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► **To cite this version:**

Eric Amar. A subordination Principle. Applications to Carleson measures and interpolating sequences in convex domains of finite type.. 2011. hal-00591845v1

**HAL Id: hal-00591845**

**<https://hal.science/hal-00591845v1>**

Preprint submitted on 10 May 2011 (v1), last revised 14 Jan 2014 (v4)

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# A subordination principle. Applications to Carleson measures and interpolating sequences in convex domains of finite type in $\mathbb{C}^n$ .

Eric Amar

May 10, 2011

## Abstract

The subordination principle states roughly: if a property is true for Hardy spaces in some kind of domains in  $\mathbb{C}^n$  then it is also true for the Bergman spaces of the same kind of domains in  $\mathbb{C}^{n-1}$ .

We give applications of this principle to Bergman-Carleson measures, interpolating sequences for Bergman spaces,  $A^p$  Corona theorem and characterization of the zeros set of Bergman-Nevanlinna class.

## 1 Introduction.

The subordination principle states roughly: if a property is true for Hardy spaces in some kind of domains in  $\mathbb{C}^n$ , then it is also true for the Bergman spaces of the same kind of domains in  $\mathbb{C}^{n-1}$ .

Let us start with some definitions. In all the sequel, domain will mean bounded domain in  $\mathbb{C}^n$  with smooth  $\mathcal{C}^\infty$  boundary defined by a real valued function  $r \in \mathcal{C}^\infty(\mathbb{C}^n)$ ,

i.e.  $\Omega = \{z \in \mathbb{C}^n :: r(z) < 0\}$ ,  $\forall z \in \partial\Omega$ ,  $\partial r(z) \neq 0$ .

Associate to it the "lifted" domain  $\tilde{\Omega}$  in  $(z, w) \in \mathbb{C}^{n+k}$  with defining function

$$\tilde{r}(z, w) := r(z) + |w|^2.$$

This operation keeps the nature of the domain :

- if  $\Omega$  is pseudo-convex,  $\tilde{\Omega}$  is still pseudo-convex ;
- if  $\Omega$  is strictly pseudo-convex, so is  $\tilde{\Omega}$  ;
- if  $\Omega$  is convex, so is  $\tilde{\Omega}$  ;
- if  $\Omega$  is of finite type  $m$ , so is  $\tilde{\Omega}$ .

Let  $dm(z)$  be the Lebesgue measure in  $\mathbb{C}^n$  and  $d\sigma(z)$  be the Lebesgue measure on  $\partial\Omega$ .

For  $z \in \Omega$ , let  $\delta(z) := d(z, \Omega^c)$  be the distance from  $z$  to the boundary of  $\Omega$ .

Define the Bergman, Hardy and Nevanlinna spaces as usual:

**Definition 1.1** Let  $f$  be a holomorphic function in  $\Omega$  ; we say that  $f \in A_k^p(\Omega)$  if

$$\|f\|_{k,p}^p := \int_{\Omega} |f(z)|^p \delta(z)^k dm(z) < \infty.$$

We say that  $f \in \mathcal{N}_k(\Omega)$  if

$$\|f\|_{\mathcal{N}_k} = \int_{\Omega} \log^+ |f(z)| \delta(z)^k dm(z) < \infty,$$

We say that  $f \in H^p(\Omega)$  if

$$\|f\|_p^p := \sup_{\epsilon > 0} \int_{\{r(z) = -\epsilon\}} |f(z)|^p d\sigma(z) < \infty.$$

Finally we say that  $f \in \mathcal{N}(\Omega)$  if

$$\|f\|_{\mathcal{N}} = \sup_{\epsilon > 0} \int_{\{r(z) = -\epsilon\}} \log^+ |f(z)| d\sigma(z) < \infty.$$

This is meaningful because, for  $\epsilon$  small enough, the set  $\{r(z) = -\epsilon\}$  is a smooth manifold in  $\Omega$ .

Now we can state our subordination lemma [3]:

**Lemma 1.2** (Subordination lemma) Let  $F(z, w) \in H^p(\tilde{\Omega})$  we have  $f(z) := F(z, 0) \in A_{k-1}^p(\Omega)$  and  $\|f\|_{A_{k-1}^p(\Omega)} \lesssim \|F\|_{H^p(\tilde{\Omega})}$  ;

if  $F(z, w) \in \mathcal{N}(\tilde{\Omega})$ , then  $f(z) := F(z, 0) \in \mathcal{N}_{k-1}(\Omega)$  and  $\|f\|_{\mathcal{N}_{k-1}(\Omega)} \lesssim \|F\|_{\mathcal{N}(\tilde{\Omega})}$ .

A function  $f$  holomorphic in  $\Omega$  is in the Bergman space  $A_{k-1}^p(\Omega)$  (resp. in the Nevanlinna Bergman space  $\mathcal{N}_{k-1}(\Omega)$ ) if and only if the function  $F(z, w) := f(z)$  is in the Hardy space  $H^p(\tilde{\Omega})$  (resp. in the Nevanlinna class  $\mathcal{N}(\tilde{\Omega})$ ) and we have  $\|f\|_{A_{k-1}^p(\Omega)} \simeq \|F\|_{H^p(\tilde{\Omega})}$  (resp.  $\|f\|_{\mathcal{N}_{k-1}(\Omega)} \simeq \|F\|_{\mathcal{N}(\tilde{\Omega})}$ ).

In the section 2 we prove the subordination lemma as a consequence of a disintegration of Lebesgue measure.

In the section 3 we introduce the notion of a "good" family of polydiscs, directly inspired by the work of Catlin [12] and introduced in [5]. This notion allows us to define geometric Carleson measure, denoted as  $\Lambda^1(\Omega)$ , for Hardy spaces and denoted as  $\Lambda_k^1(\Omega)$ , for Bergman spaces and to put it in relation with the Carleson embedding theorem still for these two classes of spaces.

**Definition 1.3** We shall say that the domain  $\Omega$  has the  $p$ -Carleson embedding property,  $p$ -CEP, if

$$\forall \mu \in \Lambda^1(\Omega), \exists C = C_\mu > 0 :: \forall f \in H^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{H^p(\Omega)}^p.$$

And the same for the Bergman spaces.

**Definition 1.4** We shall say that the domain  $\Omega$  has the  $(p, k)$ -Bergman-Carleson embedding property,  $(p, k)$ -BCEP, if

$$\forall \mu \in \Lambda_k^1(\Omega), \exists C = C_\mu > 0 :: \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{A_{k-1}^p(\Omega)}^p.$$

Still in this subsection we define a homogeneous hypothesis, (Hg), on a good family  $\mathcal{P}$ .

In subsection 3.1 we apply the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding one.

This leads to the following theorems.

**Theorem 1.5** *Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\tilde{\Omega}$  its lift ; suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypotheses (Hg). If the lifted domain  $\tilde{\Omega}$  has the  $p$ -CEP then  $\Omega$  has the  $(p, k)$ -BCEP.*

Now let  $\Omega$  be a domain in  $\mathbb{C}^n$ . We say that the  $H^p$ -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then

$$\forall f \in H^p(\Omega), \exists (f_1, \dots, f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the same vein, we say that the  $A_{k-1}^p(\Omega)$ -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0 \quad (1.1)$$

then

$$\forall f \in A_{k-1}^p(\Omega), \exists (f_1, \dots, f_m) \in (A_{k-1}^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the section 5 we apply again the subordination principle to get:

**Corollary 1.6** *We have the  $A_k^p(\Omega)$ -Corona theorem in the following cases :*

- with  $p = 2$  if  $\Omega$  is a bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;
- with  $1 < p < \infty$  if  $\Omega$  is a bounded strictly pseudo-convex domain in  $\mathbb{C}^n$ .

In section 4.1 we define and study the interpolating sequences in a domain  $\Omega$ . We also define the notion of dual bounded sequences in  $H^p(\Omega)$  and in  $A_k^p(\Omega)$ , and applying the subordination principle to the result I proved for  $H^p(\Omega)$  interpolating sequences [5], we get

**Theorem 1.7** *If  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$  and if  $S \subset \Omega$  is a dual bounded sequence of points in  $A_k^p(\Omega)$  then, for any  $q < p$ ,  $S$  is  $A_k^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

Finally in the section 6 we study zeros set for Nevanlinna Bergman functions.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $u$  a holomorphic function in  $\Omega$ . Set  $X := \{z \in \Omega :: u(z) = 0\}$  the zero set of  $u$  and  $\Theta_X := \partial\bar{\partial} \log u$  its associated  $(1, 1)$  current of integration.

**Definition 1.8** *A holomorphic divisor  $X$  in the domain  $\Omega$  is in the Blaschke class,  $X \in \mathcal{B}(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \left| \int_{\Omega} (-r(z)) \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty, \quad (1.2)$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}(\Omega)$  then it is well known that  $X$  is in the Blaschke class of  $\Omega$ .

We do the analogue for the Bergman spaces :

**Definition 1.9** A holomorphic divisor  $X$  in the domain  $\Omega$  is in the Bergman-Blaschke class,  $X \in \mathcal{B}_k(\Omega)$ , if there is a constant  $C > 0$  such that

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\overline{\Omega}), \quad \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \partial \bar{\partial} r \wedge \beta \right| \leq C \|\beta\|_\infty, \quad (1.3)$$

where  $\Lambda_{n-1, n-1}^\infty(\overline{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\overline{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}_k(\Omega)$  then it is well known that  $X$  is in the Bergman-Blaschke class of  $\Omega$ .

Hence exactly as for the Corona theorem we can set the definitions :

we say that the *Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

And the same for the Bergman spaces :

we say that the *Bergman-Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

We get, by use of the subordination lemma applied to the corresponding Nevanlinna Hardy results,

**Corollary 1.10** *The Bergman-Blaschke characterization is true in the following cases :*

- if  $\Omega$  is a strictly pseudo-convex domain in  $\mathbb{C}^n$  ;
- if  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$ .

I stated and proved the subordination lemma for the ball in  $\mathbb{C}^n$  in 1978 [3], and, since then, I gave seminars and conferences about it in the general situation.

I am indebted to Marco Abate for an interesting discussion on Bergman-Carleson measures in january 2010, which makes me realizing that we can apply the subordination principle not only to interpolating sequences, zero sets of holomorphic functions and Corona theorems, but also to the Carleson measure theory.

I thank him also for sending me his preprint with A. Saraco [1] on Carleson measures in strongly pseudo-convex domains.

## 2 The subordination principle.

In all the sequel, domain will mean bounded domain in  $\mathbb{C}^n$  with smooth  $\mathcal{C}^\infty$  boundary defined by a real valued function  $r \in \mathcal{C}^\infty(\mathbb{C}^n)$ , i.e.

$$\Omega = \{z \in \mathbb{C}^n :: r(z) < 0\}, \quad \forall z \in \partial\Omega, \quad \partial r(z) \neq 0.$$

We have associated to it the "lifted" domain  $\tilde{\Omega}$  in  $(z, w) \in \mathbb{C}^{n+k}$  with defining function

$$\tilde{r}(z, w) := r(z) + |w|^2.$$

Recall that  $dm(z)$  will denote the Lebesgue measure in  $\mathbb{C}^n$  and  $d\sigma(z, w)$  the Lebesgue measure on  $\partial\tilde{\Omega}$ . For  $z \in \Omega$ , let  $\delta(z) := d(z, \Omega^c)$  be the distance from  $z$  to the boundary of  $\Omega$ .

We shall need the following lemma on the "disintegration" of the Lebesgue measure on  $\partial\Omega$  with respect to a defining function  $r$ .

**Lemma 2.1** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary, defined by a function  $r \in \mathcal{C}^\infty$ , i.e.

$$\Omega := \{x \in \mathbb{R}^n :: r(x) < 0\}, \quad \forall x \in \partial\Omega, \quad \text{grad } r(x) \neq 0.$$

Then the Lebesgue measure  $\sigma$  on  $\partial\Omega$  is given by

$$\forall g \in \mathcal{C}(\partial\Omega), \quad \int_{\partial\Omega} g d\sigma = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\{-\eta \leq r(x) < 0\}} \frac{1}{|\text{grad } r(\alpha)|} \tilde{g}(x) dm(x),$$

where  $\alpha = \pi(x)$  is the normal projection of  $x$  on  $\partial\Omega$  and  $\tilde{g}(x)$  is any continuous extension of  $g$  near  $\partial\Omega$ , for instance  $\tilde{g}(x) := g(\alpha)$ .

Proof.

set  $\partial\Omega_\eta := \{x \in \Omega :: d(x, \partial\Omega) \leq \eta\}$ , where  $d(x, \partial\Omega)$  is the euclidean distance. Let  $g \in \mathcal{C}(\partial\Omega)$  and extend  $g$  continously in a  $\partial\Omega_\eta$ . By its very definition, the measure on  $\partial\Omega$  induced by the Lebesgue measure of  $\mathbb{R}^n$  is

$$\int_{\partial\Omega} g d\sigma = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\partial\Omega_\eta} g(x) dm(x).$$

But at a point  $\alpha \in \partial\Omega$ , we have  $r(x) = r(\alpha) + (x - \alpha) \cdot \text{grad } r(\zeta)$ , for a  $\zeta$  between  $\alpha$  and  $x$ . This implies, because  $r(\alpha) = 0$ ,

$$r(x) = (x - \alpha) \cdot \text{grad } r(\zeta),$$

hence the thickness  $\lambda = \lambda(\alpha)$  of the strip  $\{-\eta \leq r(x) < 0\}$  at a point  $x$ ,  $r(x) = -\eta$ , on the normal at the point  $\alpha$ , i.e.  $x - \alpha = \lambda \text{grad } r(\alpha) / |\text{grad } r(\alpha)|$ , is

$$|\lambda \text{grad } r(\alpha) \cdot \text{grad } r(\zeta) / |\text{grad } r(\alpha)|| = \eta \Rightarrow \lambda(\alpha) = \frac{|\text{grad } r(\alpha)|}{\text{grad } r(\alpha) \cdot \text{grad } r(\zeta)} \eta.$$

Because we want a uniform thickness, we have to correct it :

$$\lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\partial\Omega_\eta} g(x) dm(x) = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\{-\eta \leq r(x) < 0\}} \frac{1}{|\text{grad } r(\alpha)|} g(x) dm(x),$$

because  $\text{grad } r(\zeta) \rightarrow \text{grad } r(\alpha)$  as  $\eta \rightarrow 0$  and this lemma by the dominated convergence theorem of Lebesgue. ■

Now we can prove our subordination lemma [3] stated in the introduction.

We copy from [3], and adapt from the ball to this general case. If  $F(z, w) = f(z) \in H^p(\tilde{\Omega})$  then we have

$$\|F\|_p^p := \sup_{\epsilon > 0} \int_{\{\tilde{r}(z, w) = -\epsilon\}} |F(z, w)|^p d\sigma(z, w) < \infty.$$

By the previous lemma, the Lebesgue measure on  $\partial\tilde{\Omega}$  is

where  $\tilde{r}(z, w) := r(z) + |w|^2$ . To simplify notations set  $A(z, w) := \frac{1}{|\text{grad } \tilde{r}(\alpha)|}$ , then clearly

$$0 < A_1 \leq A(z, w) \leq A_2 < \infty,$$

because  $A(z, w)$  is smooth and  $\tilde{\Omega}$  is compact.

By Fubini we get

$$\begin{aligned} & \int_{\{-\eta \leq \tilde{r}(z, w) < 0\}} A(z, w) g(z, w) dm(z, w) \\ &= \int_{\{z :: r(z) < 0\}} \left( \int_{\{w \in \mathbb{C}^k :: -\eta - r(z) \leq |w|^2 < -r(z)\}} A(z, w) g(z, w) dm(w) \right) dm(z). \end{aligned}$$

Suppose that the function  $g$  is (real positive) subharmonic in  $w$  for  $z$  fixed. Then

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k :: -\eta - r(z) \leq |w|^2 < -r(z)\}} A(z, w) g(z, w) dm(w) \geq$$

$$\begin{aligned} &\geq A_1 \frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} g(z, w) dm(w) \geq \\ &\geq g(z, 0) \frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} dm(w), \end{aligned}$$

because  $\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}$  is a corona centered at 0 and  $g$  is subharmonic in  $w$ . Now we have

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} dm(w) = \frac{(-r(z))^k - (-r(z) - \eta)^k}{\eta} \rightarrow k(-r(z))^{k-1}, \quad \eta \rightarrow 0.$$

Hence we get for  $g$  a (real positive) subharmonic function in  $w$  for  $z$  fixed

$$\int_{\partial \tilde{\Omega}} g(z, w) d\sigma(z, w) \geq A_1 k \int_{\{z : r(z) < 0\}} g(z, 0) (-r(z))^{k-1} dm(z) = A_1 k \int_{\Omega} g(z, 0) (-r(z))^{k-1} dm(z).$$

Now apply this for  $\tilde{\Omega}_\epsilon := \{r(z) + |w|^2 = -\epsilon\}$  instead of  $\tilde{\Omega}$ , and with  $g(z, w) := |F(z, w)|^p$ , (resp.  $g(z, w) := \log^+ |F(z, w)|$ ) which is pluri-subharmonic, hence in particular subharmonic in  $w$  for  $z$  fixed, and continuous up to  $\partial \tilde{\Omega}_\epsilon$  because  $\epsilon > 0$ . So

$$\int_{\partial \tilde{\Omega}_\epsilon} |F(z, w)|^p d\sigma(z, w) \geq A_1 k \int_{\Omega_\epsilon} |F(z, 0)|^p (-r(z))^{k-1} dm(z).$$

Respectively

$$\int_{\partial \tilde{\Omega}_\epsilon} \log^+ |F(z, w)| d\sigma(z, w) \geq A_1 k \int_{\Omega_\epsilon} \log^+ |F(z, 0)|^p (-r(z))^{k-1} dm(z).$$

But we have  $\delta(z) \simeq -r(z)$  near the boundary. Hence by Fatou's lemma with  $\epsilon \rightarrow 0$ ,

$$A_1 k \|F(\cdot, 0)\|_{A_{k-1}^p(\Omega)} = A_1 k \int_{\Omega} |F(z, 0)|^p \delta(z)^{k-1} dm(z) \leq \|F\|_{H^p(\tilde{\Omega})}^p,$$

respectively

$$A_1 k \|F(\cdot, 0)\|_{\mathcal{N}_{k-1}(\Omega)} = A_1 k \int_{\Omega} \log^+ |F(z, 0)| \delta(z)^{k-1} dm(z) \leq \|F\|_{\mathcal{N}(\tilde{\Omega})};$$

hence

$$\|F(\cdot, 0)\|_{A_{k-1}^p(\Omega)} \leq \frac{1}{A_1 k} \|F\|_{H^p(\tilde{\Omega})}^p,$$

respectively

$$\|F(\cdot, 0)\|_{\mathcal{N}_{k-1}(\Omega)} \leq \frac{1}{A_1 k} \|F\|_{\mathcal{N}(\tilde{\Omega})}.$$

Now if we set  $f(z) := F(z, 0)$  we get

$$\|f\|_{A_{k-1}^p(\Omega)} \leq \frac{1}{A_1 k} \|F\|_p^p$$

respectively

$$\|f\|_{\mathcal{N}_{k-1}(\Omega)} \leq \frac{1}{A_1 k} \|F\|_{\mathcal{N}(\tilde{\Omega})}.$$

Hence if  $F(z, w) := f(z)$  and  $F \in H^p(\tilde{\Omega})$  we have  $f \in A_{k-1}^p(\Omega)$ , respectively

$$F \in \mathcal{N}(\tilde{\Omega}) \Rightarrow f \in \mathcal{N}_{k-1}(\Omega).$$

So we have the first part of the lemma.

Conversely if  $f \in A_{k-1}^p(\Omega)$  (resp.  $f \in \mathcal{N}_{k-1}(\Omega)$ ), setting  $F(z, w) := f(z)$  and reversing the previous computations, using equalities this time,

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} |f(z)|^p dm(w) \rightarrow |f(z)|^p k(-r(z))^{k-1},$$

respectively

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} \log^+ |f(z)| dm(w) \rightarrow \log^+ |f(z)| k(-r(z))^{k-1},$$

we get  $\|F\|_{H^p(\tilde{\Omega})} \leq A_2 k \|f\|_{A_{k-1}^p(\Omega)}$  (resp.  $\|F\|_{\mathcal{N}(\tilde{\Omega})} \leq A_2 k \|f\|_{\mathcal{N}_{k-1}(\Omega)}$ ) and this lemma.  $\blacksquare$

**Corollary 2.2** *For any  $a \in \Omega$  the Bergman kernel  $B_k(z, a)$  and the Szegő kernel  $\tilde{S}((z, w), (a, 0))$  for the lifted domain  $\tilde{\Omega}$ , verify, with  $\forall z \in \Omega, \tilde{z} := (z, 0) \in \tilde{\Omega}$  :*

$$\forall a \in \Omega, \forall z \in \Omega, B_k(z, a) = \tilde{S}(\tilde{z}, \tilde{a}).$$

Moreover we have

$$\forall a \in \Omega, \left\| \tilde{S}(\cdot, \tilde{a}) \right\|_{H^p(\tilde{\Omega})} \gtrsim \|B_k(\cdot, a)\|_{A_{k-1}^p(\Omega)}.$$

Proof.

Let  $f \in A(\Omega)$  be a holomorphic function in  $\Omega$ , continuous up to  $\partial\Omega$ . Let

$$\forall (z, w) \in \tilde{\Omega}, F(z, w) := f(z).$$

We have

$$\int_{\Omega} f(z) \overline{B}_k(z, a) dm_k(z) = f(a) = F(a, 0) = \int_{\partial\tilde{\Omega}} F(z, w) \overline{\tilde{S}((z, w), \tilde{a})} d\sigma(z, w),$$

by the reproducing property of these kernels. But  $F$  does not depend on  $w$  and  $\overline{\tilde{S}((z, w), \tilde{a})}$  is anti-holomorphic in  $w$  for  $z$  fixed in  $\Omega$ , so

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} \overline{\tilde{S}((z, w), \tilde{a})} dm(w) \rightarrow \overline{\tilde{S}((z, 0), \tilde{a})} k(-r(z))^{k-1},$$

by the previous proof, hence

$$\int_{\Omega} f(z) \overline{B}_k(z, a) dm_k(z) = \int_{\Omega} f(z) \overline{\tilde{S}((z, 0), \tilde{a})} k(-r(z))^{k-1} dm(z) = \int_{\Omega} f(z) \overline{\tilde{S}((z, 0), \tilde{a})} dm_k(z).$$

So we have

$$\forall f \in A(\Omega), \int_{\Omega} f(z) (\overline{\tilde{S}((z, 0), \tilde{a})} - \overline{B}_k(z, a)) dm_k(z) = 0,$$

hence  $\overline{\tilde{S}((z, 0), \tilde{a})} - \overline{B}_k(z, a) \perp A(\Omega)$  in  $A_k^2(\Omega)$ . But  $\overline{\tilde{S}((z, 0), \tilde{a})} - \overline{B}_k(z, a)$  is holomorphic in  $z$ , hence

$$\forall z \in \Omega, \overline{\tilde{S}((z, 0), \tilde{a})} = \overline{B}_k(z, a).$$

The second part is a direct application of the first part in the subordination lemma 1.2, and the corollary.  $\blacksquare$



### 3 Geometric Carleson measures and $p$ -Carleson measures.

In order to define precisely the geometric Carleson measures, we need the notion of a "good" family of polydiscs, directly inspired by the work of Catlin [12] and introduced in [5].

Let  $\mathcal{U}$  be a neighbourhood of  $\partial\Omega$  in  $\Omega$  such that the normal projection  $\pi$  onto  $\partial\Omega$  is a smooth well defined application.

Let  $a \in \Omega$  and let  $b(a) = (L_1, L_2, \dots, L_n)$  be an orthonormal basis of  $\mathbb{C}^n$ , such that  $(L_2, \dots, L_n)$  is a basis of the tangent complex space  $T_a^{\mathbb{C}}$  of the level set  $\partial\Omega_a := \{z \in \Omega :: r(z) = r(a)\}$  at  $a$ , hence  $L_1$  is the complex normal at  $a$  to  $\partial\Omega_a$ .

Let  $\alpha \in \partial\Omega$  and let  $b(\alpha) = (L_1, L_2, \dots, L_n)$  be an orthonormal basis of  $\mathbb{C}^n$  such that  $(L_2, \dots, L_n)$  is a basis of the tangent complex space  $T_\alpha^{\mathbb{C}}$  of  $\partial\Omega$  at  $\alpha$ ; hence  $L_1$  is the complex normal at  $\alpha$  to  $\partial\Omega$ . Let  $m(\alpha) = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$  be a multi-index at  $\alpha$  with  $m_1 = 1, \forall j \geq 2, m_j \geq 2$ . For  $a \in \mathcal{U}$  and  $t > 0$  set  $\alpha = \pi(a)$  and  $P_a(t) := \prod_{j=1}^n tD_j$ , the polydisc such that  $tD_j$  is the disc centered at  $a$ , parallel to  $L_j \in b(\alpha)$ , with radius  $t|r(a)|^{1/m_j}$  (in  $\mathcal{U}$  we have  $|r(a)| \simeq d(a, \Omega^c)$  is the distance from  $a$  to the boundary of  $\Omega$ ).

Set  $b(a) := b(\pi(a)), m(a) := m(\pi(a))$ , for  $a \in \mathcal{U}$ .

This way we have a family of polydiscs  $\mathcal{P} := \{P_a(t)\}_{a \in \mathcal{U}}$  defined by the family of basis  $\{b(a)\}_{a \in \mathcal{U}}$ , the family of multi-indices  $\{m(a)\}_{a \in \mathcal{U}}$  and the number  $t$ . Notice that the polydisc  $P_a(2)$  always overflows the domain  $\Omega$ .

It will be useful to extend this family to the whole of  $\Omega$ . In order to do so let  $(z_1, \dots, z_n)$  be the canonical coordinates system in  $\mathbb{C}^n$  and for  $a \in \Omega \setminus \mathcal{U}$ , let  $P_a(t)$  be the polydisc of center  $a$ , of sides parallel to the axis and radius  $td(a)$  in the  $z_1$  direction and  $td(a)^{1/2}$  in the other directions. So the points  $a \in \Omega \setminus \mathcal{U}$  have automatically a "minimal" multi-index  $m(a) = (1, 2, \dots, 2)$ .

Now we can set

**Definition 3.1** *We say that  $\mathcal{P}$  is a "good family" of polydiscs for  $\Omega$  if the  $m_j(a)$  are uniformly bounded on  $\Omega$  and if it exists  $\delta_0 > 0$  such that all the polydiscs  $P_a(\delta_0)$  of  $\mathcal{P}$  are contained in  $\Omega$ . In this case we call  $m(a)$  the multi-type at  $a$  of the family  $\mathcal{P}$ .*

We notice that, for a good family  $\mathcal{P}$ , by definition the multi-type is always finite. Moreover there is no regularity assumptions on the way that the basis  $b(a)$  varies with respect to  $a \in \Omega$ .

We can see easily that there is always good families of polydiscs in a domain  $\Omega$  in  $\mathbb{C}^n$ : for a point  $a \in \Omega$ , take any orthonormal basis  $b(a) = (L_1, L_2, \dots, L_n)$ , with  $L_1$  the complex normal direction, and the "minimal" multitype  $m(a) = (1, 2, \dots, 2)$ . Then, because the level sets  $\partial\Omega_a$  are uniformly of class  $\mathcal{C}^2$  and compact, we have the existence of a uniform  $\delta_0 > 0$  such that the family  $\mathcal{P}$  is a good one. But, as seen in [5], this one is actually good just for the strictly pseudo-convex domains.

The domain  $\Omega$  is still defined by the function  $r \in \mathcal{C}^\infty, \forall z \in \partial\Omega, \text{grad } r(z) \neq 0$  and we still note  $\sigma$  the Lebesgue measure on  $\partial\Omega$  and  $m$  the Lebesgue measure on  $\mathbb{C}^n$ .

We can give the definitions relative to Carleson measures.

**Definition 3.2** *A positive borelian measure  $\mu$  on  $\Omega$  is a geometric Carleson measure,  $\mu \in \Lambda^1(\Omega)$ , if  $\exists C = C_\mu > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C\sigma(\Omega \cap P_a(2))$ .*

**Definition 3.3** A positive borelian measure  $\mu$  on  $\Omega$  is a  $p$ -Carleson measure in  $\Omega$  if

$$\exists C > 0 :: \forall f \in H^p(\Omega), \int_{\Omega} |f(z)|^p d\mu(z) \leq C^p \|f\|_{H^p(\Omega)}^p.$$

And analogously for the Bergman spaces.

**Definition 3.4** A positive borelian measure  $\mu$  on  $\Omega$  is a  $k$ -geometric Bergman-Carleson measure,  $\mu \in \Lambda_k^1(\Omega)$ , if

$$\exists C = C_{\mu} > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_k(\Omega \cap P_a(2)).$$

with  $dm_k(z) := \delta(z)^{k-1} dm(z)$  is the weighted Lebesgue measure.

Notice the gap  $k \rightarrow k - 1$ .

**Definition 3.5** A positive borelian measure  $\mu$  is  $(p, k)$ -Bergman-Carleson measure in  $\Omega$  if

$$\exists C > 0 :: \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f(z)|^p d\mu(z) \leq C^p \|f\|_{A_{k-1}^p(\Omega)}^p.$$

**Definition 3.6** We shall say that the domain  $\Omega$  has the  $p$ -Carleson embedding property,  $p$ -CEP, if

$$\forall \mu \in \Lambda^1(\Omega), \exists C = C_{\mu} > 0 :: \forall f \in H^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{H^p(\Omega)}^p.$$

And the same for the Bergman spaces.

**Definition 3.7** We shall say that the domain  $\Omega$  has the  $(p, k)$ -Bergman-Carleson embedding property,  $(p, k)$ -BCEP, if

$$\forall \mu \in \Lambda_k^1(\Omega), \exists C = C_{\mu} > 0 :: \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{A_{k-1}^p(\Omega)}^p.$$

### 3.1 Application of the subordination lemma to Carleson measures.

We shall fix  $k \in \mathbb{N}$  and lift the measure on the domain  $\tilde{\Omega} := \{\tilde{r}(z, w) := r(z) + |w|^2 < 0\}$ , with  $w = (w_1, \dots, w_k) \in \mathbb{C}^k$ . We already know how to lift the function, the lifted measure  $\tilde{\mu}$  of a measure  $\mu$  is just

$$\tilde{\mu} := \mu \otimes \delta,$$

with  $\delta$  the delta Dirac measure of the origin in  $\mathbb{C}^k$ . We shall need a lemma linking Bergman and Hardy geometric Carleson measures.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\tilde{\Omega}$  be its usual lift, and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$ .

**Definition 3.8** We shall say that the good family of polydiscs  $\tilde{\mathcal{P}}$  on the domain  $\tilde{\Omega}$  is "homogeneous" if

(Hg)  $\exists t > 0, \exists C > 0 :: \forall b \in \Omega \cap P_a(2), P_b(t) \supset P_a(2) ; \sigma(P_b(t)) \leq C \sigma(P_a(2))$ ,  
where  $\Omega = \tilde{\Omega} \cap \{w = 0\} \subset \tilde{\Omega}$ .

Naturally the domain  $\Omega$  is equipped with the family  $\mathcal{P}$  induced by  $\tilde{\mathcal{P}}$  the following way

$$\forall a \in \Omega, P_a(u) := P_{(a,0)}(u) \cap \{w = 0\},$$

which is also a good family for  $\Omega$ .

As examples we have the stricly pseudo-convex domains and the convex domains of finite type, because both are domains of homogeneous type in the sense of Coifman-Weiss [13].

**Lemma 3.9** *Let  $(\Omega, \tilde{\Omega})$  be as above and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypothesis (Hg). The measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$  iff the measure  $\tilde{\mu}$  is a geometric Carleson measure in  $\tilde{\Omega}$ .*

Proof of the lemma.

Suppose that  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ , we want to show :

$$\exists C > 0 :: \forall (a, b) \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap P_{(a,b)}(2)) \leq C\sigma(\tilde{\Omega} \cap P_{(a,b)}(2)),$$

with  $P_c$  the polydisc of center  $c = (a, b) \in \tilde{\Omega}$  of the family  $\tilde{\mathcal{P}}$ . Let us see first the case where  $b = 0$ , i.e.  $(a, b) \in \Omega \subset \tilde{\Omega}$ . Then, by definition of  $\tilde{\mu}$ , we have

$$\tilde{\mu}(\tilde{\Omega} \cap P_{(a,0)}(2)) = \mu(\Omega \cap P_a(2)).$$

On the other hand, we have, exactly as in the proof of the subordination lemma,

$$\sigma(\tilde{\Omega} \cap \tilde{P}_{(a,0)}(2)) = \int_{\Omega \cap P_a(2)} k(-r(z))^{k-1} dm(z) = m_k(\Omega \cap P_a(2)),$$

with  $dm_k := k(-r)^{k-1} dm$ .

But if  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ , we have

$$\exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq Cm_k(\Omega \cap P_a(2)),$$

so

$$\tilde{\mu}(\tilde{\Omega} \cap P_{(a,0)}(2)) = \mu(\Omega \cap P_a(2)) \leq Cm_k(\Omega \cap P_a(2)) = \frac{C}{k}\sigma(\tilde{\Omega} \cap P_{(a,0)}(2)).$$

Now take a general  $P_{(a,b)}(2)$ . In order for  $\tilde{\mu}(\tilde{\Omega} \cap P_{(a,b)}(2))$  to be non zero, we must have

$$P_{(a,b)}(2) \cap \{w = 0\} \neq \emptyset \Rightarrow \exists (c, 0) \in P_{(a,b)}(2) \cap \{w = 0\}.$$

By the (Hg) hypotheses, this means that we have  $P_{(c,0)}(t) \supset P_{(a,b)}(2)$  with the uniform control  $\sigma(\tilde{\Omega} \cap P_{(c,0)}(t)) \lesssim \sigma(\tilde{\Omega} \cap P_{(a,b)}(2))$ .

We apply the above inequality

$$\tilde{\mu}(\tilde{\Omega} \cap P_{(a,b)}(2)) \leq \tilde{\mu}(\tilde{\Omega} \cap P_{(c,0)}(t)) \leq Cm_k(\Omega \cap P_c(t)) = \frac{C}{k}\sigma(\tilde{\Omega} \cap P_{(c,0)}(t)) \lesssim \sigma(\tilde{\Omega} \cap P_{(a,b)}(2)),$$

hence  $\tilde{\mu}$  is a geometric Carleson measure on  $\tilde{\Omega}$ .

Conversely suppose that  $\tilde{\mu}$  is a geometric Carleson measure on  $\tilde{\Omega}$ , this means

$$\forall (a, b) \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap P_{(a,b)}(2)) \leq C\sigma(\tilde{\Omega} \cap P_{(a,b)}(2)),$$

hence, in particular for  $b = 0$ ,

$$\forall a \in \Omega, \tilde{\mu}(\tilde{\Omega} \cap P_{(a,0)}(2)) \leq C\sigma(\tilde{\Omega} \cap P_{(a,0)}(2)),$$

but then by definition of  $\tilde{\mu}$  and with the previous computation of  $\sigma(\tilde{\Omega} \cap P_{(a,0)}(2))$ , we get

$$\forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq Cm_k(\Omega \cap P_a(2)),$$

hence the measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ , and the lemma. ■

Now we shall use the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding theorem.

**Theorem 3.10** *Let  $(\Omega, \tilde{\Omega})$  be as usual and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypotheses (Hg). If the lifted domain  $\tilde{\Omega}$  has the  $p$ -CEP then  $\Omega$  has the  $(p, k)$ -BCEP.*

Proof.

Suppose the positive measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure ; by the previous lemma, we have that the lifted measure  $\tilde{\mu}$  is a geometric Carleson measure in  $\tilde{\Omega}$ . By the  $p$ -CEP we have

$$\forall F \in H^p(\tilde{\Omega}), \int_{\tilde{\Omega}} |F|^p d\tilde{\mu} \leq C_{\mu}^p \|F\|_{H^p(\tilde{\Omega})}^p.$$

Choose  $f(z) \in A_{k-1}^p(\Omega)$  and set  $\forall (z, w) \in \tilde{\Omega}$ ,  $F(z, w) = f(z)$ . By the subordination lemma we have

$$\|f\|_{A_{k-1}^p(\Omega)} \simeq \|F\|_{H^p(\tilde{\Omega})},$$

and by definition of  $\tilde{\mu}$ , we have

$$\int_{\Omega} |f|^p d\mu = \int_{\tilde{\Omega}} |F|^p d\tilde{\mu} \leq C_{\mu}^p \|F\|_{H^p(\tilde{\Omega})}^p \lesssim \|f\|_{A_{k-1}^p(\Omega)}^p,$$

hence  $\mu$  is a  $(k, p)$ -Bergman-Carleson measure in  $\Omega$ . ■

**Theorem 3.11** *Let  $(\Omega, \tilde{\Omega})$  be as usual and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypotheses (Hg). If  $p$ -Carleson implies geometric Carleson in  $\tilde{\Omega}$ , then  $(p, k)$ -Bergman-Carleson implies geometric  $k$ -Bergman-Carleson in  $\Omega$ .*

Proof.

If the positive measure  $\mu$  is  $(p, k)$ -Bergman-Carleson in  $\Omega$  then  $\tilde{\mu}$  is a  $p$ -Carleson measure in  $\tilde{\Omega}$  by lemma 3.9 hence a geometric Carleson measure in  $\tilde{\Omega}$  by the assumption of the theorem. Then applying lemma 3.9 we get that  $\mu$  is a  $k$ -geometric Carleson measure in  $\Omega$  hence the theorem. ■

**Remark 3.12** *In the case of the unit ball  $\Omega$  of  $\mathbb{C}^n$ ,  $\tilde{\Omega} \subset \mathbb{C}^{n+1}$  N. Varopoulos showed me an alternative proof for the fact that  $F(z, w) \in H^p(\tilde{\Omega}) \Rightarrow F(z, 0) \in A_p(\Omega)$  : the Lebesgue measure on  $\{w = 0\} \cap \tilde{\Omega}$  is easily seen to be a geometric Carleson measure in  $\tilde{\Omega}$ , hence by the Carleson-Hörmander embedding theorem [17] we have*

$$\int_{\Omega} |F(z, 0)|^p dm(z) \leq C \|F\|_{H^p(\tilde{\Omega})}^p,$$

and the assertion. Of course this is still valid in codimension  $k \geq 1$ , with the weighted Lebesgue measure on  $\Omega$ , and for strictly pseudo-convex domains because the Carleson-Hörmander embedding theorem is still valid there. But this is just one direction of the lemma, it works only if there is a Carleson embedding theorem and this proof is much less elementary than the previous one.

In fact we can reverse things and say that one part of the subordination lemma asserts that the weighted Lebesgue measure on  $\Omega$  is always a Carleson measure in  $\tilde{\Omega}$ ,  $\Omega$  strictly pseudo-convex or not.

## 4 Application to convex domains of finite type in $\mathbb{C}^n$ .

In [5] we prove a Carleson embedding theorem for the convex domains of finite type in  $\mathbb{C}^n$ .

**Theorem 4.1** *Let  $\Omega$  be a convex domains of finite type in  $\mathbb{C}^n$  ; if the measure  $\mu$  is a geometric Carleson measure we have*

$$\forall p > 1, \exists C_p > 0, \forall f \in H^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C_p^p \|f\|_{H^p}^p.$$

*Conversely if the positive measure  $\mu$  is  $p$ -Carleson for a  $p \in [1, \infty[$ , then it is a geometric Carleson measure, hence it is  $q$ -Carleson for any  $q \in ]1, \infty[$ .*

We already know that if  $\Omega$  is a convex domain of finite type, so is  $\tilde{\Omega}$  with the same type. Moreover the hypotheses (Hg) are true for these domains equipped with a (slightly modified) McNeal family of polydiscs, so we can apply what precedes in this case to get from the Carleson embedding theorem the Bergman-Carleson embedding one.

**Theorem 4.2** *Let  $\Omega$  be a convex domains of finite type in  $\mathbb{C}^n$  ; if the measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure we have*

$$\forall p > 1, \exists C_p > 0, \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C_p^p \|f\|_{A_{k-1}^p(\Omega)}^p.$$

*Conversely if the positive measure  $\mu$  is  $(p, k)$ -Bergman-Carleson for a  $p \in [1, \infty[$ , then it is a  $k$ -geometric Bergman-Carleson measure, hence it is  $(q, k)$ -Bergman-Carleson for any  $q \in ]1, \infty[$ .*

## 4.1 Interpolating sequences for the Bergman spaces.

For  $k \in \mathbb{N}$  and  $a \in \Omega$ , let  $b_{k,a}(z) = B_k(z, a)$  denotes the Bergman kernel of  $\Omega$  at the point  $a$ . It is the kernel of the orthogonal projection from  $L^2(\Omega, m_k)$  onto  $A_k^2(\Omega)$  ; it is also the reproducing kernel for  $A_k^2(\Omega)$ , i.e.

$$\forall a \in \Omega, \forall f \in A_k^2(\Omega), f(a) = \int_{\Omega} f(z) b_{k,a}(z) dm_k(z) = \langle f, b_{k,a} \rangle.$$

Now we set  $\|b_{k,a}\|_p := \|b_{k,a}\|_{A_k^p(\Omega)}$  and:

**Definition 4.3** *We say that the sequence  $S$  of points in  $\Omega$  is  $A_k^p(\Omega)$  interpolating if*

$$\forall \lambda \in \ell^p(S), \exists f \in A_k^p(\Omega) :: \forall a \in S, f(a) = \lambda_a \|b_{k,a}\|_{p'},$$

*with  $p'$  the conjugate exponent for  $p$ ,  $1/p' = 1 - 1/p$ .*

We say that  $S$  has the linear extension property if  $S$  is  $A_k^p(\Omega)$  interpolating and if moreover there is a bounded linear operator  $E : \ell^p(S) \rightarrow A_k^p(\Omega)$  making the interpolation.

A weaker notion is the dual boundedness:

**Definition 4.4** *We shall say that the sequence  $S$  of points in  $\Omega$  is dual bounded in  $A_k^p(\Omega)$  if there is a bounded sequence of elements in  $A_k^p(\Omega)$ ,  $\{\rho_a\}_{a \in S} \subset A_k^p(\Omega)$  which dualizes the associated sequence of reproducing kernels, i.e.*

$$\exists C > 0 :: \forall a \in S, \|\rho_a\|_p \leq C, \forall a, c \in S, \langle \rho_a, b_{k,c} \rangle = \delta_{a,c} \|b_{k,a}\|_{p'}.$$

If  $S$  is  $A_k^p(\Omega)$  interpolating then it is dual bounded in  $A_k^p(\Omega)$  : just interpolate the element of the basic sequence in  $\ell^p(S)$ .

The converse is the crux of the characterization by Carleson [11] of  $H^\infty(\mathbb{D})$  interpolating sequences and the same by Shapiro & Shields [18] for  $H^p(\mathbb{D})$  interpolating sequences in  $\mathbb{D}$ .

In [5], I proved

**Theorem 4.5** *If  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$  and if  $S \subset \Omega$  is a dual bounded sequence of points in  $H^p(\Omega)$ , then, for any  $q < p$ ,  $S$  is  $H^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

So we get

**Theorem 4.6** *If  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$  and if  $S \subset \Omega$  is a dual bounded sequence of points in  $A_{k-1}^p(\Omega)$  then, for any  $q < p$ ,  $S$  is  $A_{k-1}^p(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

Proof.

Let  $\tilde{\Omega}$  be the lifted convex domain of finite type obtained as in section 2. Let  $\tilde{S}$  be the sequence  $S$  viewed in  $\tilde{\Omega}$ ,  $\tilde{S} := S \subset \Omega \subset \tilde{\Omega}$ . Let us denote by  $k_{\tilde{a}}(z, w) := S((z, w), \tilde{a})$  the Szegö kernel of  $\tilde{\Omega}$ , for  $\tilde{a} = (a, 0)$ . Then we have, by corollary 2.2,

$$\forall \tilde{a} = (a, 0), a \in \Omega, \|b_{k-1, a}\|_{A_k^p(\Omega)} \lesssim \|k_{\tilde{a}}\|_{H^p(\Omega)}.$$

First we have to show that  $S$  dual bounded in  $A_k^p(\Omega)$  implies  $\tilde{S}$  dual bounded in  $H^p(\tilde{\Omega})$ .

Let  $\{\rho_a\}_{a \in S} \subset A_{k-1}^p(\Omega)$  be the dual sequence to the sequence  $\{b_{k-1, a}\}_{a \in S}$ ; extend it to  $\tilde{\Omega}$  :

$$\forall a \in S, \Gamma_a(z, w) := \rho_a(z),$$

then the subordination lemma gives us that  $\|\Gamma_a\|_{H^p(\tilde{\Omega})} \simeq \|\rho_a\|_{A_k^p(\Omega)}$  and we have, using corollary 2.2,

$$\forall a, b \in S, \langle \Gamma_a, k_b \rangle = \left\langle \Gamma_a, S((\cdot, 0), \tilde{b}) \right\rangle = \langle \rho_a, B(\cdot, b) \rangle = \delta_{ab},$$

hence  $\tilde{S}$  is dual bounded in  $H^p(\tilde{\Omega})$ .

Because  $\tilde{S}$  is dual bounded in  $H^p(\tilde{\Omega})$  then we have, for  $q < p$ , that  $\tilde{S}$  is interpolating in  $H^q(\tilde{\Omega})$  provided that  $p = \infty$  or  $p \leq 2$ .

Now let  $\mu = \{\mu_a\}_{a \in S} \in \ell^q(S)$  and set  $\lambda = \{\lambda_a\}_{a \in S}$  with  $\forall a \in S, \lambda_a := \mu_a \times \frac{\|b_{k, a}\|_{A_{k-1}^{q'}(\Omega)}}{\|k_{\tilde{a}}\|_{H^{q'}(\Omega)}}$ ; then

$$\lambda \in \ell^q(S), \|\lambda\|_q \leq \|\mu\|_q.$$

So  $\tilde{S}$  is interpolating in  $H^q(\tilde{\Omega})$  hence we have a linear operator  $\tilde{E} : \ell^q(\tilde{S}) \rightarrow H^q(\tilde{\Omega})$  such that  $F(z, w) := \tilde{E}(\lambda)(z, w)$  verifies :

$$F \in H^q(\tilde{\Omega}), \forall a \in S, F(\tilde{a}) = \lambda_a \|k_{\tilde{a}}\|_{H^{q'}(\tilde{\Omega})}, \|F\|_{H^q(\tilde{\Omega})} \lesssim \|\lambda\|_q.$$

Set  $f(z) := F(z, 0) = \tilde{E}(\lambda)(z, 0) =: E(\mu)(z)$ , then clearly  $E$  is linear in  $\mu$  and then still using the subordination lemma we have

$$\|f\|_{A_{k-1}^p(\Omega)} \lesssim \|F\|_{H^q(\tilde{\Omega})} \lesssim \|\lambda\|_q \lesssim \|\mu\|_q$$

and

$$\forall a \in S, f(a) = \lambda_a \|k_{\tilde{a}}\|_{H^{q'}(\tilde{\Omega})} = \mu_a \|b_{k, a}\|_{A_{k-1}^{q'}(\Omega)}.$$

Hence  $\mu \rightarrow E(\mu)$  is bounded from  $\ell^q(S)$  in  $A_{k-1}^p(\Omega)$  and  $S$  is  $A_{k-1}^p(\Omega)$  interpolating with the linear extension property, and the theorem.  $\blacksquare$

**Remark 4.7** *I applied the subordination principle since 1978 [2], [3] essentially in this case. For instance in [2] we used it to show that the interpolating sequences for  $H^p(\mathbb{B})$ , with  $\mathbb{B}$  the unit ball in  $\mathbb{C}^n$ ,  $n \geq 2$ , are different for different values of  $p$ , opposite to the one variable case of  $H^p(\mathbb{D})$ .*

## 5 The $H^p$ -Corona theorem for Bergman spaces.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ . We say that the  $H^p$ -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then

$$\forall f \in H^p(\Omega), \exists (f_1, \dots, f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the same vein, we say that the  $A_{k-1}^p(\Omega)$ -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0 \quad (5.1)$$

then

$$\forall f \in A_{k-1}^p(\Omega), \exists (f_1, \dots, f_m) \in (A_{k-1}^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

We then have

**Theorem 5.1** *Suppose that the  $H^p$ -Corona is true for the domain  $\tilde{\Omega}$ , then the  $A_{k-1}^p(\Omega)$ -Corona theorem is also true for  $\Omega$ .*

*Proof.*

Let  $\tilde{\Omega}$  be the lifted domain ; then set

$$\forall j = 1, \dots, m, g_j \in H^\infty(\Omega), f \in H^p(\Omega), G_j(z, w) := g_j(z), F(z, w) := f(z).$$

Clearly the  $G_j$  are in  $H^\infty(\tilde{\Omega})$  and by the subordination lemma,  $F \in H^p(\tilde{\Omega})$ . Moreover, if the

condition (5.1) is true we have  $\forall (z, w) \in \tilde{\Omega}, \sum_{j=1}^m |G_j(z, w)| \geq \delta$  with the same  $\delta$ . So we can apply

the hypothesis :

$$\exists (F_1, \dots, F_m) \in (H^p(\Omega))^m :: F = \sum_{j=1}^m F_j G_j.$$

Now set  $f_j(z) = F_j(z, 0)$  then applying again the subordination lemma, we have this theorem. ■

**Corollary 5.2** *We have the  $A_{k-1}^p(\Omega)$ -Corona theorem in the following cases :*

- with  $\Omega$  the unit ball in  $\mathbb{C}^n$ , 2 generators and  $1 < p < \infty$  ;
- with  $p = 2$  if  $\Omega$  is a bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;
- with  $1 < p < \infty$  if  $\Omega$  is a bounded strictly pseudo-convex domain in  $\mathbb{C}^n$ .

The first case because I proved [4] ( with [7] already in 1980) the  $H^p$  Corona theorem for two generators in the ball ;

the second one because Andersson [8] (with a preprint in 1990) proved the  $H^2$  Corona theorem for  $\Omega$  bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;

the last one for any number of generators because Andersson & Carlsson [9] (see also [6] proved the  $H^p$  Corona theorem in this case. ■

## 6 Zeros set of the Nevanlinna-Bergman class

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $u$  a holomorphic function in  $\Omega$ . Set  $X := \{z \in \Omega :: u(z) = 0\}$  the zero set of  $u$  and  $\Theta_X := \partial\bar{\partial} \log u$  its associated  $(1, 1)$  current of integration.

**Definition 6.1** *A holomorphic divisor  $X$  in the domain  $\Omega$  is in the Blaschke class,  $X \in \mathcal{B}(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \left| \int_{\Omega} (-r(z)) \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty, \quad (6.1)$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}(\Omega)$  then it is well known that  $X$  is in the Blaschke class of  $\Omega$ .

We do the analogue for the Bergman spaces :

**Definition 6.2** *A holomorphic divisor  $X$  in the domain  $\Omega$  is in the Bergman-Blaschke class,  $X \in \mathcal{B}_k(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \quad \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \partial \bar{\partial} r \wedge \beta \right| \leq C \|\beta\|_\infty, \quad (6.2)$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}_k(\Omega)$  then it is well known that  $X$  is in the Bergman-Blaschke class of  $\Omega$ .

Hence exactly as for the Corona theorem we can set the definitions :

we say that the *Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

And the same for the Bergman spaces :

we say that the *Bergman-Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

**Theorem 6.3** *Suppose that the Blaschke characterization is true for the domain  $\Omega$ , then the Bergman-Blaschke characterization is also true for  $\Omega$ .*

Proof.

Let  $\tilde{\Omega}$  be the lifted domain ; then set  $X = u^{-1}(0)$ ,  $\Theta_X$  its associated current and suppose that  $X \in \mathcal{B}_{k-1}(\Omega)$ .

This means that

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \quad \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty. \quad (6.3)$$

Let  $U(z, w) := u(z)$ ,  $\tilde{X} := U^{-1}(0) \subset \tilde{\Omega}$ ,  $\tilde{\Theta}_{\tilde{X}} = \partial \bar{\partial} \log |U|$  ; we shall show that  $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$ . We have that  $\tilde{\Theta}_{\tilde{X}}$  does not depend on  $w$ , hence

$$\forall \tilde{\beta} \in \Lambda_{n-1, n-1}^\infty(\bar{\tilde{\Omega}}), \quad A := \int_{\tilde{\Omega}} (-\tilde{r}(z, w)) \tilde{\Theta}_{\tilde{X}} \wedge \tilde{\beta} = \int_{\Omega} \Theta_X(z) \wedge \int_{|w|^2 < -r(z)}^{-(r(z) + |w|^2)} \tilde{\beta}(z, w). \quad (6.4)$$

hence  $A = A_1 + A_2$  with

$$A_1 := \int_{\Omega} \Theta_X \wedge (-r(z)) \left( \int_{|w|^2 < -r(z)} \tilde{\beta}(z, w) \right);$$

$$A_2 := \int_{\Omega} \Theta_X \wedge \left( \int_{|w|^2 < -r(z)} (-|w|^2) \tilde{\beta}(z, w) \right);$$



But  $\beta_1(z) := \int_{|w|^2 < -r(z)} \tilde{\beta}(z, w)$  is clearly in  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$ , because  $\tilde{\beta}(z, w)$  has bounded coefficients and the ball  $|w|^2 < -r(z)$  is uniformly bounded hence  $\|\beta_1\|_\infty \lesssim \|\tilde{\beta}\|_\infty$ .

Let  $\beta_2(z) := \int_{|w|^2 < -r(z)} |w|^2 \tilde{\beta}(z, w)$ , then

$$|A_2| = \left| \int_{\Omega} \Theta_X \wedge \beta_2 \right| \leq \left| \int_{\Omega} \Theta_X \wedge (-r(z))\beta_1 \right|,$$

because  $|w|^2 < -r(z)$  hence  $|\beta_2| \leq -r(z)|\beta_1|$ , recall that  $\Theta_X$  is positive. Hence we can apply the hypothesis to these integrals :

$$|A| \leq |A_1| + |A_2| \lesssim \|\beta_1\|_\infty + \|\beta_2\|_\infty \lesssim \|\tilde{\beta}\|_\infty,$$

and  $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$ .

Now we apply the hypothesis of the theorem,

$$\exists V \in \mathcal{N}(\tilde{\Omega}) :: \tilde{X} = V^{-1}(0),$$

and clearly  $X = V^{-1}(0) \cap \{w = 0\}$ , because if  $z \in X$  then  $\forall w :: |w|^2 < -r(z)$ ,  $(z, w) \in \tilde{X}$ . Hence we set

$$v(z) := V(z, 0) \in \mathcal{N}_k(\Omega),$$

by the subordination lemma, and we are done. ■

**Corollary 6.4** *The Bergman-Blaschke characterization is true in the following cases :*

- if  $\Omega$  is a strictly pseudo-convex domain in  $\mathbb{C}^n$  ;
- if  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$ .

Proof.

The first case is true by the famous theorem proved by Henkin [16] and Skoda [19] which says that the Blaschke characterization is true for strictly pseudo-convex domain in  $\mathbb{C}^n$ .

The second one because the Blaschke characterization is true for convex domain of finite strict type by a theorem of Bruna-Charpentier-Dupain [10] generalized to all convex domains of finite type by Cumenge [14] and Diederich & Mazzilli [15]. ■

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