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Programming as mathematical narrative

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Abstract: This paper describes a narrative-oriented approach to the design and the analysis of a computational system and a set of activities for mathematical learning. Our central contention is that programming can offer a key to resolving the tension between the different representational structures of narrative and mathematical formalism. In the course of describing our approach, we make a distinction between the epistemic-cognitive elements of narrative and the social, cultural and affective elements. We then elaborate the theoretical grounds of the individual epistemic facets of narrative. We propose a link between narrative theories of learning and constructionist traditions, specifically the notion of situated abstraction. This link suggests the possibility of further dialogue between the two academic communities.

Keywords: collaborative learning; constructionism; Computer Supported Collaborative Learning; mathematical learning; narrative; programming; situated abstraction.

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1 Introduction

> "Actually, it is half the art of storytelling to keep a story free from explanation as one reproduces it (...). The most extraordinary things, marvellous things, are related with the greatest accuracy, but the psychological connection of the events is not forced on the reader. It is left up to him to interpret things the way he understands them, and thus the narrative achieves amplitude that the information lacks (Benjamin, 1968)".

This paper describes a narrative-oriented approach to the design and the analysis of a computational system and a set of activities for mathematical learning. The language of mathematics is often perceived as propositional; a formalism which defines terms, states axioms and rules, then derives theorems and proves them. Its structures are static, devoid of time and person. This view was demonstrated lucidly by Wittgenstein (1989).

> "In mathematics, we have propositions which contain the same symbols as, for example, "write down the integral of…", etc. with the difference that when we have a mathematical proposition, time does not enter into it and in the other it does (p.34)".

This would appear to be antithetical to narrative form, which is always personal, contextual and time-bound. By contrast, Bruner (1986, 1990) shows that narrative is a powerful cognitive and epistemological construct. The main question we explore is: how can the epistemic power of narrative be harnessed in the construction of mathematical meaning?

We approach this question from a design perspective. We are concerned with the design of platforms, tools, and activities for mathematical learning, focusing on the notion of situated abstraction (Noss and Hoyles, 1996). The idea (since developed in, for example, Noss, Healy and Hoyles, 1997) highlights the dynamics of constructing knowledge from activity, by inserting or populating an abstraction with meaning – in the shape of special cases, particular values, or familiar contexts (or, in the special case of the mathematical situation, with mathematical objects and relationships). The questions we ask include: what are the possible contributions of narrative that might facilitate such a trajectory? What is required from such narrative, and what is required from the learning activity encompassing it? In brief, we aim at elaborating the role that narrative could play in the construction of mathematical abstraction.

Our central contention is that programming can offer a key to resolving the tension between the different representational structures of narrative and the mathematical formalism. We see programming as an expressive activity, a form of writing or composing, contingent on context and used purposefully to carry out actions. We claim that programming can afford a narrative form for representing mathematical meanings. The issues we address have strong social and cultural dimensions, and occasionally we refer to these. However, our goal is to highlight the often neglected aspects of individual knowledge construction within a social environment.

The structure of this paper is as follows. We begin by presenting a review of the use of narrative in educational theory in general and in the teaching of mathematics in particular. Inter alia, we present our own perspective on the relationship between narrative, learning and technology. We then briefly describe the WebLabs project and the tools developed for it, as an example of a narrative-aware learning environment. Following this, we present three illustrative episodes from our observations, and comment on the role of narrative in students' learning and in the design of technology to

support it. Our concluding discussion highlights the potential of constructionist programming to provide students with a medium for mathematical narrative.

2 Narrative and education

The concept of narrative has been investigated extensively within a wide range of disciplines over the last few decades. To name but a few: in literary theory, Genette (1980) establishes narrative as a fundamental tool; in the social sciences, Gergen (1998) refers to it as a tenet of social construction; Carr (1986) positions it as a central concept in the philosophy of history. Since the 1980s, narrative approaches have also become popular in counselling, where the term refers to a patient's personal account of her condition (White and Epston, 1990; Roberts, 2000).

Our own interests centre on the epistemic role of narrative, in the tradition of Bruner. We focus on storytelling as a means of meaning-making, with an emphasis on the structural and the semantic components of narrative. In reality, it is hard to separate the individual epistemic aspects of narrative from the social, affective and cultural aspects. Nevertheless, the emphasis in this paper is on the former. In his theory of learning and education, Bruner (1986, 1990, 1991, 1996) and Bruner and Lucariello (1989) identified narrative as the predominant vernacular form of representing and communicating meaning. Humans use narrative as a means of organising their experiences and making sense of them. Parents use narrative as a means of sharing knowledge with their children. Schank and Abelson (1995) argue that stories about one's experiences, and the experiences of others, are the fundamental constituents of human memory, knowledge, and social communication. They call for a shift towards a functional view of knowledge, as Schank (1995) explains: "intelligence is really about understanding what has happened well enough to be able to predict when it may happen again" (p.1). Such knowledge is constructed by indexing narratives of self and others' experiences, and mapping them to structures already in memory. While Schank and Abelson (1995) come from an AI perspective, their theory is supported by recent psychological studies. Atance and O'Neill (2005) define episodic future thinking as the ability to project oneself into the future to pre-experience an event. This, they claim, is a uniquely human phenomenon which precedes semantic future thinking (Atance and Meltzoff, 2005), and provides the developmental basis for skills such as planning and causal reasoning. They found that episodic future thinking emerges around the age of four, and is related to children's abilities to construct and comprehend verbal accounts of experiences. Recent developments suggest a neural basis for the role of narrative in the abstraction of daily experience to knowledge (Mar, 2004). Narrative comprehension engages a widely distributed network of brain regions, and is clearly distinct from basic language comprehension (Nichelli et al., 1995; Ferstl, Rinck and von Cramon, 2005; Xu et al., 2005).

Following Bruner, we define narrative as a progression of statements describing something happening to someone in some circumstances. This view entails a form of language which includes a context (setting) and a plot: a sequence of events bound by temporal – and implicitly causal – relationships. Likewise, Mar (2004) identifies the presence of a causal-temporal event structure as imperative, and notes:

> "The most basic elements of a story include a setting, and an agent who holds a certain goal […] and whose progress towards that goal is impeded […] or facilitated by certain events" (p.1415).

In this paper, we explore three constituents of narrative: context, plot and moral. The context includes the background information assumed or conveyed explicitly with a narrative. The plot denotes its temporal and causal structure. We use the term moral to refer to the implicit endpoint of a narrative, the purpose for which it is told.

A narrative is always contextualised. An important contextual element is the exposition, which lays out the context: time, location, props and characters. Such an exposition is not limited to imaginative narrative: it also appears in scientific texts (Bruner, 1986). One particular element of context we focus on is the idea of voice, which relates to the presence of the speaker. Even in allegedly 'de-humanised' arenas, such as scientific or legal writing, great significance is attached to the voice of a document's author. When approaching a scientific paper, one draws on knowledge of the author: past publications, close collaborators, institution, etc. Likewise, when writing a paper, one is advised to imagine its readers and engage in a dialogue with them. Familiarity with the writer's personal style makes the writing much easier to interpret and understand. A clear sense of authorship promotes responsibility for the text.

A well-formed narrative must maintain coherence of temporality and causality (Gergen, 1998). Temporality refers to the chronological ordering of events. In the light of narrative intelligence theory (Mateas and Sengers, 1999), it is clear that maintaining the temporal structure is crucial to the reader's ability to comprehend a story. The identification of temporal affinity of events also plays a strong role in learners' inferences of causality, an important component in the construction of meanings. The sequencing of events is referred to as the plot. Gergen (1998) adds that events are carefully selected to support an endpoint.

Yet perhaps the most important part of a narrative is typically left unstated: its moral. We use this term with an expanded meaning, referring to the narrative's implicit endpoint. A story is told for a purpose – establishing norms, conveying knowledge, or raising a question. It is the implicit layer that holds the narrative together – the causal relationships along the way and the climactic moral at the end. Without them, all we have is an arbitrary list of events. As Mar (2004) asserts,

> "If a well-crafted story contains mention of an event or a character, it is assumed that this element is in some way relevant to the goals of the protagonist" (p.1416).

Recent advances in neural psychology ground these observations in new understandings of the brain's inner working (Holyoak and Krogen, 1995; Young and Saver, 2001; Addis et al., 2004; Mar, 2004; Mason, 2004; Mar et al., in press). Xu et al. (2005) link context to brain regions responsible for global semantic processes such as inference, coherence, conceptual association and text integration. Other findings point to a strong link between narrative comprehension and theory-of-mind processing (Mar, 2004), suggesting that the cognitive modelling of the storyteller and the protagonists is a critical constituent in understanding a story. A detailed discussion of the relations between neural and cultural theories is called far, but is beyond the scope of this paper.

Closer to home, the mechanism described above is consistent with the ideas of situated abstraction and webbing (Noss and Hoyles, 1996; Noss et al., 1997). The concept of situated abstraction focuses attention on the process of making meanings through

activity. It highlights the fact that this process is situated in a context, and thus the linguistic and conceptual resources made available for expressing meaning are rooted in that context. Abstraction is achieved within, not above, context. Mathematical knowledge is constructed and expressed with available tools (physical, linguistic, digital or social) that may not map trivially to standard mathematical notation. People situate abstraction by *webbing* together meaning from artefacts, actions, symbols and context. Our current perspective on narrative as an epistemic vehicle elaborates the ideas of situated abstraction, by demonstrating one mechanism by which this layering and webbing works. The neuropsychological evidence gives direct support to the ideas of layering and webbing, albeit using a different terminology. For example, Addis et al. (2004) talk about 'specific and general autobiographical memories' (p.1740), and show that these activate the same regions in the brain, or, in the words of Mason and Just (2006):

> "Text attributes at the discourse level enter into combinations with other information to allow a reader to weave individual sentences into an integrated narrative structure. The resulting conceptual structure incorporates pragmatic information and connects the text with the reader's world knowledge".

Coming from a design research methodology (Mor and Winters, 2007), we take the concepts of narrative both as analytical tools and as design guidelines. The analytical dimension asserts that we interpret learners' expressions as mathematical narratives, i.e. narratives which are intended to communicate or construct mathematical meanings. This approach lends itself naturally to verbal and written expressions; our argument is that it could also be applied to other modalities – including graphics and programming. A similar claim is voiced eloquently by Healy and Sinclair (2007).

As for the role of narrative in design, the challenge is to engender situations for mathematical learning that acknowledge forces such as voice, context and plot. Our argument is that it is necessary to find interpretations of these terms which have inherent mathematical meanings. In other words, the context has to be a mathematical context, the plot has to be a mathematical plot and its moral – the implicit endpoint – must be mathematical.

Given the strong cultural and neurological grounding of narrative, it seems that we should strive to embed narrative structure in the design of systems or activities which are aimed at meaning-making. However, narrative approaches to computer-enhanced learning are often focused on designing systems that support narrative-based learning (Mott et al., 1999; Decortis and Rizzo, 2002; Decortis, 2004), i.e. systems that support the production of imaginary narrative as the site of learning. Nehaniv (1999) argues for a broader view, claiming that any design that does not acknowledge the 'narrative grounding' of humans will appear to its users as bizarre, unintelligent and unintelligible.

> "…it is desirable to take into account that humans are temporally grounded, narratively intelligent beings. Their evolutionary heritage leads them to expect that the actions of others are embedded in a context of past history and future events" (Nehaniv, 1999, p.102).

Likewise, Laurillard et al. (2000) highlight the importance of embedding narrative structure in the design of multi-media resources, where non-linearity risks impeding learners from maintaining a personal narrative line and thus increasing cognitive costs. It is the responsibility of teachers and designers to maintain narrative flow in order to allow learners to maintain a focus on the development of sound arguments:

> "With such design features, the non-linear medium is able to afford something more than mere browsing: it will afford structured, meaningful learning" (p.18).

3 Mathematics and narrative in education

When we say 'two plus two equals four', the truth value of this statement is independent of when we say it or who 'we' are. Yet how do we present such a statement to a young child? One might say:

> "You had two marbles, and I gave you two more, so now you have four".

When we attempt to humanise the mathematical statement, we unconsciously transform it from the propositional form to the narrative. Something (transfer of marbles) happened to someone (the child and me) under some circumstances (say, sitting around the kitchen table). In that event, two groups of two were magically exchanged for one group of four.

Such conversions from propositional to narrative do not disappear as the subject matter becomes more elaborate. Let us review one more example:

$$\{S_i\} \rightarrow C \equiv \text{for each } \varepsilon \text{ there exists } N \text{ such that for every } n > N, |S_n - C| < \varepsilon$$

How do we explain such a statement to a student? Perhaps:

> "Let's look at the sequence we had yesterday {1, 1/2, 1/3,…}. Go far enough along the sequence and you will reach a point such that all subsequent terms within a very small range of 0. Now, we can make that small range as small as we want".

Again, in our attempt to make the mathematical idea accessible, the propositional is rendered as a narrative. A static structure, 'devoid of time and person', is placed in a specific context, and becomes a string of events happening to 'you'. However, this symbiosis is short lived. Very quickly, the student is asked to abandon the narrative discourse and pick up the propositional form, to use algebraic symbolism in its static interpretation, a demand expressed by Solomon and O'Neill (1998):

> "Mathematics can be embedded in a variety of texts in a variety of styles from dialogue [...]. This, however, is quite distinct from linguistic features constitutive of mathematical discourse itself: mathematics cannot be narrative for it is structured around logical and not temporal relations" (p.217).

Solomon and O'Neill (1998) reject the idea that "Children could re-invent mathematics by abstracting it from the world around them" (p.217): for them mathematics is a strict social practice, with distinct rules of genre. This requirement, they readily admit, gives rise to a dissonance between the students' interpretation of the symbols and the one expected by their teachers. We ask: is the static, disembodied form a necessary feature of mathematical language? A historical perspective suggests that there are other possibilities for mathematics' notational infrastructure, and that the static formalism may have been optimised for static media (Kaput, Noss and Hoyles, 2002). Do new media offer new opportunities – can there be a representational system that allows us to express mathematical concepts adequately in a narrative form?

We are not alone in challenging the static view of mathematics. Indeed, Healy and Sinclair (2007), in a studied response to Solomon and O'Neill (1998), argue that the latter's position overlook the possible role of narrative in more personal acts of

understanding. Many testimonies show an alienated experience of mathematics. This barrier can be breeched by allowing space for learners' personal narratives, relating mathematical meanings to their own experiences and reflecting on their individual learning trajectories. We contend that the chasm runs deeper: it is not, as Solomon and O'Neill (1998) phrase it, a debate between "an emphasis on authorship and creativity versus an emphasis on understanding genre" (p.210). It is a question of what is the mathematics we wish to teach: a practice, or a phenomenon, a noun or a verb? Should children learn to see mathematics or to do mathematics? Perhaps both, but then – which comes first? Solomon and O'Neill (1998) present an example of two texts by William Rowan Hamilton, one from his published letters and the other from his more formal publications. But while they see the former as literature and the latter as mathematics, Healy and Sinclair (2007) see one as a window on the process of doing mathematics, and the other as the output of that process. They inspect various reports of mathematician's personal experiences, and find that all have temporal structure, and carry a strong sense of voice.

Bruner (1991) distinguishes between scientific knowledge, which is organised by logical principles, and cultural assets, what he calls 'folk psychology', which he argues are 'organised narratively' (p.21). He calls for a shift of attention which would honour both forms of knowledge. Nevertheless, this distinction does not preclude representing and learning of scientific and logical knowledge in narrative forms. Indeed, Bruner (1986) notes two modes of thinking, mapped to two genres of narrative – paradigmatic and imaginative. Paradigmatic narrative is top down, seeks generality and demands consistency. Imaginative narrative is bottom up, seeks specificity and demands coherence. Several researchers have suggested that in order to provide learners with tools for coping with unfamiliar problems, they need to share the experiences of those who posses such tools. Burton (1996) argues that this points to a need to facilitate learners' authoring of their accounts of how they came to know mathematics. These narratives are personal, i.e. imaginative, as they are general and paradigmatic. Livingston (2006) calls for an educational approach to mathematical proof that acknowledges the context in which proofs are constructed and the personal path taken by those who prove. Although he does not refer explicitly to the notion of narrative, we find many parallels in his situated view of proof. Morgan (2001) also distinguishes between mathematical 'facts' and 'activity'. Inspecting several mathematical texts, she identifies elements of temporality and personalisation, similar to the constituents of narrative we noted. Morgan argues that rather than rejecting such style as 'inappropriate', we should ask: what are the criteria for a personal narrative to qualify as an account of mathematical activity?

The model of narrative comprehension we presented above provides further support for these arguments. We saw how developing a theory-of-mind is fundamental in narrative comprehension. Likewise, if we want children to learn to think and act like mathematicians they need to develop a theory of mathematical mind: the ability to imagine "how a mathematician approaches this problem", and what better way then through mathematical narratives? Furthermore, our minds are geared towards extracting causal structures from the temporal sequencing of a narrative. "The queen died, then the king died" is transformed to "if queen dies, then king dies" (with apologies to Forster (1927)). So, counter to Solomon and O'Neill's claim, it may be possible that children will invent mathematical structure by abstracting it from the narratives around them – be it those they receive, or those they construct. Indeed, O'Neill, Pearce and Pick (2004) find a surprising correlation between children's performance in generating narratives at the age

of three to four, and their mathematical abilities 2 years later. This correlation is unique: general language skills were neither predictive of mathematical achievement nor narrative skills predictive of spelling skills or general knowledge. They suggest that the same skills which underlie narrative comprehension form the basis of mathematical thinking: inference of relationships and logical chains.

Our approach is in agreement with many of the assertions of the emerging discourse approach to mathematics (Kieran, Forman and Sfard, 2002). Perhaps the main distinction is that we focus on the micro, individual, epistemic facets of discourse while most of the research in this framework emphasises the social and cultural aspects of cognition as communication.

4 Narrative learning environments and mathematics

We wish to differentiate between three types of systems: interactive narrative games, narrative learning environments and environments with narrative elements.

The first group includes interactive storytelling and interactive drama environments, such as Façade (Stern and Mateas, 2005) and Storytron (Crawford, 2004), that are designed to engage participants in an enhanced dramatic experience. Most of the work in this field stems from a gaming and game design tradition. While learning is acknowledged, other qualities are highlighted, such as aesthetic experience and pleasure.

Narrative learning environments, the second group, are designed from the premise of narrative as a defining factor in learning (Mott et al., 1999; Dettori et al., 2006). In this case, learning is the aim and narrative is the primary means, manifested in technological tools. Many of the efforts in this category come from an AI background, with an emphasis on narrative agents. Systems in this category, such as Teatrix (Paiva, Machado and Prada, 2001), often share the interactivity and dramatic qualities of the first, but with a shift of emphasis from playing to learning. Some even attempt to appropriate these characteristics to a mathematical domain (Alexandre, 2006).

The third class of technologies are interactive learning environments which acknowledge narrative elements (Back, 2005; Sarmiento, Trausan-Matu and Stahl, 2005; Stahl et al., 2006; Yukawa, 2006), either in their design, in their use or in their analysis. These studies typically emerge from a general Computer Supported Collaborative Learning (CSCL) tradition, assimilating ideas of narrative into existing frameworks of technology enhanced learning. The systems in this category may be traditional interactive learning environments with an added discursive or reflective element. In other cases, we find pedagogical innovations using ubiquitous social software, such as blogs (Makri, 2006) or wikis (Yukawa, 2006).

Our work falls into the last category. The reason for this is quite simply that we set out to create an environment for mathematical learning. Our initial notions of narrative were, in all honesty, vague and naïve. It is through the iterative process of design research that these ideas were refined, and their presence in the design amplified. This process inevitably left us with some rough edges, but it also led us to discover narrativity in unexpected places.

In the remainder of this paper, we demonstrate our approach, based on creating situations in which students have an incentive to make formal arguments and to challenge the validity of each others' statements, adopting narrative forms that are themselves embedded in a formal expressive system that allows mathematical ideas to be developed

and shared. We begin by describing the context of our work. We then review three illustrative episodes.

5 WebLabs

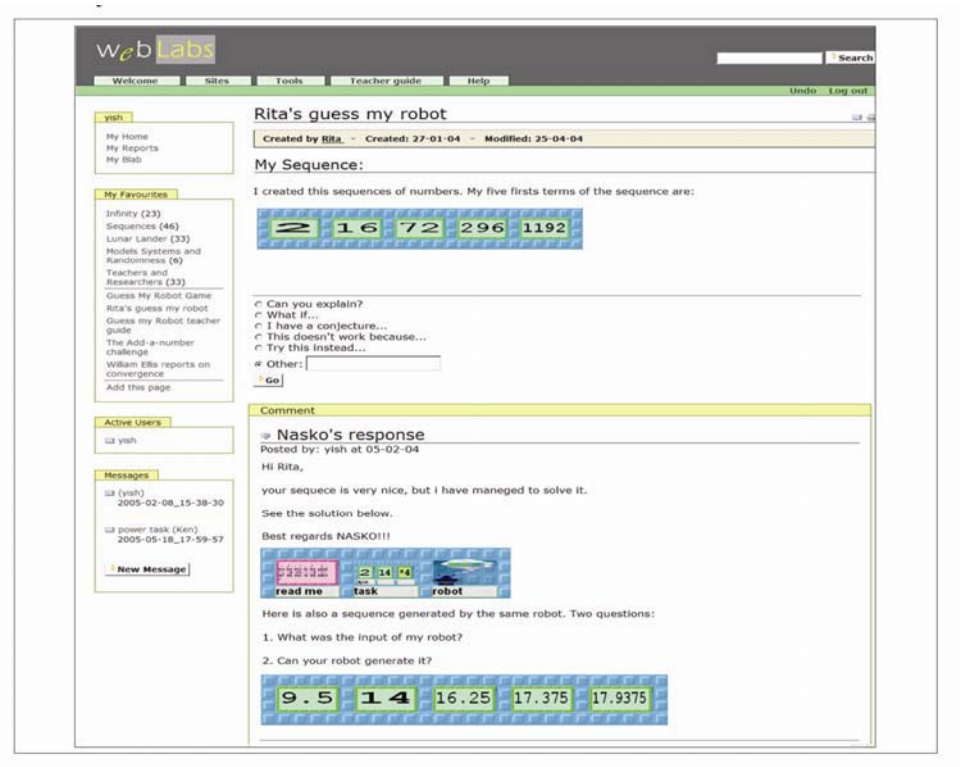
The examples below are derived from the WebLabs Project (www.weblabs.eu.com), which has been described in detail elsewhere (Mor et al., 2006; Simpson et al., 2006). We will only mention briefly those elements that are essential for the topic at hand. The project aimed at exploring new ways of constructing and expressing mathematical and scientific knowledge in communities of young learners. The WebLabs project involved several hundred students, aged 10–14, across 16 schools and clubs in six European countries. Our approach brought together two traditions: constructionist learning as described by Papert and Harel (1991) and collaborative knowledge-building in the spirit of Scardamalia and Bereiter (1994). The former was largely supported by the programming language ToonTalk (Kahn, 1996, 2004) (www.toontalk.com), whereas for the latter we have designed and built a web-based collaboration system called WebReports (Mor, Tholander and Holmberg, 2006). The central design intention of our approach is that students should simultaneously build and share models of their emerging mathematical knowledge.

ToonTalk (Figure 1) is a language and a programming environment designed to be accessible by children of a wide range of ages, without compromising computational and expressive power. Following a video game metaphor, the programmer is represented by an avatar that acts in a virtual world. Through this avatar the programmer can operate on objects in this world, or can train a robot to do so. Training a robot is the ToonTalk equivalent of programming. The programmer leads the robot through a sequence of actions, and the robot will then repeat these actions whenever presented with the right conditions. ToonTalk programs are animated: the robot displays its actions as it executes them.

Figure 1 The ToonTalk programming environment. The programmer's avatar is on the right, and a robot generating a sequence of numbers in the centre

The WebReports system (Figure 2) was set up to serve both as a personal memory aid and as a communication tool. A web report is a document that is composed and displayed online, through which a learner can share experiences, questions and ideas derived from her activities. The uniqueness of our system is that it allows the author to share her ideas not just as text, but also graphics and animated ToonTalk models. This last point is crucial: rather than simply discussing what each other may be thinking, students can share what they have built, and rebuild each others' attempts to model any given task or object.

Figure 2 The WebReports collaboration environment. A student's report and peer comments, both incorporating embedded ToonTalk objects



A main concern was the careful design of a set of activities, aiming to foster learning of specific mathematical topics, such as sequences, infinity and randomness. The choice and design of technologies was subordinate to this cause. In that sense, our environment is not a narrative learning environment *per se*, but rather a narrative-aware learning environment. It supports construction, collaboration and exploration by providing learners with a *Narrative space*: a medium, integrated with the activity design, which allows learners to express and explore ideas in a narrative form. The examples in the next section aim to elucidate these ideas.

6 A few illustrative episodes

We will review several episodes from a strand of activities on number sequences. While an extensive narrative-oriented analysis of our data has yet to be presented, we will use these examples to demonstrate our two-way approach: on the one hand, interpret learners' expressions as narrative, and use this lens to understand their learning process. On the other hand, we will identify the relationship between the learners' modes of expression and the design of the learning environment.

6.1 Episode I: adding up

Our first snapshot is taken from an experiment conducted in London in autumn 2004. This experiment involved a group of ten boys, aged 13–14, for six one-hour sessions and a full day workshop. One of the first activities we conducted focused on generating and understanding partial sum series, using the *Streams* design pattern (Mor et al., 2006). Participants were asked to create one robot which generates a sequence of numbers, and feed its output to a second robot which sums the terms. Traditionally in such contexts, a sequence would be represented by a list: a static array of the first *n* terms of the sequence. The streams pattern replaces this structure with a dynamic process that generates the terms on the fly, and passes them from one module to the next, in a manner similar to a factory production line. Rather than seeing the sequence as a fixed and finite array of numbers, students observe and manipulate a continuous, dynamic and potentially infinite entity. The standard representation of numbers and commas is replaced by a string of events with a temporal and casual structure.

The following fragment is taken from a group discussion. Using the electronic whiteboard, Alan had just demonstrated how he constructed and connected the two robots. The first robot, called add-a-num, generates the natural numbers by iteratively adding 1 to the current term. By replacing this 1 with a variable, it can be generalised to any arithmetic sequence. The second robot sums the terms of the first sequence as they are generated, producing a series of partial sums. In the case of natural numbers, this is the sequence {1, 3, 6, 10,…}.

As we watched Alan's robots in action, Peter was asked to provide a commentary on their actions.

Peter: Ok, huh, well, the robot's taking the numbers from the nest.

Researcher: Which robot?

Peter: The 'add up' robot is taking the numbers from the nest which says numbers I think, and the numbers in the numbers nest are coming from the other sequence which the other robot is doing so he's taking these numbers and he's adding them on to the total creating a different sequence out of the other sequence.

Researcher: What is this different sequence that it's created? This last sequence what is it, can you describe it?

Peter: It's, (pause) it adds, it's going up I think, (laugh) it's going up one and adding that number on each time to the total.

Phrase (1) shows some confusion and hesitation. Phrase (3) exhibits a specific narrative of the events on the screen. We see a simultaneous process of narrative comprehension and construction. Peter observes Alan's animated narrative, and reconstructs it in words.

Reading Peter's expressions, several issues emerge. First, notice the narrative structure of both phrase (3) and (5). In (3), the 'add up' robot is the protagonist, going through a string of events. The purpose of add up's actions is to 'create a different sequence out of the other sequence'. But what is that sequence? This is left unsaid. In phrase (5) we see a temporal structure and a protagonist, except that the identity has changed: now it is the sequence. What we see is the rule of the sequence expressed in narrative form. Replace 'going up in one' with 'natural numbers' and 'adding that number to the total' with 'partial sums of' and we get the standard definition.

Peter's narrative in phrase (3) is already an abstraction. Children who have not constructed such a robot would describe what they see in a procedural manner: 'the bird brings in the number', 'the mouse bams it on to the other number'. In constructing his narrative, Peter chooses the events that are worth noting, those that serve the mathematical moral of the tale: creating one sequence from another. It is Peter's own experience in modelling this idea which allows him to connect the events he sees before him to his own episodic memories, and shift from a specific narrative to a generic one. On the other hand, capturing this idea as a tale of two robots gives the mathematical concept a narrative body.

It is important to note the blending of the technology into the classroom culture. While the use of programming and the display of animated code on the whiteboard are technologically advanced, the discussion itself – the narrative space, is conducted in a traditional classroom environment. When designing digital environments for collaborative learning, such a narrative space needs to be preserved.

6.2 Episode 2: Joe999's robot

The partial sums activity was followed by a game called Guess my Robot (GmR). In this game, students challenge each other to reconstruct the robots they used to produce complex number sequences. A mathematical analysis of this game is presented in Mor et al. (2004), and a more elaborate narrative-oriented analysis is available in Mor and Noss (2004).

Joe999 was the self-adopted WebReports nickname of an 11-year-old boy from London. His group worked on a different activity, and he was not initially involved in the Guess my Robot game. Having found his way to the game in a round-about manner, he started from a relatively advanced challenge: {11, 7.5, 5.75, 4.875, 4.4375}, and responded by posting a textual comment:

1. Joe999: Yish. After 10 min I figured out how to do the sequence. You take away 3.5. Then you find half of 3.5 and take that away from 11 and continue this sequence.

Such a response would be disqualified by many teachers. It is unclear what you 'take away 3.5' from, where the 11 comes from, and how you 'continue this sequence'. Reading this as a narrative, one can infer that Joe999 knows the answer. He assumes that the context is known. This context includes the box with the initial term of the sequence, and the fact that ToonTalk robots repeat the action they were trained to do. Nevertheless, our goal is to lead him to express his knowledge in rigorous form. Thus, YM responded:

2. YM: Don't just talk. ToonTalk. Instead of telling me you figured it out, build a robot (or chain of robots) that produces this sequence.

Joe999 took up the gauntlet, and trained a robot. To our surprise, this robot did not produce the challenge sequence – it acted out the story of how Joe999 had solved the puzzle! The robot takes the differences of the sequence, arranges them in a box and labels them: ('0.4375' is half of, '0.875' is half of, '1.75' is half of, '3.5' this number]. Then, the robot proudly prints:

3. Joe999: Conclusion: You are halving the number you halved before. I have shown this in this box. Good sequence though Yish (^_^).

Joe999 appropriated ToonTalk to create his own narrative medium. Without any guidance from the researchers, he had used programming as a way of making a mathematical argument. He has retold the narrative from excerpt (1), yet in a form that is precise and succinct in nature, leaving no room for ambiguity. Joe999 does not have the linguistic tools to express himself accurately in text, but when programming – one has no choice but to be mathematical.

The genre dimension in Joe99's work is fascinating: taking ToonTalk programming as a shared cultural asset, he uses the execution of a program as the framework for telling his story. As Bruner notes, "it is by virtue of this embeddedness in genre…that narrative particulars can be 'filled in' when they are missing from an account." (Bruner, 1991). Indeed, the narrative of Joe999's robot would make little sense to a reader unfamiliar with ToonTalk programming.

The context of Joe999's narrative is given by the facilities of the WebReports and ToonTalk environment, and then enhanced by Joe999 in his packaging of the robot. Referring to emergent conventions, he positions the robot and its inputs in a manner that will ease the entry of potential readers into his narrative.

Joe999 meticulously assembles his plot. The robot goes through a carefully chosen sequence of actions and events. As with any good plot, Joe999's code has a moral. The purpose of the protagonist's (the robot's) actions in the story is not their immediate outcome (a box of numbers, a block of text), but the implicit transfer of an idea from Joe999 to his 'readers': to convince them that Joe999 has uncovered the structure of the sequence.

Even the message that the robot prints has narrative characteristics: there is a protagonist (you are…), a progression of events (halving the number you halved before), and a sense of personal voice. The robot acts as an avatar for Joe999, expressing his conviction and emotion when typing "I have shown this in this box. Good sequence though Yish". Yet, the vague description in (1) has been replaced by the more generic and precise 'halving the number you halved before'.

6.3 Episode 3: 'fatal mistake'

The last phase of our number sequence activities focused on partial sums of converging sequences. Students constructed robots to produce various converging sequences they proposed. They then conjectured about the behaviour of the partial sums of these sequences, and used the add-up robot from the first activity to test these conjectures. The students published their constructions along with their observations as they progressed. In this episode, we focus on one of the students in the group mentioned in episode one. We

refer to him by the nickname he chose for himself: Sodapop. His report streamlines text, graphics, excel charts and ToonTalk robot. We include only the text here, but refer the readers to the original at: http://www.weblabs.org.uk/wlplone/Members/sodapop/my_reports/Report.2005-03-22.2632.

Sodapop was exploring the reciprocals sequence {1, 1/2, 1/3, 1/4…} and its partial sums. After constructing the sequence, he plotted its terms and used a paint program to overlay his prediction of the graph of the partial sums on the image.

1. Sodapop: This is my prediction of what will happen. (Sodapop embeds prediction graph here)

2. Sodapop: This is the real graph that was produced by the cumulate total of the halving-a-number robot. It looks like the top of my graph but I made the fatal mistake of thinking it started at zero. I also said it wouldn't go over 100, which was very wrong. (Sodapop embeds function graph here)

3. Sodapop: After lengthy research and a detailed experiment, I have concluded that if the primary source was an integer between 99 and 101 (not including those numbers) that the cumulative total can never go above 200. This is because if you have 0.1 and you double it and add it together you will get 0.15 so every time you do this you will get another number after the decimal place. So you will constantly get more numbers after the decimal place, but the numbers closest to the decimal place will not be getting any larger.

The difference in style between this excerpt and verbal expressions by Sodapop and his friends suggest that putting his words into a public medium may have enhanced Sodapop's audience awareness (Mor et al., 2004). Some of the stylistic decorations are just that (primary source between 99 and 101), and indeed in a follow-up interview Sodapop admitted they were there to impress. Yet stripped of those, the text makes an interesting argument – expressed in narrative form. It is 'you' who 'have a number' and then progress through a line of actions. Unfortunately, the argument itself is flawed. But as with every good narrative, the important part is left unsaid. Sodapop is not telling the tale of the harmonic sequence. He is reporting on a process of inquiry: he had a theory; he tested it and found a counter-example, and consequently searched for a new theory.

While learners may be led through such a process many times, it is hard to get them to reflect on it and adopt it as a meta-cognitive strategy. Perhaps by giving Sodapop an opportunity to organise this experience as a narrative, we have allowed him to reflect on it at a higher level. Here is an important lesson in design: had we built strong scaffolding into the WebReports system, forcing Sodapop's expression into a structural mould, the narrative space would have been lost, and with it an opportunity for learning – both for him and for his peers who read his account.

7 Conclusions

This paper attempts to bridge the divide between narrative and formal language, by positioning programming – or rather, a particular form of programming – as a mediating linguistic form. We have explored this question by examining three episodes from the WebLabs project in which children use programming and web-based discussion to conduct mathematical investigations. We have proposed a narrative-oriented framework

for design and analysis of mathematical learning activities and the computer-enhanced means of supporting them. The main elements of this framework are context, plot and moral (in the sense of an implicit conclusion). This framework was used to analyse a software system and the activities it affords.

We opened with the question of how to harness the epistemic power of narrative in the construction of mathematical meaning, arguing that computer programming holds that potential. Like narrative, computer programs operate in a specified context, have a temporal structure (or 'plot') with underlying causal reasoning, and involve 'actors' and 'objects' – in fact, such terms are habitually used in software design. Yet like mathematical language, computer programs do not tolerate ambiguity and inconsistency. They are no less valid than algebraic formulae as a means of mathematical expression. Writing a program means taking the story of a phenomenon and restructuring it into formal statements. Once this is done, the programmer's ideas are reified in an object that can be passed around, examined, manipulated and argued about. This is not to say that all programming is narrative, and certainly not that narrative is all that programming is. Our intention is to highlight the narrative dimension of programming and its contribution to mathematical learning. Thus, one could argue that a procedural language such as Pascal is higher on the narrative scale than a declarative language such as Prolog, and that this difference may provide an insight as to their suitability as educational tools. From this perspective ToonTalk is a rather special case. It is, as we have seen, a system which is based on the idea of a narrative: the objects are characters, some with well-defined characters, programs can only be 'read' in real time, and the running of a program involves a story unfolding on the screen.

Let us recall one of the examples we gave earlier, so that we can now view it through the window of a ToonTalk program.

$$\{S_i\} \rightarrow C \equiv \text{for each } \varepsilon \text{ there exists } N \text{ such that for every } n > N,\ |S_n - C| < \varepsilon$$

and, in contrast,

> "If you go far enough along with the sequence, you reach a point such that all subsequent terms are within a very small range of some constant, and that small range can be as small as we want."

There is little doubt that the essence of the first statement is captured in the second, and that in terms of grasping the key abstraction involved, the latter is much more intelligible. Its intelligibility lies partly in the lack of symbols (why symbols render a text unintelligible is another matter, but we should recall the advice given to Stephen Hawking by his editor that every equation in his book would halve the number of readers!). Clearly, the narrative nature of the sentence is, as we have pointed out earlier, also a key consideration.

Nevertheless, something is lost. It is extremely difficult to manipulate the latter expression, to use it to, say, prove some alternative theorem, to quantify how small ε can be for a given N. It is, in fact, rigorous only as a narrative, and it is this limitation that leads to the mathematician's privileging the symbolic text as exemplified in the first formulation. The problem is that while both are abstractions, the latter is an abstraction that is situated in time, and for that luxury one has to sacrifice the utility of the expression for generating new results.

The examples of ToonTalk programming above suggest a way to avoid having to make this difficult choice. By situating abstraction in time and space, abstraction can be

given meaning, situated within a narrative. So what, in formal settings is regarded as unnecessary 'noise' in terms of narrative gained at the expense of utility and rigour, becomes constitutive of mathematical meaning. Indeed, this is, we think, a special case of a more general observation, that the noise of a situation – be it contextual cues, social setting, or implicit narrative – is crucial to meaning making and thus to learning. By embedding narrative elements in the design of the WebReports collaborative system, utilising the narrative features of the ToonTalk programming language and applying a narrative-oriented approach to the design of activities, we have enabled students to utilise their narrative intelligence in constructing mathematical knowledge.

We do not claim to have resolved the questions we raised. At most, we hope we have convinced the reader that they are worth consideration. We argued for the value of observing narrative as a cognitive structure, yet some work still needs to be done to obtain a clear definition of narrative from a cognitive perspective, identify the atomic unit of narrative, and gain a better understanding of the relations between narrative, embodiedness and neuropsychology.

There is also a notable difference between the narrative present in most of the literature and that expressed in programmed code. The former is predominantly a recount of past events, whereas the latter is a recipe for affecting future events. This distinction needs to be elaborated. In a way, programming is a form of fantasy: but perhaps so is mathematics?

We have distinguished between social, cultural, affective and epistemic facets of narrative, and limited our discussion to the latter. While we feel it is important to highlight the often neglected individual aspect of narrative, this separation is somewhat induced. It remains to be explored how these different facets interact. In fact, even programming needs to be considered in a social context. After all, assembly languages aside, code is written to be read by humans. The notion of theory-of-mind is a possible link between the social context and the individual construction of knowledge. This link needs to be explored by combining diverse perspectives. One particular aspect is learners' anthropomorphism of code in the process of constructing or reading of programs, with ToonTalk robots as a special case.

The surprising correlations between social theories of narrative, situated abstraction and recent neurological models need to be explored. On one hand, being able to ground the social theory in the workings of the brain provides depth and credibility. On the other hand, most neurological models of learning seem largly based on individualistic or even behavioural frameworks, and might be enriched by social and cultural dimensions.

While most of the literature focuses on narrative comprehension, the nature of our activities led us to emphasise narrative construction. It would be interesting to complement that with observations regarding the epistemic effects of 'reading' ToonTalk code and peer reports. A hint at this potential is given in Mor and Noss (2004), but more needs to be done. Similarly, while we believe that we have demonstrated the potential of programming as mathematical narrative, we do not know how to prescribe a method to manifest and exploit this potential. Future research needs to identify design patterns (Mor and Winters, 2007) for creating narrative spaces.

A call for narrativity or situatedness of any kind should not be taken as an excuse for lack of rigour. A solid pedagogy informed by the idea of situated abstraction strives to design settings in which the desired mathematical concepts can be derived as a necessity of the learners' activity, generating situations in which phenomena 'beg to be organised' (Freudenthal, 1983). The site of learning has to have integrity as a narrative – but it must

also have mathematical integrity. In order for abstraction to take place, the learner must be able to relate to the story. In order for it to foster the mathematical concepts we are interested in, these need to be the moral, or consequence of this story, not artificially grafted on top of standard mathematical pedagogical rituals.

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