directly construct chaotic open maps.

What do we call a “quantization” of the map κ ? If $\Sigma = T^*\mathbb{R}^d$, it is a family of operators $(\mathcal{M}(\hbar))_{\hbar \rightarrow 0}$ on $L^2(\mathbb{R}^d)$, with the following asymptotic properties when $\hbar \rightarrow 0$.

$\mathcal{M}(\hbar)$ is an \hbar -Fourier Integral Operator (FIO) associated with κ , which is unitary microlocally near \mathcal{K} . This denomination means the following (see the Appendix for more details). For any smooth observable $a \in C_c^\infty(T^*\mathbb{R}^d)$, the quantized observable $\text{Op}_\hbar(a)$ is transformed as follows when conjugated by $\mathcal{M}(\hbar)$:

$$(29) \quad \mathcal{M}(\hbar)^* \text{Op}_\hbar(a) \mathcal{M}(\hbar) = \text{Op}_\hbar(b) + \mathcal{O}_{L^2 \rightarrow L^2}(\hbar^\infty),$$

where $b(x, \xi; \hbar)$ is a semiclassical symbol admitting the expansion

$$(30) \quad b \sim \sum_{j \geq 0} \hbar^j b_j, \quad \text{with } b_0 = |\alpha|^2 \times a \circ \kappa,$$

and all b_j are supported on $\text{supp } \alpha$. The function $\alpha \in C_c^\infty(V)$ is the *symbol* of the FIO $\mathcal{M}(\hbar)$ (we will sometimes note this operator $\mathcal{M}(\alpha, \hbar)$). For \hbar small enough, the operator $\mathcal{M}(\alpha, \hbar)$ is uniformly bounded on L^2 , with norm

$$(31) \quad \|\mathcal{M}(\alpha, \hbar)\| = \|\alpha\|_\infty + \mathcal{O}(\hbar).$$

Equation (29) is a form of Egorov theorem. It implies that $\mathcal{M}(\hbar)$ transforms a wavepacket ψ_0 microlocalized at a point $(\rho_0) \in V$ into a wavepacket $\mathcal{M}(\hbar)\psi_0$ microlocalized at $\kappa(\rho_0)$, with a norm modified by

$$\frac{\|\mathcal{M}(\hbar)\psi_0\|}{\|\psi_0\|} = |\alpha(\rho_0)| + \mathcal{O}(\hbar).$$

The coefficient $|\alpha(\rho_0)|$ can be interpreted as an absorption (resp. gain) factor if $|\alpha(\rho_0)| < 1$ (resp. $|\alpha(\rho_0)| > 1$). The compact support of α shows that the particles outside $\text{supp } \alpha$ are fully absorbed by $\mathcal{M}(\hbar)$.

For $\mathcal{M}(\alpha, \hbar)$ to be an *open quantum map*, one generally requires that it is microlocally unitary inside V . This implies that the symbol $|\alpha(\rho)|$ is a smoothed version of the characteristic function $\mathbb{1}_V$. Typically, one can construct a family of neighbourhoods $\mathcal{K} \Subset W_\hbar \subset V$ converging to V , say, $W_\hbar = \{\rho \in V, \text{dist}(\rho, \mathbb{C}V) \leq r(\hbar)\}$, with $r(\hbar) \sim |\log \hbar|^{-1}$, and require that the symbol α satisfies

$$(32) \quad \alpha(\rho) = 0 \quad \text{outside } V, \quad |\alpha(\rho)| = 1 \quad \text{inside } W_\hbar, \quad \text{and } |\alpha(\rho)| \in [0, 1] \text{ inbetween.}$$

Such α depends on \hbar , but in a mild enough way (precisely, it belongs to a symbol class S_δ^0 , see the Appendix). To insist on the fact that α remains a nice symbol, we will call such an operator a *smooth open quantum map*.

5.2. Open quantum maps of finite rank. A priori, the operators $\mathcal{M}(\hbar)$ have infinite rank, even though they have a finite *essential rank*, of the order of $\hbar^{-d} \text{Vol}(\text{supp } \alpha)$, when $\hbar \rightarrow 0$. This essential rank corresponds to the dimension of a subspace $\mathcal{H}_\hbar \subset L^2(\mathbb{R}^d)$ of states which are not fully absorbed. For practical reasons it can be convenient to replace $\mathcal{M}(\hbar)$ by a finite rank operator $M(\hbar)$, by composing $\mathcal{M}(\hbar)$ with a projector $\Pi(\hbar)$ microlocally equal to the identity in some neighbourhood of $\text{supp } \alpha$, and of rank $\sim C\hbar^{-d}$. One then obtains a family of operators

$$(33) \quad M(\hbar) \stackrel{\text{def}}{=} \mathcal{M}(\hbar) \Pi(\hbar) = \mathcal{M}(\hbar) + \mathcal{O}_{L^2 \rightarrow L^2}(\hbar^\infty).$$

Such a projection onto a subspace of finite dimension will be useful in the construction of quantum monodromy operators in §5.4.

A practical way to construct an open quantum map is to start from a symplectomorphism $\tilde{\kappa}$ defined on a compact phase space, say the torus $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$, and then restrict it to

a proper open subset $V \Subset \mathbb{T}^{2d}$, that is take $\kappa = \tilde{\kappa} \upharpoonright_V$. There exist recipes to quantize the “closed” map $\tilde{\kappa}$ into a quantum map [21], that is a family $(U(\hbar))_{\hbar \rightarrow 0}$ of *unitary* operators acting on the family of quantum spaces \mathcal{H}_\hbar associated with \mathbb{T}^{2d} , and enjoying a Egorov property similar with (29), with $\alpha \equiv 1$. The spaces \mathcal{H} have dimensions $\sim \hbar^{-d}$ due to the compactness of \mathbb{T}^{2d} .

To “open” this quantum map, one can truncate $U(\hbar)$ by a projector, or at least a quasiprojector $\Pi(\hbar)$ quantizing $\mathbb{1}_V$, and get the operator

$$(34) \quad M(\hbar) = U(\hbar)\Pi(\hbar).$$

The rank (or at least essential rank) of $\Pi(\hbar)$ then scales as $\hbar^{-d} \text{Vol}(V)$. We will speak of a *smooth open quantum map* if $\Pi(\hbar)$ is a pseudodifferential operator, that is the quantization of a nice symbol α as in (32). This choice allows to avoid diffraction problems near δV , present if $\Pi(\hbar)$ is a too steep cutoff, which will inevitably spoil the semiclassical properties of $\mathcal{M}(\hbar)$.

The above construction was implemented for various chaotic maps on the 2-torus, chosen such that the map κ admits a hyperbolic trapped set. In all cases the cutoff $\Pi(\hbar)$ was a sharp projector (in position or momentum). The first such map was, to my knowledge, the “kicked rotor” with absorption [8]; it was already aimed at studying the the statistics of quantum lifetimes, defined in terms of the spectrum $\{\lambda_j(\hbar)\}$ of $M(\hbar)$:

$$(35) \quad e^{-\tau_j(\hbar)/2} = |\lambda_j(\hbar)|, \quad j = 1, \dots, \text{rank}(M(\hbar)).$$

This formula shows that the “long living” spectrum of $M(\hbar)$, say $\{\lambda_j(\hbar)\} \cap \{|\lambda| \geq r\}$ for some fixed $r > 0$, is seen as a model for the resonances of $P(\hbar)$ in some box $[E - C\hbar, E + C\hbar] - i[0, \gamma\hbar]$, with the connection $r \equiv e^{-\gamma}$.

Further studies lead to the verification of the fractal Weyl law [69, 53, 54] (see §6.1). Such discrete time models have several advantages. Firstly, the long time dynamics of the classical map κ is sometimes very easy to analyze; this is the case for instance for the open baker’s map studied in [53, 54], which we will study explicitly in §6.1.1. Secondly, the corresponding open quantum maps are often very explicit matrices, which can be numerically diagonalized, much more easily so than Schrödinger operators like $P_\theta(\hbar)$. A third reason to study these open quantum maps lies in the quantum monodromy operators associated with a Schrödinger operator $P(\hbar)$, establishing a direct connection between a family of open quantum maps and a “true” scattering system. (see §5.4). To explain this construction we need to recall the definition of a Poincaré section of a smooth flow.

5.3. Poincaré section. We are back in the setting of §2, with a Hamiltonian flow Φ^t on $T^*\mathbb{R}^d$ (more generally T^*X for some manifold X). A Poincaré section for Φ^t near the trapped set K_E is a finite union of hypersurfaces $\Sigma = \sqcup_{i=1}^I \Sigma_i$ in $p^{-1}(E)$, uniformly transverse to the flow, such that for each point ρ sufficiently close to K_E the trajectory $\Phi^t(\rho)$ intersects Σ (in the future and the past) after a uniformly bounded time. This

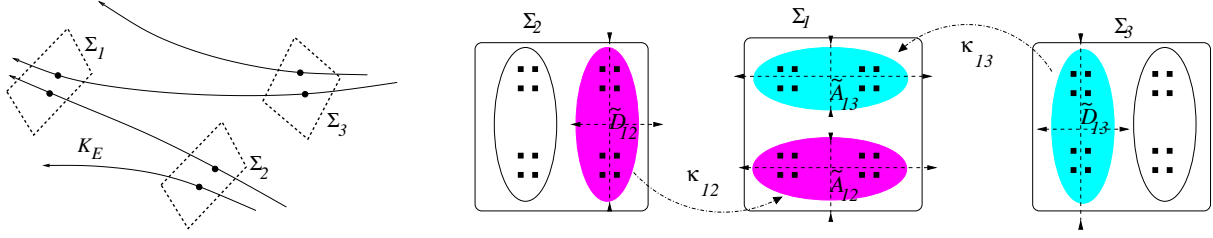


FIGURE 6. Left: schematic representation of a Poincaré section Σ near the trapped set K_E . Right: the induced return map $\kappa = (\kappa_{ij})$ on Σ . Vertical/horizontal axes indicate the stable/unstable directions, and the trapped set \mathcal{K} is sketched by the black squares.

property allows to define

$$\text{a return map } \kappa : V \subset \Sigma \rightarrow \kappa(V) \subset \Sigma, \quad \text{and a return time } \tau : V \subset \Sigma \rightarrow \mathbb{R}_+,$$

where V is a neighbourhood of the reduced trapped set

$$\mathcal{K} \stackrel{\text{def}}{=} K_E \cap \Sigma = \sqcup_{i=1}^I \mathcal{K}_i.$$

Since our flow Φ^t is symplectic on T^*X , the Poincaré section Σ can be given a natural symplectic structure, which is preserved by κ . Notice the dimensional reduction: Σ has dimension $2d - 2$. The flow Φ^t in the neighbourhood of K_E is fully described by the pair (κ, τ) . In particular, $\kappa \upharpoonright_{\mathcal{K}}$ is (uniformly) hyperbolic if $\Phi^t \upharpoonright_{K_E}$ is so. Analyzing the ergodic properties of such a hyperbolic map has proved easier than directly analyzing the flow. Indeed, the thermodynamic formalism, which allows to construct nontrivial invariant measures, and analyze its mixing properties, is based on such a Poincaré reduction [10].

Since Σ is a union of disjoint hypersurfaces Σ_i locally equivalent with $T^*\mathbb{R}^{2d-2}$, the map κ can be seen as a collection of symplectic maps $\kappa_{ij} : D_{ij} \subset \Sigma_j \mapsto \Sigma_i$, where the open set D_{ij} consists of the points in Σ_j , the trajectories of which next intersect Σ_i .

A open quantum map corresponding to κ is then an operator valued matrix $\mathcal{M}(\hbar) = (\mathcal{M}_{ij}(\hbar))_{i,j=1,\dots,I}$, such that $\mathcal{M}_{ij}(\hbar) = 0$ if $D_{ij} = \emptyset$, otherwise $\mathcal{M}_{ij}(\hbar)$ is an open map quantizing κ_{ij} . The operator $\mathcal{M}(\hbar)$ acts on $L^2(\mathbb{R}^{d-1})^I$, and satisfies a vector-valued Egorov property similar of the form (29), where $a = (a^i)_{i=1,\dots,I}$ and $b = (b^i)_{i=1,\dots,I}$ are observables on Σ .

One can modify the quantum map $\mathcal{M}(\hbar)$ into a finite rank operator $M(\hbar)$ as in (33), using a vector of orthogonal projectors $\Pi(\hbar) = (\Pi^i(\hbar))_{i=1,\dots,I}$.

5.4. Quantum monodromy operators. It turns out that, under a mild condition on the trapped set K_E , there exists a specific choice of quantum map associated with the Poincaré map κ , which allows to directly recover the resonance spectrum of the quantum Hamiltonian $P(\hbar)$. This family of operators depends on \hbar , but also depends holomorphically

on the spectral parameter $z \in D(E, C\hbar)$, so we denote these operators by $M(z, \hbar)$. The operator $(\mathcal{M}(z, \hbar))_{z \in D(E, C\hbar)}$ is called a *quantum monodromy operator* associated with $P(\hbar)$.

Theorem 6. [51, 52] *Let $P(\hbar)$ be as in Thm. 3, and assume that the trapped set set K_E is totally disconnected transversely to the flow⁷.*

Alternatively, take $P(\hbar) = -\frac{\hbar^2 \Delta}{2}$ the Dirichlet Laplacian outside J convex obstacles satisfying a no-eclipse condition, and take $E = 1/2$.

Consider a Poincaré section $\Sigma = \sqcup_{i=1}^I \Sigma_i$ transverse to $\Phi^t \upharpoonright_{K_E}$. Then, there exists a family of operators $(M(z, \hbar))_{z \in D(E, C\hbar)}$ on $L^2(\mathbb{R}^{d-1})^I$ with the following properties:

- (i) $M(E, \hbar)$ is an open quantum map quantizing κ , of finite rank $\asymp \hbar^{-d+1}$.
- (ii) $M(z, \hbar)$ depends holomorphically in $z \in D(E, C\hbar)$, and

$$(36) \quad M(z, \hbar) = M(E, \hbar) \text{Op}_{\hbar}(e^{-i(z-E)\tau/\hbar}) + \mathcal{O}(\hbar^{1-\epsilon}),$$

where τ is the return time (smoothly continued outside V).

- (iii) the resonances of $P(\hbar)$ in $D(E, C\hbar)$ are the roots (with multiplicities) of the equation

$$(37) \quad \det(1 - M(z, \hbar)) = 0.$$

Properties (i), (ii) ensure that for all $z \in D(E, C\hbar)$, $M(z, \hbar)$ remains an FIO associated with κ , but for $z \notin \mathbb{R}$ it is no more unitary near \mathcal{K} .

The crucial property (iii) exhibits the direct connection between the spectrum of $M(z, \hbar)$ and the resonances of $P(\hbar)$. It has transformed a linear spectral problem $(P_{\theta}(\hbar) - z)u = 0$, into a problem $M(z, \hbar)v = v$ of finite rank, depending nonlinearly in z . The construction

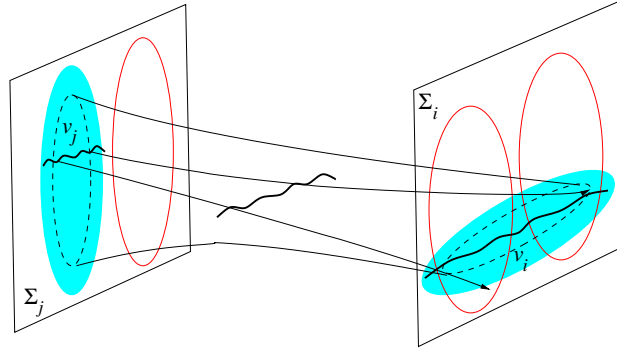


FIGURE 7. Schematic construction of the monodromy operator $M_{ij}(z, \hbar)$, by “following” a microlocal solution of $(P(\hbar) - z)u = 0$ from a neighbourhood of Σ_j to a neighbourhood of Σ_i .

of the monodromy operators $M(z, \hbar)$ is not unique, and rather implicit. Roughly speaking,

⁷This condition is probably generic within the family of chaotic scattering flows we are considering. It can be relaxed a bit, see [51]

each component $M_{ij}(z, \hbar)$ is obtained by expressing a microlocal solutions to the equation $(P_\theta(\hbar) - z)u = 0$ near \mathcal{K}_j in terms of its “local transverse data” $v_j \in L^2(\mathbb{R}^{d-1})$, using a choice of coordinates near Σ_i . The microlocal solution u can be continued up to Σ_i , where it is analyzes it in terms of local transverse data v_i . The linear operator mapping v_j to v_i makes up the operator $M_{ij}(z, \hbar)$, at least microlocally near \mathcal{K} . This explains the denomination of “monodromy operator” (see Fig. 7). The technical difficulty consists in transforming this microlocal characterization into a globally defined, finite rank operator.

A monodromy operator had been constructed, and used to study the resonance spectrum of a scattering operator $P(\hbar)$, in the case where the trapped set consists in a single hyperbolic orbit [30]. In a different framework, a microlocal form of monodromy operator associated with an isolated periodic orbit was used in [75] to compute the contribution of this orbit to Gutzwiller’s semiclassical trace formula.

In the physics literature, Bogomolny [5] formally defined a “quantum transfer operator” $T(E, \hbar)$ associated with the Hamiltonian $P(\hbar)$ of a closed system: this operator quantizes the return map through a certain spatial hypersurface, and in the semiclassical limit the eigenvalues of $P(\hbar)$ are (formally) given by the roots of the equation $\det(1 - T(E, \hbar)) = 0$. This approach was adapted by Doron and Smilansky to study the spectrum of closed Euclidean billiards [24], and was also implemented in a nonsemiclassical setting by Prosen [63].

In the case of the scattering by J convex obstacles satisfying a no-eclipse condition [29, 36], similar operators were constructed. Here Σ is selected by the problem: it consists in the union of the cotangent bundles of the obstacle boundaries, $\Sigma_i = T^*\partial\mathcal{O}_i$. To each obstacle \mathcal{O}_i one associates a Poisson operator $H_i(k) : C^\infty(\partial\mathcal{O}_i) \mapsto C^\infty(\mathbb{R}^d \setminus \mathcal{O}_i)$, such that

$$\begin{cases} (\Delta + k^2)H_i(k)v = 0 & \text{on } \mathbb{R}^d \setminus \mathcal{O}_i, \\ H_i(k)v \text{ is outgoing} \\ (H_i(k)v) \upharpoonright_{\partial\mathcal{O}_i} = v. \end{cases}$$

Then, the scattering problem by the J obstacles can be expressed in terms of the “quantum boundary map” $\mathcal{M}(k) = (\mathcal{M}_{ij}(k))_{i,j=1,\dots,J}$ defined by

$$\mathcal{M}_{ij}(k) : C^\infty(\partial\mathcal{O}_j) \mapsto C^\infty(\partial\mathcal{O}_i), \quad \begin{cases} \mathcal{M}_{ij}(k) = 0, & i = j \\ \mathcal{M}_{ij}(k)v = (H_j(k)v) \upharpoonright_{\partial\mathcal{O}_i} & i \neq j. \end{cases}$$

One can show that, in the high frequency limit $k \rightarrow \infty$, the operator $\mathcal{M}_{ij}(k)$ has the structure of an open quantum map associated with the boundary map of the billiard flow, away from the glancing orbits. In [52] we show how to reduce these boundary operators $\mathcal{M}(k)$ to finite rank monodromy operators $M(k)$, which have the properties expressed in the above theorem (the correspondence with the semiclassical formalism reads $\hbar \sim |k|^{-1}$, $z = \hbar^2 k^2 / 2 = 1/2 + \mathcal{O}(\hbar)$).

6. FROM FRACTAL WEYL UPPER BOUND TO FRACTAL WEYL LAW ?

In our attempts to address the question (1) in §1.2, we have so far only obtained upper bounds for the number of resonances. Lower bounds are more tricky to obtain, due to the fact that the spectral problem we are dealing with is effectively nonselfadjoint. These upper bounds are generally obtained by first counting the *singular values* of some operator related with $P_\theta(\hbar)$, which is a selfadjoint spectral problem; after controlling the distribution of singular values one can then apply Weyl's inequalities to bound (from above) the density eigenvalues of $P_\theta(\hbar)$.

The difficulty to obtain a lower bound (that is, ensure that there are indeed about as many eigenvalues as what is permitted by the upper bound) may be traced to the possible strong sensitivity of the spectrum w.r.to perturbations. In our semiclassical setting, the only access to lower bound is provided by using a form of Gutzwiller's trace formula. Using this strategy, lower bounds on the number of resonances have been obtained in the case of convex co-compact manifolds $\Gamma \backslash \mathbb{H}^{n+1}$.

Theorem 7. [33, 60] *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$, with Γ convex co-compact and torsion-free. Then, for any small $\epsilon > 0$, there exists $\gamma_\epsilon > 0$ such that*

$$\#\{s_j \in [-\gamma_\epsilon, n/2] + i[0, r]\} = \Omega(r^{1-\epsilon}) \quad \text{for } r \rightarrow \infty.$$

(we recall that the notation $f(r) = \Omega(g(r))$ means that $\frac{f(r)}{g(r)}$ takes arbitrarily large values when $r \rightarrow \infty$). The proof of this lower bound uses an exact, Selberg-like trace formula, which connects the resonance spectrum on one side, with a sum over the closed geodesics on the other side. Applying a well-chosen test function on this trace formula, one exhibits a singularity on the "geodesics side", which implies (on the "spectral side") the presence of many resonances.

We notice a gap between this lower bound and the upper bound of Thm 5, which implies that the left hand side is bounded above by $C_\gamma r^{1+\delta}$, with $\delta > 0$ the dimension of the limit set. In [33] the authors conjecture that the actual number of resonances in the strip is of the order of the fractal upper bound. Similar conjectures for various other systems could be split into two forms.

Definition 6.1. Let $P(\hbar)$ be a Schrödinger operator as in (18), and assume that for some $E > 0$ the trapped set K_E is a hyperbolic repeller of pure Minkowski dimension $1 + 2\nu$. We define the weak, resp. strong form of *fractal Weyl law conjecture* as follows.

(i) **Weak form.** For $C, \gamma > 0$ large enough (for a very weak form, take $C \sim \delta \hbar^{-1}$), there exists $C_\gamma > 0$, $\hbar_{C,\gamma} > 0$ such that

$$\#\{\text{Res } P(\hbar) \cap [E - C\hbar, E + C\hbar] - i[0, \gamma\hbar]\} \geq C C_\gamma \hbar^{-\nu}, \quad \forall \hbar < \hbar_{C,\gamma}.$$

(ii) **Strong form.** There exists an increasing function $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$, nonidentically vanishing, such that, for any $C, \gamma > 0$,

$$(38) \quad \#\{\text{Res } P(\hbar) \cap [E - C\hbar, E + C\hbar] - i[0, \gamma\hbar]\} = C F(\gamma) \hbar^{-\nu} + o(\hbar^{-\nu}), \quad \text{when } \hbar \rightarrow 0.$$

In order to test either form of the conjecture, resonance spectra have been numerically computed for the three types of systems: the 3-bump potential [42, 43] or a modified Hénon-Heiles Hamiltonian [64], the 3-disk scattering on the plane [45] (see Fig. 8), or a scattering by 4 spheres on the 3-dimensional space [25], and several convex co-compact surfaces [34]⁸. In all cases, the counting was compatible with the fractal Weyl law, although the convergence to the asymptotic behaviour was difficult to ascertain. More recently,

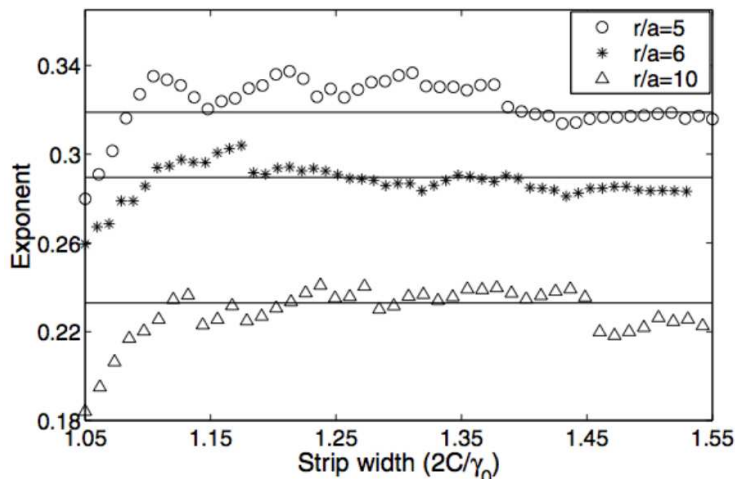


FIGURE 8. Check of the fractal Weyl law for the scattering by 3 disks of radii a located in an equilateral triangle of sidelength r , for 3 values of the ratio r/a . In each case a fractal exponent $\nu(C)$ was extracted from counting the resonances in a long strip of depth C for various values of C larger than $2\gamma_{cl}$, and compared with the geometrical exponent ν (horizontal lines). Reprinted figure with permission from W. Lu, S. Sridhar, M. Zworski, Phys. Rev. Lett. **91**, 154101 (2003). Copyright 2003 by the American Physical Society.

attempts have been made to extract the long living resonances of the 3-disk scattering system from an *experimental signal* on a microwave (quasi)-2D billiard by the Marburg group [41]. Yet, computing high frequency resonances in such an experiment presents many difficulties: a noisy and discrete signal, the presence of antennas which perturb the ideal system, the difficulties to go to sufficiently high frequencies, and the delicate implementation of the harmonic inversion method used to extract the “true” resonances.

The fractal Weyl conjecture was actually much easier to test numerically on the toy model of *open quantum maps*.

⁸In the last two cases, resonances were obtained by computing the zeros of the Selberg/Gutzwiller zeta function: this procedure exactly provides the resonances in the convex co-compact case, while in the obstacle case the zeros are believed to be good approximations of the resonances.

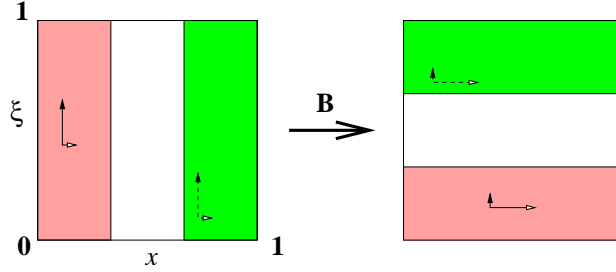


FIGURE 9. Sketch of the symmetric 3-baker's map with a hole in the central rectangle.

6.1. Fractal Weyl law for open quantum maps. As explained above, in the quantum map framework the statistics of long-living states is obtained by fixing some radius $r > 0$, and counting the number of eigenvalues $\lambda_j(\hbar)$ of $M(\hbar)$ in the annulus $\{r \leq |\lambda| \leq 1\}$. This task is easy to implement numerically for operators (matrices) $M(\hbar)$ of reasonable dimensions. Schomerus and Tworzydło implemented it on the kicked rotor [69] in a strongly chaotic régime⁹, with a sharp opening along a vertical strip (namely, the opening is realized by a projector on $\{x \in [a, b]\}$). A very good agreement with the “strong” fractal Weyl law was observed: most eigenvalues accumulate near $\lambda = 0$, while a small fraction of them have moduli $\geq r$. Their numerics hint at the existence of a *nontrivial* profile function $r \in (0, 1] \mapsto F(r) \geq 0$, such that

$$(39) \quad \forall r > 0, \quad \sharp(\text{Spec } M(\hbar) \cap \{|\lambda| \geq r\}) = F(r) \hbar^{-\nu} + o(\hbar^{-\nu}),$$

with $\nu = \dim \mathcal{K}/2$.

6.1.1. The open baker's map. The spectra of several types of quantum open baker's maps were analyzed in [53, 54, 50]. Let us recall the definition and basic properties of this family of chaotic maps on \mathbb{T}^2 .

A baker's map $\tilde{\kappa}$ is defined by splitting \mathbb{T}^2 into D “Markov rectangles” $R_i = \{x_i \leq x < x_{i+1}, 0 \leq \xi < 1\}$, $i = 0, \dots, D-1$, $x_0 = 0$, $x_D = 1$, and mapping the points in R_i as follows:

$$(40) \quad (x, \xi) \in R_i \mapsto \tilde{\kappa}(x, \xi) = \left(\frac{x - x_i}{\ell_i}, \ell_i \xi + x_i \right), \quad \text{with } \ell_i \stackrel{\text{def}}{=} x_{i+1} - x_i.$$

Since $\ell_i < 1$, the stable/unstable directions are the vertical/horizontal axes. This map is invertible, but discontinuous along the ∂R_i . There is an obvious symbolic dynamics: to a point $\rho = (x, \xi)$ one associates the sequence $\dots \epsilon_{-1} \epsilon_0 \epsilon_1 \dots$, $\epsilon_j \in \{0, \dots, D-1\}$, such that

⁹As far as I know, this chaotic régime has not been proved rigorously, but seems plausible in view of numerics.

$\tilde{\kappa}^j(\rho) \in R_{\epsilon_j}$ for each time $j \in \mathbb{Z}$. Conversely, to any bi-infinite sequence corresponds a point ρ , and this map is “almost” one-to-one¹⁰.

In order to take advantage of this symbolic dynamics, the opening $\mathbb{T}^2 \setminus V$ was chosen to consist in the union of $D-n$ of the Markov rectangles, $0 < n < D$. The trapped set \mathcal{K} is then easy to describe (see Fig. 11): it consists in the sequences ϵ with all $\epsilon_j \in I = \{i_1, \dots, i_n\}$ the set of “kept rectangles”. This set is the cartesian product $\text{Can} \times \text{Can}$, where Can is a Cantor set on the unit interval; the Hausdorff or Minkowski dimension ν of Can is explicitly given by the only real root of the equation

$$\ell_{i_1}^s + \ell_{i_2}^s + \dots + \ell_{i_n}^s = 1.$$

For instance, if we keep n rectangles in the symmetric D -baker’s map, we get $\nu = \frac{\log n}{\log D}$.

The quantization of κ was using the quantizations of Balasz-Voros and Saraceno [4, 65] for the closed map $\tilde{\kappa}$, and projecting out the column (positions) corresponding to the opening. Assuming the ℓ_i are rational, for quantum dimensions $N = (2\pi\hbar)^{-1}$ such that $N\ell_i$ are all integer, we start from the unitary matrix

$$U(\hbar) = U_N = F_N^{-1} \begin{pmatrix} F_{N\ell_0} & & \\ & \dots & \\ & & F_{N\ell_{D-1}} \end{pmatrix},$$

where F_* is the $*$ -dimensional discrete Fourier transform, and kill the $D-n$ blocks $F_{N\ell_j}$ corresponding to the opening.

A strong form of fractal Weyl law was observed for an asymmetric baker’s map (see Fig. 10). For symmetric baker’s maps (that is, taking $x_i = i/D$, see Fig 9), the fractal scaling seems satisfied, but we observed that different profile functions occurred along different geometric sequences $(N = N_o D^k)_{k \geq 0}$, a manifestation of the number theoretic properties of such symmetric maps. Apart from these specific number theoretic issues, the form of the profile function for both the kicked rotor and the baker maps looks similar: $F(r)$ decays regularly from $r \approx 0$ and approximately vanishes around some value $r_{\max} < 1$, with a “dip” before r_{\max} , showing a (mild) peak of the density of states around some value $r_{\text{peak}} \leq r_{\max}$. The position of this peak seems close to the classical decay rate, $r_{\text{peak}} \approx e^{-\gamma_{\text{cl}}/2}$ [70].

A random matrix model Ansatz was proposed in [69] to account for the profile function $F(r)$, hinting at a certain “universality” of this profile, but the validity of this Ansatz remains unclear.

6.1.2. *A solvable model satisfying the fractal Weyl law.* A “toy-of-the-toy” model was studied in [53, 54], in the form of a nonstandard quantization of the symmetric D -baker’s map.

¹⁰The defect of injectivity comes from points with sequences ending by infinite strings of 0, on either end. For instance, the point $(0, 0)$ can be represented by the constant sequences $\overline{0}$ or $\overline{D-1}$.

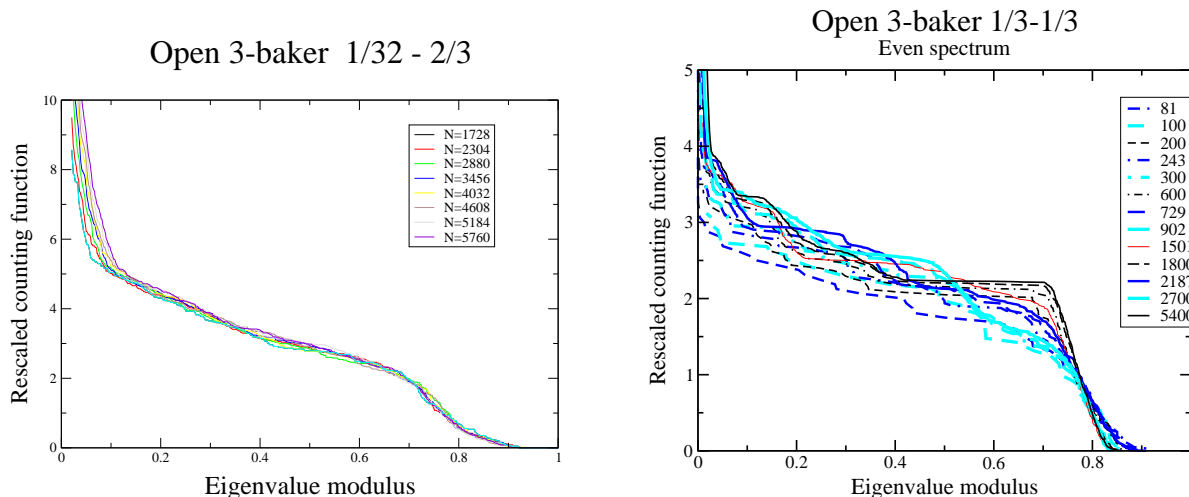


FIGURE 10. Rescaled counting function for an asymmetric (left) and a symmetric (right) open 3-baker opened by removing the central rectangle (the fractions in the title denote the contraction factors ℓ_0, ℓ_2). In both cases the rescaling consists in dividing the counting function by the factor N^ν , where $\nu = \dim(\mathcal{K})/2$.

In the case of quantum dimension $N = D^k$, this quantization amounts to replacing the discrete Fourier transform on F_N by the Walsh-Fourier transform, that is the Fourier transform on $(\mathbb{Z}_D)^k$. This quantization $M_N = M_{D^k}$ of the open baker's map then admits a very simple tensor product representation on the Hilbert space $\mathcal{H}_N \equiv (\mathbb{C}^D)^{\otimes k}$.

This property allows to *explicitly* compute the spectrum of M_N : the latter is given in terms of the $D \times D$ matrix Ω_D , obtained by removing from inverse Fourier transform F_D^* the $D - n$ columns corresponding to the opening. Generally, this matrix has a $D - n$ -dimensional kernel and n nontrivial eigenvalues, which are the eigenvalues of the $n \times n$ square matrix $\tilde{\Omega}_D$ extracted from Ω_D . This results in $n^k = N^\nu$ nontrivial eigenvalues (counted with multiplicities) for M_N , and proves the strong form of fractal Weyl law (39). The profile function $F(r)$ has the form of a step function at some value $r_c = |\det \tilde{\Omega}_D|^{1/2}$.

Yet, we noticed in [53, Remark 5.2] that for some choices of parameters¹¹, the spectrum of $\tilde{\Omega}_D$ may present an “accidental” kernel. In that case, the counting function is $\mathcal{O}(\hbar^{-\nu'})$ with $\nu' < \dim(\mathcal{K})/2$, so the even the weak form of fractal Weyl law is not satisfied. This accidental degeneracy seems due to the very special tensor product structure of the Walsh quantization, in any case it seems very nongeneric.

¹¹It is the case, for instance, if we kill the second and fourth rectangles from the symmetric 4-baker: in that case, M_N has a single, simple nontrivial eigenvalue. One can cook up an even more dramatic example (with $D = 16, n = 2$), for which $\text{Spec}(M_N) = \text{Spec}(\tilde{\Omega}_D) = \{0\}$.

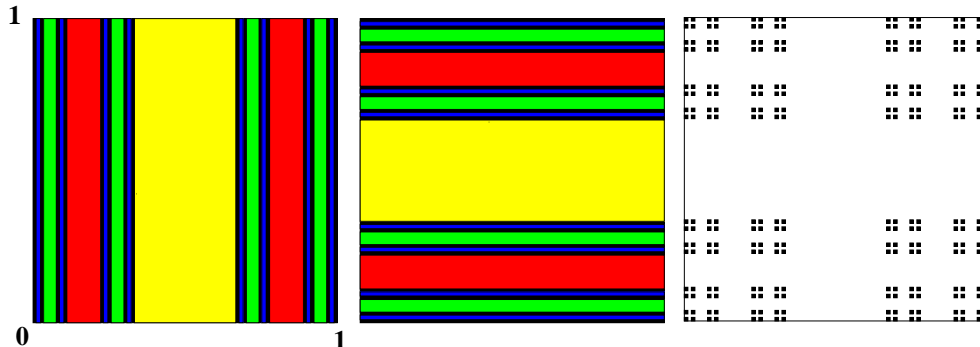


FIGURE 11. Construction of the trapped set (right) and its incoming (left) and outgoing (center) tails for the symmetric 3-baker. Each color corresponds to a specific escape time (in the futur or past) $n = 1 - 4$, the trapped set (and its tails) being approximated by union of black intervals/squares.

It was checked [28] that this accidental degeneracy disappears if we modify the matrix M_N by multiplying it by a diagonal matrix of random, or even deterministic phases. Nevertheless, this problem may indicate that any attempt to prove the fractal Weyl law in any setting might require some *genericity* assumption, or the introduction of some random parameters in the system.

7. INTERPRETATION OF THE FRACTAL WEYL UPPER BOUND FOR OPEN QUANTUM MAPS

After reviewing the numerical (and some analytical) results regarding the optimality of the fractal Weyl upper bound, let us present a couple of heuristic proofs of this upper bound in the case of open quantum maps, as well as a rigorous proof for “smooth quantum maps”. All proofs use a reduction of the dynamics to an effective propagator of “minimal rank”, which accounts for the quantum dynamics near the trapped set.

7.1. Heuristic explanations. A semiclassical mechanism explaining the fractal Weyl upper bound for an open chaotic map κ has been put forward in [69]. The idea is that the (essential) generalized kernel of the $M(\hbar)$ is larger than its kernel (associated with the hole), due to the presence of (approximate) Jordan blocks reflecting the transient dynamics of points which wander through V before escaping.

For any $n \geq 1$, consider the sets $D_n \in \mathbb{T}^2$ of points escaping before the time n . This set consists in a union of finitely many connected components $D_{n,j}$, most of which look like “thin tubes” aligned along the stable manifolds when $n \gg 1$ (see Fig. 11). The widths of the thin tubes decay like $e^{-n\lambda}$, where λ is the Lyapunov exponent. For fixed n , one can associate to each component $D_{n,j}$ a quantum subspace $\mathcal{H}_{\hbar,n,j}$ of dimension $(2\pi\hbar)^{-1} \text{Vol}(D_{n,j})$. Besides,

the various subspaces $\mathcal{H}_{\hbar,n,j}$ are semiclassically almost orthogonal to each other, so that the subspace $\mathcal{H}_{\hbar,n} = \bigoplus_j \mathcal{H}_{\hbar,n,j}$ has dimension $(2\pi\hbar)^{-1} \text{Vol}(D_n)$.

The semiclassical evolution implies that any state $u \in \mathcal{H}_{\hbar,n}$ will be absorbed when iterated up to time n :

$$M(\hbar)^n u = \mathcal{O}(\hbar^\infty) \|u\|, \quad u \in \mathcal{H}_{\hbar,n}.$$

This property implies that the long-living eigenvalues of $M(\hbar)$ are essentially the same as those of $M(\hbar) \upharpoonright_{\mathcal{H}_{\hbar,n}^\perp}$, so their number is at most $(2\pi\hbar)^{-1}(1 - \text{Vol}(D_n))$.

When $n \gg 1$ the set $\mathcal{C}D_n$ of the points with escape times $> n$ is a small neighbourhood of the incoming tail \mathcal{K}^- (see Fig. 11), and

$$\text{Vol}(\mathcal{C}D_n) \sim e^{-n\gamma_{cl}},$$

where $\gamma_{cl} > 0$ is the classical decay rate. To get a fractal upper bound, one needs to push the time n to infinity in a \hbar -dependent way. As long as n is smaller than the *Ehrenfest time*

$$(41) \quad T_{Ehr} \sim \frac{\log 1/\hbar}{\lambda_{\max}}, \quad \text{where } \lambda_{\max} \text{ is the largest expansion rate,}$$

the tubes $D_{n,j}$ have volumes $\gg \hbar$, which means that one can construct quantum subspaces $\mathcal{H}_{\hbar,n,j}$ microlocalized inside $D_{n,j}$. Ignoring problems due to the boundaries of $D_{n,j}$, one can push the above argument up to $n = T_{Ehr}$: the bound on the number of long-living eigenvalues then reads

$$(2\pi\hbar)^{-1} \text{Vol}(\mathcal{C}D_{T_{Ehr}}) \sim \hbar^{-1 + \frac{\gamma_{cl}}{\lambda_{\max}}}.$$

This argument is not optimal if the hyperbolicity is not homogeneous, the exponent $1 - \frac{\gamma_{cl}}{\Lambda}$ being larger than $\nu = \dim(\mathcal{K})/2$. Still, the above reasoning clearly exhibits the connection between resonance counting and small (\hbar -dependent) neighbourhoods of the trapped set (or its tail \mathcal{K}^-). It also shows the crucial Jordan structure of $M(\hbar)$, quantum analogue of the transient dynamics before T_{Ehr} .

An alternative approach to the problem was adopted by Novaes *et al.* in [57]. There the quantum dynamics was projected “by hand” on a quantum subspace microlocalized near the trapped set, resulting in an effective spectral problem of smaller dimension. Precisely, the minimal subspace was generated by a certain number ($\sim C\hbar^{-\nu}$) of (Left and Right) *scar functions* u_n^L, u_n^R , that are quasimodes of $M(\hbar)$, resp. $M(\hbar)^*$, microlocalized along periodic orbits of κ of periods $T \leq T_{Ehr}$. The authors used these scar functions to construct the generalized eigenvalue problem

$$(42) \quad \det(E(\lambda)) = 0, \quad E_{mn}(\lambda) \stackrel{\text{def}}{=} \langle u_n^L, (I - \lambda^{-1} M(\hbar)) u_m^R \rangle,$$

and checked that the solutions of this problem accurately approximated the long living spectrum of $M(\hbar)$ (see Fig. 12).

The operator $E(\lambda)$, representing the quantum dynamics on \mathcal{K} , is an (approximate) *effective propagator* for the quantum map $M(\hbar)$, and can be considered to be of minimal

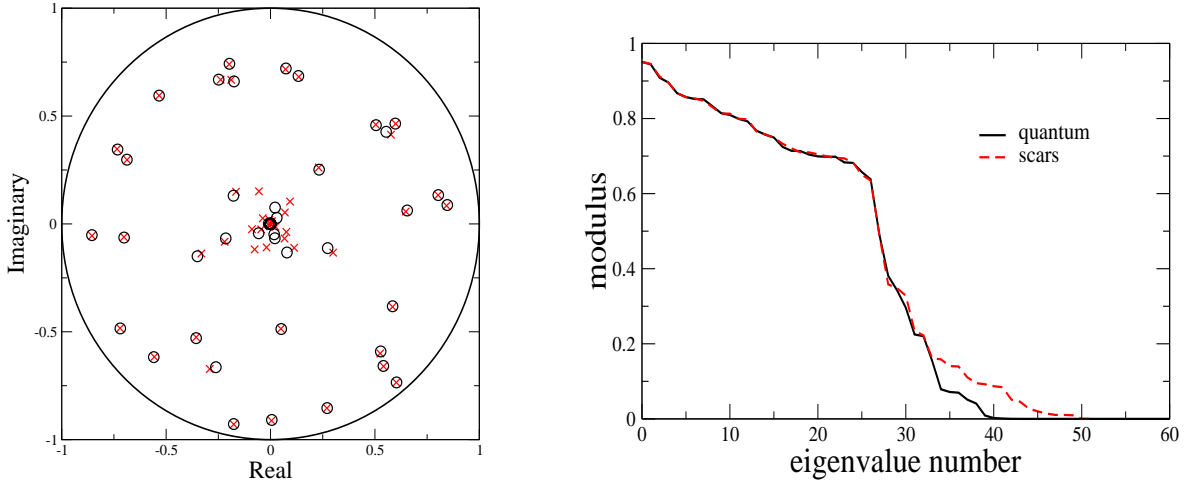


FIGURE 12. Left: spectrum of the quantum open baker's map ($D = 3$ symmetric, $N = 81$), by diagonalizing the matrix M_N (red crosses) or by using the “scar matrix” $E(z)$ from periodic orbits of length ≤ 5 , and solving (42) (black circles). Right: the corresponding radial counting (compared with Fig. 10, the two axes should be exchanged). Reprinted figure with permission from M. Novaes *et al.*, Phys. Rev. **E 80** 035202(R) (2009). Copyright 2009 by the American Physical Society.

rank, meaning that no further reduction seems possible. Another advantage of this matrix $E_{mn}(\lambda)$ is its sparsity: since quasimodes are localized near periodic orbits, each quasimode u_m^R (resp. iterated quasimode $M(\hbar)u_m^R$) interferes only with a few quasimodes u_n^R .

7.2. A proof of the fractal Weyl upper bound. A rigorous proof of the fractal upper bound for *smooth* open quantum maps¹² can be obtained using similar ideas. Notice that this smoothness condition unfortunately excludes the open quantum maps used in most numerical studies [69, 55, 70].

Theorem 8. [52] *Consider $\kappa : V \mapsto \kappa(V)$ a (smooth) open map such that its trapped set $\mathcal{K} \Subset V$ is a hyperbolic repeller of upper Minkowski dimension 2ν , and a corresponding smooth quantum map $\mathcal{M}(\hbar)$, or more generally a FIO $\mathcal{M}(\alpha, \hbar)$ with some symbol $\alpha \in C_c^\infty(V)$.*

Then, for any $r \in (0, 1)$ and any small $\epsilon > 0$, $\exists C_{r,\epsilon}, \hbar_{r,\epsilon} > 0$ s.t.

$$\#\{\text{Spec}(\mathcal{M}(\hbar)) \cap \{|\lambda| \geq r\}\} \leq C_{r,\epsilon} \hbar^{-\nu-\epsilon}, \quad \forall \hbar < \hbar_{r,\epsilon}.$$

If \mathcal{K} is of pure dimension, one can take $\epsilon = 0$.

¹²see §5.2: the map κ is smooth, and the quasiprojector $\Pi(\hbar) = \text{Op}_\hbar(\alpha)$ is a decent pseudodifferential operator

The same estimate holds for $M(\hbar)$ an open quantum map of finite rank.

The proof is an “exponentiation” of the case of Schrödinger operators presented in §4.2. One constructs an escape function $G(x, \xi)$ on $V \cup \kappa(V)$, such that

$$(43) \quad G \circ \kappa - G \geq 1 \quad \text{outside an } \varepsilon\text{-neighbourhood } \mathcal{K}^\varepsilon \text{ of the trapped set } \mathcal{K}.$$

To ensure that G is a “nice symbol”, this neighbourhood must have a width $\varepsilon \gtrsim \hbar^{1/2}$. One uses this escape function to conjugate the quantum map $\mathcal{M}(\hbar)$ into

$$\mathcal{M}_{tG}(\hbar) \stackrel{\text{def}}{=} e^{-tG^w} \mathcal{M}(\alpha, \hbar) e^{tG^w}, \quad t \gg 1.$$

The Egorov theorem and the pseudodifferential calculus show that $M_{tG}(\hbar)$ is still a Fourier Integral Operator, but with a modified symbol:

$$M_{tG}(\hbar) \approx M(\hbar) e^{-t((G \circ \kappa)^w - G^w)}, \quad \alpha \rightarrow \alpha_{tG} \approx \alpha e^{-t(G \circ \kappa - G)}.$$

Hence, for $t \gg 1$ the escape property (43) ensures that $M_{tG}(\hbar)$ strongly suppresses the states microlocalized outside \mathcal{K}^ε . One can thus construct a quantum subspace \mathcal{H}_ε of dimension $\sim C\hbar^{-\nu}$ microlocalized in \mathcal{K}_ε , such that

$$(44) \quad \|(I - \Pi_\varepsilon)M_{tG}\| \ll 1,$$

where Π_ε is the orthogonal projector on \mathcal{H}_ε .

From this remark, one can easily show (through a minimax argument) that the number of long living singular values of M_{tG} is $\mathcal{O}(\hbar^{-\nu})$, and get a similar bound for its eigenvalues using Weyl inequalities.

We present an alternative argument, which has the advantage to apply as well to the case of monodromy operators. The property (44) shows that, for any $\lambda \in \mathbb{C}$ with $|\lambda| > r$, the operator

$$(45) \quad E(\lambda) \stackrel{\text{def}}{=} (I - \lambda^{-1}\Pi_\varepsilon M_{tG}) - \lambda^{-2} \Pi_\varepsilon M_{tG} (I - \Pi_\varepsilon) M_{tG} [I - \lambda^{-1}(I - \Pi_\varepsilon) M_{tG}]^{-1} \quad \text{is well-defined,}$$

and the second term on the RHS is a small correction compared with the first one.

A little algebra shows that the long living eigenvalues of $M_{tG}(\hbar)$ can be exactly obtained (with multiplicities) by solving

$$(46) \quad \det(E(\lambda)) = 0, \quad |\lambda| \geq r.$$

This confers to $E(\lambda)$ the title of an *effective Hamiltonian* for the quantum map $M(\hbar)$. This operator “minimally” captures the long time quantum evolution, which lives on \mathcal{K} .

Applying Jensen’s formula, one easily shows that the number of roots of (46) with $|\lambda| \geq r$ is bounded from above by $\dim \mathcal{H}_\varepsilon \sim C\hbar^{-\nu}$.

The same argument can apply to count the roots of $\det(I - M(z, \hbar))$ with $M(z, \hbar)$ a quantum monodromy operator: one defines an effective Hamiltonian $E(z)$ as in (45), replacing everywhere $\lambda^{-1}M(\hbar)$ by $M(z, \hbar)$ [52]. Using Thm 6, this leads to a proof of

Thms 2, and an alternative proof for Thms.4 and 5 (under the assumption that K_E is totally disconnected transversely to the flow).

8. RESONANCE GAP FOR OPEN QUANTUM MAPS AND MONODROMY OPERATORS

The second main result on the semiclassical distribution of resonances is the criterium for a resonance gap, expressed in Theorems 1 and 3. Below we will state the corresponding result for open quantum maps and quantum monodromy operators, which can also be used to prove these theorems.

Our result will be stated in the framework of general Fourier Integral Operators $\mathcal{M}(\alpha, \hbar)$ associated with an open smooth map $\kappa : V \Subset \Sigma \mapsto \kappa(V)$ with a hyperbolic trapped set. This means that the symbol of the quantum map is not necessarily equal to unity near \mathcal{K} , but can be any function $\alpha \in C_c^\infty(V)$, such that $|\alpha(\rho)| > c > 0$ in some fixed neighbourhood of \mathcal{K} .

Theorem 9 (Spectral gap for open quantum maps). *Let $\mathcal{M}(\alpha, \hbar)$ satisfy the above conditions.*

Then, for any small enough $\epsilon > 0$ there exists $\hbar_\epsilon > 0$, such that for any $\hbar < \hbar_\epsilon$, the spectral radius of the quantum map $\mathcal{M}(\alpha, \hbar)$ satisfies

$$(47) \quad r_{sp}(\mathcal{M}(\alpha, \hbar)) \leq \exp \{ \mathcal{P}(-\varphi^+/2 + \log |\alpha|, \kappa \upharpoonright_{\mathcal{K}}) + \epsilon \}.$$

Here $\varphi^+(\rho) \in C(\mathcal{K})$ is the logarithm of the unstable Jacobian of κ , and $\mathcal{P}(\bullet)$ is the topological pressure for κ .

The same bound holds if we replace $\mathcal{M}(\alpha, \hbar)$ by a finite rank truncation $M(\alpha, \hbar)$ (see §5.2).

The norm estimate (31) implies the obvious bound

$$(48) \quad r_{sp}(\mathcal{M}(\alpha, \hbar)) \leq \|\alpha\|_\infty + \mathcal{O}(\hbar).$$

This bound may be sharper than (47), depending on both κ and α . For instance, if $\alpha \equiv 1$ near \mathcal{K} , then (47) is sharper than (48) iff the pressure $\mathcal{P}(-\varphi^+/2, \kappa \upharpoonright_{\mathcal{K}})$ is negative, a condition satisfied only provided the trapped set is “thin”.

Before sketching the proof of this theorem, let us explain how it can be used to prove Thms 1 and 3. Eq. (36) shows that the monodromy operator $M(z, \hbar)$ associated with a scattering operator $P(\hbar)$ (or an obstacle problem) has the form of an FIO associated with κ , with symbol

$$\alpha_z(\rho) = e^{-i\zeta\tau(\rho)} + \mathcal{O}(\hbar) \quad \text{near } \mathcal{K}, \quad \zeta \stackrel{\text{def}}{=} \frac{z - E}{\hbar},$$

where $\tau(\rho)$ is the return time. Applying the above theorem, we find that, if the root s_0 of the equation

$$(49) \quad \mathcal{P}(-\varphi^+/2 - s\tau, \kappa \upharpoonright_{\mathcal{K}}) = 0 \quad \text{satisfies } s_0 < 0.$$

The theory of Axiom A flows [10] shows that this root is exactly given by the topological pressure of the flow Φ^t :

$$s_0 = \mathcal{P}\left(-\frac{1}{2}\varphi_{\Phi}^+, \Phi^1 \upharpoonright K_E\right),$$

where φ_{Φ}^+ is the unstable Jacobian of the flow, (11).

If we take $\text{Im } \zeta > s_0 + \tilde{\epsilon}$ for some $\tilde{\epsilon} > 0$, the pressure $\mathcal{P}(-\varphi^+/2 + \log |\alpha_z| \tau)$ will be negative, and the above theorem implies that, for \hbar small enough,

$$r_{sp}(M(z, \hbar)) \leq \ll 1.$$

In turn, this bound shows that the equation $\det(1 - M(z, \hbar)) = 0$ has no solution in the strip $\text{Im } \zeta > s_0 + \tilde{\epsilon}$, which is the statement of Thms1 and 3.

In the next section we sketch the proof of Thm 9.

8.1. Proof of the resonance gap in terms of the pressure. The proof of the spectral bound (48), analogous to the case of the Schrödinger flow in [55], is quite different from that of the fractal Weyl upper bound, Thm 8. It uses as well the hyperbolicity of $\kappa \upharpoonright_{\mathcal{K}}$, but instead of conjugating $\mathcal{M}(\alpha, \hbar)$ through an escape function, one splits the propagator using a phase space partition of unity.

We will assume that the trapped set \mathcal{K} is totally disconnected, which is the case if we want to apply the result to monodromy operators. This restriction is not necessary, but it simplifies the proof of the Theorem.

To obtain an upper bound on the spectral radius of $\mathcal{M}(\alpha, \hbar)$ (which we will denote by $\mathcal{M}(\hbar)$ from now on), the usual strategy is to estimate $\|M(\hbar)^n\|$, with $n \gg 1$. Inspired by classical dynamical methods, we will proceed by splitting $M(\hbar)^n$ into many pieces, each one being associated with a “pencil” of classical trajectory segments. The topological pressure will then naturally arise after summing over all the “pencils”.

Let us be more precise. Using the assumption that \mathcal{K} is totally disconnected, for any small $\epsilon > 0$ we may consider a *Markov cover* $(V_a)_{a \in A_1}$ of the trapped set, such that the open sets V_a have diameters at most ϵ , and enjoy the following Markov properties: the sets V_a are disjoint, and from the cover we may construct the

$$T_{a'a} = \begin{cases} 1 & , \quad V_a \cap \kappa^{-1}(V_{a'}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

Then, for any sequence of symbols $\alpha = \alpha_0 \alpha_1 \cdots \alpha_{n-1}$, the set

$$V_{\alpha} \stackrel{\text{def}}{=} V_{\alpha_0} \cap \kappa^{-1}(V_{\alpha_1}) \cap \cdots \cap \kappa^{-n+1}(V_{\alpha_{n-1}})$$

is nonempty iff, at all steps $j = 0, \dots, n-2$, one has $T_{\alpha_{j+1}\alpha_j} = 1$. V_{α} consists of the initial points ρ which share the same symbolic history for times $j = 0, \dots, n-1$.

To each set V_a we may associate the weight

$$(50) \quad w_a = \max_{\rho \in \mathcal{K} \cap V_a} e^{-\varphi^+(\rho)/2} |\alpha(\rho)|,$$

and consider the weighted transition matrix $T_{a'a}^w = T_{a'a} w_a$, $a', a \in A_1$. The topological pressure appearing in (47) is then approximated by the largest (Perron-Frobenius) eigenvalue of the matrix T^w :

$$(51) \quad \mathcal{P}(-\varphi^+/2 + \log |\alpha|, \kappa \upharpoonright_{\mathcal{K}}) = \lim_{\epsilon \rightarrow 0} \log \lambda_{PF}(T^w).$$

We may complete this cover into an open cover of V ,

$$(52) \quad V \subset \cup_{a \in A} V_a, \quad A = A_1 \cup A_+ \cup A_-,$$

such that, for some time n_o , all sets V_{a_-} , $a_- \in A_-$ (resp. V_{a_+} , $a_+ \in A_+$) escape in the hole before the time n_o in the backwards (resp. forward) evolution, see Fig. 13. To the

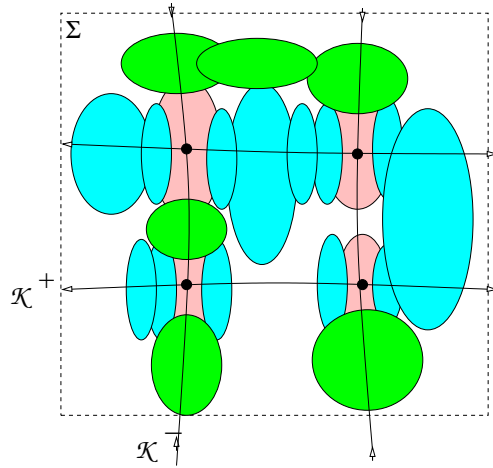


FIGURE 13. Sketch of the open cover $(V_{a_-})_{a \in A_-}$ of the trapped set (pink), completed by the open sets $(V_{a_+})_{a \in A_+}$ (cyan) and $(V_a)_{a \in A_-}$ (green) away from \mathcal{K} . Black lines indicate the tails \mathcal{K}^\pm and black dots are the trapped set.

cover (V_a) we associate a smooth partition of unity of the phase space Σ , namely a finite collection of cutoffs $\chi_a \in C_c^\infty(V_a, [0, 1])$, satisfying

$$\sum_{a \in A} \chi_a \equiv 1 \quad \text{in some neighbourhood of } V,$$

and add a component in the hole, $\chi_\infty = 1 - \sum_{a \in A} \chi_a$ to get a full partition of unity.

This smooth partition of unity is then quantized into a quantum partition

$$Id = \sum_{a \in A \cup \infty} \text{Op}_h(\chi_a),$$

which is used to split the iterated propagator $\mathcal{M}(\hbar)^n$ into elementary blocks:

$$\mathcal{M}(\hbar)^n = \sum_{|\alpha|=n} \mathcal{M}_\alpha, \quad \mathcal{M}_\alpha = \mathcal{M}_{\alpha_{n-1}} \mathcal{M}_{\alpha_{n-2}} \cdots \mathcal{M}_{\alpha_0}, \quad \mathcal{M}_a \stackrel{\text{def}}{=} \mathcal{M}(\hbar) \text{Op}_\hbar(\chi_a).$$

8.1.1. *Analyzing the components \mathcal{M}_α .* The advantage of this decomposition is to obtain an upper bound for the individual components \mathcal{M}_α which is sharper than the obvious bound

$$(53) \quad \|\mathcal{M}_\alpha\| \leq (\|\alpha\|_\infty + \mathcal{O}(\hbar))^n.$$

For this, we use the information we have on the classical dynamics. From Egorov's theorem, we know that for any finite sequence α , the element $\mathcal{M}_\alpha = \mathcal{O}(\hbar^\infty)$ unless the set $V_\alpha \neq \emptyset$; the sequence α is then called *admissible*. Also, any sequence containing $a = \infty$ leads to a negligible term.

As a result, since the sets V_{a_-} (resp. V_{a_+}) escape before the time n_o in the past (resp. in the future), we deduce that for any $n > 2n_o$ the only nonnegligible operators \mathcal{M}_α must be of the form

$$\alpha = \alpha_+ \alpha^{(1)} \alpha_-, \quad |\alpha_-| = |\alpha_+| = n_o, \quad \alpha^{(1)} \in A_1^{n-n-n_+} \quad \text{admissible.}$$

8.1.2. *Acting on a Lagrangian state: a hyperbolic dispersive estimate.* In view of this property, for $n \gg 1$ we may restrict ourselves to the admissible sequences $\alpha \in A_1^n$. We now want to use the hyperbolicity of κ near \mathcal{K} . If we apply \mathcal{M}_{α_0} to a Lagrangian (or WKB) state u_0 supported by a Lagrangian manifold Λ_{α_0} transverse to the stable direction E^- , the state will expand along the unstable direction. Assuming the state has spread outside V_{α_1} , cutting it through $\text{Op}_\hbar(\chi_{\alpha_1})$ will reduce its norm by a finite factor, while the output state will again be a WKB state along a Lagrangian $\Lambda_{\alpha_1 \alpha_0}$ transverse to E^- . The same phenomenon repeats itself, and lead to the following *hyperbolic dispersive estimate*:

$$(54) \quad \|\mathcal{M}_\alpha u_0\| \lesssim w_\alpha, \quad w_\alpha = \prod_{j=0}^{n-1} w_{\alpha_j}.$$

Because the unstable Jacobian is bounded below, $\varphi^+(\rho) \geq \Lambda^+ > 0$, the weights satisfy

$$w_a \leq e^{-\Lambda^+} \|\alpha\|_\infty,$$

so after a some time T_0 the estimate (54) becomes sharper than the trivial bound (53).

This type of estimate first appeared in the work of Anantharaman on Anosov flows [2] (without damping). It was extended to the case of scattering problems with a fractal hyperbolic repeller in [55] (see also [49, Section 4]), and to situations with a nonhomogeneous damping [66]. The proof of this estimate is a simple adaptation (taking the damping factors into account) of the proof of [55, Prop.6.3].

8.1.3. *Putting the pieces together.* One then applies the triangular inequality to get a bound on the sum of terms with $\alpha \in A_1^n$ (we take $n \leq C \log(1/\hbar)$, and omit to indicate the remainders $\mathcal{O}(\hbar^\infty)$):

$$\|\mathcal{M}(\hbar)_{(A_1^n)}^n u_0\| \stackrel{\text{def}}{=} \left\| \sum_{\alpha \in A_1^n} \mathcal{M}_\alpha u_0 \right\| \lesssim \sum_{\alpha \in A_1^n \text{ admis.}} w_\alpha = \sum_{a', a \in A_1} [(T^w)^n]_{a'a},$$

where we use the weighted transition matrix. For $n \gg 1$, the high power of the matrix T^w is dominated by its Perron-Frobenius eigenvalue, which, from (51), is close to the topological pressure:

$$[(T^w)^n]_{a'a} = \lambda_{PF}^n \Pi_{a'a} + \mathcal{O}(|\lambda_2|^n) \leq e^{n(\mathcal{P}(-\varphi^+/2 + \log|\alpha|) + \tilde{\epsilon})},$$

so we get,

$$(55) \quad \|\mathcal{M}(\hbar)_{(A_1^n)}^n u_0\| \leq C e^{n(\mathcal{P}(-\varphi^+/2 + \log|\alpha|) + \tilde{\epsilon})}.$$

So far we have considered the action of the propagator on a single Lagrangian state u_0 , localized in a Lagrangian manifold uniformly transverse to E^- . However, one can cook up a “microlocal basis” $(u_\zeta)_{\zeta \in W}$ made of WKB states associated with Lagrangian manifolds $\Lambda_\zeta \subset V_{\alpha_0}$ close to the unstable direction, such that any normalized state $u \in L^2$ microlocalized inside V_{α_0} can be expanded as

$$(56) \quad u = \int_W \frac{d\zeta}{(2\pi\hbar)^{(d-1)/2}} f(\zeta) u_\zeta,$$

with W a bounded domain in \mathbb{R}^{d-1} and $\int_W d\zeta |f(\zeta)| = \mathcal{O}(1)$.

Applying the triangular inequality to the decomposition (56), and adding the contributions of the “tails” α_\pm , we obtain for $n \gg 1$ the norm estimate

$$\|\mathcal{M}(\hbar)^n\| \leq C \hbar^{-(d-1)/2} e^{n(\mathcal{P}(-\varphi^+/2 + \log|\alpha|) + \tilde{\epsilon})}.$$

We notice that, due to the prefactor $\hbar^{-(d-1)/2}$, this upper bound may start to be nontrivial only for times $n \geq C_+ \log(1/\hbar)$ for a certain C_+ (in particular, after the Ehrenfest time). To obtain the bound (47) one actually needs to absorb this factor in a factor of the form $e^{n\epsilon}$: this can be done only by taking long logarithmic times $n \sim \epsilon^{-1} \log(1/\hbar)$. One then gets

$$(57) \quad \|\mathcal{M}(\hbar)^n\| \leq \exp\{n(\mathcal{P}(-\varphi^+/2 + \log|\alpha|) + 2\tilde{\epsilon})\},$$

which proves the spectral bound. \square

8.2. Is the pressure bound optimal? Are the upper bound (57) and the ensuing spectral bound (47) optimal?

As shown above, the pressure bound (57) is obtained by evolving certain Lagrangian states like u_0 through the truncated maps M_α , resulting in a nontrivial estimate for the norms $\|\mathcal{M}_\alpha u_0\|$, Eq. ((54)); then we applied the triangular inequality to bound the norm

of $\mathcal{M}_{(A_1^n)}^n u_0$, and got the bound (55). The question is: how much does one “lose” through this (blind) triangular inequality?

If the states $\mathcal{M}_\alpha u_0$ were (essentially) orthogonal to each other, one would get the bound

$$(58) \quad \|\mathcal{M}(\hbar)_{(A_1^n)}^n u_0\| \leq \left(\sum_{\alpha \in A_1^n \text{ admis.}} w_\alpha^2 \right)^{1/2} \leq \exp\{n(\mathcal{P}(-\varphi^+ + 2 \log |\alpha|)/2 + \tilde{\epsilon})\},$$

which is generally sharper than (55) because for any nonzero test function f one has $\mathcal{P}(2f) < 2\mathcal{P}(f)$. For instance, in the case $\alpha \equiv 1$ near \mathcal{K} , the pressure $\mathcal{P}(-\varphi^+) = -\gamma_{cl}$ is always negative, as opposed to the pressure $\mathcal{P}(-\varphi^+/2)$, which is negative only provided \mathcal{K} is “thin”.

Two Lagrangian states $\mathcal{M}_\alpha u_0$, $\mathcal{M}_{\alpha'} u_0$ will indeed be (essentially) orthogonal if the final indices $\alpha_{n-1} \neq \alpha'_{n-1}$ (because these two states are then microsupported on disjoint sets), or if they are supported on Lagrangians $\Lambda_\alpha, \Lambda_{\alpha'} \subset V_{\alpha_{n-1}}$ at distance $\gg \hbar$ from one another, so that a nonstationary phase estimate ensures that

$$\langle \mathcal{M}_{\alpha'} u_0, \mathcal{M}_\alpha u_0 \rangle = \mathcal{O}(\hbar^\infty).$$

This essential orthogonality will be the case for sequences of length $n \leq c \log(1/\hbar)$, with $c > 0$ sufficiently small. But for large logarithmic times $n \gg C_+ \log(1/\hbar)$ (which are the times necessary for our proof), many states $\mathcal{M}_\alpha u_0$ will be supported by Lagrangians \hbar -close to one another, and the corresponding off-diagonal terms are of the form

$$\langle \mathcal{M}_\alpha u_0, \mathcal{M}_{\alpha'} u_0 \rangle \approx e^{i(\theta_\alpha - \theta_{\alpha'})} w_\alpha w_{\alpha'}.$$

The phases θ_α , $\theta_{\alpha'}$ depend on the details of the trajectories along the paths α , α' , so it is tempting to believe that they are *pseudo-random*. If these phases were truly independent random variables, the sum of the off-diagonal elements would be of the same order as the sum of diagonal terms, and lead to a spectral bound $r_{sp} \leq e^{\mathcal{P}(-\varphi^+ + 2 \log |\alpha|)/2 + \tilde{\epsilon}}$.

8.2.1. *Phase cancellations in classical dynamics.* The pseudo-randomness of the phases θ_α , if true, seems very difficult to prove. In classical dynamics, *partial* phase cancellations were exhibited by Dolgopyat in his proof of exponential mixing for contact Anosov flows [23, 44]. In this situation the evolution operators $\mathcal{M}(\hbar)$ are replaced by Ruelle’s transfer operator associated with the projection T (along the stable manifold) of κ onto pieces of unstable manifold W^+ :

$$\mathcal{L}_s \psi(x) = \sum_{y: T(y)=x} e^{-s\tau(y)} \psi(y).$$

Here τ is the Poincaré return time (projected on W^+), and the parameter $s = s_0 + it$, where the imaginary part t should be compared with \hbar^{-1} . Dolgopyat shows that for t large enough a partial phase cancellation occurs in the sum for \mathcal{L}_s^n , leading to an shrinking of the spectral radius:

$$(59) \quad \exists \epsilon_0, t_0 > 0, \forall t \geq t_0, \quad r_{sp}(\mathcal{L}_{s_0+it}) \leq r_{sp}(\mathcal{L}_{s_0}) e^{-\epsilon_0}.$$

This improvement also requires to consider logarithmic times $n \sim C \log(t)$.

A similar improvement could be obtained in the case of the Laplacian on convex co-compact hyperbolic surfaces $X = \Gamma \backslash \mathbb{H}^2$ (see §3.2). Following [38], let us define the *essential spectral gap* representing the optimal resonance free strip at high frequency (as in §3.2 we use the spectral parameter s):

$$G(X) \stackrel{\text{def}}{=} \inf \{ \sigma < \delta, \{ \text{Re } s \geq \sigma \} \cap \text{Res}(\Delta_X) \text{ is finite} \}.$$

This quantity means that, for any small $\epsilon > 0$, there exist resonances of arbitrary high frequencies ($\text{Im } s_j$) such that $\text{Re } s_j \geq G(X) - \epsilon$. The “pressure” bound (20) means that $G(X) \leq \delta$.

Using the characterization of the resonances in terms of a classical Ruelle transfer operator \mathcal{L}_s , Naud [47] used the improved spectral bound (59) to show that the optimal gap could also be improved:

$$\exists \epsilon_1 > 0, \quad G(X) \leq \delta - \epsilon_1.$$

Using this resonance free strip, Guillarmou and Naud showed a long time expansion for the wave group on X with an exponentially small remainder [31].

In [38], Jakobson and Naud further investigated the location of resonances for certain *arithmetic* convex co-compact surfaces, in both cases of “thick” ($\delta > 1/2$) and “thin” ($\delta \in (0, 1/2)$) trapped sets. Analyzing Selberg’s zeta function, they managed to prove *lower bounds* for the essential spectral gap:

$$(60) \quad \text{”thick” } K: \quad G(X) \geq \frac{\delta}{2} - \frac{1}{4}, \quad \text{”thin” } K: \quad G(X) \geq \frac{\delta(1-2\delta)}{2}.$$

In this work they also conjecture that in both cases, the spectral gap is given by

$$(61) \quad G(X) = \frac{\delta}{2}.$$

This conjecture is equivalent with the value $\mathcal{P}(-\varphi^+)/2 = -\gamma_{cl}/2$ appearing in (58). Therefore, the conjecture (61) amounts to assume that the phases appearing in the off-diagonal terms of $\|\mathcal{L}_s^n \psi\|$ cancel each other sufficiently, to make their sum subdominant w.r.to the diagonal terms (or at least of the same order of magnitude).

This conjecture is inspired by the case of arithmetic surfaces of *finite* volume (e.g. the modular group $\Gamma = SL(2, \mathbb{Z})$), for which the high frequency resonances are actually eigenvalues embedded in the absolute spectrum, with $\text{Re } s_j = 1/2 = \delta/2$.

A Dolgopyat type estimate (59) was also shown by Stoyanov in the case of classical scattering by J convex obstacles in two dimensions [77] or higher dimensions (with some conditions) [78] and also for more general Axiom A flows [79]. In the case of scattering by convex obstacles, there is no exact connection between the zeros of the semiclassical (Selberg-type) zeta function and the quantum resonances. Yet, Petkov and Stoyanov [61] were able to compare the long time quantum evolution $\mathcal{M}(k)^n u_0$ for some initial Lagrangian

state u_0 , with a (modified) evolution of u_0 through a *classical* transfer operator of the form \mathcal{L}_s , $s = ik$. This connection allowed them to use the improved spectral gap for \mathcal{L}_s , $|\operatorname{Im} s| \gg 1$, to (effectively) get a smaller spectral radius for $\mathcal{M}(k)$ than predicted in (47), hence a wider resonance free strip than predicted in Thm 1.

It is very likely that this (slight) improvement on the resonance gap can be extended to the case of semiclassical Schrödinger operators (18) with an hyperbolic repeller; the main difficulty probably resides in checking that the conditions for a Dolgopyat estimate to hold (for the classical transfer operator) are met.

How large could be the resonance gap for such obstacle problems? Could it be as large as $\gamma_{cl}/2$, as conjectured above for hyperbolic surfaces? As noticed in §6, the numerics performed for the obstacle by 3 disks [45] seems to indicate a peak in the resonance density near $\operatorname{Im} k = -\gamma_{cl}/2$, which would imply that the resonance gap is smaller than $\gamma_{cl}/2$.

8.2.2. Some numerics for the open baker's map. Most numerical studies on open quantum maps were focussing on the spectral density and the fractal Weyl law, not on the spectral radii of the maps. We have mentioned in §6.1 that the numerics relative to several open maps show a mild peak in the radial spectral distribution near a value $r_{peak} \approx e^{-\gamma_{cl}/2}$, “pushes” the spectral radius to a strictly larger value. Below we provide some numerical results for the open baker's map (see §6.1.1).

We only consider symmetric baker's map with D symbols (that is the map (40) with $x_i = i/D$), so that the unstable Jacobian $\varphi^+ \equiv \log D$. We let the hole consist in the union of n Markov rectangles ($0 < n < D$). The topological pressure then is given by

$$\mathcal{P}(-s\varphi^+) = \log n - s \log D, \quad s \in \mathbb{R}.$$

In particular, this pressure does not depend on *which* of n rectangles are removed, but only on their number. The baker's map is discontinuous along the boundaries of the rectangles, and these discontinuities are believed to induce diffraction effects at the quantum level; for this reason, the open baker's map does not satisfy the assumptions of Thm 9. Yet, if the leftmost and rightmost rectangles both belong to the hole, the trapped set \mathcal{K} is at finite distance from the discontinuities, so it is reasonable to expect that the quantum spectrum should not be too sensitive to the boundaries. We have numerically computed the spectral radii of several open quantum baker's maps (see Fig. 14). The quantum dimension $N = (2\pi\hbar)^{-1}$ was taken in a range $10 \lesssim N \lesssim 5000$. The first case (top left) is a baker with $D = 5$ symbols, where the hole consists in the rectangles of indices $i = 0, 2, 4$, so that the trapped set \mathcal{K} is away from the boundaries. The spectral radii $r(N)$ seem to satisfy

$$e^{\mathcal{P}(-\varphi^+)/2} + \epsilon_1 < r(N) < e^{\mathcal{P}(-1/2\varphi^+)} - \epsilon_2,$$

for some $\epsilon_i > 0$, but keeps fluctuating for large N . Parity of the modes does not seem to play an important rôle.

On the opposite, in the case of the $D = 5$ baker with kept rectangles $i = 0, 4$, the trapped set contains the axes $x = 0$, $\xi = 0$, where the map is discontinuous. The spectral radii

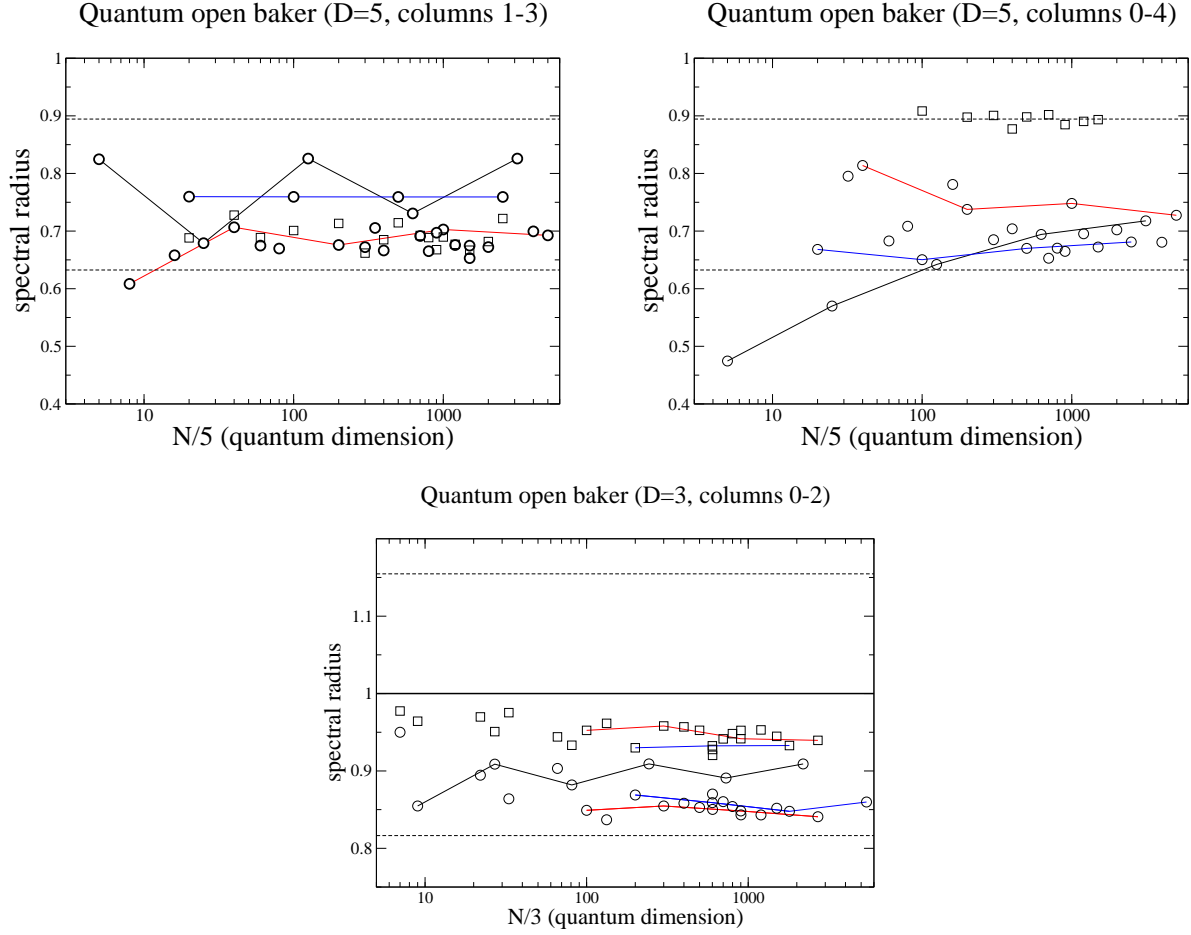


FIGURE 14. Spectral radii of various quantum symmetric open baker's maps, with $n = 2$ kept rectangles. $N = (2\pi\hbar)^{-1}$ is the quantum dimension. Eigenmodes are split according to parity: even (\circ) vs. odd (\square). Straight lines emphasize geometric series $N = N_0 D^k$, $k = 0, 1, 2, \dots$. Top left: $D = 5$, kept rectangles $i = 1, 3$. Top right: $D = 5$, kept rectangles $i = 0, 4$. Bottom: $D = 3$, kept rectangle $i = 0, 2$. The dashed horizontal lines indicate the values $e^{\mathcal{P}(-1/2\varphi^+)} > e^{\mathcal{P}(-\varphi^+)/2}$.

of the even-parity sector satisfy the same bound as above (for large enough N), but even for N large the odd parity spectral radius $r_{\text{odd}}(N) \approx e^{\mathcal{P}(-1/2\varphi^+)}$, thus barely violating the bound of Thm 9. The reason why the spectral radius for the odd sector is larger than for the even one is unclear.

In the case $D = 3$ (bottom), the trapped set also touches the discontinuities; the value $e^{\mathcal{P}(-1/2\varphi^+)} > 1$ is irrelevant due to the unitarity bound. As above, odd states show larger

eigenvalues than even ones. The radii seem to satisfy

$$e^{-\mathcal{P}(-\varphi^+)/2} + \epsilon_1 < r(N) < 1 - \epsilon_2, \quad \epsilon_i > 0 \text{ fixed},$$

indicating a gap ϵ_2 below the unitarity bound.

On the figures we follow some geometric series $N = N_0 D^k$, $k = 0, 1, 2, \dots$, because such series were shown important when studying the fractal Weyl law [53, 54]. The spectral radii along such series indeed show some regularity, especially when taking the parity of k into account.

In the case of the Walsh quantization of the symmetric D -baker's map presented in §6.1.2, the spectral radius of M_N (for $N = D^k$, $k \in \mathbb{N}$) is given by the largest eigenvalue λ of the $n \times n$ matrix $\tilde{\Omega}_D$, obtained by removing from the D -discrete Fourier transform the $D - n$ columns and lines corresponding to the hole. As a result, the spectral radius of M_N is the same for all quantum dimension $N = D^k$. The pressure bound

$$r_{sp}(M_N) \leq e^{\mathcal{P}(-1/2\varphi^+)} = \frac{n}{\sqrt{D}}$$

is then obvious, from the fact that all entries of $\tilde{\Omega}_D$ have modulus $\frac{1}{\sqrt{D}}$.

The various examples we have treated in [53, 54] show that the spectral radius is unrelated with the value $e^{-\mathcal{P}(-\varphi^+)/2} = \sqrt{\frac{n}{D}}$, it can range from 0 to $\min(1, n/\text{sqrt}D)$, depending on the explicit phases in the matrix $\tilde{\Omega}_D$. This situation is far from generic, and seems to depend on the fact that the underlying harmonic analysis is associated with the Walsh-Fourier transform, rather than the usual discrete Fourier transform.

9. PHASE SPACE STRUCTURE OF WAVEFUNCTIONS

In this last section we will investigate the structure of the “eigenfunctions” (in a generalized sense) of our scattering system introduced in §1.1. The first class of such eigenfunctions will be the metastable states $u_j(\hbar)$ associated with the (discrete) resonances $z_j(\hbar)$. They satisfy the differential equation $(P(\hbar) - z_j)u_j$, are purely outgoing and blow up exponentially at infinity.

On the opposite, the scattering functions form, for any real energy E , a complete set of functions satisfying $(P(\hbar) - E)u = 0$. They are not square-integrable either, but contain both incoming and outgoing components.

In both cases, we will focus on the structure of these functions in the interaction region, say the ball $B(0, R_0)$.

9.1. Structure of metastable states. Let $u_j(\hbar)$ be the metastable state associated with a resonance $z_j(\hbar)$ of our scattering Hamiltonian $P(\hbar)$. S “interaction region”, say $B(R_0) =$

$\{|x| \leq R_0\}$, We may (somewhat arbitrarily) normalize this state inside the interaction regions, by putting

$$\|u_j(\hbar)\|_{L^2(B(R_0))} = 1.$$

Alternatively, we may consider the corresponding eigenstates $u_{j,\theta}(\hbar)$ of the deformed operator $P_\theta(\hbar)$, which are globally square-integrable.

In order to connect oneself with the classical dynamics, it is natural to investigate the *phase space distribution* of the mode $u_j(\hbar)$. This can be done by using various semiclassical tools, like the Wigner (or Husimi) functions associated with the metastable states. The Wigner function $W_{u_j(\hbar)}(x, \xi)$ depends quadratically on u_j , depends explicitly on \hbar . The Husimi function $H_{u(\hbar)}$ is a Gaussian smoothing of $W_{u(\hbar)}$ on a scale $\sqrt{\hbar}$, which has the advantage to be positive. Both represent (in some sense) the phase space probability density of the particle in the state $u_j(\hbar)$.

It seems difficult to describe the individual functions $W_{u_j(\hbar)}$ for finite $\hbar > 0$. Instead, in the mathematical literature it has become customary to consider semiclassical sequences of eigenstates; in the present context, take any sequence $\hbar_k \rightarrow 0$, and for each $\hbar = \hbar_k$ choose some resonance $z(\hbar) = z_j(\hbar) \in D(E, C\hbar)$, the associated metastable state $u(\hbar) = u_j(\hbar)$ and construct its Wigner function $W_{u(\hbar)}$. We will then be interested in the behaviour of the sequence of Wigner functions $(W_{u(\hbar)})_{\hbar \rightarrow 0}$ in the semiclassical limit.

A result of Bony and Laurent [7, Thm 2.1] shows that, for a general trapping potential, the Wigner functions $W_{u(\hbar)}$ are semiclassically negligible away from the outgoing tail. This means that, for any test function $a \in C_c^\infty(T^*X)$ supported inside the energy layer $p^{-1}(E - \epsilon, E + \epsilon)$ but outside K_E^+ , one has

$$\int a(x, \xi) W_{u(\hbar)}(x, \xi) dx d\xi = \mathcal{O}(\hbar^\infty).$$

This result does not depend at all on the structure of the flow on K_E .

The same type of asymptotics was also obtained in the case of the open baker's map [40, 50], where some Husimi measures $H_{u(\hbar)}$ were numerically computed, and shown to concentrate on the outgoing tail \mathcal{K}^+ (see this set on Fig. 11, center, and compare with Fig. 15). In those papers, the objective was to get a more precise description of the *semiclassical measures* associated with the sequences $(H_{u(\hbar)})_{\hbar \rightarrow 0}$.

By definition, a semiclassical measure is a measure μ on phase space¹³, which is the limit (in the weak-* topology) of the sequence of distributions $(H_{u(\hbar)} dx d\xi)_{\hbar \in S}$, where S is some infinite subsequence of the original sequence ($\hbar_k \rightarrow 0$). This measure describes the asymptotic phase space behaviour of the metastable states along the sequence $(u(\hbar))_{\hbar \in S}$. A priori, several limit measures μ may be extracted from the original sequence, corresponding to different subsequences S . In the case of a closed chaotic system (say, the geodesic flow on a compact riemannian manifold of negative curvature), any semiclassical measure

¹³We will actually only consider such a measure on the interaction regions $T^*B(0, R_0)$.

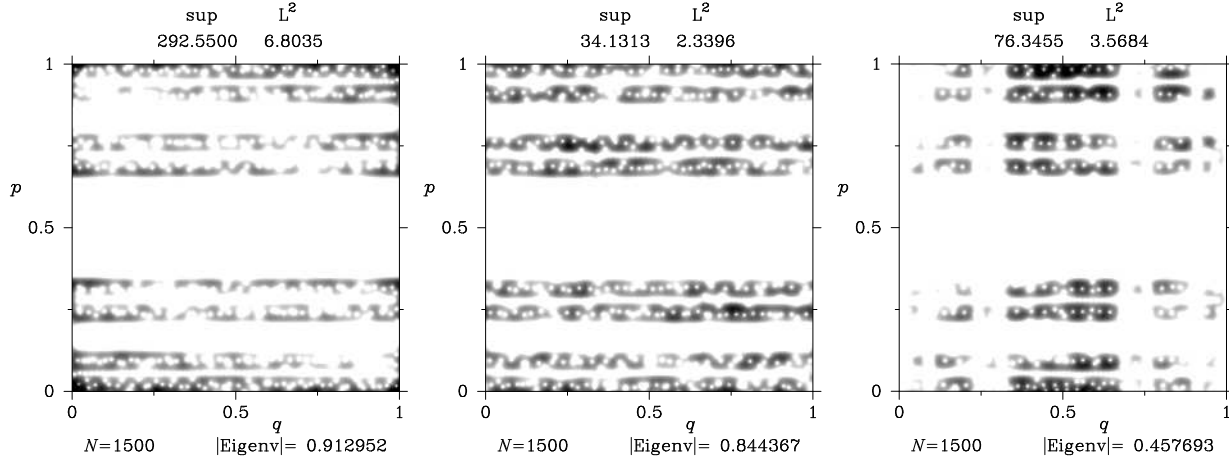


FIGURE 15. Husimi functions of three metastable states of the quantum symmetric open 3-baker (logarithmic grey scale). The high intensities (black) are clearly localized on \mathcal{K}^+ .

constructed from the sequence of Wigner distributions associated with the eigenstates of $P(\hbar) = -\hbar^2 \Delta_X / 2$ must be invariant w.r.to the classical flow (this fact is easily proved using Egorov's theorem). Furthermore, the quantum ergodicity theorem states that, as long as the flow (or the closed map) is ergodic w.r.to the Liouville measure, then one can extract a subsequence of density one¹⁴, such that $(W_{u(\hbar)} dx d\xi)_{\hbar \in S}$ converges to the Liouville measure on the energy shell $p^{-1}(1/2)$ [68, 88, 18].

In the frameworks of scattering by a potential [55] and open chaotic map [40, 50], an analogue of the invariance of semiclassical measures could be proved by similar means.

Theorem 10. [55] *Consider a Hamiltonian (18) such that for some $E > 0$ the trapped set K_E is a hyperbolic repeller. Take a sequence of resonances $(z(\hbar) \in D(E, C\hbar))_{\hbar \rightarrow 0}$, and extract a subsequence $(W_{u(\hbar)})_{\hbar \in S}$ converging to a measure μ on $T^*B(0, R_0)$. Then the limit measure μ will automatically be invariant up to a decay rate $\Lambda \geq 0$:*

$$(62) \quad \forall t \geq 0, \quad \Phi^{t*} \mu = e^{-t\Lambda} \mu \quad \text{inside the interaction region } T^*B(0, R_0).$$

Furthermore, the subsequence S must be such that the resonances $(z(\hbar))_{\hbar \in S}$ satisfy

$$(63) \quad \lim_{\hbar \in S, \hbar \rightarrow 0} \frac{\text{Im } z(\hbar)}{\hbar} = -\frac{\Lambda}{2}.$$

In other words, the quantum decay rates along the sequence S must converge to the (classical) decay rate of μ .

¹⁴This means that this sequence contains “almost all” the eigenstates

In the case of an open map $\kappa : V \mapsto \kappa(V)$, a Λ -eigenmeasure is a probability measure on $\kappa(V)$ satisfying

$$(64) \quad \kappa^* \mu = e^{-\Lambda} \mu.$$

For $\Lambda \geq 0$, any measure satisfying the decay property (62), resp. (64), is called a Λ -eigenmeasure of the flow, resp. the open map. It is easy to classify all Λ -eigenmeasures. In the case of maps, each such μ is uniquely specified by its restriction on $\mathcal{K}^+ \cap \kappa(V) \setminus V$, which can be chosen arbitrarily. All such measures are supported on K^+ , and satisfy $\mu(\mathcal{K}) = 0$ for $\Lambda > 0$, while they are supported on \mathcal{K} iff $\Lambda = 0$. In view of the quantum ergodicity result for chaotic closed systems, the following question naturally arises:

Given $\Lambda \geq 0$, and considering a sequence of resonances $(z(\hbar))_{\hbar \in S}$ satisfying (63), *which* Λ -eigenmeasures are obtained as semiclassical limits of $(W_{u(\hbar)})_{\hbar \in S}$? Is there a "favoured" limit, or even a unique one?

Such question presumes that there exist sequences of resonances satisfying (63), a fact which depends on the semiclassical distribution of resonances; in case the strong form (38) of fractal Weyl law holds, then such a sequence certainly exists if the profile function satisfies $\frac{dF}{d\gamma}(\Lambda/2) > 0$.

From various numerics, we have noticed before that the density of resonances often shows a (mild) peak near the value $\Lambda = \gamma_{cl}$. For this specific value of Λ , there exists a "natural" Λ -measure, which is obtained by iterating an initial smooth measure μ_0 (with support intersecting \mathcal{K}_-):

$$\mu_{nat} = \lim_{t \rightarrow \infty} \mathcal{N}_t \Phi^{t*} \mu_0, \quad \text{respectively} \quad \mu_{nat} = \lim_{n \rightarrow \infty} \mathcal{N}_n \kappa^{n*} \mu_0,$$

with $\mathcal{N}_t, \mathcal{N}_n$ normalization factors. Yet, the study of [50] did not reveal that this measure played a particular rôle for the open quantum baker's map.

In [40], the authors computed averages of the spatial densities $|u_j(x)|^2$ over a few eigenstates with comparable decay rates, for the symmetric open 3-baker. They noticed strong self-similar properties of the densities, depending on the decay rates. Some of the individual Husimi functions of [50] were also hinting at a selfsimilar behaviour in both the momentum and position directions.

Analytical results on semiclassical measures were obtained in the case of the Walsh-quantized open baker [39, 50], using explicit formulas for the eigenstates. In this case most eigenvalues $\lambda_j(\hbar)$ have large multiplicities, leaving a lot of freedom to construct eigenstates. In [50, 39] we showed that, for the Walsh-quantized symmetric 3-baker, any semiclassical sequence of eigenstates $(u(\hbar))_{\hbar \rightarrow 0}$ with eigenvalues converging towards the outer circle $|\lambda(\hbar)| \rightarrow r_{\max}$ (resp. the inner circle $|\lambda(\hbar)| \rightarrow r_{\min}$) of the nontrivial spectrum, is associated with a single semiclassical measure μ_{\max} (resp. μ_{\min}), which is of Bernoulli type, therefore perfectly selfsimilar. This is a form of "quantum unique ergodicity" at the edges of the nontrivial spectrum. On the opposite, for any value $r \in (r_{\min}, r_{\max})$, we exhibited

many semiclassical measures associated with sequences $(u(\hbar))_{\hbar \rightarrow 0}$ of asymptotic decay rates $|\lambda(\hbar)| \rightarrow r$. For $r = e^{-\gamma_{cl}/2}$ we showed that the natural measure μ_{nat} is not a semiclassical measure for this map.

9.2. Distribution of scattering states. Metastable states appear in expansions for the resolvent of $P(\hbar)$, and consequently in expansions for the time dynamics [82, 13]. Another class of generalized eigenstates is more natural from the point of view of scattering theory, namely the scattering states, used to define the scattering matrix. In the semiclassical setting of an operator $P(\hbar)$ on a manifold X , a scattering state at energy $E > 0$ is a wavefunction $u_E(x)$ satisfying the differential equation $(P(\hbar) - E)u_E = 0$, with some conditions at infinity. These conditions depend on the structure of X at infinity.

If $X \equiv \mathbb{R}^d$ outside a ball $B(0, R_0)$, one can expand $u_E(x)$ using a basis of incoming and outgoing waves, as in (3,(4)). Fixing the incoming part of $u_{E,in}$ near infinity uniquely determines the full wavefunction u_E , and in particular determines its outgoing part where the relation between $u_{E,in}$ and $u_{E,out}$ is given by the scattering matrix $S(E)$.

Given $u_{E,in}(x)$ near infinity, what is the spatial (or phase space) structure of $u_E(x)$ inside the interaction region?

In the semiclassical/high-frequency limit, the usual basis states for the incoming wave $u_{E,in}$ (e.g. the angular momentum eigenstates) are locally Lagrangian states, associated with a certain Lagrangian submanifold of the energy shell (for instance, a spherically symmetric incoming wave sits on the Lagrangian submanifold $\{x, \xi = -\sqrt{2E}x/|x|\}$). Most of the incoming trajectories will be scattered inside the interaction region and then expelled back towards infinity after a short, transient time. Still, a small fraction of the incoming trajectories may be trapped during a long time in this region along trajectories very close to K_E , or even trapped for ever if they arrive exactly along the incoming tail K_E^- . One may wonder how these trapped (or long transient) trajectories influence the structure of u_E . This question has been studied numerically by Ishio and Keating [37] in a different geometry, namely the case of a 2d cavity opened by two infinite ‘‘leads’’ (or waveguides). In this case, the structure of $u_E(x)$ at infinity is given by propagating waves inside each lead, of the form $u_E(x, y) = \sin(k_n y)e^{ik_l x}$, where the longitudinal and transverse wavevectors satisfy

$$k_n = \pi n/L, \quad E = \frac{\hbar^2}{2}(k_l^2 + k_n^2), \quad n \in \mathbb{N} \setminus 0, \quad L \text{ the width of the lead.}$$

This incoming wave semiclassically corresponds to a pencil of trajectories coming out of the lead with an angle $\pm\theta_n$, $\theta_n = \arcsin(k_n/k_l)$. Two scattering states were numerically computed in [37]; in both cases, the density $|u_E(x)|^2$ is strongly imprinted by relatively short scattering orbits. The authors also derived approximative semiclassical expression for $u_E(x)$, as a sum over classical trajectories, and showed that this expression is quite accurate for the two examples of states they have computed. In the course of the derivation, they distinguished between two complementary situations: for *almost closed* situations,

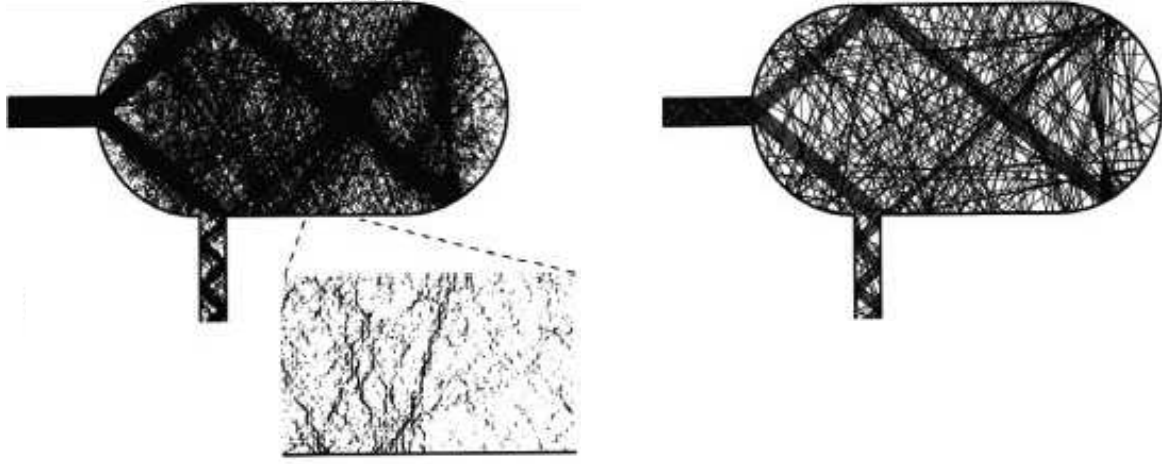


FIGURE 16. Left: density plot of a scattering state $|u_E(x)|^2$, incoming in the left lead into the stadium billiard. Right: pencil of classical trajectories issued from the left opening with angles $\pm\theta_n$. Reprinted with permission from H. Ishio and J.P. Keating, *Semiclassical wavefunctions in chaotic scattering systems*, J. Phys. A **37** (2004), 015401. Copyright 2004 by the Institute of Physics.

the contributions of long trajectories is large, and can be reduced only by averaging in energy; on the opposite, for *very open systems*, the contribution of long trajectories decays exponentially fast, so that the wavefunction is mainly influenced by short the trajectories. This dichotomy is of course reminiscent of the one mentioned in §1.2.

More recently, Guillarmou and Naud [32] studied the scattering states for convex co-compact manifolds $X = \Gamma \backslash \mathbb{H}^{n+1}$, also called the Eisenstein functions in this context. The high frequency functions are associated with spectral parameters $s = n/2 + it$, $t \gg 1$. A convenient basis for these functions consists in functions $u_{s,y}(x)$ which become singular when x converges to a given point y of the boundary ∂X ; semiclassically, the incoming wave is supported by the unstable manifold of the geodesics issued from y . In such a homogeneous situation, the wavefunction $u_{s,y}(x)$ can be simply expressed by a sum over the group Γ . The authors are able to precisely describe the functions $u_{s,y}$ provided the trapped set is “thin”, that is the dimension of the limit set satisfies $\delta < n/2$ (or equivalently, $\mathcal{P}(-\varphi^+/2) < 0$). This is now a precise criterium for the “very open” situation described in [37].

One can then explicitly compute the semilassical measure associated with the family $(u_{n/2+it,y})_{t \rightarrow \infty}$: it is an invariant measure supported by the full unstable manifold of y ; above any point x , this manifold is given by an infinite set of unstable leaves. Their result relies on the fact that the sum over the many unstable leaves converges exponentially fast (due to the negative pressure), so that the nonnegligible leaves do not interfere with one another.

They also show that, when averaging the densities $|u_{s,y}(x)|^2$ over the point y , one recovers the uniform (Haar) measure on X , plus a semiclassically small correction given by a sum over periodic orbits, similarly with Gutzwiller's trace formula for closed systems (one difference being that the sum over the orbits is absolutely convergent).

This description of scattering states can certainly be extended to more general geometries or systems with a “thin” hyperbolic trapped set.

10. CONCLUSION

We have presented several analytical methods used to analyze the spectral properties of scattering operators in the semiclassical/high frequency limit, in cases where the set of classically trapped trajectories is a hyperbolic repeller. In particular, the number of long living metastable states near some classical energy $E > 0$ was bounded from above by a fractal power of the semiclassical parameter, reflecting the fact that these long living states must be supported on the trapped set, which is a fractal subset of the energy shell. We stated two types of “fractal Weyl law” conjectures under which this upper bound would be sharp, and presented some numerical results in favor of these laws, both for scattering flows and for the model of open quantum maps.

A second type of result is the presence of a “resonance gap” (or a uniform lower bound on the quantum decay rates), provided the instability of the flow exceeds its complexity (precisely, provided the topological pressure $\mathcal{P}(-\varphi^+/2)$ is negative). This criterium allows to split such chaotic scattering systems into “very open” vs. “mildly open” systems. We showed that this dichotomy was also relevant in the precise description of scattering wavefunctions.

At the technical level, we presented an auxiliary family of quantum monodromy operators which can be used to investigate this spectral problem; these operators, which contain the full long living quantum dynamics, can be reduced to live in a “minimal” neighbourhood of the trapped set. In a (somewhat pompous) sense, they embody the quantum mechanics on the trapped set. These operators are connected with Ruelle transfer operators appearing in classical dynamics; one can hope that a more precise analysis of these operators could deliver some nontrivial information on the resonance spectrum, like a proof of the fractal Weyl law (under some genericity assumption) or a sharper criterium for a resonance free strip.

In the above study the resonances were analyzed as eigenvalues of a nonselfadjoint pseudodifferential operator. The techniques presented above can also be used in a different context, namely the study of a “closed” quantized chaotic system, like damped waves propagating on a manifold of negative curvature. In that case the “escape to infinity” is replaced by a continuous damping, but the spectral problem associated with the wave semigroup at high frequency is quite similar [73]. Using similar techniques as above, fractal Weyl upper bounds were obtained for such systems [3], and a pressure criterium for the

presence of a semiclassical spectral gap was also proved [66, 67], with applications to the stabilization (energy decay) of the damped wave system.

The same type of ideas are also relevant when describing the scattering by a *dielectric cavity*, relevant in the description of quasi-2D microlasers (see e.g. [86, 71, 6] and references therein). In such situations, the damping is due to the fact that a wavepacket propagating inside the cavity loses a fraction of its energy when being reflected through the boundary of the cavity (the rest of the energy being refracted outside the cavity and escaping to infinity). The mathematical study of such cavities is more subtle than in the case of boundaryless manifolds; to my knowledge, has been so far restricted so to strictly convex cavities with smooth boundaries [14], for which the ray dynamics cannot be purely chaotic.

APPENDIX A. A BRIEF REVIEW OF \hbar -PSEUDODIFFERENTIAL CALCULUS

In this Appendix we recall some definition and basic properties of Weyl's quantization, in the semiclassical setting. For simplicity, we will only consider operators on the Euclidean space \mathbb{R}^d . See [22, Chapter 7] for a detailed discussion of semiclassical quantization, and [26, Appendix D.2] for the pseudodifferential calculus for the symbol classes presented below, and its implementation on manifolds.

A.1. Weyl quantization and pseudodifferential calculus. Weyl's quantization associates to a smooth function on phase space $a(x, \xi) \in C^\infty(T^*\mathbb{R}^d)$ (the *classical observable*, or *symbol*) an operator acting on $u \in C_c^\infty(\mathbb{R}^d)$ as follows:

$$(65) \quad \begin{aligned} a^w u(x) &= [\text{Op}_\hbar(a)u](x) \\ &\stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^d} \int \int a\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y, \xi \rangle / \hbar} u(y) dy d\xi, \end{aligned}$$

In these notations, $\hbar \in (0, 1]$ is Planck's "constant" (which is a small parameter). The above integral converges absolutely only if $a(x, \xi)$ decays fast enough w.r.to ξ , but by integrating by parts one can easily extend the definition to functions growing algebraically in ξ . The classes of symbols presented below will ensure that the above formula makes sense.

The advantage of taking \hbar small is due to the following fact: the product of two operators $\text{Op}_\hbar(a) \text{Op}_\hbar(b)$ can be analyzed by studying the symbols a, b . More precisely, that product is itself an operator of the form $\text{Op}_\hbar(c)$, with a symbol $c(x, \xi)$ given by the Moyal product of a and b :

$$(66) \quad \begin{aligned} c &= a \#_\hbar b \stackrel{\text{def}}{=} a \exp\left(\frac{i\hbar}{2}(\langle \overleftarrow{D}_\xi, \overrightarrow{D}_x \rangle - \langle \overleftarrow{D}_x, \overrightarrow{D}_\xi \rangle)\right) b \\ &\sim ab + \frac{i\hbar}{2}\{a, b\} + \sum_{j \geq 2} \frac{(i\hbar/2)^j}{j!} a \left(\langle \overleftarrow{D}_\xi, \overrightarrow{D}_x \rangle - \langle \overleftarrow{D}_x, \overrightarrow{D}_\xi \rangle\right)^j b, \end{aligned}$$

where $D_\bullet = -i\partial_\bullet$, and $\{a, b\}$ is the Poisson bracket. The above expansion is a good asymptotic expansion, in the sense that the sum up to \hbar^{N-1} gives a good approximation of the symbol c up to a remainder $\mathcal{O}(\hbar^N)$. It is at the heart of *pseudodifferential calculus*.

Notice that, even if a, b are independent of \hbar , the symbol $c = a \sharp_\hbar b$ does depend on \hbar . It thus makes sense to define classes of \hbar -dependent symbols, such that the above makes sense. These classes are characterized by the regularity property of $a(x, \xi; \hbar)$, uniformly in the limit $\hbar \rightarrow 0$.

One standard class of symbols is the following: for $k \in \mathbb{R}$, let

$$(67) \quad S^k(T^*\mathbb{R}^d) = \left\{ a \in C^\infty(T^*\mathbb{R}_{x,\xi}^d \times (0, 1]_\hbar) : \forall \alpha, \beta \in \mathbb{N}^d, \right. \\ \left. |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar)| \leq C_\alpha \langle \xi \rangle^{k-|\beta|} \right\},$$

where we use the standard notation $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. The improved decay in ξ upon differentiation is necessary for the class to be invariant upon a smooth change of coordinates; this invariance is crucial when considering symbols on manifolds. The corresponding operators $\text{Op}_\hbar(a)$ belong to operator classes denoted by $\Psi^k(\mathbb{R}^d)$. For instance, the Schrödinger operator (18) is the Weyl quantization of the symbol $p(x, \xi) = \frac{|\xi|^2}{2} + V(x) \in S^2(T^*\mathbb{R}^d)$.

These symbol classes are closed under composition: for $a \in S^k$, $b \in S^\ell$, the product operator $\text{Op}_\hbar(a) \text{Op}_\hbar(b) = \text{Op}_\hbar(c)$ belongs to $\Psi^{k+\ell}$. Another important property of these classes is their action on $L^2(\mathbb{R}^d)$. For $a \in S^0(T^*\mathbb{R}^d)$, the operators $\text{Op}_\hbar(a)$ are bounded on $L^2(\mathbb{R}^d)$, with norms

$$(68) \quad \|\text{Op}_\hbar(a)\|_{L^2 \rightarrow L^2} = \|a(\hbar)\|_{L^\infty} + \mathcal{O}(\hbar).$$

Also, if $a(x, \xi)$ is real valued, $\text{Op}_\hbar(a)$ will be self-adjoint on L^2 . In this case, one can also analyze functions of $\text{Op}_\hbar(a)$ using their symbols: for a smooth functions $f : \mathbb{R} \rightarrow \mathbb{R}$, the operator $f(\text{Op}_\hbar(a))$ belongs to $\Psi^0(\mathbb{R}^d)$, with symbol $f(a) + \mathcal{O}(\hbar)$. For instance, in §4.2 we consider an escape function $G \in C_c^\infty(T^*\mathbb{R}^d)$, and consider the operators $e^{\pm tG^w}$. These still belong to $\Psi^0(\mathbb{R}^d)$, and the composition rule (66) shows that the conjugated operator $e^{-tG^w} P_\theta(\hbar) e^{tG^w}$ belongs to $\Psi^2(\mathbb{R}^d)$, with a symbol of the form (24).

A.2. Exotic symbol classes. The symbols $a \in S^0(T^*\mathbb{R}^d)$ fluctuate on distances ~ 1 . For our aims, it is also necessary to consider more singular families of symbols, made of functions fluctuating on microscopic distances. For $k \in \mathbb{R}$ and $\delta \in [0, 1/2]$, let us consider the “exotic” symbol classes

$$(69) \quad S_\delta^k(T^*\mathbb{R}^d) = \left\{ a \in C^\infty(T^*\mathbb{R}^d \times (0, 1]_\hbar) : \forall \alpha, \beta \in \mathbb{N}^d, \right. \\ \left. |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar)| \leq C_\alpha \hbar^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{k-|\beta|} \right\},$$

This class contains $S^k(T^*\mathbb{R}^d)$, but it also encompasses symbols fluctuating on distances $\sim \hbar^\delta$. For $\delta < 1/2$, the expansion (66) still makes sense, and we can still use the symbol to

analyze the operators. The action of $\text{Op}_\hbar(a) \in \Psi_\delta^0(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$ satisfies

$$(70) \quad \|\text{Op}_\hbar(a)\|_{L^2 \rightarrow L^2} = \|a(\hbar)\|_{L^\infty} + \mathcal{O}(\hbar^{1-2\delta}).$$

These exotic classes are used when constructing the exponential weights of §4.2. If we take $\varepsilon = \hbar^\delta$, the escape function $G(x, \xi)$ belongs to the class $\log(1/\hbar)S_\delta^\varepsilon(T^*X)$ (see the model $G_1(x, \xi)$ of (28)), and the corresponding functional calculus allows to analyze the operators $e^{\pm tG_1^w}$ and $e^{-tG^w}P(\hbar)e^{tG^w} \in \Psi_\delta^2(\mathbb{R}^d)$.

A.3. Fourier integral operators. A real valued, time dependent Hamiltonian $p(t, x, \xi) \in C([0, 1]_t, S^2(T^*\mathbb{R}^d))$ generates a symplectic flow $(\kappa_t)_{t \in [0, 1]}$ through Hamilton's equations

$$\frac{d\kappa_t}{dt} = (\kappa_t)_* H_{p(t)}, \quad \kappa_0 = Id, \quad t \in [0, 1].$$

Then, the family of unitary operators $U(t)$ defined by

$$i\hbar\partial_t U(t) = U(t)p^w(t), \quad U(0) = Id,$$

defines a time-dependent family of quantum propagators, which are unitary Fourier Integral Operators associated with the diffeomorphisms κ_t . These operators can be expressed as oscillatory integrals, with phases given by the generating actions for κ_t . The quantum maps

Consider the propagator $U = U(1)$ associated with $\kappa = \kappa(1)$. U maps a wavepacket microlocalized at (x_0, ξ_0) to a wavepacket localized at $\kappa(x_0, \xi_0)$. Its action on a quantum observable satisfies a quantum-classical correspondence (called Egorov's theorem in the mathematical literature). Namely, for any symbol $a \in S_\delta^0(\mathbb{R}^d)$ of compact support, one has

$$(71) \quad U^{-1} \text{Op}_\hbar(a) U = \text{Op}_\hbar(b), \quad b \in S_\delta^0, \quad b = a \circ \kappa + \hbar^{1-\delta} S_\delta^0(T^*\mathbb{R}^d).$$

More generally, an FIO associated with κ will be an operator of the form

$$(72) \quad \mathcal{M}(\alpha, \hbar) = U \text{Op}_\hbar(\alpha),$$

with $\alpha \in S_\delta^0(T^*\mathbb{R}^d)$ for some $\delta \in [0, 1/2)$. The open quantum maps of §5.2 are of this type. One easily shows that the above Egorov property, together with pseudodifferential calculus, leads to the “nonunitary” Egorov property (29). The L^2 norm estimate (31) is obtained from (70).

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