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SUMMARY ABSTRACT
The effects of dynamic traffic information on the network assignment of equipped and uninformed users are modelled analytically to address various patterns of cooperative behaviour among the user classes, including User Equilibrium, System Optimum and Informed Cooperation by minimization of travel cost among the informed users only.

ABSTRACT
An analytical model is developed for a two link network subject to congestion both recurrent and incidental on link travel times, dynamic traffic information (DTI), and two classes of users either informed or not to choose their route. Closed-form formulas are provided for the main model variables, including average link flow and trip cost by user class, as well as marginal external cost by user class, for each of the two basic patterns of cooperation among the network users: User Equilibrium (UE) and System Optimum (SO). Based on parametric analysis, it is shown that under UE the informed users benefit from dynamic information in a decreasing way with respect to demand volume and equipment rate, whereas the uninformed users get benefit only if the link average times are unequal. The potential benefits of system optimization include several terms, of which the balance depends on the system state. Their split between the two classes of users depends on: equal or unequal average link social time; demand volume and equipment rate; and a network configuration parameter that synthesizes the free flow time and capacity features of the network routes. To harvest the benefits of system optimization, while maintaining user acceptability, a pattern of Informed Cooperation is put forward for cost minimization limited to the set of informed users: its effectiveness is demonstrated by numerical simulation.

KEYWORDS
- Dynamic Traffic Information
- Multi-class Traffic Assignment
- Bi-layer Equilibrium
- Cooperative Behaviour
- User Equilibrium
- System Optimum
- Partial Cooperation

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INTRODUCTION

In a road network, the congestion delays include the recurrent delays determined by the flow pattern and the laws of traffic, and the non-recurrent delays due to traffic incidents, accidents or other random events, such as weather hazard. The non-recurrent, “incidental” causes cannot be forecasted by the individual user. To tackle congestion, the network operator can avail itself of a wide set of tools and resources, including long run capacity planning, medium run operations planning and short run or even real-time traffic management by dynamic adaptation of traffic signals, of the local limit speed, of the fare of HOT lanes etc. Traffic management also includes demand management through quality setting, pricing and information provision which are closely related with Dynamic Traffic Information (DTI). Actually the user chooses his route and/or departure time on the basis of his knowledge about the network state. The disposal of sharp information enables the road user to react in real-time to traffic disturbances and to cooperate in some way to the overall performance.

The provision of DTI makes a complex issue of which objective, contents, target, diffusion medium, equipment type, equipment rate etc. In particular the interplay of the congestion sources and physical laws with the provision of DTI calls for a model, purported to analyse and simulate the phenomena hence to gain a better understanding. The distinction of congestion sources and their interplay with DTI has been addressed correctly for the choice of departure time by Small and Noland (1995), Leurent (2001, 2004), although the analytical approach does not include the feedback of user re-timing due to congestion. Concerning route choice, dynamic simulation in the 2000s (e.g. Lo and Szeto, 2004) has taken the same line as static simulation in the 1990s (Van Vuren and Watling, 1991; Maher and Hughes, 1995) by assuming that informed users would perceive a mean travel time, whereas uninformed users would experience disturbed travel times. In fact, this stands exactly opposite to the essence of dynamic disturbances and information (Sétra, 1996). Recently, Fernandez et al (2009) introduced an analytical model in which traffic disruptions are taken into account by assuming a limited set of four alternative system states. In our first paper (Leurent and Nguyen, 2008), two classes of users either informed or not are considered; the informed users have DTI about the actual travel time conditions, whereas the unequipped users only know about the average travel times. This is a novel assignment model since it distinguishes two layers of user equilibrium in traffic, associated with two time scales in demand perception and behaviour: an upper layer of short run decision-making by informed users as opposed to a lower level of long run decision-making by uninformed users, each layer constraining the other.

The objective of the paper is to study the interplay of congestion recurrent or incidental, dynamic information, demand equipment, user route choice and cooperative behaviour, on the basis of an analytical model that extends our User Equilibrium model (Leurent and Nguyen, 2008) to capture various patterns of cooperation among the user classes. In each class the user behaviour is assumed to be either selfish, or class-cooperative, or cross-class altruistic, or fully cooperative. This makes a set of 16 cooperative patterns, ranging from User Equilibrium (UE) where both classes are selfish to System Optimum (SO) where both classes are fully cooperative, passing by a partial cooperation within the informed class, called IC for Informed Cooperation. A simple, two-link network is considered with linear travel time functions of link flow, additive random disturbances in travel times and two classes of users, respectively informed of disturbances in travel time or not. Then closed-form formulas are provided for the main variables of interest: this enables one to analyze clearly their sensitivity to the model parameters.
The extended model enables us to provide a clear definition of system optimization in the presence of dynamic information, by including a long term component and a short term component. On assuming Gaussian disruptions, analytical formulae are derived for the average (resp. marginal) individual trip cost by user class for both UE and SO. Based on these formulae, some specific qualitative properties are established:

- The marginal external cost of a user trip to the whole traffic depends on whether the user is informed or not: the difference makes a collective information gain which could justify subsidizing the user to get an equipment.

- Under UE, an informed user obtains an average trip cost (un-subsidized) which is lower than that of an uninformed user. This private information gain increases with the dispersion of the disruptions; it decreases with the volume of demand and the equipment rate. The overall average trip cost decreases with the equipment rate up to a given threshold, beyond which it increases.

- Under SO, an informed user obtains an average trip cost (un-subsidized) which may be larger than that of an uninformed user: this is because his dynamic reaction is aimed to improve the overall situation, possibly by “sacrificing” oneself.

Then the implementation of System Optimization would raise the issue of its acceptability to the informed user – who might benefit on an individual basis from not using the information! This motivated us to define the pattern of Informed Cooperation that would be acceptable since it would deliver the benefits from dynamic adaptation primarily to the informed users. In a numerical experiment we found that the overall results are very close to those of System Optimization.

The body of the paper is organized into eight parts. Section 1 sets the modelling assumptions. Section 2 provides the equilibrium solution for any pattern of cooperation among the user classes: analytical formulae are available for the main variables of interest under some more assumptions. Section 3 is focused on the economic analysis of the user classes within traffic: an individual user gets an average trip cost and causes a marginal external congestion cost to the rest of traffic. These costs are the key variables to characterize the relationships between the actors in the transportation system. We then analyze parametrically the UE pattern in Section 4 and the SO pattern in Section 5. Section 6 compares the two patterns in an analytical way. In Section 7, we use numerical simulation to investigate the Informed Cooperation pattern as well as the UE and SO patterns, for two network configurations that mimic either motorway versus urban road, or urban road versus urban arterial. Lastly, Section 8 concludes by assessing the model limitations and pointing to potential developments.
<table>
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<tr>
<td>$u, v \in {I, N}$</td>
<td>User class Informed or Not</td>
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<tr>
<td>$Q$ (resp. $q^I, q^N$)</td>
<td>Demand volume (resp. of informed users, of uninformed users)</td>
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<td>$\beta$</td>
<td>Equipment rate in DTI device</td>
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<td>$a$</td>
<td>Link index $\in A = {1,2}$</td>
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<td>$\omega$</td>
<td>Random occurrence $\in \Omega$</td>
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<td>$x_{a\omega}^u$</td>
<td>Flow of class $u$ on link $a$ under occurrence $\omega$</td>
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<td>$\zeta_{a\omega} = \zeta_a(\omega)$</td>
<td>Disturbance on link time</td>
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<td>$t_a(x_a)$</td>
<td>Link travel time function w.r.t. link flow $x_a$</td>
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<td>$i_a$</td>
<td>Derivative of $t_a$</td>
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<td>$T_{a\omega} = t_a(x_{a\omega}) + \zeta_{a\omega}$</td>
<td>Effective travel time of link $a$ under $\omega$</td>
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<tr>
<td>$\tau_{a\omega}$</td>
<td>Link pseudo time that dictates the route choice behaviour of a user of class $u$ under circumstance $\omega$</td>
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<td>$Z = \zeta_2 - \zeta_1$</td>
<td>Difference in link times – a random variable</td>
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<td>$z = Z(\omega)$</td>
<td>Value taken by $Z$ under circumstance $\omega$</td>
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<tr>
<td>$\sigma$</td>
<td>Standard deviation of $Z$</td>
</tr>
<tr>
<td>$F$ (resp. $\hat{F}$)</td>
<td>Cumulative Distribution Function of $Z$ (resp. PDF)</td>
</tr>
<tr>
<td>$\tilde{F}(x) = \int x^\omega dF(z)$</td>
<td>Truncated first moment of variable $Z$</td>
</tr>
<tr>
<td>$\tilde{F}(x) = \int x^2 dF(z)$</td>
<td>Truncated second moment of variable $Z$</td>
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<td>Related function $\tilde{F}$</td>
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<td>$\alpha_a + \gamma_a x_a$</td>
<td>Linear specification of link travel time function</td>
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<td>$\gamma = \gamma_1 + \gamma_2$</td>
<td>Parameter of overall sensitivity to congestion</td>
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<tr>
<td>$\gamma^u_a$</td>
<td>Parameter of consideration given by a user of class $u$ to his marginal effect on the class $v$ flow on link $a$ (excepting himself if $v = u$)</td>
</tr>
<tr>
<td>$P$</td>
<td>A cooperation pattern</td>
</tr>
<tr>
<td>$C_p^u$</td>
<td>Average trip cost effectively borne by a user of class $u$</td>
</tr>
<tr>
<td>$\Gamma_p^{self} = C_p^N - C_p^I$</td>
<td>User gain of getting equipped in a DTI device, per trip</td>
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<tr>
<td>$K_p^{uv}$</td>
<td>Average marginal cost inflicted by a user of class $u$ on class $v$</td>
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<tr>
<td>$\Gamma_p^{ext}$</td>
<td>External social gain of providing DTI to a marginal user</td>
</tr>
<tr>
<td>$\Gamma_p^{sys} = \Gamma_p^{self} + \Gamma_p^{ext}$</td>
<td>System gain of marginal DTI equipment</td>
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(To be continued)
\[
\begin{array}{|c|c|}
\hline
\theta & = \gamma_1 \gamma_2 Q + \alpha_1 \gamma_2 + \alpha_2 \gamma_1 \\
& = \frac{\gamma_1 + \gamma_2}{\gamma_1 + \gamma_2} \text{ Average user cost at UE with no DTI} \\
\hline
\theta' & = \theta - \Delta \alpha^2 / 4 \gamma Q \text{ Average user cost at SO with no DTI} \\
\hline
\gamma^# & = \gamma^I_1 + \gamma^I_2 \\
\hline
\alpha^I_a & = \alpha_a + \gamma^N_a \bar{x}_a^N \\
\hline
\alpha^N_a & = \alpha_a + \gamma^N_a \bar{x}_a^I \\
\hline
\alpha^I_{a0} & = \alpha_a + \gamma_a \bar{x}_a^N \\
\hline
B_0 & = \alpha^I_{a0} - \alpha^I_{a2} + \gamma^I_1 q^I \\
\hline
A & = \alpha^I_1 - \alpha^I_2 - \gamma^I_2 q^I \\
\hline
B & = \alpha^I_1 - \alpha^I_2 + \gamma^I_1 q^I \\
\hline
\end{array}
\]

1. MODELLING ASSUMPTIONS

Let us come to the modelling assumptions that pertain to, respectively: (i) the supply side of the network and its routes, with the local travel times, congestion effects and random disturbances; (ii) the demand side of the trip-makers whether or not equipped with a device for receiving dynamic information; (iii) the interaction of supply and demand.

1.1 Supply side assumptions

Let us consider a transport network made up of arcs \( a \in A \) the arc set, with endpoints \( n \in N \) the node set. Our application here is restricted to a classroom case of two parallel arcs linking an origin node to a destination node, as in Figure 1a: hence \( A = \{1,2\} \) and \( N = \{O,D\} \).

On each network arc \( a \), the arc flow \( x_a \) induces an individual travel time of \( T_a \) which is subject to congestion effects on the basis of a travel time function \( T_a = t_a(x_a) \), an increasing function of the arc flow. For instance let us take a linear affine function as follows, \( t_a(x_a) = \alpha_a + \gamma_a x_a \) in which \( \alpha_a \) denotes a free-flow travel time and \( \gamma_a \) the sensitivity of the individual time to the flow (see Figure 1b). This assumption is consistent with the unqueued state of traffic, not with the queued state in which the flow is constrained by a flowing capacity.

Despite our flow-based model is basically a stationary model, dynamic effects are considered from period to period, by modelling the inter-period variability as a random variable. We assume that there is a set \( \Omega \) of circumstances (or periods) \( \omega \), each of which has arc flow \( x_{a\omega} = x_a(\omega) \) and arc travel time

\[
T_{a\omega} = t_a(x_{a\omega}) + \zeta_a(\omega) .
\]  

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\[
T_{a\omega} = t_a(x_{a\omega}) + \zeta_a(\omega) .
\]
The random variable \( \zeta_a(\omega) \) models the eventual variation in travel time that may arise due to exogenous disturbances. It is assumed to have variance of \( \sigma_a^2 \) and null mean, which is consistent with the no bias assumption on the travel time function, that is

\[
t_a(x) = E_\omega[T_{a0} : x_{a0} = x].
\]

For simplicity it can be assumed that the variables \( \zeta_a \) are independently distributed.

\[
T_{a0} = t_a(x_{a0}) + \zeta_{a0},
\]

\[
x_{a0} = x_a^N + x_a^I
\]

since the class \( I \) users react to dynamic information hence to situation \( \omega \) while the \( N \) users do not react in the short run.

The travel cost that is known prior to route choice by a user of class \( u \in \{ I, N \} \) is \( t_{a0}^u \) such that

\[
t_{a0}^I = t_{a0},
\]

\[
t_{a0}^N = E_\omega[T_{a0}].
\]

Let us assume that every user behaves in a class-specific way to be specified from among:

- Selfish behaviour: the individual user chooses by himself his route so as to minimize his own perceived travel cost: \( C_{a0}^{u,\text{self}} = t_{a0}^u \).
Cooperative behaviour within his own class, in which case the class users are assigned to the available routes so as to minimize their mean class cost:

\[ G_{a0,\text{class}}^u = i_{a0}^u + x_{a0}^u i_a, \text{letting } i_a \equiv \frac{dt_a}{dx_a}. \]

- Cooperative behaviour with the other class only, i.e. every class user takes a route so as to minimize his travel cost augmented by the external cost of congestion which he puts on the users of the other class: 

\[ G_{a0,\text{other}}^u = i_{a0}^u + (x_{a0} - x_{a0}^u) i_a. \]

- Fully cooperative behaviour with all traffic, i.e. every class user takes a route so as to minimize his travel cost augmented by the external cost which he puts on all other users: 

\[ G_{a0,\text{all}}^u = i_{a0}^u + x_{a0} i_a. \]

This paper is not concerned with how to implement any of the four behaviours. The focus is on deriving the consequences of assuming a given behaviour for each user class. There are \( 16 = 4 \times 4 \) pairs of class behaviours, making up 16 patterns of trip-maker cooperation.

1.3 Pseudo cost functions

In network assignment theory it is well-known that a problem of system optimum may be cast into an equivalent problem of user equilibrium, by internalizing the external congestion cost into a pseudo function of individual travel cost: to any cost function \( t_a(x_a) \) considered for system optimum is associated a pseudo cost function as follows:

\[ t_a^\#(x_a) \equiv t_a(x_a) + x_a i_a(x_a). \]

By assuming that the individual user perceives \( t_a^\# \) as his own cost along route \( a \) and chooses the route of minimum perceived cost to himself, then a user equilibrium flow state for the pseudo cost functions \( t_a^\# \) coincides with a system optimum flow state for the original cost functions \( t_a \).

We adapt this transformation principle to multiple user classes and various patterns of cooperation: a user of class \( u \) cooperates with class \( v \) either actively \((\epsilon_{uv} = 1)\) if he considers his congestion effect on that class or passively \((\epsilon_{uv} = 0)\) if he is indifferent to class \( v \). Then the pseudo cost function to user \( u \) is – notwithstanding the stochastic disturbance:

\[ t_a^\#(x_a) \equiv t_a(x_a) + [\sum_v \epsilon_{uv} x_v^\#] i_a. \]

A related notation in italics, \( t_a^{\#u} \), stands for the dependency on the random occurrence \( \omega \) through \( \zeta_{a0} \) and \( x_{a0} \).

Table 1 provides the pseudo cost function that corresponds to any behaviour of any user class \( u \). In Table 2, linear affine functions \( t_a(x_a) = \alpha_a + \gamma_a x_a \) and centred disturbances i.e. \( E_{\omega}[\zeta_a] = 0 \) are assumed, yielding simplification. Given a cooperation pattern, we shall accordingly define the \( \gamma^u_v \) coefficients that stand for the consideration given by a user of class \( u \) to the flow of class \( v \), hence:

\[ t_a^\#(x_{a0}) = \alpha_a + \gamma^u_a x_{a0} N_a + \gamma^u_a x_{a0} I_a. \]
1.4 A pseudo user equilibrium formulation

Every user is assumed to choose his network route from origin to destination under a rational behaviour of cost minimization, subject to his knowledge of the costs. An informed user is assumed to know the pseudo cost accurately in any circumstance:

$$t_{a0}^I = t_a(x_{a0}) + \zeta_{a0}, \quad (7a)$$

while a non-equipped user is assumed to possess only an average knowledge:

$$t_{a0}^N = E[ t_a(x_{a0}) + \zeta_{a0} ] \quad (7b)$$

Here the user cost is restricted to the travel time component – neglecting tolls, comfort and other quality criteria in order to focus on the disturbances which make our primary concern.

Let us denote by $Q$ the total demand volume, assumed constant whatever the circumstance.

Let $\beta = q^l/Q$ denote the equipment rate within the demand i.e. the ratio of the volume of equipped users $q^l$, to $Q$, and let $q^N = Q - q^l$ be the volume of non-equipped users.

The conditions for user equilibrium are: (i) the non negativity of flows; (ii) the assignment of all volume to the network routes; (iii) the assignment of every user to a route of minimum pseudo cost to him; (iv) the formation of the travel times, on the basis of Eqn. (6) and (7).

By arc $a$, occurrence $\omega$ and user class $u \in \{I, N\}$, $x_a^u(\omega)$ denotes the flow on arc $a$. Equilibrium condition (i) is stated as:
\[ x^u_a(\omega) \geq 0, \ \forall a, u, \omega. \]  
\[ \bar{x}^u_a = E[x^u_{a\omega} : \omega \in \Omega], \ \forall a, u. \]  

Equilibrium condition (ii) is stated as:
\[ \sum_{a \in \{1, 2\}} x^u_a(\omega) = q^u, \ \forall u, \omega. \]

Equilibrium condition (iii) involves dual variables \( \mu^I_{\omega} \), \( \mu^N \) that correspond to the optimum pseudo cost by user class – by occurrence \( \omega \) for class \( I \) or on the average for class \( N \):
\[ t^I_{a\omega} - \mu^I_{\omega} \geq 0, \ x^I_a(\omega)(t^I_{a\omega} - \mu^I_{\omega}) = 0 : \ \forall \omega \in \Omega, \ \forall a \in A. \]  
\[ t^N_{a\omega} - \mu^N \geq 0, \ x^N_a(\omega)(t^N_{a\omega} - \mu^N) = 0 : \ \forall a \in A. \]

1.5 The macroscopic structure of pseudo equilibrium

Let us chain the assumptions about supply and demand into the following statement:
- In any occurrence \( \omega \) every network user chooses his route: his travel along that route makes a piece of flow.
- Considering the population of trip-makers, their individual choices induce the arc flows onto the network.
- The arc flows determine the arc travel times on the basis of the congestion functions.
- The travel times determine the individual route choice.

Thus there is a cyclical chaining in the interaction of supply and demand. Figure 2 makes the statement more precise by addressing each user class in a specific way: a class \( I \) user reacts to any particular occurrence \( \omega \) by adapting his route choice to the dynamic context – thus leading to \( x^I_a(\omega) \) flows that vary with \( \omega \) hence in the short run, while a class \( N \) user makes his route choice only in the long run on the basis of the route performance averaged over the distribution \( \Omega \) of cases \( \omega \), leading to \( \bar{x}^N_a \) flows that do not vary with \( \omega \).

FIGURE 2 Model composition by chaining the assumptions
Our model should not be mistaken as a two-class assignment model of a deterministic user class together with a Probit user class: as shown in Leurent and Nguyen (2008), in a Probit model the arc time is \( t_a(x_a) + \zeta_{a0} \) i.e. the travel time function has an average flow \( \bar{x}_a \) as argument, instead of an occurrence flow \( x_a(\omega) \).

2. EQUILIBRIUM ANALYSIS

Following the logical structure in Figure 2, we shall first assume a given assignment of class \( N \) and focus on the assignment of class \( I \) in any occurrence \( \omega \) (§2.1); then we will average the informed flows over all cases (§2.2). Next the class \( N \) users will be assigned conditional on the class \( I \) average assignment (§2.3). After having characterized the equilibrium as a fixed point problem (§2.4), we shall indicate the “symmetry simplification” that plays an important role in the analytical treatment (§2.5).

2.1 Occurrence assignment of equipped users

Let \( \alpha_a^I = \alpha_a + \gamma_a^N x_a^N \) : then \( t_{a0}^I = \alpha_a^I + \gamma_a^{IL} x_{a0} + \zeta_{a0} \) is the travel time of arc \( a \) under occurrence \( \omega \). If there were only one equipped user, he would choose the route of minimum \( t_{a0}^I(0) = \alpha_a^I + \zeta_{a0} \). However the dynamic reassignment of informed users will tend to increase the travel time of that route due to its congestion function: this effect may yield partial compensation i.e. \( t_{a0}^I \) remains less than \( t_{b0}^I \) for \( b = 3 - a \), or total compensation i.e.

\[
\alpha_a^I + \gamma_a^{IL} x_{a0} + \zeta_{a0} = \alpha_b^I + \gamma_b^{IL} x_{b0} + \zeta_{b0} \tag{9a}
\]

In the latter case, combining (9a) to (8b) leads to:

\[
x_{a0}^I = \frac{\gamma_b^{IL} q^I + \alpha_b^I + \zeta_{b0} - \alpha_a^I - \zeta_{a0} \gamma_a^{IL}}{\gamma_a^{IL} + \gamma_b^{IL}}, \tag{9b}
\]

\[
t_{a0}^I = \frac{\gamma_a^{IL} \gamma_b^{IL} q^I + \gamma_a^{IL} \alpha_b^I + \gamma_b^{IL} \alpha_a^I + \gamma_a^{IL} \zeta_{a0} + \gamma_b^{IL} \zeta_{b0} \gamma_a^{IL}}{\gamma_a^{IL} + \gamma_b^{IL}}. \tag{9c}
\]

In the former case, \( x_{a0}^I = q^I \) and \( x_{b0}^I = 0 \), hence \( t_{a0}^I = \alpha_a^I + \gamma_a^{IL} q^I + \zeta_{a0} \) and \( t_{b0}^I = \alpha_b^I + \zeta_{b0} \) with \( t_{a0}^I \leq t_{b0}^I \) hence:

\[
\zeta_{b0} - \zeta_{a0} \geq \alpha_a^I - \alpha_b^I + \gamma_a^{IL} q^I. \tag{10}
\]

2.2 From conditional to average assignment of equipped users

Let \( B \equiv \alpha_1^I - \alpha_2^I + \gamma_1^{IL} q^I \), \( A \equiv \alpha_1^I - \alpha_2^I - \gamma_2^{IL} q^I \) and define random variable \( Z = \zeta_2 - \zeta_1 \). In any occurrence \( \omega \), conditional on the value \( z = Z(\omega) \) the informed users are assigned according to only one of the following three cases:

- if \( z > B \) then \( t_{10}^I(q^I) \leq t_{20}^I(0) \) hence \( x_1^I(\omega) = q^I \) and \( x_2^I(\omega) = 0 \), \( \tag{11a} \)

- if \( z < A \) then \( t_{10}^I(0) \geq t_{20}^I(q^I) \) hence \( x_1^I(\omega) = 0 \) and \( x_2^I(\omega) = q^I \), \( \tag{11b} \)
If \( z \in [A, B] \) then \( I_{10}^# = I_{20}^# \) at flows \( x^I_1(\omega) = \frac{z - A}{\gamma_1^I + \gamma_2^I} \) and \( x^I_2(\omega) = \frac{B - z}{\gamma_1^I + \gamma_2^I} \). (11c)

This deterministic process amounts to defining a function \( \chi^I_1(z) \equiv x^I_{10} \).

On the basis of the distribution function \( F \) of \( Z \) and its truncated moment function \( \tilde{F}(\omega) = \int x(z) dF(z) \), we get that:

\[
\chi^I_1(z) = \int_{-\infty}^{\infty} \frac{B - z}{A \gamma_1^I + B \gamma_2^I} dF(z) + q^I . \int_{-\infty}^{\infty} dF(z) \]

Some computation yields the complementary flow \( \chi^I_2 = q^I - \chi^I_1 \) as a simple function of \( A \) and \( B \), in which \( G(x) = xF(x) - \tilde{F}(x) = \int F(z) dz \):

\[
\chi^I_2 = \frac{G(B) - G(A)}{\gamma_1^I + \gamma_2^I} . \quad (12b)
\]

### 2.3 The assignment of non-informed users

Let \( \alpha^N_a = \alpha^N_a + \gamma^N_a \chi^I_a \) and compare \( \alpha^N_a \) with \( \alpha^N_2 \):

- if \( \alpha^N_1 \geq \alpha^N_2 + \gamma^N_2 q^N \) then \( \chi^N_2 = q^N \) and \( \chi^N_1 = 0 \). (13a)

- if \( \alpha^N_1 + \gamma^N_1 q^N \leq \alpha^N_2 \) then \( \chi^N_1 = q^N \) and \( \chi^N_2 = 0 \). (13b)

- if \( -\gamma^N_1 q^N \leq \alpha^N_2 - \alpha^N_1 \) then \( \chi^N_a = \frac{\alpha^N_b - \alpha^N_a + \gamma^N b q^N}{\gamma_1^N + \gamma_2^N} \). (13c)

The last equation stems from (7b) and the equilibrium condition of user class \( N \): \( I_{a}^# = I_{b}^# \) that holds in the sub-domain.

### 2.4 Fixed-point characterization of supply-demand equilibrium

Linking together the previous formulae, we obtain that the unequipped flows \( (\chi^N_a) \) induce the reference informed travel times \( (\alpha^I_a) \), which in turn determine the average equipped flows \( (\chi^I_a) \) (Eqn 12), which in turn determine the reference un-equipped travel times \( (\alpha^N_a) \), which in turn determine the unequipped flows \( (\chi^N_a) \) (Eqn 13). This cycle states that any set of variables either \( (\chi^N_a) \), \( (\chi^I_a) \), \( (\alpha^I_a) \) or \( (\alpha^N_a) \) solves a specific problem of fixed point. In the two-link case that kind of problem is easy to solve since it involves only one real unknown; a relaxation algorithm would be appropriate, for instance a convex combination algorithm in the unequipped flows. So, the fixed-point problem for \( x^N_2 \) is formulated as:

\[
x^N_2 = X^N_2 \circ X^I_2(x^N_2), \quad \text{in which:} \quad (14a)
\]
\[
- X_2^I (x_2^N) = \frac{G(B) - G(A)}{\gamma_{II}^H + \gamma_{I2}^H} \quad \text{wherein } B(x_2^N) \text{ and } A(x_2^N) \text{ through the } \alpha_{ui}^N, \quad (14b)
\]

\[
- X_2^N = \min\{q^N, \max\{0, \frac{\alpha_{NN}^N - \alpha_{II}^N + \gamma_{II}^N q^N}{\gamma_{NN}^N + \gamma_{I2}^N}\}\}. \quad (14c)
\]

### 2.5 The “symmetry” simplification

If the class interactions are symmetric i.e. \( \gamma_{II}^H = \gamma_{II}^N = \gamma_{I2}^N = \gamma_{II}^N \equiv \gamma_{a} \), which includes the UE and SO cases, then

\[
B = t_{1^N}^I - \gamma_{1}^I x_1^I - t_{2^N}^I + \gamma_{2}^I x_2^I - \gamma_{I}^N q^I = t_{1^N}^I - t_{2^N}^I + (\gamma_{2}^I - \gamma_{I}^N) x_2^I.
\]

Assuming equal average pseudo times for the two links and combining with (12b) yields

\[
B = G(B) - G(B - (\gamma_{2} + \gamma_{I}) q^I ). \quad (14)
\]

Assuming further that the random variable \( Z \) is centred and symmetrically distributed, then \( G(-x) = G(x) - x \) for all \( x \). This implies that (14) is solved by \( B \) such that

\[-B = B - (\gamma_{2} + \gamma_{I}) q^I , \quad \text{in other words } B = \frac{1}{2} (\gamma_{2} + \gamma_{I}) q^I = -A. \]

Substituting in the definition of \( B \), this implies in turn that

\[
\frac{q^I}{2} = \gamma_{a}.
\]

indeed a remarkable property.

### 3. ACTORS AND INDICATORS

Let us now focus on the economic analysis of the actors, namely any user of a given class and the whole traffic which we assimilate to the society.

#### 3.1 Actors and relationships

Under any cooperation pattern \( P \), let \( C_{P}^{I} \) denote the average trip cost per user of class \( u \). A network user of class \( u \) makes one trip per occurrence \( \omega \). Throughout the occurrences he incurs an average effective cost of

\[
\bar{C}_{P}^{I} = \frac{1}{q_{u}} \sum_{a \in \{1,2\}} x_{a0}^{u} (t_{a}(x_{a0}) + \zeta_{a0}). \quad (16)
\]

This average cost involves the effective occurrence cost, not the pseudo cost function.

An individual user of class \( u \) makes up one unit of flow: if he uses link \( a \) under occurrence \( \omega \) then he imposes on any other user of that link a delay of \( t_{a}(x_{a0}) \), hence on the flow of class \( v \) (excepting himself if \( u = v \)) an overall delay of

\[
\chi_{a0}^{uv} = x_{a0}^{v} t_{a}(x_{a0}). \quad (17a)
\]

A marginal user of class \( u \) would use any link \( a \) in a proportion of \( x_{a0}^{u} / q_{u} \) since he makes one unit of flow to be split between the links in accordance with the assignment of \( q_{u} \) to the links under that occurrence, yielding an occurrence, marginal cost to class \( v \) of
\[ \chi_{\omega}^{uv} = \sum_a x_{a\omega}^v \hat{i}_a(x_{a\omega}) \frac{x_{a\omega}^u}{q^u}. \] (17b)

Thus the average marginal cost of a user of class \( u \) on the flow of class \( v \) is

\[ \bar{\chi}_p^{uv} = E_{\omega}[\sum_a x_{a\omega}^v \hat{i}_a(x_{a\omega}) \frac{x_{a\omega}^u}{q^u}] = \frac{1}{q^u} E_{\omega}[\sum_a x_{a\omega}^v x_{a\omega}^u \hat{i}_a(x_{a\omega})]. \] (17c)

The relationship between user \( u \) and the society is that the user incurs cost \( \bar{C}_P^u \) whereas he induces cost \( \bar{C}_P^u + \sum_v \bar{\chi}_p^{uv} \). The part \( \bar{\chi}_{aP} = \sum_v \bar{\chi}_p^{uv} \) is a marginal, external cost of congestion.

### 3.2 Average costs

The evaluation of \( \bar{C}_P^N \) is straightforward since \( x_{a\omega}^N \) keeps to value \( \bar{x}_a^N \):

\[ \bar{C}_P^N = [\bar{x}_1^N (\alpha_1 + \gamma_1 \bar{x}_1) + \bar{x}_2^N (\alpha_2 + \gamma_2 \bar{x}_2)]/q^N. \] (18)

To evaluate the mean cost of a trip to an informed user, \( \bar{C}_P^I = E_{\omega}[x_{1\omega}^I t_{1\omega} + x_{2\omega}^I t_{2\omega}]/q^I \), let us decompose it conditionally on \( Z \):

\[ \bar{C}_P^I = \frac{1}{q^I} \int E[X_1^I(z)T_{1\omega} + (q^I - X_1^I(z))T_{2\omega}] \, dF(z). \] (19)

From the detailed computation in the Appendix, it comes out that

\[ \bar{C}_P^I = (\alpha_2^I + \gamma_2 q^I) F_A + (\alpha_1^I + \gamma_1 q^I)(1 - F_A) \]

\[ + \frac{\gamma_1 \gamma_2 q^I}{\gamma_2^2 q^I} \left( F_B - F_A \right) \]

\[ + \frac{\gamma_2 - \gamma_2}{\gamma_2^2 q^I} \left( F_B - F_A \right) \]

wherein \( \gamma = \gamma_1 + \gamma_2, \gamma^I = \gamma_1^I + \gamma_2^I, \alpha_a^I = \alpha_a + \gamma_a^I \bar{x}_a^N, B_0 \equiv \alpha_1^I - \alpha_2^I + \gamma_1 q^I \) and function \( \tilde{F}(x) = \int x - z^2 \, dF(z) \).

Aggregating over the user classes, we get the average trip cost per user:

\[ \bar{C}_P = \beta \bar{C}_P^I + (1 - \beta) \bar{C}_P^N. \] (21)

### 3.3 Private information gain

The private gain associated with the disposal of dynamic information is equal to the difference in average trip cost between non-equipped and informed users:

\[ \Gamma_{P}^{\text{self}} = \bar{C}_P^N - \bar{C}_P^I. \] (22)
3.4 Marginal external cost

The probabilistic nature of the model requires some caution in evaluating the marginal cost of a given user class \( u \), \( \delta(QC_p^u)/\delta q^u \), which splits into an internal part of \( C_p^u \) and an external part of \( \delta(QC_p^u)/\delta q^u - C_p^u \).

Let us decompose the total travel cost by user class:

\[
QC_p = q^I C_p^I + q^N C_p^N = \sum_{v \in \{1, N\}} q^v C_p^v,
\]

in which

\[
q^v C_p^v = E_{\omega}[\sum_{a \in \{1, 2\}} x_{a \omega} T_{a \omega}(x_{a \omega})].
\]

This enables us to derive the marginal cost in two ways. In a top-down approach, based on the left-hand side:

\[
\frac{\partial}{\partial q^u}(q^v C_p^v) = \delta_{uv} C_p^u + q^v \frac{\partial}{\partial q^u}(C_p^v).
\] (23a)

In a bottom-up approach on the basis of the right-hand side, as

\[
\frac{\partial}{\partial q^u} E_{\omega}[\sum_{a \in \{1, 2\}} x_{a \omega} T_{a \omega}(x_{a \omega})] = \delta_{uv} E[\sum_{a} x_{a \omega} i(x_{a \omega})] + E[\sum_{a} x_{a \omega} i(x_{a \omega})] - \delta_{uv} C_p^u + \chi_p^uv
\]

by identification based on (17c)

On comparing (23a) and (23b), we obtain that

\[
\chi_p^uv = q^v \frac{\partial}{\partial q^u}(C_p^v).
\] (23c)

The marginal cost of a class \( u \) user to the system is:

\[
\frac{\partial}{\partial q^u}(QC_p) = C_p^u + \chi_p^ul + \chi_p^un
\] (23d)

**Lemma.** To evaluate an external cost \( \chi_p^uv \), assume that an analytical function \( \overline{C}_p^v = \psi(Q, \beta) \) is available for the average cost of class \( v \) with respect to parameters \( Q \) and \( \beta \). The derivatives of \( \psi \) with respect to \( Q \) and \( \beta \) can be used to evaluate \( \chi_p^uv \) as

\[
\chi_p^uv = q^v \left[ \frac{\partial \overline{C}_p^v}{\partial Q} + \frac{\delta_{ul} - \beta}{Q} \frac{\partial \overline{C}_p^v}{\partial \beta} \right].
\]

**Proof.** A variation \( \delta Q^l \) induces \( \delta Q = \delta Q^l \) hence \( \delta Q/\delta Q^l = 1 \). The induced variation \( \delta \beta \) is such that \( \beta + \delta \beta)(Q + \delta Q) = Q^l + \delta Q^l \), hence \( Q^l \delta \beta + \beta \delta Q = \delta Q^l \) which yields that \( \delta \beta/\delta Q^l = (1 - \beta)/Q \). Similarly, a variation \( \delta Q^N \) induces \( \delta Q = \delta Q^N \) hence \( \delta Q/\delta Q^N = 1 \). The induced variation \( \delta \beta \) is such that \( (\beta + \delta \beta)(Q + \delta Q) = Q^l \), hence \( Q^l \delta \beta + \beta \delta Q^N = 0 \) which yields that \( \delta \beta/\delta Q^N = -\beta/Q \). Overall, \( \delta Q/\delta Q^l = 1 \) and \( \delta \beta/\delta Q^N = (\delta_{ul} - \beta)/Q \), in which \( \delta_{ul} = 1 \) if \( u = 1 \) or \( 0 \) if \( u = N \). By the chain rule of derivation,
\[
\frac{\partial \psi}{\partial q^a} = \frac{\partial \psi}{\partial Q} \frac{\partial Q}{\partial q^a} + \frac{\partial \psi}{\partial \beta} \frac{\partial \beta}{\partial q^a} = \frac{\partial \psi}{\partial Q} + \frac{\delta u}{Q} \frac{\partial \psi}{\partial \beta}.
\]

### 3.5 Marginal external social information gain

Let us denote by \( J_{ab} = \sum_v J_{P}^{uv} \) the external cost of a user of class \( u \) under cooperation pattern \( P \). To the whole traffic, the difference in external cost between a non-equipped user and an informed user, \( \Gamma_P^{\text{ext}} = J_{NP} - J_{IP} \), is the system external gain associated with the provision of dynamic information to a marginal user, e.g. with the equipment of a previously uninformed user. This social information gain could justify subsidizing a user to get an information device.

The marginal system gain of equipping a previously non-equipped user is composed of the private gain plus the external gain:

\[
\Gamma_P^{\text{sys}} = \frac{\partial}{\partial q^u}(Q \bar{C}_P) - \frac{\partial}{\partial q^u}(Q \bar{C}_P) = \Gamma_P^{\text{self}} + \Gamma_P^{\text{ext}}.
\]

### 4. ON THE USER EQUILIBRIUM PATTERN

Not only do the User Equilibrium (UE) and System Optimum (SO) patterns possess major economic significance, but they also are endowed with nice analytical properties. This enables us to study the basic patterns in a parametric way that yields controlled and generic properties.

#### 4.1 Average trip cost by user class

Under the UE pattern, every user behaves in a selfish way, yielding \( \gamma_a^{uv} = \gamma_a \) for every pair \((u,v)\) of user classes. Then \( \gamma_a = \gamma \) and \( B_0 = B \) in Eqn (20), which is simplified into

\[
\bar{C}_{UE}^I = \bar{t}_1 - A \Gamma_A + \int_{A}^{A} \zeta_{2/z} \ dF + \int_{-\infty}^{+\infty} \zeta_{1/z} \ dF,
\]

in which \( \zeta_{a/z} = E[\zeta_{z\omega} : Z(\omega) = z] \), provided that \( \bar{t}_1 \leq \bar{t}_2 \) i.e. link 1 is used by the non-informed users. As \( Z = \zeta_2 - \zeta_1 \) and the disturbance variables are centred, then Appendix 1 applies and (25a) is reduced to

\[
\bar{C}_{UE}^I = \bar{t}_1 - G(A).
\]

As \( \bar{t}_1 = \bar{C}_{UE}^N \), this demonstrates that the private information gain is indeed positive, since

\[
\Gamma_{UE}^{\text{self}} = \bar{C}_{UE}^N - \bar{C}_{UE} = G(A) \text{ and } G \text{ is a positive function.}
\]

More precise properties are available in the case of equal average link travel times (see Appendix): then

\[
\bar{C}_{UE}^N = \frac{\gamma_1 \gamma_2 Q + \alpha_1 \gamma_2 + \alpha_2 \gamma_1}{\gamma_1 + \gamma_2} \equiv \theta
\]

which is also the equilibrium average travel cost in the absence of disturbances and dynamic information. Furthermore, \( A = -\frac{1}{2} \gamma \beta /Q \) hence
\[ \bar{C}_{\text{UE}} = \theta - G(-\frac{1}{2} \gamma \beta Q), \]  
\[ \Gamma_{\text{UE}}^{\text{self}} = G(-\frac{1}{2} \gamma \beta Q). \]  

The average trip cost is

\[ \bar{C}_{\text{UE}} = \theta - \beta G(-\frac{1}{2} \gamma \beta Q), \]  

which shows that the system benefits of DTI under UE amount to

\[ Q \bar{C}_{\text{UE}}(\beta = 0) - Q \bar{C}_{\text{UE}}(\beta) = \beta Q G(A) = \beta Q G(-\frac{1}{2} \gamma \beta Q). \]  

This function is positive; it is first increasing then decreasing with respect to \( \beta Q \), the number of informed users.

### 4.2 Sensitivity analysis of the private information gain

Since \( G \) is a positive, increasing function, formula (28) implies that the private information gain decreases with the demand volume, \( Q \), the equipment rate, \( \beta \), and the sensitivity to congestion, \( \gamma = \gamma_1 + \gamma_2 \).

If the disruptions are Gaussian then \( G(z) = \sigma g(z/\sigma) \), in which \( \sigma \) is the standard deviation of \( Z \) and \( g(x) = \phi(x) + x\Phi(x) \) a positive, increasing function depicted in Figure 3. Then

\[ \Gamma_{\text{UE}}^{\text{self}} = \sigma g(-\frac{\gamma \beta Q}{2\sigma}), \]  

which implies that the private information gain is an increasing function of the dispersion \( \sigma \) of the disruption in travel times.

![FIGURE 3- Graph of function g](image)

### 4.3 Marginal social costs

Assuming equal average link times, it holds that \( \partial \bar{C}_{\text{UE}}^N / \partial \beta = 0 \) and

\[ \frac{\partial}{\partial \gamma} \bar{C}_{\text{UE}} = \frac{\partial}{\partial \gamma} \theta = \frac{\partial}{\partial \gamma} \theta = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} \equiv \hat{\theta}. \]  

As \( \partial \bar{C}^l / \partial \lambda = \hat{\theta} + \frac{1}{2} \gamma \hat{\beta} F(A) \) and \( \partial \bar{C}^l / \partial \beta = \frac{1}{2} \gamma \beta F(A) \), on the basis of §3.4 we obtain the marginal external cost of a trip by user class:
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\[ \mathcal{X}_{UE}^I = q^N \dot{\theta} + q^I \{ \dot{\theta} + \frac{\gamma^N B}{2} F_A + \frac{1-B}{2} (\gamma^N \mathcal{F}_A) \} = Q \dot{\theta} + \frac{\gamma^N B}{2} F(-\frac{\gamma^N B}{2}), \]  

(33a)

\[ \mathcal{X}_{UE}^N = Q \dot{\theta}. \]  

(33b)

Thus the marginal external information gain amounts to

\[ \Gamma_{UE}^{ext} = \mathcal{X}_{UE}^N - \mathcal{X}_{UE}^I = -\frac{\gamma^N B}{2} F(-\frac{\gamma^N B}{2}). \]  

(33c)

This is a negative quantity since \( F \) is positive: noticing that under the equal link time assumption the dynamic information of one particular user does not benefit to the uninformed users, the rationale is that the more informed users, the less individual profit is derived by each of them from being equipped. Recalling (28) and the definition of \( G \), the total marginal social information gain is as follows – showing that its sign is expected to vary with \( \gamma^N B \):

\[ \Gamma_{UE}^{sys} = \Gamma_{UE}^{self} + \Gamma_{UE}^{ext} = -\gamma^N B F(-\frac{\gamma^N B}{2}) - \tilde{F}(-\frac{\gamma^N B}{2}). \]

5. ON THE SYSTEM OPTIMUM PATTERN

The system optimum pattern provides a reference for any alternative pattern, since it corresponds to overall cost minimization. This Section is focused on the definition and properties of the SO pattern under Dynamic Traffic Information. The comparison of UE and SO is dealt with in Section 6, whereas the related issues of implementation and user acceptability are addressed in Section 7.

5.1 A bi-layer definition of system optimum

A given assignment of demand consists in flow vectors \( (x_a^N)_{a}, (x_{a0})_{a0} \). It yields an overall travel cost of

\[ J = E_\omega \{ \sum_a x_{a0} T_{a0} \}, \]  

(34)

wherein \( x_{a0} = x_a^N + x_{a0}^I \) and \( T_{a0} = t_a(x_{a0}) + \zeta_{a0} \).

The SO pattern involves a demand assignment that minimizes the function \( J \). Under given \( (x_a^N)_a \), the problem to find an SO state can be decentralized into one problem per occurrence \( \omega \):

\[ \min_{x_{a0}} J_\omega(x_{a0} | x_a^N) = \sum_{a \in \{1,2\}} (\mathcal{X}_a^N + x_{a0}) t_a(\mathcal{X}_a^N + x_{a0}) + \zeta_{a0}. \]

Given \( \omega \), the term \( \zeta_{a0} \) is a constant hence every decentralized problem is a classical, undisrupted SO problem: this legitimates the short-run approach taken in §2.2 for assigning the informed users conditional on the uninformed flows.

On aggregating the optimized decentralized travel costs \( J_\omega^s(\mathcal{X}_a^N) \equiv J_\omega(x_{a0}^I | x_a^N) \), we obtain an overall cost function with respect to the uninformed link flows only:

\[ \bar{J}(\mathcal{X}_a^N) = E_\omega [J_\omega^s(\mathcal{X}_a^N)] \]  

(35)

Derivation of \( \bar{J} \) with respect to \( x_b^N \) yields
\[
\frac{\partial \bar{J}}{\partial x_b^N} = \sum_a E_{ao}(\delta_{ao} + \frac{\partial x_{10}^I}{\partial x_b^N} + \frac{\partial x_{20}^I}{\partial x_b^N})[t_b(x_{b0}) + \zeta_{b0} + x_{b0} i_b(x_{b0})] + \frac{\partial x_{10}^I}{\partial x_b^N} + \frac{\partial x_{20}^I}{\partial x_b^N}
\] (36)

For any \( \omega \), \( \frac{\partial x_{10}^I}{\partial x_b^N} = 0 \) if \( Z(\omega) \leq A' \) or \( \geq B' \), \( A' \leq B' \) being the integration thresholds conditional on \( (x_a^N)_a \). On the intermediary domain \( Z(\omega) \in [A', B'] \), the social assignment of the informed users yields equal social times \( t_{a0}^I \); then

\[
\frac{\partial x_{10}^I}{\partial x_b^N} + \frac{\partial x_{20}^I}{\partial x_b^N} = t_{a0}^I \frac{\partial (x_{10}^I + x_{20}^I)}{\partial x_b^N} = t_{a0}^I \frac{\partial q^I}{\partial x_b^N} = 0.
\]

This enables us to drop the terms in \( \frac{\partial x_{10}^I}{\partial x_b^N} \) from further consideration. Simplifying in (36) we get that:

\[
\frac{\partial \bar{J}}{\partial x_b^N} = E_{ao}[t_b(x_{b0}) + \zeta_{b0} + x_{b0} i_b(x_{b0})],
\]

which is consistent with the long-run approach taken in §2.3 for assigning the uninformed users conditional on the average informed flows. To summarize, the pseudo cost functions introduced in §1.3 to deal with the SO pattern are rigorous.

### 5.2 Average trip cost per user class

Knowing the equilibrium solution, formulae (18) and (20) yield \( C^N_{SO} \) and \( C^I_{SO} \), respectively. The parameter values specific to SO are \( \gamma^N_uv = 2\gamma_a \forall u, v \in \{N, I\} \) which induce \( \gamma^N_# = 2\gamma \), \( \alpha^N_0 = (\alpha_a + \alpha_a)/2 \) and \( \gamma^N_B - \gamma^N_B = \gamma(\alpha_1 - \alpha_2) \), providing some simplification:

\[
\begin{align*}
C^I_{SO} = & \frac{1}{2} t^I + \frac{1}{2} \alpha_1 - \frac{x^I_2}{2q^I} (\alpha_1 - \alpha_2) - \frac{1}{2} A F_A - \frac{1}{4q^I} (F_B - F_A - B(F_B - F_A))
+ \int_{-\infty}^{A} \tilde{\zeta}_{2/z} \; dF + \int_{A}^{\infty} \tilde{\zeta}_{1/z} \; dF
\end{align*}
\] (38a)

Under centred disturbances, \( \int_{-\infty}^{A} \tilde{\zeta}_{2/z} \; dF + \int_{A}^{\infty} \tilde{\zeta}_{1/z} \; dF = \bar{F}_A \) (see § 10.1) yielding:

\[
\begin{align*}
C^I_{SO} = & \frac{1}{2} t^I + \frac{1}{2} \alpha_1 - \frac{x^I_2}{2q^I} (\alpha_1 - \alpha_2) - \frac{1}{2} G(A) + \frac{1}{2} \bar{F}_A - \frac{1}{4q^I} (\bar{F}_B - \bar{F}_A - B(\bar{F}_B - \bar{F}_A))
\end{align*}
\] (38b)

Assuming further that the disruptions have Gaussian distributions, the indicators of average trip costs are detailed below, in the case of equal average link social time:

\[
\begin{align*}
C^I_{SO} = & \theta + \frac{(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)}{4\gamma} - \frac{\sigma^2}{2\gamma^2 Q}(1 - F(\gamma Q))/2 - F(-\gamma Q)] - \frac{1}{2} G(-\gamma Q)
\end{align*}
\] (38c)

\[
\begin{align*}
\bar{C}^N_{SO} = & \theta - \frac{(\alpha_2 - \alpha_1)^2}{4\gamma(1 - \beta) Q} - \frac{\beta(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)}{4\gamma(1 - \beta)}
\end{align*}
\] (39)
Parameter $\theta$ is equal to \((\gamma_1 \gamma_2 \gamma + \alpha_1 \gamma_2 + \alpha_2 \gamma_1)/(\gamma_1 + \gamma_2)\) as in the UE pattern, where it denotes the equilibrium private link time. Parameter $\theta' \equiv \theta - \Delta \alpha^2/4 \gamma Q$ would indicate the average trip time per user under the SO pattern with no dynamic information i.e. $\beta = 0$.

If both links have the same social time, then $x_{a}^I = q^I/2$ as in the UE pattern under equal link private times, which is remarkable again.

### 5.3 Sensitivity analysis

In the case of equal link social times the average cost to an uninformed user is:

$$\overline{C}_{SO}^N = \theta' - \frac{\beta}{4 \gamma (1 - \beta)} \left[ \frac{\Delta \alpha^2}{Q} + \Delta \alpha \Delta \gamma \right].$$

The sign of $\overline{C}_{SO}^N - \theta'$ depends on the bracketed term. When the configuration parameter $\Delta \alpha \Delta \gamma$ is positive then DTI improves the SO cost to the uninformed users.

To an informed user, the average trip cost given in (38c) involves $\theta$ plus three additional terms which are respectively network dependent, negative and negative.

The average trip cost per user is easier to analyze:

$$\overline{C}_{SO} = \theta' - \frac{\sigma^2}{2 \gamma Q} \left[ \frac{1}{2} - F(-\gamma \beta Q) \right] - \frac{\beta}{2} G(-\gamma \beta Q),$$

which is less than $\theta' = \overline{C}_{SO}$ at point $\beta = 0$. The difference gets larger as $\sigma$ increases, but it diminishes with respect to $\gamma$ and $Q$. The effect of the equipment rate $\beta$ depends on the other parameters.

### 5.4 Private information gain

Under the SO pattern with equal average link social times, the private gain from using dynamic information is a sum of three terms, of which the respective signs are positive (second and third terms) or may depend on the network configuration and the level of demand (first term):

$$\Gamma_{SO}^{self} = -\frac{1}{4 \gamma (1 - \beta)} \left[ \frac{\Delta \alpha^2}{Q} + \Delta \alpha \Delta \gamma \right] + \frac{1}{2} G(-\gamma \beta Q) + \frac{\sigma^2}{2 \beta Q} \left[ \frac{1}{2} - F(-\gamma \beta Q) \right]$$

(41)

When $\alpha_1 \leq \alpha_2$ and $\gamma_1 \leq \gamma_2$ e.g. when link 1 is a motorway and link 2 an urban arterial, then the configuration parameter is $\Delta \alpha \Delta \gamma \geq 0$ and $\Gamma_{SO}^{self}$ is likely to be negative.

### 5.5 Marginal external cost of a trip

As the expressions of $\Gamma_{SO}^{uv} = q^v \delta \overline{C}_{SO} / \delta q^u$ are quite involved, let us focus on the external social gain of providing DTI to a marginal trip (i.e. by supplying a previously non-equipped user with an information device):
\[
\Gamma_{SO}^{ext} = q^N \left( \frac{\partial c^N}{\partial q^N} - \frac{\partial c^I}{\partial q^I} \right) + q^I \left( \frac{\partial c^I}{\partial q^N} - \frac{\partial c^I}{\partial q^I} \right) \\
= -(1 - \beta) \frac{\partial c^N}{\partial \beta} - \beta \frac{\partial c^I}{\partial \beta}
\]

after simplification. Thus

\[
\Gamma_{SO}^{ext} = \frac{\Delta \alpha \Delta \gamma + \Delta \alpha^2 / Q}{4 \gamma(1 - \beta)} + \frac{1}{2} \sigma^2 \tilde{F}(\gamma \beta Q) - \frac{1}{2} \gamma \beta Q \tilde{F}(\gamma \beta Q) - \frac{\sigma^2}{2 \gamma \beta Q} \left[ \frac{1}{2} - \tilde{F}(\gamma \beta Q) \right]. \tag{42}
\]

The overall sign depends on the various parameters and may change. The addition of \(\Gamma_{SO}^{self}\) and \(\Gamma_{SO}^{ext}\) yields the system marginal gain, which is positive – as could be expected:

\[
\Gamma_{SO}^{sys} = \frac{1}{2} \left[ G(A) + AF(A) \right] + \frac{1}{2} \sigma^2 \tilde{F}(A) \\
= G(-\gamma \beta Q). \tag{43}
\]

This is indeed another remarkable analytical property, to be compared to the private gain under UE, \(\Gamma_{SO}^{self} = G(-\frac{1}{2} \gamma \beta Q)\): as function \(G\) is positive and increasing, it comes out that the collective gain of a marginal DTI equipment under SO is less than the private gain under UE.

6. USER EQUILIBRIUM VERSUS SYSTEM OPTIMUM

Let us now compare the patterns of UE and SO, which sets a reference minimum travel cost but is less easy to implement than UE. We shall restrict ourselves to a parametric analysis, on assuming that in each pattern there is equality of the average link pseudo costs. This is the most likely situation when there is significant congestion. The case of unequal link pseudo costs will be studied by numerical simulation in the next Section, with outcomes that may differ from the mainstream case in the share of the SO benefits among the user classes.

6.1 Average trip cost

Let us measure the benefits of cooperation by the variation in average trip cost between SO and UE: namely \(V \equiv \overline{C}_{UE} - \overline{C}_{SO}\).

From our previous results, \(V\) splits into three terms:

\[
V = \left( \frac{\alpha_2 - \alpha_1}{4 \gamma Q} \right)^2 + \frac{\sigma^2}{2 \gamma Q} \left[ \frac{1}{2} - \tilde{F}(\gamma \beta Q) \right] - \beta \left[ G(-\frac{1}{2} \gamma \beta Q) - \frac{1}{2} G(-\gamma \beta Q) \right]. \tag{44}
\]

The first term, a gain \(V_0 \equiv \Delta \alpha^2 / 4 \gamma Q\), is the same as in an undisturbed network without DTI. The second term is a gain that is specific to disruptions and DTI: it increases with \(\sigma^2\), the variance of the disturbances in travel times. The last term is a loss specific to DTI from UE to SO, of second order magnitude except at very small \(\gamma \beta Q\). The overall effect is a gain of SO over UE, from the very definition of SO.

To analyze the interplay of SO and DTI, we may compare four traffic states, namely: UE(\(\beta = 0\)), SO(\(\beta = 0\)), UE(\(\beta\)) and SO(\(\beta\)). The gain \(V_0\) from UE(\(\beta = 0\)) to SO(\(\beta = 0\)) does not involve DTI and is independent of \(\beta\). Between UE(\(\beta = 0\)) and UE(\(\beta\)), DTI provides an average benefit of \(-\beta G(-\frac{1}{2} \beta \gamma Q)\) per user. Between SO(\(\beta = 0\)) and SO(\(\beta\)), DTI provides an
average benefit per user of \( \frac{\sigma^2}{2\gamma Q} \left[ \frac{1}{2} - F(-\beta\gamma Q) \right] + \frac{\beta}{2} G(-\beta\gamma Q) \), of which the second term is less than half its counterpart gain under UE but the first term provides an additional benefit related to \( \sigma^2 \).

### 6.2 Overall system cost

The saving in overall travel cost from UE to SO, \( Q V \), is bounded by \( (\Delta\alpha^2 + \sigma^2)/4\gamma \), a constant that does not depend on \( Q \). This stems from the linearity assumption on the travel time functions. The upper bound provides a good approximation to the benefits when \( \gamma\beta Q \) is high: it shows that DTI enables the society to save a supplementary proportion \( (\sigma/\Delta\alpha)^2 \) of travel cost.

### 6.3 Average trip cost per user class

To an uninformed user, SO provides a per trip benefit of \( V_0 + \frac{\beta}{1-\beta} (V_0 + \frac{\Delta\alpha\Delta\gamma}{4\gamma}) \): DTI is specifically involved in the second term, with guaranteed positive effect if \( \Delta\alpha\Delta\gamma \geq 0 \).

To an informed user, SO provides a per trip benefit of \( \sigma^2 \left[ \frac{1}{2} - F(-\beta\gamma Q) \right]/2\gamma\beta Q \), plus a term \( \Delta\alpha\Delta\gamma/4\gamma \) of which the sign depends on the network configuration, plus a last term of \( \frac{1}{2} G(-\beta\gamma Q) - G(-\frac{1}{2}\beta\gamma Q) \) which is a loss. The magnitude \( \sigma \) of the disruptions influence functions \( F \) and \( G \). The overall effect depends on the second term, which is crucial in the interplay of the two user classes since under SO a cost of \( \beta Q\Delta\alpha\Delta\gamma/4\gamma \) is transferred from the uninformed users to the informed users, with algebraic sign determined by the network configuration. In the motorway versus urban arterial network, this is a net transfer of cost from class \( N \) to class \( I \).

### 7. ON THE ACCEPTABILITY AND EFFICIENCY OF DTI POLICIES

Having assessed in an analytical way the gap in travel cost between the SO and UE states, let us now investigate the issue of how to implement SO using DTI, i.e. of how to use DTI to get the system state as close as possible to the system optimum.

A basic requirement is that the DTI operation be acceptable to the informed user, by supplying him with a positive advantage over an uninformed user.

#### 7.1 On the acceptability of SO

When the system state under SO delivers a positive advantage to every informed user, then it should be possible to implement the SO pattern in the following way:

- to the equipped users, deliver dynamic guidance with social cost as routing criterion. If this improves upon an uninformed strategy, every informed user has an advantage to follow the advice.
- To the unequipped users, deliver static guidance (through e.g. fixed signs) along the routes of assignment under SO.
This implementation should work well in cases where the long-run SO routes are advantageous, or at least competitive, to the individual non-equipped user. This happened in most but not all of the numerical experiments described hereafter.

The network operator has also a “lazy” option at his disposal: since DTI under UE indeed improves on the travel cost, the diffusion of selfish user optimizing information will deliver some benefits.

An intermediary option is to let the uninformed users behave in a selfish manner, and provide dynamic information to the equipped users so as to minimize their own aggregate travel cost in any circumstance \( \omega \). This is the IC pattern of Information Cooperation, i.e. cooperation restricted to the informed users. At \( \beta = 0 \) the IC pattern corresponds to UE, whereas at \( \beta = 1 \) the IC and SO patterns are identical. We can expect this option to extend the scope for DTI traffic optimization.

As we were not able to endow the IC pattern with nice analytical properties, we had to investigate it through numerical simulation.

### 7.2 The setting of numerical experiments

Let us set up two configurations for a two link network:

- **Net1:** “Motorway versus City Arterial” in which one route dominates the other in both link capacity and free flow link travel time. The numerical values of the parameters are set to \( \alpha_1 = 40, \gamma_1 = 1 \) and \( \alpha_2 = 80, \gamma_2 = 2 \).

- **Net2:** “City Arterial versus City Road” in which each of the two routes has its own advantage either in capacity or free flow travel time. The numerical values of the parameters are set to \( \alpha_1 = 40, \gamma_1 = 2 \) and \( \alpha_2 = 80, \gamma_2 = 1 \).

The random disturbances on link travel times are taken as independent, centred Gaussian variables, each one with variance of \( \sigma^2 / 2 \) to ensure that the difference variable \( Z = \zeta_2 - \zeta_1 \) has variance \( \sigma^2 \). The value of \( \sigma \) is set to 40, which supplies a link time with a relative dispersion of \( \sigma / \sqrt{\sigma} \) that decreases from 2/3 to 1/5 as the link flow increases.

Two levels of travel demand are taken: first a low volume \( Q = 10 \) in which case the network users are assigned only to the first route in the undisrupted situation; second a high volume \( Q = 100 \) in which case the network users are assigned to the two routes. The equipment rate \( \beta \) is varied from 0% to 100% by increment of 1%.

### 7.3 The overall performance of DTI by cooperation pattern

The average trip cost per network user, taken as the indicator of the overall network performance, is depicted in Fig. 4 with respect to equipment rate \( \beta \) under each of the three cooperation patterns, by test network and level of demand. It comes out that when the demand volume is low, whatever the network, all of the DTI policies yield similar results: the average trip cost decreases with respect to the equipment rate. However, under high demand volume the alternative policies yield specific effects: under SO, the larger the equipment rate, the more savings in average trip cost, due to the larger scope for dynamic traffic optimization. The IC policy also delivers increasing overall benefits but along a two-stage path. The UE policy delivers positive benefits at any equipment rate on Net1 but only a sub-interval on Net2; furthermore, the benefits are positive with respect to a reference state with no DTI, whereas there is a particular value \( \beta \in ]0,1[ \) that yields optimal benefits and beyond...
which any additional equipment is detrimental or sub-optimal, as would be any lesser rate of equipment.

![Graph showing average trip cost per network user under low (top) or high (bottom) demand.](image)

**FIGURE 4** Average trip cost per network user under low (top) or high (bottom) demand

### 7.4 Acceptability criteria and the efficiency of DTI policies

Each actor in the system has his own condition for acceptability:

- To the informed user, the private gain of being equipped must be positive: this makes the “I” criterion.

- To the un-equipped user, the average travel cost must be equal to or less than a reference cost, namely that of the UE pattern at $\beta = 0$. This makes the “U” criterion.

- To the network operator who represents the whole traffic, the marginal system gain of equipping a previously un-equipped user must be positive. This makes the “O” criterion.

The three criteria ought to be met jointly, making up a OUI positive assessment. For given network and DTI strategy, the values of the equipment rate $\beta$ that fulfil a given criterion make up a conditional interval: the OUI condition is satisfied at the intersection of the three conditional intervals. Fig. 5 (resp. 6) shows the conditional and joint intervals for network Net1 (resp. Net2) under high demand volume and for each DTI strategy, by depicting the average travel cost, private gain ($T_p^{\text{self}}$) and marginal system gain ($T_p^{\text{sys}}$) with respect to the equipment rate.

On Net1, the OUI interval is quite small under the UE pattern (about $[0\%, 18\%]$), somewhat extended under the SO pattern (about $[0\%, 28\%]$) but much larger under the IC pattern (up to $[0\%, 70\%]$).
On Net2, the OUI interval is limited again under the UE pattern (about [0%, 18%]) but larger under the IC pattern, [0%, 85%]. The SO policy has a OUI range restricted to $\beta = 0$ since any $\beta > 0$ yields results unwelcome to the non-informed user.

**FIGURE 5** Acceptability and efficiency of DTI policy on Net1 at Q=100

**FIGURE 6** Acceptability and efficiency of DTI policy on Net2 at Q=100
7.5 Discussion

On one hand, our experiments confirm that there is a gap in travel cost between UE and SO. Numerically, $\Delta \alpha^2 / 4\gamma \approx 133$ hence at $Q = 100$ the individual benefit is about 1 at $\beta = 0$: a larger parameter $\Delta \alpha$ would emphasize the potential for cost minimization. Traffic disruptions increase that potential in proportion of $(\sigma / \Delta \alpha)^2$. So there is room for an information policy more advanced than just disseminating the information device and delivering UE information, which is not always beneficial to the overall network performance.

On the other hand, it seems difficult to implement an SO state, not only due to the acceptability constraints, but also because the optimum state depends on $Q$, $\beta$ and also the network configuration. As $Q$ is likely to vary throughout the day as well as from day to day, the implementation of SO would require at least variable message signs along the network. This could correspond to our model with equipment rate of 100%: in this case parameter $\beta$ could model the compliance rate among the users.

The IC pattern of cooperation sets a fair compromise between the stakes of cost reduction and the difficulty to implement SO through DTI. It stands as an attractive information policy which grants the users for cooperating and entices a user to get access to the dynamic information. As users of the same class may undergo different costs in a given circumstance $\omega$, the acceptability of IC still requires to ensure that in the long run every user gets the average cost of his class. This could be fulfilled by keeping a user account of the difference between his individual cost and the class average cost, cumulated over the past circumstances. Based on the user accounting system, individual guidance could be targeted at the individual user so as to equalize their individual balance across each user class.

8. CONCLUSION

An analytical model was provided to capture the effects of dynamic traffic information in a two link network subject to both recurrent and incidental congestion, where two classes of users either informed or not choose their path in the short or long run, respectively. Alternative patterns of user cooperation were studied, specifically User Equilibrium (UE), System Optimum (SO) and Informed Cooperation (IC), which involves cost minimization among the informed users. By simplifying the assumptions to two parallel links, linear link travel times with respect to link flow and additive random disturbances on the link times, we obtained analytical formulae for the main variables of interest: average link flow and trip cost by user class, marginal external cost of a class trip by user class, total travel time.

The total travel time under System Optimum sets a reference for optimizing a network with dynamic disturbances and information. We conducted a parametric analysis of the UE pattern, the SO pattern and the UE-SO gap, with respect to demand volume and equipment rate. Then a numerical experiment was performed to study these patterns in relation to the IC pattern and to demonstrate the effect of a network configuration parameter on the relationship between the two user classes. Under User Equilibrium, the informed users get benefits from DTI that decrease with the equipment rate; the effect on the uninformed users is positive if the average link travel times are equal or neutral otherwise. Under System Optimum, the sign of the network configuration parameter determines that of a transfer of cost from the uninformed users to the informed users, yielding variable effects. The IC pattern stands as a fair compromise between the stakes of minimizing the system costs and the user acceptability of dynamic information aimed at overall cost minimization.
Apart from simplifying the network structure to two parallel links, our treatment is subject to several limitations. First, recurrent congestion is modelled in a static way by assuming that the link travel time is a linear function of the link volume: this corresponds to the range of light-to-medium congestion effects, not to the medium-to-severe range. Second, the dynamic disruptions in link travel times are modelled as random variables, involving specific assumptions of, respectively: (i) independence between the two links; (ii) Gaussian distribution; (iii) homoskedasticity. It is easy to deal with dependency and heteroskedasticity under the Gaussian assumption, by specific adaptation of the difference in link travel times, our $Z$ variable. Furthermore, the Gaussian assumption was shown not to differ significantly from the assumption of Bernoulli-Exponential disturbances by network link if the property of a symmetric distribution is maintained for the $Z$ variable.

Further work about our model and the associated economic analysis may be targeted at the following topics:

- Numerical investigation of dependent, heteroskedastic disturbances, for instance in the case of Bernoulli-Exponential variables.
- Analytical study of the marginal external social information gain in the case of unequal link times. We expect this gain to be positive, which could justify for subsidizing the equipment of individual users.
- A more realistic model of link congestion, by replacing the travel time function with a bottleneck model.
- Analytical study of the interplay of congestion recurrent and incidental, DTI and capacity management.
- Analytical study of the interplay of congestion recurrent and incidental, DTI and toll setting, in order to assess the complementariness and eventual redundancy of the two tools to improve on the system state.
- Bi-layer equilibrium with state dependent traffic disturbances, e.g. making $\sigma_a$ to depend on $x_a$.

9. REFERENCES


European Transport Conference, The Netherlands, 6-8 October. 16 pages. CD Rom edition also available at: http://www.etcproceedings.org/


10. APPENDIX

10.1 The central property for the difference in link disturbance

Assuming that the variables $\zeta_1$ and $\zeta_2$ are centred, then so is $Z \equiv \zeta_2 - \zeta_1$, then it holds that:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta),
\]

yielding that:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta)
\]

Hence

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta^{-1} \zeta^{-2} dF(\zeta) = \tilde{F}(A)
\]

10.2 The average travel cost to an informed user

Our aim is to evaluate the average travel cost to an informed user, which was stated in (19) as follows:

\[
\bar{C}_P^I = E_\omega[x_{10}^I T_{10} + x_{20}^I T_{20}]/q^I
\]

To compute the average, our method is to make a conditional analysis on the basis of the net disturbance $Z \equiv \zeta_2 - \zeta_1$. This is because in any occurrence $\omega$ the split of informed flow
between the two routes stems from the value \( z \) of \( Z \) compared to average quantities that do not depend on \( \omega \).

Let us also decompose Eqn (A2a) into three parts associated with the sub-domains \( z \in ]-\infty, A[ \), \( z \in ]A, B[ \) and \( z \in ]B, +\infty[ \), respectively:

\[
\overline{C}_p = \int_{-\infty}^{A} \tilde{f}_{2/z} dF(z) + \frac{1}{q} \int_{A}^{B} [x_{10f10}^f + x_{20f20}^f] dF(z) + \int_{B}^{+\infty} \tilde{f}_{1/z} dF(z),
\]

(A2b)

in which \( \tilde{f}_{a/z} \) denotes \( \mathbb{E}[f_{a0} \cdot \zeta_{z} - \zeta_{10} = z] \) i.e. the expectation conditional on the value \( z \) of \( Z \). In fact Eqn (A2b) stems from the fact that:

- if \( z \in ]-\infty, A[ \) then \( x_{10}^f = 0 \), \( x_{20}^f = q^f \) hence \( x_{10f10}^f + x_{20f20}^f = q^f t_{20} \).
- if \( z \in ]B, +\infty[ \) then \( x_{10}^f = q^f \), \( x_{20}^f = 0 \) hence \( x_{10f10}^f + x_{20f20}^f = q^f t_{10} \).

The endpoint parts in (A2b) are easy to evaluate since, letting \( \alpha_{a}^{f0} \equiv \alpha_{a} + \gamma_{a} x_{a}^{N} \), it holds that if \( z \in ]-\infty, A[ \) then \( \tilde{t}_{2/z} = \alpha_{2}^{f0} + \gamma_{2} q^{1} + \zeta_{2/z} \), whereas if \( z \in ]B, +\infty[ \) then \( \tilde{t}_{1/z} = \alpha_{1}^{f0} + \gamma_{1} q^{1} + \zeta_{1/z} \). Thus

\[
\int_{-\infty}^{A} \tilde{f}_{2/z} dF(z) + \int_{B}^{+\infty} \tilde{f}_{1/z} dF(z) = \left( \alpha_{2}^{f0} + \gamma_{2} q^{1} \right) F_{A} + \left( \alpha_{1}^{f0} + \gamma_{1} q^{1} \right) (1 - F_{B}) + \int_{B}^{+\infty} \tilde{F}_{2/z} dF(z) + \int_{-\infty}^{A} \tilde{F}_{1/z} dF(z).
\]

(A3a)

About the middle part in (A2b), notice that when \( z \in ]A, B[ \) the informed users are assigned so as to equalize \( t_{10}^f = t_{20}^f \), yielding that \( x_{10}^f = (z - A) / \gamma_{#} \) and \( x_{20}^f = (B - z) / \gamma_{#} \) where \( \gamma_{#} \equiv \gamma_{1}^f + \gamma_{2}^f \).

Since \( x_{10f10}^f + x_{20f20}^f = q^f t_{10} - x_{20}^f (t_{10} - t_{20}) \), we restate the middle term as:

\[
\mathbb{E}\left[ \frac{1}{q} \int_{A}^{B} \left[ x_{10f10}^f + x_{20f20}^f \right] dF \right] = \mathbb{E}\left[ t_{10} dF \right] - \mathbb{E}\left[ \frac{1}{q} \int_{A}^{B} x_{20}^f (t_{10} - t_{20}) dF \right] = \int_{A}^{B} \tilde{t}_{1/z} dF - \int_{A}^{B} \frac{1}{q} \int_{A}^{B} x_{20}^f (t_{10} - t_{20}) dF(z)\].

(A3b)

In the first term, \( \tilde{t}_{1/z} = \alpha_{1}^{f0} + \gamma_{1} x_{10/z} + \zeta_{1/z} \) with \( x_{10}^f = (z - A) / \gamma_{#} \). Then:

\[
\int_{A}^{B} \tilde{t}_{1/z} dF = \left( \alpha_{1}^{f0} - \frac{\gamma_{1}}{\gamma_{#}} A \right) (F_{B} - F_{A}) + \frac{\gamma_{1}}{\gamma_{#}} (\tilde{F}_{B} - F_{A}) + \int_{A}^{B} \tilde{F}_{1/z} dF
\]

As \( -A = \gamma_{#} q^{1} - B \), it also holds that

\[
\int_{A}^{B} \tilde{t}_{1/z} dF = \left( \alpha_{1}^{f0} + \gamma_{1} q^{1} \right) (F_{B} - F_{A}) + \frac{\gamma_{1}}{\gamma_{#}} [\tilde{F}_{B} - \tilde{F}_{A} - B (F_{B} - F_{A})] + \int_{A}^{B} \tilde{F}_{1/z} dF
\]

(A3c)

Adding that to the extreme parts in (A3a) yields that
\begin{align*}
\bar{C}^l &= (\alpha_2^{l0} + \gamma_2 q^l) F_A + (\alpha_1^{l0} + \gamma_1 q^l)(1-F_A) + \frac{\gamma_1}{l} [\tilde{F}_B - \tilde{F}_A - B(F_B - F_A)] \quad (A4) \\
- \frac{1}{q_l} B \int_{A}^{B} E_z [x^l_{20}(t_{10} - t_{20})] dF + \int_{-\infty}^{A} \zeta_{2}^l dF + \int_{A}^{\infty} \zeta_{1/z} dF
\end{align*}

Let us now develop \( x^l_{20}(t_{10} - t_{20}) \), knowing that \( x^l_{20} = (B-z)/\gamma_# \) and that

\[
t_{10} - t_{20} = \alpha_1^{l0} - \alpha_2^{l0} + \gamma_1(q^l - x^l_{20}) + \zeta_1 - \gamma_2 x^l_{20} - \zeta_2
\]

\[
= B_0 - \gamma x^l_{20} - z \quad \text{where } B_0 \equiv \alpha_1^{l0} - \alpha_2^{l0} + \gamma_1 q^l \quad \text{and} \quad \gamma \equiv \gamma_1 + \gamma_2
\]

Thus \( x^l_{20}(t_{10} - t_{20}) = \gamma_#^2 (B-z)[\gamma_# B_0 - \gamma B + (\gamma - \gamma_#)z] \).

Integration from A to B yields that (recall that \( \tilde{F}(x) = \int_{-\infty}^{x} z^2 dF \)):

\[
\int_{A}^{B} E_z [x^l_{20}(t_{10} - t_{20})] dF = \gamma_#^2 [\gamma B - \gamma_# B_0] (\tilde{F}_B - \tilde{F}_A - B(F_B - F_A))
\]

\[
- \gamma_#^2 (\gamma - \gamma_#)(\tilde{F}_B - \tilde{F}_A - B(F_B - F_A))
\]

This ends up the computation of the average cost to an informed user:

\[
\bar{C}_p^l = (\alpha_2^{l0} + \gamma_2 q^l) F_A + (\alpha_1^{l0} + \gamma_1 q^l)(1-F_A) + \frac{\gamma_1}{l} [\tilde{F}_B - \tilde{F}_A - B(F_B - F_A)]
\]

\[
+ \frac{\gamma - \gamma_#}{\gamma_# q^l} (\tilde{F}_B - \tilde{F}_A - B(F_B - F_A)) + \int_{-\infty}^{\infty} \zeta_{2/z} dF + \int_{-\infty}^{\zeta_{1/z}} dF
\]

\[ A6 \]

10.3 The pattern of true user equilibrium

In the pattern of true user equilibrium, it holds that \( \gamma_# = \gamma \) and \( B_0 = B \). Thus \( \bar{C}_UE^l \) is simplified into:

\[
\bar{C}_UE^l = (\alpha_2^{l0} + \gamma_2 q^l) F_A + (\alpha_1^{l0} + \gamma_1 q^l)(1-F_A) + \frac{\gamma_1}{\gamma_#} [\tilde{F}_B - \tilde{F}_A - B(F_B - F_A)]
\]

\[ A7 \]

Using the formulae for A, B and \( x^l_a \), this is further simplified into:

\[
\bar{C}_UE^l = \bar{l}_a - A F_A + \int_{-\infty}^{\infty} \zeta_{2/z} dF + \int_{-\infty}^{\infty} \zeta_{1/z} dF
\]

\[ A8a \]

Moreover, if \( \zeta_1 \) and \( \zeta_2 \) are centred then property (A1) implies that:

\[
\bar{C}_UE^l = \bar{l}_a - G(A)
\]

\[ A8b \]

Average link time \( \bar{l}_a \) stems from the conditions of long run equilibrium, following one out of two scenarios: either equality or inequality of link average times.
Equal average travel times

The equality condition yields that $\bar{t}_1 = \bar{t}_2 = \bar{t} = \frac{\gamma_1 \gamma_2 Q + \alpha_1 \gamma_2 + \alpha_2 \gamma_1}{\gamma_1 + \gamma_2}$ and $A = -\gamma Q \beta / 2$. Hence:

$$\bar{C}_{UE}^{L} = \bar{t} - G(A) \quad (A9a)$$

The average travel time to an uninformed user is equal to the average time on each link:

$$\bar{C}_{UE}^{N} = \bar{t} \quad (A9b)$$

This yields the average travel time to a network user:

$$\bar{C}_{UE} = \bar{t} - \beta G(A) \quad (A9d)$$

Unequal average travel times

Assuming that $\bar{t}_1^N < \bar{t}_2^N$, class $N$ is assigned only to link 1: $\bar{x}_1^N = q^N$; $\bar{x}_2^N = 0$. It holds that $\alpha_1^l = \alpha_1 + \gamma_1 q^N$; $\alpha_2^l = \alpha_2$. Then $B = \alpha_1 - \alpha_2 + \gamma_1 Q$ which is fixed under given network parameters and travel demand volume, $Q$. Furthermore, $A = B - \gamma q^l = B - \gamma Q \beta$.

The formulae for $\bar{t}_a^l$, $\bar{t}_a^l$ imply that: $\bar{t}_a = \alpha_1 + \gamma_1 Q - \frac{\gamma_1}{\gamma} G(B) + \frac{\gamma_1}{\gamma} G(A)$. Then the average travel cost to an informed user may be derived as follows:

$$\bar{C}_{UE}^{L} = \alpha_1 + \gamma_1 Q - \frac{\gamma_1}{\gamma} G(B) - \frac{\gamma_2}{\gamma} G(A) \quad (A10a)$$

The average travel time to an uninformed user is:

$$\bar{C}_{UE}^{N} = \alpha_1 + \gamma_1 Q - \frac{\gamma_1}{\gamma} G(B) + \frac{\gamma_1}{\gamma} G(A) \quad (A10b)$$

This yields the average travel time to a network user:

$$\bar{C}_{UE} = \alpha_1 + \gamma_1 Q - \frac{\gamma_1}{\gamma} G(B) - \left( \frac{\gamma_1}{\gamma} - \beta \right) G(A) \quad (A10c)$$

10.4 The pattern of system optimum

As system optimum means fully cooperative users, the following properties hold: $\gamma_# = 2 \gamma$, $\alpha_# = (\alpha_1^l + \alpha_2^l) / 2$ and $\gamma_# B_0 - \gamma B = \gamma (\alpha_1 - \alpha_2)$.

Let us decompose $\bar{C}_{SO}^{L} = \bar{C}_{SO,a}^{L} + \bar{C}_{SO,b}^{L} + \bar{C}_{SO,c}^{L}$ along the lines of the formula (A2b):

$$\bar{C}_{SO,a}^{L} = (\frac{\alpha_1^l + \alpha_2^l}{2} + \gamma_2 q^l) F_A + (\frac{\gamma_1}{2} + \frac{\alpha_1}{2} + \gamma_1 q^l) (1 - F_A) \quad (A11a)$$

Recalling that $A = \alpha_1^l - \alpha_2^l - 2 \gamma q^l$, we get:

$$\bar{C}_{SO,a}^{L} = \frac{\alpha_1(1 - F_A) + \alpha_2 F_A + \alpha_1^l + 2 \gamma q^l}{2} - \frac{1}{2} A F_A - \gamma q^l F_A \quad (A11b)$$
Similarly \((\tilde{F}_b - \tilde{F}_A - B(F_b - F_A))/\gamma_# = -\gamma_2 x_2' + \gamma_1 q' F_A\). Then:

\[
\overline{C}_{SO,b}^{\prime} = \frac{\gamma_2 q' \gamma_1 + \gamma_1 \gamma_2 (a_1 - a_2)}{\gamma_# q'} (-\tilde{x}_2' + q' F_A) \\
= -\gamma_2 \tilde{x}_2' + \gamma_1 q' F_A + \left(\frac{F_a}{2} - \frac{\tilde{x}_2'}{2q'}\right) (a_1 - a_2) 
\]

(A11c)

By summing the three terms and recalling that \(\tilde{r}_1' = a_1' + 2\gamma_1 \tilde{x}_1'\), we obtain:

\[
\overline{C}_{SO}^{\prime} = \frac{1}{2} \tilde{r}_1' + \frac{a_1}{2} - \frac{\tilde{x}_2'}{2q'} (a_1 - a_2) - \frac{1}{2} A F_A - \frac{1}{4} q' \gamma_1 \left(\bar{F}_B - \bar{F}_A - B(\bar{F}_B - \bar{F}_A)\right) \\
+ \int_{-\infty}^{\gamma} \xi_2/\zeta \ dF + \int_{\gamma}^{+\infty} \xi_1/\zeta \ dF 
\]

(A11d)

Under a symmetric distribution of \(Z\), the formula becomes simpler:

\[
\overline{C}_{SO}^{\prime} = \frac{1}{2} \tilde{r}_1' + \frac{a_1}{2} - \frac{\tilde{x}_2'}{2q'} (a_1 - a_2) - \frac{1}{2} G(A) + \frac{1}{2} \tilde{F}_A - \frac{1}{4} q' \gamma_1 \left(\bar{F}_B - \bar{F}_A - B(\bar{F}_B - \bar{F}_A)\right) 
\]

(A11e)

Following the same approach as previously, let us derive more specific formulae by distinguishing two scenarios of long run equilibrium: either equality or inequality of pseudo average travel times.

**Equal average pseudo travel times**

When the pseudo average travel times are equal, it holds that: 
\(-A = B = \gamma q'\), \(\tilde{F}_B = \tilde{F}_A\), \(F_A + F_B = 1\), \(\tilde{x}_2' = 2q'\) and \(\tilde{r}_1' = \tilde{r}_0 = 2\gamma_1 \gamma_2 Q + \gamma_1 a_2 + \gamma_2 a_1\)

\[
\overline{C}_{SO}^{\prime} = \frac{1}{2} \tilde{r}_1' + \frac{a_1 + a_2}{2} - \frac{1}{2} G(A) - \frac{1}{4} q' \gamma_1 \left(\bar{F}_B - \bar{F}_A\right) + \frac{1}{2} \tilde{F}_A 
\]

(A12a)

As \(\frac{1}{2} \tilde{r}_1' + \frac{a_1 + a_2}{4} = \tilde{\theta} + \frac{(\gamma_2 - \gamma_1)(a_2 - a_1)}{4\gamma}\), we have:

\[
\overline{C}_{SO}^{\prime} = \tilde{\theta} + \frac{(\gamma_2 - \gamma_1)(a_2 - a_1)}{4\gamma} - \frac{1}{2} G(A) - \frac{1}{4} q' \gamma_1 \left(\bar{F}_B - \bar{F}_A\right) + \frac{1}{2} \tilde{F}_A 
\]

(A12b)

If \(Z\) is Gaussian, it holds that:

\[
\frac{1}{4} q' \gamma_1 \left(\bar{F}_B - \bar{F}_A\right) = -\frac{\sigma^2}{4q' \gamma_1} \frac{\sigma^2}{2q'} + \frac{\sigma^2}{4A} \frac{\sigma^2}{2A}\]

(A12c)

Using \(A = -\gamma \beta Q\) for substitution, we have:

\[
\overline{C}_{SO}^{\prime} = \tilde{\theta} + \frac{(\gamma_2 - \gamma_1)(a_2 - a_1)}{4\gamma} + \frac{\sigma^2}{A} - \frac{\Delta G(A) + \sigma^2 F_A}{2A} 
\]

(A12d)
Coming back to the equality condition, it holds that:
\[ x_i^N = \frac{\alpha_2 - \alpha_1}{2\gamma} + \frac{\gamma_2 Q - \frac{1}{2} q^I}{\gamma}, \]
so:
\[ \alpha_1 x_i^N + \alpha_2 x_2^N = \frac{\alpha_1 \gamma_2 + \alpha_2 \gamma_1}{\gamma} q^N - \frac{(\alpha_2 - \alpha_1)^2}{2\gamma} - \frac{(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)}{2\gamma} q^I \]  
\hspace{1cm} (A13a)

Combining to
\[ \mathcal{C}_N^{SO} = \frac{\pi^N_1 + \pi^N_2}{q^N} = \frac{\alpha_1 x_i^N + \alpha_2 x_2^N}{2q^N}, \]
we have:
\[ \mathcal{C}_N^{SO} = \frac{\alpha_1 \gamma_2 + \alpha_2 \gamma_1}{2(\gamma_1 + \gamma_2)} - \frac{(\alpha_2 - \alpha_1)^2}{4(\gamma_1 + \gamma_2)q^N} - \frac{q^I(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)}{4(\gamma_1 + \gamma_2)q^N} \]  
\hspace{1cm} (A13b)

As
\[ \frac{\bar{\theta}_#}{2} = \frac{\gamma_1 \gamma_2 Q}{\gamma_1 + \gamma_2} + \frac{\gamma_1 \alpha_2 + \gamma_2 \alpha_1}{2(\gamma_1 + \gamma_2)}, \]
we get the final formula for the average cost to an uninformed user:
\[ \bar{C}_N^{SO} = \bar{\theta} - \frac{(\alpha_2 - \alpha_1)^2}{4\gamma Q} - \frac{\beta(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)}{4\gamma(1 - \beta)} \]  
\hspace{1cm} (A14a)

This yields the average travel cost to the whole traffic:
\[ \bar{C}_SO = \bar{\theta} - \frac{(\alpha_2 - \alpha_1)^2}{4\gamma Q} - \frac{\sigma^2}{4\gamma Q} + AG(A) + \sigma^2 F_A \]  
\hspace{1cm} (A14b)

**Unequal average pseudo travel times**

Assuming that \( \bar{t}_1^N < \bar{t}_2^N \), class N is assigned only to link 1: \( x_i^N = q^N \); \( x_2^N = 0 \); \( \alpha_1^I = \alpha_1 + 2\gamma_1 q^N \); \( \alpha_2^I = \alpha_2 \). Then \( B = \alpha_1 - \alpha_2 + 2\gamma_1 Q \) which is fixed under given network parameters and travel demand \( Q \) and \( A = B - 2\gamma q^I = B - 2\gamma Q \).

Assuming that the link disturbances have Gaussian distributions, it holds that:
\[ \tilde{F}_B - \tilde{F}_A - B(\tilde{F}_B - \tilde{F}_A) = \sigma^2 (F_B - F_A) + 2\gamma Q \beta F_A. \]

So we get finally:
\[ \bar{C}_SO^{I} = \alpha_1 + \gamma_1 O - \gamma_1 x_2^I - \frac{x_2^I}{2q^I}(\alpha_1 - \alpha_2) - \frac{1}{2} G(A) - \frac{\sigma^2}{4\gamma Q^I}(F_B - F_A) \]  
\hspace{1cm} (A15a)

The average travel time to an uninformed user is:
\[ \bar{C}_SO^{N} = \alpha_1 + \gamma_1 (x_1^I + q^N) \quad \text{with} \quad x_1^I = q^I - x_2^I \quad \text{and} \quad x_2^I = (G(B) - G(A))/2\gamma \]  
\hspace{1cm} (A15b)

The average travel time to a road user is:
\[ \overline{c}_{SO} = \alpha_1 + \gamma_1 Q - \left( \frac{\gamma_1}{2} + \frac{\alpha_1 - \alpha_2}{4\gamma Q} \right) (G(B) - G(A)) - \frac{1}{2} \beta G(A) - \frac{\sigma^2}{4\gamma Q}(F_B - F_A) \]

\[ = \alpha_1 + \gamma_1 Q - \frac{BG(B) - AG(A)}{4\gamma Q} - \frac{\sigma^2(F_B - F_A)}{4\gamma Q} \quad \text{(A16a)} \]

\[ \overline{c}_{SO} \] can be alternatively stated as:

\[ \overline{c}_{SO} = \alpha_1 + \gamma_1 Q - \frac{BG(B) + \sigma^2 F_B}{4\gamma Q} + \frac{AG(A) + \sigma^2 F_A}{4\gamma Q} \quad \text{(A16b)} \]

10.5 Alternative Fixed point problem

In the case of equal average pseudo travel time, let \( \theta^N = T_a^#N = T_b^#N \). As \( B = \alpha_1 + \gamma_1^N x_1^N - \alpha_2 - \gamma_2^N x_2^N + \gamma_1^H q^I \), we obtain:

\[ B = \alpha_1 + \gamma_1^N \frac{\theta^N - \alpha_1 - \gamma_1^N x_1^N}{\gamma_1^{NN}} - \alpha_2 - \gamma_2^N x_2^N \quad \text{(A17)} \]

As \( \gamma_1^{uv} / \gamma_1 = \gamma_2^{uv} / \gamma_2 \forall u,v \in \{N, I\} \) and \( \overline{x}_1^I + \overline{x}_2^I = q^I \), we have:

\[ B = (\alpha_1 - \alpha_2)(1 - \frac{\gamma_1^{NI}}{\gamma_1^{NN}}) + q^I \gamma_1^H \left( 1 - \frac{\gamma_1^{NI}}{\gamma_1^{NN}} \right) + \frac{\gamma_1^H}{\gamma_1^{NN}} - \frac{\gamma_1^{NI}}{\gamma_1^{NN}} \quad \text{(A17)} \]

We derive \( \overline{x}_2^I \) from the above equation as:

\[ \gamma_# \overline{x}_2^I = mB - n \quad \text{(A18)} \]

wherein: \( m = \frac{\gamma_1^{NN} \gamma_1^{II}}{\gamma_1^{NI}} \); \( n = (\alpha_1 - \alpha_2)(1 - \frac{\gamma_1^{NI}}{\gamma_1^{NN}}) + q^I \gamma_1^H \left( \frac{\gamma_1^{NN} \gamma_1^{II}}{\gamma_1^{NI}} - 1 \right) \)

Combining Eqns (12b) and (A18), we get an equation in a single variable B:

\[ G(B) - G(B - \gamma_# q^I) - mB + n = 0 \quad \text{(A19)} \]

In the case of homogenous behavior, then \( \gamma_#^H / \gamma_a = \gamma_a^{NN} / \gamma_a = \gamma_a^{NI} / \gamma_a = \gamma_a^{NN} / \gamma_a \) which lead to \( m = 1 \) and \( n = 1 \). Then Eqn (A20) can be solved analytically as \( B = D/2 \), yielding \( \overline{x}_2^I = q^I / 2 \).