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Abstract : As in preceding papers in which we studied the limits of penalized one-dimensional Wiener measures with certain functionals Γ_t, we obtain here the existence of the limit, as $t \to \infty$, of d-dimensional Wiener measures penalized by a function of the maximum up to time t of the Brownian winding process (for $d = 2$), or in $d \geq 2$ dimensions for Brownian motion prevented to exit a cone before time t.
Various extensions of these multidimensional penalisations are studied, and the limit laws are described.
Throughout this paper, the skew-product decomposition of d-dimensional Brownian motion plays an important role.

Keywords : Skew-product decomposition, Brownian windings, Dirichlet problem, spectral decomposition.

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1 Introduction

a) Let $\{\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, P_x\}$ denote the canonical d-dimensional Brownian motion with dimension $d \geq 2$. Ω is the space of continuous functions defined on $\mathbb{R}_+$, and taking values in $\mathbb{R}^d$, $(X_t, t \geq 0)$ is the coordinate process on Ω and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$, and P_x denotes the Wiener measure on $(\Omega,\, \mathcal{F}_\infty)$ such that $P_x(X_0 = x) = 1$.

b) We consider $(\Gamma_t,\, t \geq 0)$ an $\mathbb{R}_+$-valued, $(\mathcal{F}_t)$ adapted process such that : $0 < E_x[\Gamma_t] < \infty$. Our aim in this work is to show the existence and some properties of the limit, as $t \to \infty$, of $P_x^{(t)}$, which is defined by :

$$P_x^{(t)}(\Lambda) := \frac{E_x[1_\Lambda \Gamma_t]}{E_x[\Gamma_t]} \qquad (\Lambda \in \mathcal{F}_\infty), \tag{1.1}$$

for a certain process (Γ_t).
In a series of preceding papers ([10], [11], [12], [15], [16], [17]), we have shown that for a large class of processes $(\Gamma_t,\, t \geq 0)$, one has :

i) for every $s \geq 0$, and every $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} P_x^{(t)}(\Lambda_s) \quad \text{exists.} \tag{1.2}$$

ii) This limit is of the form

$$E_x[1_{\Lambda_s}\, M_s^\Gamma], \tag{1.3}$$

where $(M_s^\Gamma,\, s \geq 0)$ is a $((\mathcal{F}_s)_{s\geq 0},\, P_x)$ $\mathbb{R}_+$-valued martingale.
A survey of our main results involving various processes (Γ_t) is given in [14]; see also [13] for some complements.
Our main tool used to prove (1.2) and (1.3) is the following

Theorem 1.1. *Assume that, for every fixed $s \geq 0$:*

$$\frac{E_x[\Gamma_t|\mathcal{F}_s]}{E_x[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} M_s^\Gamma \qquad a.s. \tag{1.4}$$

and

$$E_x[M_s^\Gamma] = 1. \tag{1.5}$$

Then :

(i) $\forall\, s \geq 0,\ \forall\, \Lambda_s \in \mathcal{F}_s$

$$\frac{E_x[1_{\Lambda_s}\,\Gamma_t]}{E_x[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} E_x[1_{\Lambda_s}\, M_s^\Gamma] \tag{1.6}$$

(ii) $(M_s^\Gamma,\, s \geq 0)$ *is a* $((\mathcal{F}_s)_{s\geq 0},\, P_x)$ $\mathbb{R}_+$*-valued martingale such that* $M_0^\Gamma = 1$.

The proof of Theorem 1.1 - which is true independently of this Brownian scheme and, in particular, of the dimension d - is quite elementary. It hinges upon Scheffé's lemma (see [5], p. 37, T21).

c) We now assume that the hypotheses of Theorem 1.1 are satisfied. Then, the formula :

$$Q_x(\Lambda_s) = E_x[1_{\Lambda_s} M_s^\Gamma], \quad s \geq 0,\, \Lambda_s \in \mathcal{F}_s \tag{1.7}$$

induces a family of probabilities $(Q_x,\, x \in \mathbb{R}^d)$ on the canonical space $(\Omega,\, \mathcal{F}_\infty)$. In the articles ([10], [11], [12], [15], [16], [17]), we described precisely the main properties of the canonical process $(X_t,\, t \geq 0)$ under Q_x. The aim of the present work is to study several penalisations with respect to $(\Gamma_t,\, t \geq 0)$ in a multidimensional framework, i.e : we assume $d \geq 2$.

d) For this purpose, for $x \neq 0$, we shall use the skew-product decomposition of $(X_t, t \geq 0)$:

$$X_t = R_t \Theta_{H_t} \quad , \tag{1.8}$$

where

- **(i)** $(\Theta_u, u \geq 0)$ is a Brownian motion on the unit sphere S_{d-1} in $\mathbb{R}^d$. Recall that $(\Theta_u, u \geq 0)$ is the diffusion process with the infinitesimal generator $\frac{1}{2}\tilde{\Delta}$, where $\tilde{\Delta}$ denotes the Laplace-Beltrami operator on S_{d-1};
- **(ii)** the process $(R_t := |X_t|, t \geq 0)$ is a Bessel process with dimension d, or index $\nu = \frac{d}{2} - 1$ which is independent from $(\Theta_u, u \geq 0)$;
- **(iii)** $H_t = \displaystyle\int_0^t \frac{ds}{R_s^2}$.

When $d = 2$, formula (1.8) may be written:

$$X_t = R_t \exp\left(i\beta_{H_t}\right) \tag{1.9}$$

where, now $(\beta_u, u \geq 0)$ is a standard real-valued Brownian motion, independent from $(R_t, t \geq 0)$, a two dimensional Bessel process. The process

$$\theta_t := \beta_{H_t} = \theta_0 + \mathrm{Im}\Big(\int_0^t \frac{dX_s}{X_s}\Big), \quad t \geq 0,$$

shall be called the winding process of X around 0. (We may choose $\theta_0 \in [0, 2\pi)$, with $x = |x| \exp(i\theta_0)$).

e) Notation : Throughout the paper, we shall use the notation $(X_u; u \geq 0)$ for the process X indexed by $u \in \mathbb{R}_+$ or $(X(u); u \geq 0)$ when the latter notation may be more convenient.

f) The paper is organized as follows : it is devoted to the penalisations of d-dimensional Brownian motion by the functionals $(\Gamma_t,\, t \geq 0)$, displayed below in (1.10)-(1.12).

(i) In Section 2 we restrict ourselves to $d = 2$. We first consider in Theorem 2.1 the case where Γ_t is a function of the one-sided maximum of the winding process :

$$\Gamma_t = \varphi(S_t^\theta) \quad \text{with} \quad S_t^\theta = \sup_{s \leq t} \theta_s = \sup_{s \leq t} \beta_{H_s} \tag{1.10}$$

We also study in Theorem 2.9 the penalisation with the more general functionals

$$\Gamma_t = \varphi(S_t^\theta)\, \exp(-\lambda(S_t^\theta - \theta_t))\ , \tag{1.11}$$

for some Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, and $\lambda \geq 0$ (see also Theorem 2.14).

(ii) Section 3 is devoted to the penalisations related to a cone $\mathcal{C}$ in $\mathbb{R}^d$ with $d \geq 2$. More precisely, if $\mathcal{C}$ is a cone with vertex the origin, and basis $\mathcal{O}$ (where $\mathcal{O}$ is an open set of the unit sphere S_{d-1}), we study the penalisation with :

$$\Gamma_t = 1_{\{T_{\mathcal{C}} > t\}}\, \exp\Big(\frac{\gamma}{2} H_t + \rho R_t\Big) \qquad (\gamma \in \mathbb{R},\, \rho \geq 0) \tag{1.12}$$

where $T_{\mathcal{C}} = \inf\{u \geq 0 : X_u \notin \mathcal{C}\}$ is the exit time of the cone $\mathcal{C}$.

At the end of Section 3, we study the case when $d = 2$, and the functional Γ_t equals $f(\overline{\theta}_t,\, \underline{\theta}_t)$, with $\overline{\theta}_t = S_t^\theta = \sup_{s \leq t} \theta_s$, $\underline{\theta}_t = \inf_{s \leq t} \theta_s$. Thus, Γ_t is a function of the maximum and minimum of the winding process. In fact, we only study the particular case : $f(s,\, i) = 1_{s \leq \alpha_1,\, i > \alpha_0}$, with : $\alpha_0 < 0 < \alpha_1$.

g) Another penalisation study for Brownian motion in $\mathbb{R}^2$ is discussed in [18]; it involves the penalisation process :

$$\Gamma_t := \exp\Big(-\frac{1}{2} \int_0^t V(X_s) ds\Big) \tag{1.13}$$

where V is a function with compact support from $\mathbb{R}^2$ to $\mathbb{R}_+$. Note that such penalisations have been studied in [11], when $(X_s,\, s \geq 0)$ is a one-dimensional Brownian motion, or more generally, a Bessel process with index $\mu \in]-1, 0[$. Thus, our extension in [18] complements the Bessel studies in [11] and corresponds to the case $\mu = 0$.

2 Penalisation with a function of the one-sided maximum of the continuous winding of planar Brownian motion

a) We keep the notation from the Introduction. We write the skew-product representation of the canonical 2-dimensional Brownian motion (X_t), starting at $x \neq 0$, as :

$$X_t = R_t \exp(i\beta_{H_t}), \qquad\qquad t \geq 0, \tag{2.1}$$

where :

- $R_t = |X_t|$ is a 2-dimensional Bessel process starting at $r = |x|$ i.e : $d = 2, \nu = 0$ is the corresponding Bessel index (2.2)
- $H_t = \displaystyle\int_0^t \frac{ds}{R_s^2}$; (2.3)
- $(\beta_u, u \geq 0)$ is a linear Brownian motion; (2.4)
- the processes $(\beta_u, u \geq 0)$ and $(R_t, t \geq 0)$ are independent. (2.5)

In fact,

$$\theta_t := \beta_{H_t}, \quad t \geq 0, \tag{2.6}$$

is the process of continuous windings of $(X_t, t \geq 0)$ around 0; we denote :

$$S_t^\theta = \sup_{s \leq t} \theta_s \equiv \sup_{u \leq H_t} (\beta_u) \tag{2.7}$$

the one-sided maximum process of θ.

b) Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel function such that $\displaystyle\int_0^\infty \varphi(y)dy = 1$, and define

$$\Phi(u) = \int_0^u \varphi(y)dy \tag{2.8}$$

We now describe the limiting laws obtained by the penalisations of P_x with the functionals $\Gamma_t = \varphi(S_t^\theta)$, $t \geq 0$.

Theorem 2.1. *1) Suppose that the starting point x is a positive real number; we take $\beta_0 = 0$. Let φ as above. For every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$,*

$$\lim_{t\to\infty} \frac{E_x[1_{\Lambda_s}\varphi(S_t^\theta)]}{E_x[\varphi(S_t^\theta)]} \quad \textit{exists} \tag{2.9}$$

2) This limit is equal to

$$E_x[1_{\Lambda_s} M_s^\varphi] \tag{2.10}$$

where :

$$M_s^\varphi = \varphi(S_s^\theta)(S_s^\theta - \theta_s) + 1 - \Phi(S_s^\theta). \tag{2.11}$$

Moreover, $(M_s^\varphi, s \geq 0)$ is a $((\mathcal{F}_s, s \geq 0), P_x)$ positive martingale which converges to 0 P_x a.s., as $s \to \infty$.
3) The formula

$$Q_x^\varphi(\Lambda_s) = E_x[1_{\Lambda_s} M_s^\varphi] \tag{2.12}$$

induces a probability on $(\Omega, \mathcal{F}_\infty)$. Under Q_x^φ $(x \neq 0)$, the canonical process $(X_t, t \geq 0)$ satisfies :

(i) *the random variable S_∞^θ is finite a.s. and admits φ as its probability density;*

(ii) *let $\tilde{g} = \inf\{s \geq 0 : S_s^\theta = S_\infty^\theta\} = \sup\{s \geq 0 : \theta_s = S_\infty^\theta\}$, then, $Q_x^\varphi(0 < \tilde{g} < \infty) = 1$;*

(iii) *the process $(X_t, t \geq 0)$ admits the skew-product representation (2.1), where :*

- *(a) $R_t := |X_t|, t \geq 0$, is a 2-dimensional Bessel process, independent from the process $(\beta_s, s \geq 0)$,*
- *(b) Let $(A_u, u \geq 0)$ denote the inverse of $(H_t, t \geq 0)$, i.e : $A_u = \inf\{t : H_t > u\}$ and define $g = A_{\tilde{g}}$, then*
  - *i. $(\beta_s, s \leq g)$ and $(\beta_g - \beta_{g+s}, s \geq 0)$ are independent;*
  - *ii. $(\beta_g - \beta_{g+s}, s \geq 0)$ is a 3-dimensional Bessel process;*
  - *iii. Conditionally on $S_\infty^\theta = y$, $(\beta_s, s \leq g)$ is a Brownian motion considered up to the first time when it reaches y.*

Remark 2.2. To deal with any $x \in \mathbb{R}^2, x \neq 0$, we should start with $\varphi : \mathbb{R} \to]0, \infty[$ such that $\displaystyle\int_{\mathbb{R}} \varphi(y)dy = 1$. The associated function Φ is $\displaystyle\Phi(u) = \int_{-\infty}^{u} \varphi(y)dy$.

Note that when $x = \rho e^{i\theta_0}$ $(0 \leq \theta_0 < 2\pi)$ is the starting point of (X_t), then we take $\beta_0 = \theta_0$. It can be shown that (2.9) and (2.10) hold with :

$$M_s^{\varphi} = [\varphi(S_s^{\theta})(S_s^{\theta} - \theta_s) + 1 - \Phi(S_s^{\theta})] \frac{1}{1 - \Phi(\theta_0)} \tag{2.13}$$

□

We state in Remark 2.8 below an extension of Theorem 2.1.
To prove Theorem 2.1, we first present two lemmas.

Lemma 2.3. *Let $(R_t, t \geq 0)$denote a 2-dimensional Bessel process starting from $r \neq 0$, and $H_t = \int_0^t \frac{ds}{R_s^2}$. Then, for every $m > 0$, one has :*

$$E_r\left[\left(\frac{\log t}{2\sqrt{H_t}}\right)^m\right] \underset{t\to\infty}{\rightarrow} E[|N|^m] = \left(\frac{2^m}{\pi}\right)^{\frac{1}{2}} \Gamma\left(\frac{m+1}{2}\right) \tag{2.14}$$

where N denotes a standard centered Gaussian random variable.

Remark 2.4. Lemma 2.3 is in fact equivalent to the celebrated asymptotic result due to Spitzer ([19], see also Durrett [2], and e.g. Pap-Yor [6], Pitman-Yor ([7], [8]) for many complements) :

$$\frac{2\theta_t}{\log t} \underset{t\to\infty}{\overset{(law)}{\rightarrow}} C\,, \tag{2.15}$$

where C denotes a standard Cauchy variable .
In fact, due to the skew-product representation of $(\theta_t, t \geq 0)$, (2.15) is equivalent to :

$$\frac{4H_t}{(\log t)^2} \underset{t\to\infty}{\overset{(law)}{\rightarrow}} T_1 \overset{(law)}{=} \frac{1}{N^2} \tag{2.16}$$

and (2.14) expresses the convergence of negative moments of the LHS of (2.16) to the corresponding ones of the RHS. (In (2.16), T_1 denotes the first hitting time of level 1 by a standard Brownian motion starting from 0).

□

Proof of Lemma 2.3
1) We note :

$$\alpha_t = \left(\frac{\log t}{2}\right)^2 \tag{2.17}$$

and we use the "elementary identity" :

$$\frac{1}{x^{m/2}} = \frac{1}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty e^{-ux} u^{\frac{m}{2}-1} du, \quad x > 0. \tag{2.18}$$

Thus, we obtain :

$$\begin{aligned} E_r\left[\left(\frac{\log t}{2\sqrt{H_t}}\right)^m\right] &= E_r\left[\left(\frac{\alpha_t}{H_t}\right)^{m/2}\right] = \frac{1}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty E_r\left[e^{-u\frac{H_t}{\alpha_t}}\right] u^{\frac{m}{2}-1} du \\ &= \frac{1}{\Gamma(\frac{m}{2}) 2^{\frac{m}{2}-1}} \int_0^\infty v^{m-1} E_r\left[e^{-\frac{\nu_t^2}{2} H_t}\right] dv. \end{aligned} \tag{2.19}$$

where we have denoted : $\nu_t = \dfrac{v}{\log \sqrt{t}}$.

2) Let $E_r^{(\gamma)}$ be the expectation for a Bessel process with index γ, starting from r. Recall the absolute continuity formula (see [9], Ex (1.22), p.450) :

$$E_r^{(\mu)}\left[\xi_t \exp\left\{\frac{\mu^2 - \nu^2}{2} H_t\right\}\right] = E_r^{(\nu)}\left[\xi_t \left(\frac{r}{R_t}\right)^{\nu-\mu}\right], \tag{2.20}$$

where ξ_t is any non-negative $\sigma(R_s, s \leq t)$-measurable r.v.
Applying (2.20) with $\mu = 0$, $\nu = \nu_t$, and $\xi_t = 1$ leads to :

$$E_r^{(0)}\left[\exp\left\{-\frac{\nu_t^2}{2} H_t\right\}\right] = E_r^{(\nu_t)}\left[\left(\frac{r}{R_t}\right)^{\nu_t}\right]. \tag{2.21}$$

Plugging (2.21) in (2.19) we obtain :

$$E_r\left[\left(\frac{\log t}{2\sqrt{H_t}}\right)^m\right] = \frac{1}{\Gamma(\frac{m}{2}) 2^{\frac{m}{2}-1}} \int_0^\infty v^{m-1} \psi(v,t) dv, \tag{2.22}$$

where :

$$\psi(v,t) := E_r^{(\nu_t)}\left[\left(\frac{r}{R_t}\right)^{\nu_t}\right]. \tag{2.23}$$

Using the scaling property of Bessel processes we get :

$$\psi(v,t) = \left(\frac{r}{\sqrt{t}}\right)^{\nu_t} E_{r/\sqrt{t}}^{(\nu_t)}\left[\left(\frac{1}{R_1}\right)^{\nu_t}\right]. \tag{2.24}$$

a) Since the density function of R_1 under $P_{r/\sqrt{t}}^{(\nu_t)}$ is explicitly known (see for instance [9], p 446) we have :

$$E_{r/\sqrt{t}}^{(\nu_t)}\left[\left(\frac{1}{R_t}\right)^{\nu_t}\right] = \int_0^\infty y \exp\left\{-\frac{1}{2}\left(y + \frac{r}{\sqrt{t}}\right)^2\right\} \left(\frac{\sqrt{t}}{r}\right)^{\nu_t} I_{\nu_t}\left(\frac{ry}{\sqrt{t}}\right) dy, \tag{2.25}$$

with

$$I_\mu(z) = \left(\frac{z}{2}\right)^\mu \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\mu+1)} \left(\frac{z}{2}\right)^{2k}. \tag{2.26}$$

Since

$$\lim_{t\to\infty} \nu_t = 0, \tag{2.27}$$

it is easy to deduce from (2.26) that

$$\lim_{t\to\infty} y \exp\Big\{ -\frac{1}{2}\big(y + \frac{r}{\sqrt{t}}\big)^2 \Big\} \Big(\frac{\sqrt{t}}{r}\Big)^{\nu_t} I_{\nu_t}\big(\frac{ry}{\sqrt{t}}\big) = y e^{-\frac{y^2}{2}}. \tag{2.28}$$

It is clear that (2.26) implies that :

$$I_\mu(z) \le \left(\frac{z}{2}\right)^\mu I_0(z), \quad (\mu > 0, z > 0). \tag{2.29}$$

Therefore, for any $t \ge 1$, we have :

$$y \exp\Big\{ -\frac{1}{2}\big(y + \frac{r}{\sqrt{t}}\big)^2 \Big\} \Big(\frac{\sqrt{t}}{r}\Big)^{\nu_t} I_{\nu_t}\big(\frac{ry}{\sqrt{t}}\big) \le 2\Big(\frac{y}{2}\Big)^{1+\nu_t} I_0(ry) e^{-\frac{y^2}{2}}. \tag{2.30}$$

Since v is fixed and $I_0(z) \underset{z\to\infty}{\sim} \frac{1}{\sqrt{2\pi z}} e^{-z}$ (cf [4], p123), we may apply the dominated convergence theorem in the right-hand side of (2.25) :

$$\lim_{t\to\infty} E^{(\nu_t)}_{r/\sqrt{t}}\Big[\Big(\frac{1}{R_t}\Big)^{\nu_t} \Big] = \int_0^\infty y e^{-\frac{y^2}{2}} dy = 1. \tag{2.31}$$

Note that :

$$\lim_{t\to\infty} \Big(\frac{r}{\sqrt{t}}\Big)^{\nu_t} = e^{-v}. \tag{2.32}$$

As a result :

$$\lim_{t\to\infty} \psi(v,t) = e^{-v}. \tag{2.33}$$

b) Using the definition of ν_t, it is clear that :

$$\Big(\frac{r}{\sqrt{t}}\Big)^{\nu_t} \le e^{-\frac{v}{2}}, \quad \text{for any} \quad t \ge r^4. \tag{2.34}$$

Note that $1+\nu_t \le 2$ as soon as $v \le \log(\sqrt{t})$, then, using (2.24), (2.25), (2.30) and (2.34), we get :

$$\begin{aligned} \psi(v,t) &\le 2e^{-\frac{v}{2}} \int_0^\infty (1+y)^2 I_0(ry) e^{-\frac{y^2}{2}} dy \\ &\le K e^{-\frac{v}{2}} \quad (v \le \log(\sqrt{t}),\ t \ge r^4). \end{aligned} \tag{2.35}$$

Consequently, applying the dominated convergence theorem leads to :

$$\lim_{t\to\infty}\int_0^\infty v^{m-1}\psi(v,t)1_{\{v\le\log(\sqrt{t})\}}dv = \int_0^\infty v^{m-1}e^{-v}dv. \tag{2.36}$$

c) We claim that :

$$\lim_{t\to\infty}\int_0^\infty v^{m-1}\psi(v,t)1_{\{v>\log(\sqrt{t})\}}dv = 0. \tag{2.37}$$

We define :

$$A(t) := \int_0^\infty v^{m-1}\psi(v,t)1_{\{v>\log(\sqrt{t})\}}dv.$$

Using $\Gamma(k+\nu_t+1)\ge\Gamma(\nu_t+1)$ $(k\ge 0)$ and (2.26) we get :

$$I_{\nu_t}\Big(\frac{ry}{\sqrt{t}}\Big) \le \frac{1}{\Gamma(\nu_t+1)}\Big(\frac{r}{\sqrt{t}}\Big)^{\nu_t}\Big(\frac{y}{2}\Big)^{\nu_t}\exp\Big\{\frac{r^2y^2}{2t}\Big\}. \tag{2.38}$$

Then, it is easy to deduce from (2.24), (2.25), (2.34) and (2.38) that :

$$\begin{aligned} A(t) \le 2\int_0^\infty & \frac{v^{m-1}}{\Gamma(\nu_t+1)}e^{-\frac{v}{2}}1_{\{v>\log(\sqrt{t})\}} \\ & \times\Big(\int_0^\infty \Big(\frac{y}{2}\Big)^{1+\nu_t}\exp\Big\{-\big(1-\frac{r^2}{2t}\big)\frac{y^2}{2}\Big\}dy\Big)\,dv. \end{aligned} \tag{2.39}$$

Let $t\ge r^2$, then $1-\dfrac{r^2}{2t}\ge\dfrac{1}{2}$ and

$$\begin{aligned} A(t) &\le 2\int_0^\infty \frac{v^{m-1}}{\Gamma(\nu_t+1)}e^{-\frac{v}{2}}1_{\{v>\log(\sqrt{t})\}}\Big(\int_0^\infty \Big(\frac{y}{2}\Big)^{1+\nu_t}e^{-\frac{y^2}{4}}\Big\}dy\Big)\,dv \\ &\le 2\int_0^\infty v^{m-1}\frac{\Gamma(1+\nu_t/2)}{\Gamma(\nu_t+1)}e^{-\frac{v}{2}}1_{\{v>\log(\sqrt{t})\}}dv \\ &\le 2\int_0^\infty v^{m-1}e^{-\frac{v}{2}}1_{\{v>\log(\sqrt{t})\}}dv. \end{aligned}$$

This shows (2.37).

d) Using (2.36) and (2.37) and passing to the limit in (2.22) as $t\to\infty$, we obtain :

$$\begin{aligned} \lim_{t\to\infty}E_r\Big[\Big(\frac{\log t}{2\sqrt{H_t}}\Big)^m\Big] &= \frac{1}{\Gamma(\frac{m}{2})2^{\frac{m}{2}-1}}\int_0^\infty v^{m-1}e^{-v}\,dv = \frac{\Gamma(m)}{\Gamma(\frac{m}{2})2^{\frac{m}{2}-1}} \\ &= \frac{1}{\sqrt{\pi}}\,2^{\frac{m}{2}}\,\Gamma\Big(\frac{m+1}{2}\Big) = E(|N|^m) \end{aligned} \tag{2.40}$$

from the Legendre duplication formula (see [4], p. 4); (2.40) is precisely the statement of Lemma 2.3.

$\blacksquare$

The next Lemma is a corollary of Lemma 2.3.

Lemma 2.5. *For every integrable function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, one has :*

$$\lim_{t\to\infty} (\log t)\, E_x \left[\Psi(S_t^\theta)\right] = \frac{4}{\pi} \int_0^\infty \Psi(y) dy. \tag{2.41}$$

Proof of Lemma 2.5

1) The identity :

$$E_x \left[\Psi(S_t^\theta)\right] = E_x \big[\Psi\big(\sqrt{H_t}\, |N|\big)\big] \tag{2.42}$$

holds, since :

$$S_u^\beta := \sup_{s\le u} \beta_s \tag{2.43}$$

is distributed as $\sqrt{u}|N|$, and $(H_t, t \ge 0)$ is independent from $(\beta_u, u \ge 0)$. Hence :

$$\begin{aligned}
(\log t)\, E_x \left[\Psi(S_t^\theta)\right] &= (\log t) \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-z^2/2} E_x \big[\Psi(z\sqrt{H_t})\big] dz \\
&= (\log t) \sqrt{\frac{2}{\pi}} E_x \Big[\int_0^\infty \frac{1}{\sqrt{H_t}} \exp\Big\{ -\frac{y^2}{2H_t} \Big\} \Psi(y) dy \Big] \\
&= 2 \sqrt{\frac{2}{\pi}} \int_0^\infty E_x \Big[Z_t \exp\Big\{ -\frac{y^2}{2H_t} \Big\} \Big] \Psi(y) dy,
\end{aligned} \tag{2.44}$$

where

$$Z_t := \frac{\log t}{2\sqrt{H_t}}. \tag{2.45}$$

We have :

$$\int_0^\infty E_x \Big[Z_t \exp\Big\{ -\frac{y^2}{2H_t} \Big\} \Big] \Psi(y) dy = E_x \big[Z_t\big] \int_0^\infty \Psi(y) dy + \delta(t), \tag{2.46}$$

where

$$\delta(t) := \int_0^\infty E_x \Big[Z_t \Big(\exp\Big\{ -\frac{y^2}{2H_t} \Big\} - 1 \Big) \Big] \Psi(y) dy. \tag{2.47}$$

First, observe that Lemma 2.3 implies that

$$\lim_{t\to\infty} E_x[Z_t] = \sqrt{\frac{2}{\pi}} \tag{2.48}$$

Next, we claim that :

$$\lim_{t\to\infty} \delta(t) = 0. \tag{2.49}$$

Applying Cauchy Schwarz inequality we obtain :

$$\left|E_x\Big[Z_t\Big(\exp\Big\{-\frac{y^2}{2H_t}\Big\}-1\Big)\Big]\right| \leq (E_x[Z_t^2])^{1/2}\left(E_x\Big[\Big(1-\exp\Big\{-\frac{y^2}{2H_t}\Big\}\Big)^2\Big]\right)^{1/2}.$$

According to Lemma 2.3, $t \mapsto E_x[Z_t^2]$ is a bounded function. Since $H_t \underset{t\to\infty}{\longrightarrow} \infty$ a.s., we may conclude that :

$$\lim_{t\to\infty} (E_x[Z_t^2])^{1/2}\left(E_x\Big[\Big(1-\exp\Big\{-\frac{y^2}{2H_t}\Big\}\Big)^2\Big]\right)^{1/2} = 0.$$

It is now clear that (2.49) follows from the dominated convergence theorem. As a result, (2.44), (2.46), (2.48) and (2.49) show (2.41).

■

Corollary 2.6. *Let φ as in Theorem 2.1 and Φ be defined by (2.8). Then :*

$$\lim_{t\to\infty} (\log t) P_x(S_t^\theta < c) = \frac{4}{\pi}c \qquad (c > 0) \tag{2.50}$$

and

$$\lim_{t\to\infty} (\log t) E_x\big[\varphi(a+S_t^\theta)1_{\{S_t^\theta > b-a\}}\big] = \frac{4}{\pi}(1-\Phi(b)) \quad (b > a). \tag{2.51}$$

Proof of Corollary 2.6
It is an immediate consequence of Lemma 2.5, which we apply by choosing as functions Ψ respectively $\Psi(u) = 1_{[0,c]}(u)$, and $\Psi(u) = \varphi(a+u)1_{[b-a,\infty[}(u)$.

■

Remark 2.7. Note that the rates of decay of $t \to P_x(S_t^\theta < c)$ and $t \to P_0(S_t^\beta < c)$ as $t \to \infty$ are very different (due to the time-change (H_t)). Indeed, it is classical, and it has been used in [12] that :

$$\lim_{t\to\infty} (\sqrt{t}) P_0(S_t^\beta < c) = c\sqrt{\frac{2}{\pi}}. \tag{2.52}$$

□

Proof of Theorem 2.1
a) Let us first prove points 1) and 2) of Theorem 2.1.
For $x \neq 0$, for every $s \geq 0$,

$$E\big[\varphi(S_t^\theta)|\mathcal{F}_s\big] = A(X_s, \theta_s, S_s^\theta, t-s),$$

with :

$$A(y,a,b,u) = E_y\Big[\varphi\big(b \vee (a+S_u^\theta)\big)\Big].$$

Thus :

$$A(y,a,b,u) = \varphi(b)E_y\left[1_{\{S_u^\theta < b-a\}}\right] + E_y\left[\varphi(a+S_u^\theta)1_{\{S_u^\theta > b-a\}}\right].$$

Hence, from Corollary 2.6 :

$$E\big[\varphi(S_t^\theta)|\mathcal{F}_s\big] \underset{t\to\infty}{\sim} \frac{4}{\pi}\Big(\varphi(S_s^\theta)\big((S_s^\theta - \theta_s\big) + 1 - \Phi(S_s^\theta)\Big)\frac{1}{\log(t-s)} \quad (2.53)$$

$$E\big[\varphi(S_t^\theta)\big] \underset{t\to\infty}{\sim} \frac{4}{\pi}\frac{1}{\log t}. \quad (2.54)$$

Consequently (2.53) and (2.54) imply that (1.4) holds with $\Gamma_t = \varphi(S_t^\theta)$ and $M_t^\Gamma = M_t^\varphi$, with $M_t^\varphi = \varphi(S_t^\theta)(S_t^\theta - \theta_t\big) + 1 - \Phi(S_t^\theta)$.
It has been already proved (see Proposition 3.1 in [12]) that
(M_t^φ) is P_x-martingale. Therefore $E_x\big[M_t^\varphi\big] = 1$. This shows (1.5). Applying Theorem 1.1 gives 1) and 2) of Theorem 2.1.
b) The end of the proof of Theorem 2.1 is then quite similar to that of Theorem 4.6 in [12], modulo the change of clock $(H_t, t \geq 0)$. We refer the reader to that proof.

■

Remark 2.8. We note that the penalisation with $f(S_t^\theta)$ where (θ_t) denotes the winding number of our $\mathbb{C}$-valued Brownian motion $X_t = U_t + iV_t, t \geq 0$, is the limiting case of penalisations with respect to $f(S_t^{\theta^{(\alpha)}})$, where :

$$\theta_t^{(\alpha)} := \int_0^t \frac{U_s dV_s - dV_s dU_s}{R_s^\alpha}, \quad t \geq 0,$$

for $0 < \alpha < 2$, for which the discussion is in fact easier than for $\alpha = 2$.
We claim that Theorem 2.1 is still valid when S^θ is replaced by $S^{\theta^{(\alpha)}}$.
Indeed, we still have :

$$\theta_t^{(\alpha)} = \gamma\Big(\int_0^t R_s^{2(1-\alpha)} ds\Big),$$

where (γ_u) is a Brownian motion independent of $(R_s, s \geq 0)$, but now we also have :

$$E_x\Big[\int_0^t R_s^{2(1-\alpha)} ds\Big] \underset{t\to\infty}{\sim} C_\alpha \int_0^t s^{1-\alpha} ds = \frac{C_\alpha}{2-\alpha} t^{2-\alpha},$$

for an universal constant C_α, independent of the starting position x (which now may be taken equal to 0).
Moreover, for some probability density $f : \mathbb{R}_+ \to \mathbb{R}_+$, we obtain, with the same kind of arguments as previously :

$$E_x\big[f(S_t^{\theta^{(\alpha)}})\big] \underset{t\to\infty}{\sim} C'_\alpha \frac{1}{t^{1-\alpha/2}}, \tag{2.55}$$

where C'_α is a universal multiple (depending on α) of $E_0\Big[\Big(\int_0^1 R_s^{2(1-\alpha)}ds\Big)^{-1/2}\Big]$.
Due to ([9], Corollary (1.12), Chap. XI), it is easy to prove that the last expectation is finite.
Note that in the case $\alpha = 2$, the rate of decay of $E_x\big[f(S_t^{\theta^{(\alpha)}})\big]$ is drastically different as (2.41) shows .
To be complete, it would be of some interest to consider also the penalisations with

$$\exp\Big\{-\int_0^t R_s^{-\alpha}ds\Big\}, \quad or \quad f(S_t^{\theta^{(\alpha)}}),$$

for $\alpha > 2$. We leave this question to the interested reader.

□

The end of this section is devoted to two generalisations of Theorem 2.1. We start with the first one. The notation is the same as previously. Let now $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\lambda > 0$ such that :

$$\int_0^\infty (1+\lambda y)\psi(y)dy = 1 \tag{2.56}$$

We shall now study the penalisation with $\Gamma_t = \psi(S_t^\theta)\exp(-\lambda(S_t^\theta - \theta_t))$ (Theorem 2.1 corresponds to the case $\lambda = 0$).

Theorem 2.9. *Suppose that x is a positive real number. Then, for every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$,*

$$\frac{E_x\big[1_{\Lambda_s}\psi(S_t^\theta)\exp\big\{-\lambda(S_t^\theta - \theta_t)\big\}\big]}{E_x\big[\psi(S_t^\theta)\exp\big\{-\lambda(S_t^\theta - \theta_t)\big\}\big]} \underset{t\to\infty}{\longrightarrow} E_x\big[1_{\Lambda_s}M_s^\varphi\big), \tag{2.57}$$

with :

$$\varphi(y) = \psi(y) + \lambda\int_y^\infty \psi(u)du, \quad y \geq 0. \tag{2.58}$$

Remark 2.10. It follows clearly from (2.56) that $\displaystyle\int_0^\infty \varphi(y)dy = 1$; moreover, Φ, the primitive of φ such that $\Phi(0) = 0$, satisfies :

$$1 - \Phi(u) = \int_u^\infty \psi(y)(1 + \lambda(y - u))dy, \quad u \geq 0. \tag{2.59}$$

$\square$

Proof of Theorem 2.9
1) Let $a \in \mathbb{R}$, $b \geq a_+ (= \max(a, 0))$ and $t \geq 0$. Define :

$$\begin{aligned} N(a,b,t) \;&:=\; \psi(b)e^{-\lambda(b-a)}E\big[e^{\lambda\theta_t}1_{(S_t^\theta \leq b-a)}\big] \\ &\quad + E\big[\psi(a + S_t^\theta)e^{-\lambda(S_t^\theta - \theta_t)}1_{\{S_t^\theta > b-a\}}\big]. \end{aligned} \tag{2.60}$$

Since $\theta_t = \beta_{H_t}$ and $(H_t,\, t \geq 0)$ is independent from $(\beta_u,\, u \geq 0)$, we obtain from the explicit knowledge ([9], Ex (3.14), Chap. III, see also [3]) of the law of the pair $(S_u^\beta := \sup_{s \leq u} \beta_s,\, \beta_u)$ under P_0 :

$$P_0(S_u^\beta \in dy,\, \beta_u \in dx) = \frac{2(2y - x)}{\sqrt{2\pi\, u^3}} e^{-\frac{(2y-x)^2}{2u}}\, 1_{(x<y,\, y>0)}\; dx\, dy, \tag{2.61}$$

$$\begin{aligned} N(a,b,t) \;&=\; E\Bigg[\sqrt{\frac{2}{\pi\xi^3}}\Big\{\psi(b)e^{-\lambda(b-a)} \int_0^{b-a} dy \int_{-\infty}^{y} e^{\lambda x}\,(2y - x)e^{-\frac{(2y-x)^2}{2\xi}}dx \\ &\quad + \int_{b-a}^{\infty} \psi(a+y)dy \int_{-\infty}^{y} e^{-\lambda(y-x)}\,(2y - x)e^{-\frac{(2y-x)^2}{2\xi}}dx\Big\}\Bigg], \end{aligned}$$

with $\xi = H_t$.
Setting $r = 2y - x$ in the last integral, we obtain :

$$\begin{aligned} N(a,b,t) \;&=\; E\Bigg[\sqrt{\frac{2}{\pi\xi^3}}\Big\{\psi(b)e^{-\lambda(b-a)} \int_0^{b-a} e^{2\lambda y}dy \int_y^\infty re^{-\frac{r^2}{2\xi} - \lambda r}dr \\ &\quad + \int_{b-a}^{\infty} \psi(a+y)e^{\lambda y} \int_y^\infty re^{-\frac{r^2}{2\xi} - \lambda r}dr\Big\}\Bigg]. \end{aligned} \tag{2.62}$$

But, from Lemma 2.3 we have :

$$\lim_{t\to\infty} (\log t)^3\, E\left[\frac{1}{H_t^{\frac{3}{2}}}\right] = 16\sqrt{\frac{2}{\pi}}. \tag{2.63}$$

Using moreover the fact that $H_t \to \infty$ as $t \to \infty$, we get :

$$\lim_{t\to\infty} (\log t)^3\, N(a,b,t) := N^\dagger(a,b), \tag{2.64}$$

with :

$$
\begin{aligned}
N^{\dagger}(a,b) &= \frac{32}{\pi}\Big\{\psi(b)e^{-\lambda(b-a)}\int_0^{b-a} e^{2\lambda y}dy\int_y^{\infty} re^{-\lambda r}dr \\
&\quad + \int_{b-a}^{\infty}\psi(a+y)e^{\lambda y}dy\int_y^{\infty} re^{-\lambda r}dr\Big\} \\
&= \frac{32}{\pi\lambda^2}\Big\{\big(\psi(b)+\lambda\int_b^{\infty}\psi(y)dy\big)(b-a) \\
&\quad + \int_b^{\infty}\psi(y)\big(1+\lambda(y-b)\big)dy\Big\} \\
&= \frac{32}{\pi\lambda^2}\big((b-a)\varphi(b)+1-\Phi(b)\big) \qquad (2.65)
\end{aligned}
$$

(the notation (2.58) and property (2.59) have been used to obtain the last equality).
2) Then, conditioning with respect to $\mathcal{F}_s$, and separating the cases when S_t^{θ} is attained before, or after s, we obtain :

$$E_x\Big[\psi(S_t^{\theta})\exp\big\{-\lambda(S_t^{\theta}-\theta_t)\big\}\Big|\mathcal{F}_s\Big] = N(\theta_s, S_s^{\theta}, t-s).$$

From (2.64) and (2.65) we deduce :

$$
\begin{aligned}
\frac{N(\theta_s, S_s^{\theta}, t-s)}{N(0,0,t)} &\underset{t\to\infty}{\sim} \left(\frac{\log t}{\log(t-s)}\right)^3 [(S_s^{\theta}-\theta_s)\varphi(S_s^{\theta})+1-\Phi(S_s^{\theta})] \\
&\underset{t\to\infty}{\rightarrow} (S_s^{\theta}-\theta_s)\varphi(S_s^{\theta})+1-\Phi(S_s^{\theta}) = M_s^{\varphi}.
\end{aligned}
$$

Theorem 2.1 implies that $E_x[M_s^{\varphi}]=1$; thus, Theorem 2.9 follows directly from Theorem 1.1.

■

We now prepare some material for our second generalisation of Theorem 2.1. The notation is the same as previously. Let $0<r<R$ two real numbers and define :

$$
\begin{aligned}
\theta_t^{-,r} &= \int_0^t 1_{\{R_s<r\}}\, d\theta_s \qquad (2.66) \\
\theta_t^{+,R} &= \int_0^t 1_{\{R_s>R\}}\, d\theta_s \qquad (2.67) \\
H_t^{-,r} &= \int_0^t 1_{\{R_s<r\}}\, \frac{ds}{R_s^2} \qquad (2.68) \\
H_t^{+,R} &= \int_0^t 1_{\{R_s>R\}}\, \frac{ds}{R_s^2} \qquad (2.69)
\end{aligned}
$$

The process $(\theta_t^{-,r}, t \geq 0)$ (resp. $(\theta_t^{+,R}, t \geq 0)$) is the process of small (resp. big) windings.
The following result may be found in Pitman-Yor ([7]) :

Theorem 2.11. *The 4-dimensional vector :*

$$\left(\frac{4}{(\log t)^2}(H_t^{-,r}, H_t^{+,R}), \frac{2}{\log t}(\theta_t^{-,r}, \theta_t^{+,R})\right) \text{ converges in law,}$$

as $t \to \infty$ to :

$$\left(\int_0^{T_1} 1_{\{\alpha_s \leq 0\}} ds,\ \int_0^{T_1} 1_{\{\alpha_s > 0\}} ds,\ \gamma^-\left(\int_0^{T_1} 1_{\{\alpha_s \leq 0\}} ds\right),\ \gamma^+\left(\int_0^{T_1} 1_{\{\alpha_s > 0\}} ds\right)\right) \tag{2.70}$$

where $(\alpha(t), t \geq 0)$, $(\gamma^-(t), t \geq 0)$ and $(\gamma^+(t), t \geq 0)$ are three independent one-dimensional Brownian motions and $T_1 = T_1(\alpha) := \inf\{s \geq 0; \alpha_s = 1\}$.

We shall use the following lemma, whose proof is postponed to the end of this subsection.

Lemma 2.12. *Let $(\alpha_s, s \geq 0)$ be a real-valued Brownian motion starting from 0, and let $T_1 := \inf\{s; \alpha_s = 1\}$. We denote :*

$$A_{T_1}^- := \int_0^{T_1} 1_{\{\alpha_s < 0\}}\, ds, \quad A_{T_1}^+ := \int_0^{T_1} 1_{\{\alpha_s > 0\}}\, ds.$$

Then, for $a, b \in \mathbb{R}$:

$$E[(A_{T_1}^-)^a (A_{T_1}^+)^b] < \infty$$

if and only if : $-\frac{1}{2} < a < \frac{1}{2}$.

Proposition 2.13. *We define :*

$$S_t^{\theta^{-,r}} := \sup_{s \leq t} \theta_s^{-,r}, \quad S_t^{\theta^{+,R}} := \sup_{s \leq t} \theta_s^{+,R}. \tag{2.71}$$

Let $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function such that $\int_{\mathbb{R}_+^2} \psi(u,v) du\, dv < \infty$. Let m, n two reals, with $0 < m < 1$. Then :

$$\lim_{t \to \infty} \frac{4^{m+n-1}}{(\log t)^{2m+2n-2}} E_x\Big[(H_t^{-,r})^m (H_t^{+,R})^n \psi(S_t^{\theta^{-,r}}, S_t^{\theta^{+,R}})\Big]$$
$$= \Big(\frac{2}{\pi}\int_{\mathbb{R}_+^2} \psi(u,v) du\, dv\Big)\, E\Big[(A_{T_1}^-)^{m-1/2} (A_{T_1}^+)^{n-1/2}\Big]. \tag{2.72}$$

Proof of Proposition 2.13

We may write $\theta_t^{-,r} = \gamma^-_{H_t^{-,r}}$, $\theta_t^{+,R} = \gamma^+_{H_t^{+,R}}$, with γ^- and γ^+ two independent real valued Brownian motions independent from $(R_s, s \geq 0)$. Thus :

$$
\begin{aligned}
& E_x\Big[(H_t^{-,r})^m (H_t^{+,R})^n \psi(S_t^{\theta^{-,r}}, S_t^{\theta^{+,R}})\Big] \\
= \; & E_x\Big[(H_t^{-,r})^m (H_t^{+,R})^n \psi\big(\sqrt{H_t^{-,r}}|N^-|, \sqrt{H_t^{+,R}}|N^+|\big)\Big] \\
& \text{(where } N^- \text{ and } N^+ \text{ are two independent gaussian variables,} \\
& \text{independent from } (R_s, s \geq 0)) \\
= \; & \frac{2}{\pi} \int_{\mathbb{R}_+^2} e^{-\frac{u^2+v^2}{2}} E_x\Big[(H_t^{-,r})^m (H_t^{+,R})^n \psi(\sqrt{H_t^{-,r}}u, \sqrt{H_t^{+,R}}v)\Big] du\, dv \\
= \; & \frac{2}{\pi} \int_{\mathbb{R}_+^2} \psi(u,v) du\, dv\, E_x\Big[(H_t^{-,r})^{m-1/2} (H_t^{+,R})^{n-1/2} \exp\Big\{ - \frac{u^2}{2H_t^{-,r}} - \frac{v^2}{2H_t^{+,R}} \Big\}\Big],
\end{aligned}
$$

and so, by Theorem 2.11 and Lemma 2.12 and because $H_t^{-,r}$ and $H_t^{+,R}$ converge a.s. to ∞ as $t \to \infty$, the quantity :

$$
\frac{4^{m+n-1}}{(\log t)^{2m+2n-2}} \; E_x\Big[(H_t^{-,r})^m (H_t^{+,R})^n \psi(S_t^{\theta^{-,r}}, S_t^{\theta^{+,R}})\Big]
$$

converges, as $t \to \infty$, to

$$
\Big(\frac{2}{\pi} \int_0^\infty \int_0^\infty \psi(x,y) dx\, dy\Big)\, E\big[(A^-_{T_1})^{m-1/2} (A^+_{T_1})^{n-1/2}\big].
$$

Note that $E[(A^-_{T_1})^{m-1/2}(A^+_{T_1})^{n-1/2}] < \infty$ by Lemma 2.12, because $0 < m < 1$ and so $-\frac{1}{2} < m - \frac{1}{2} < \frac{1}{2}$.

■

We may now state our second generalisation of Theorem 2.1.
Let $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be integrable and : $\displaystyle\int_0^\infty \int_0^\infty \psi(u,v)du\,dv = 1$.
We study penalisation by :

$$\Gamma_t^{m,n,\psi} := (H_t^{-,r})^m (H_t^{+,R})^n \psi(S_t^{\theta^{-,r}}, S_t^{\theta^{+,R}}), \tag{2.73}$$

where n is real and $0 < m < 1$.

Theorem 2.14. *1) For any $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:*

$$\widetilde{Q}_x^\psi(\Lambda_s) = \lim_{t\to\infty} \frac{E_x[1_{\Lambda_s}\,\Gamma_t^{m,n,\psi}]}{E_x(\Gamma_t^{m,n,\psi})} \quad \textit{exists.} \tag{2.74}$$

2) This limit is equal to :

$$\widetilde{Q}_x^\psi(\Lambda_s) = E_x[1_{\Lambda_s}\,\widetilde{M}_s^\psi] \tag{2.75}$$

where :

$$\begin{aligned} \widetilde{M}_s^\psi \;\; &= \;\; \psi(S_s^{\theta^{-,r}}, S_s^{\theta^{+,R}})(S_s^{\theta^{-,r}} - \theta_s^{-,r})(S_s^{\theta^{+,R}} - \theta_s^{+,R}) \\ &\quad + \int_{S_s^{\theta^{-,r}}}^\infty dx \int_{S_s^{\theta^{+,R}}}^\infty \psi(x,y)dy \\ &\quad +(S_s^{\theta^{-,r}} - \theta_s^{-,r}) \int_{S_s^{\theta^{-,R}}}^\infty \psi(S_s^{\theta^{-,r}}, y)dy \\ &\quad +(S_s^{\theta^{+,R}} - \theta_s^{+,R}) \int_{S_s^{\theta^{+,r}}}^\infty \psi(x, S_s^{\theta^{+,R}})dx. \end{aligned} \tag{2.76}$$

3) $(\widetilde{M}_s^\psi, s \geq 0)$ is a positive martingale.

4) The formula (2.74) induces a probability on $(\Omega, \mathcal{F}_\infty)$. Under $\widetilde{Q}_x^\psi$, the couple $(S_\infty^{\theta^{-,r}}, S_\infty^{\theta^{+,R}})$ is finite a.s and admits ψ as its probability density.

Note the remarkable feature that the martingale $(\widetilde{M}_s^\psi, s \geq 0)$ and the probability $\widetilde{Q}_x^\psi$ do not depend on m, n.

Proof of Theorem 2.14
The proof of Theorem 2.14 is very similar to that of Theorem 2.1 and some details are left to the reader. However, it hinges mainly on the relation (which follows from a simple application of the Markov property) :

$$E_x[\Gamma_t^{m,n,\psi}|\mathcal{F}_s] = e\big(X_s, H_s^{-,r}, H_s^{+,R}, S_s^{\theta^{-,r}}, S_s^{\theta^{+,R}}, \theta_s^{-,r}, \theta_s^{+,R}, t-s\big)$$

where the function e, which depends on eight arguments, is defined as :

$$e(x, h^-, h^+, s^-, s^+, \theta^-, \theta^+, u) = E_x\Big[(h^- + H_u^{-,r})^m (h^+ + H_u^{+,R})^n$$
$$\times \psi(s^- \vee (\theta^- + S_u^{\theta^-,r}), s^+ \vee (\theta^+ + S_u^{\theta^+,R}))\Big].$$

Since :

$$\lim_{t\to\infty} H_t^{-,r} = \lim_{t\to\infty} H_t^{+,R} = \infty,$$

we deduce from Proposition 2.13 that :

$$\frac{E_x[\Gamma_t^{m,n,\psi} | \mathcal{F}_s]}{E_x[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} \int_0^\infty \int_0^\infty \psi(S_s^{\theta^-,r} \vee (\theta_s^{-,r} + u), S_s^{\theta^+,R} \vee (\theta_s^{+,R} + v)) du\, dv.$$

It is easy to verify that the limit equals $\widetilde{M}_s^\psi$.

■

Remark 2.15. Of course, it is tempting to use Theorem 2.14 with $m = n = 0$. Unfortunately, we do not know whether the conclusion holds in this case, since, from Lemma 2.12 the quantity which then appears in (2.72) is :

$$E[(A_{T_1}^-)^{-1/2}(A_{T_1}^+)^{-1/2}] = \infty.$$

However, we conjecture that the conclusion of Theorem 2.14 still holds in this case.

□

Proof of Lemma 2.12

1) It is known (see [7]) that :

$$A_{T_1}^- = A_{\tau(\frac{1}{2}L_{T_1})}^- = V_{L_{T_1}} \tag{2.77}$$

where :

- $(L_u, u \geq 0)$ denotes the local time process at 0 of the 1-dimensional Brownian motion $(\alpha_u, u \geq 0)$, and $(\tau_\ell, \ell \geq 0)$ is its right-inverse :

$$\tau_\ell = \inf\{u > 0 : L_u > \ell\} ;$$

- $(V_s, s \geq 0)$ is a stable (1/2) subordinator, independent of the pair $(L_{T_1}, A_{T_1}^+)$; to be precise :

$$E[\exp(-\lambda V_s)] = \exp(-s\sqrt{2\lambda})$$

- L_{T_1} is exponentially distributed, with parameter $(1/2)$.

Therefore,

$$(A^-_{T_1}, A^+_{T_1}) \stackrel{\text{law}}{=} \left(\left(\frac{1}{2}L_{T_1}\right)^2 \frac{1}{N^2}, A^+_{T_1}\right)$$

where N denotes a standard Gaussian variable independent of the pair $(L_{T_1}, A^+_{T_1})$; hence, for $a, b \in \mathbb{R}$:

$$E\big[(A^-_{T_1})^a (A^+_{T_1})^b\big] = E\Big[\frac{1}{N^{2a}}\Big]\, E\Big[\Big(\frac{1}{2}L_{T_1}\Big)^{2a} (A^+_{T_1})^b\Big]. \tag{2.78}$$

2) We also recall (cf [7]) that :

- $$\left(\frac{1}{2}L_{T_1}, A^+_{T_1}\right) \stackrel{\text{law}}{=} (L_{T_1^*}, T_1^*) \tag{2.79}$$

- $T_1^* \stackrel{\text{law}}{=} \left(\sup_{t\le 1} |\alpha_t|\right)^{-2}$ admits positive and negative moments of all orders,

with $T_1^* = \inf\{s \ge 0; |\alpha_s| = 1\}$.
Hence, for any $m \in \mathbb{R}$:

$$E[(A^+_{T_1})^m] < \infty. \tag{2.80}$$

3) Observe that the density of occupation formula implies that $A^+_t = \int_0^\infty L^x_t dx$, where (L^x_t) is the local time process associated with (α_t). From Ray-Knight theorem (see [9], Chap XI, Theorem 2.2) :

$$\big(L^{1-x}_{T_1}, 0 \le x \le 1\big) \stackrel{\text{law}}{=} \big(R^2_x, 0 \le x \le 1\big), \tag{2.81}$$

where $(R^2_s, s \ge 0)$ is a squared Bessel process with dimension 2 started at 0. Consequently :

$$(L_{T_1}, A^+_{T_1}) \stackrel{\text{law}}{=} \big(R^2_1, \int_0^1 R^2_s\, ds\big). \tag{2.82}$$

Hence, from Lévy's formula (see [9], Chap XI, Cor. 3.3) :

$$\begin{aligned} E\Big[\exp\big\{-\frac{v^2}{2}A^+_{T_1}\big\} | L_{T_1} = \ell\Big] &= E\Big[\exp\big\{-\frac{v^2}{2}\int_0^1 ds R^2_s\big\} | R^2_1 = \ell\Big] \\ &= \frac{v}{\sinh v}\exp\big\{-\frac{\ell}{2}(v\coth v - 1)\big\}. \end{aligned} \tag{2.83}$$

4) Let us assume $b < 0$. Replacing in the elementary formula :

$$r^b = \frac{2^{1+b}}{\Gamma(-b)}\int_0^\infty e^{-\frac{v^2}{2}r} v^{-2b-1} dv$$

r by $A^+_{T_1}$, we get :

$$(A^+_{T_1})^b = \frac{2^{1+b}}{\Gamma(-b)} \int_0^\infty v^{-2b-1} \exp\big\{ -\frac{v^2}{2} A^+_{T_1} \big\}\, dv$$

Since L_{T_1} is exponentially distributed with parameter $(1/2)$, using (2.83) we obtain :

$$\begin{aligned}
E\Big[\big(L_{T_1}\big)^{2a}\big(A^+_{T_1}\big)^b\Big] &= \frac{1}{2}\int_0^\infty \ell^{2a} E\big[\big(A^+_{T_1}\big)^b | L_{T_1} = \ell\big] e^{-\ell/2} d\ell \\
&= \frac{2^b}{\Gamma(-b)} \int_{\mathbb{R}_+^2} \ell^{2a} v^{-2b-1} e^{-\frac{\ell}{2}} \\
&\qquad\qquad \times E\big[\exp\big\{-\tfrac{v^2}{2}A^+_{T_1}\big\} | L_{T_1} = \ell\big] d\ell\, dv \\
&= \frac{2^b}{\Gamma(-b)} \int_{\mathbb{R}_+^2} \frac{\ell^{2a} v^{-2b}}{\sinh v}\, e^{-\frac{\ell}{2} v \coth v}\, d\ell\, dv \\
&= \frac{2^{b+2a+1}\Gamma(2a+1)}{\Gamma(-b)} \int_0^\infty \frac{v^{-2b}}{\sinh v} \Big(\frac{\tanh v}{v}\Big)^{1+2a} dv.
\end{aligned}$$

Hence :

$$E\big[(A^-_{T_1})^a (A^+_{T_1})^b\big] = E\Big[\frac{1}{N^{2a}}\Big] \frac{2^{1+b}\Gamma(2a+1)}{\Gamma(-b)} \int_0^\infty \frac{v^{-2b}}{\sinh v} \Big(\frac{\tanh v}{v}\Big)^{1+2a} dv \tag{2.84}$$

It is now clear that, for $b < 0$:

$$E\big[(A^-_{T_1})^a (A^+_{T_1})^b\big] < \infty \quad \text{if and only if} \quad -\frac{1}{2} < a < \frac{1}{2}. \tag{2.85}$$

In particular, we recover :

$$E\big[(A^+_{T_1})^b\big] < \infty, \qquad \text{for any} \ \ b < 0, \tag{2.86}$$

which also follows from (2.79).
5) Now, we assume : $b > 0$.
If $|a| > 1/2$ let $p > 1$ such that $|a/p| > 1/2$. Denote q the conjugate exponent of p (i.e. $1/p+1/q = 1$), $a' := a/p$ and $b' := b/p$. Applying Hölder's inequality leads to :

$$\begin{aligned}
E\big[(A^-_{T_1})^{a'} (A^+_{T_1})^{-b'}\big] &= E\big[(A^-_{T_1})^{a'} (A^+_{T_1})^{b'} (A^+_{T_1})^{-2b'}\big] \\
&\leq \Big(E\big[(A^-_{T_1})^{a} (A^+_{T_1})^{b}\big]\Big)^{1/p} \Big(E\big[(A^+_{T_1})^{-2b'q}\big]\Big)^{1/q}.
\end{aligned}$$

Consequently (2.85) and (2.86) imply that $E\big[(A^-_{T_1})^a (A^+_{T_1})^b\big] = \infty$.

When $|a| < 1/2$, choosing $p > 1$ such that $|ap| < 1/2$ we obtain :

$$E\big[(A^-_{T_1})^a (A^+_{T_1})^b\big] \le \Big(E\big[(A^-_{T_1})^{ap}(A^+_{T_1})^{-bp}\big]\Big)^{1/p}\Big(E\big[(A^+_{T_1})^{2bq}\big]\Big)^{1/q}.$$

with $1/p + 1/q = 1$. It then suffices to apply (2.85) together with (2.86) to conclude that the left-hand side in the above inequality is finite.

■

3 Penalisation related to a cone

1) We keep the notation concerning the d-dimensional canonical Brownian motion, as given in the Introduction, d); in particular, if $X_0 = x \neq 0$, there is the skew-product decomposition :

$$X_t = R_t \Theta_{H_t} \quad , \quad t \ge 0 \tag{3.1}$$

where $(R_t,\, t \ge 0)$ is a Bessel process with index $\mu = \dfrac{d}{2} - 1$. We suppose here that $d \ge 2$.

2) Let $\mathcal{O}$ be a connected, regular, open set of S_{d-1}. Let $0 < \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \ldots$, and $\varphi_1, \ldots, \varphi_n, \ldots$ denote a spectral decomposition of $\widetilde{\Delta}$ in $\mathcal{O}$, associated with the Dirichlet problem, i.e :

$$\begin{array}{rl} \textbf{i)} & \widetilde{\Delta}\varphi_n = -\lambda_n^2 \varphi_n \\ \textbf{ii)} & \varphi_n : \overline{\mathcal{O}} \to \mathbb{R},\ \varphi_n = 0 \text{ on } \partial\mathcal{O},\ \varphi_n \text{ is } C^\infty \text{ in } \mathcal{O} \\ \textbf{iii)} & (\varphi_n,\ n \ge 1) \text{ is an orthonormal basis of } L^2(\mathcal{O}) \\ & \text{(for the Riemannian measure on } \mathcal{O}) \\ \textbf{iv)} & \varphi_1 > 0 \text{ in } \mathcal{O} \end{array} \tag{3.2}$$

Note that we denoted by λ_n^2 (and not by λ_n) the eigenvalues of $\widetilde{\Delta}$, for "aesthetical" reasons which will appear below.

3) We denote by $\mathcal{C}$ the cone in $\mathbb{R}^d$ with vertex at the origin, and basis $\mathcal{O}$, and we define :

$$T_{\mathcal{C}} = \inf\{t \ge 0 : X_t \notin \mathcal{C}\} \tag{3.3}$$

$$T^{\Theta}_{\mathcal{O}} = \inf\{u \ge 0 : \Theta_u \notin \mathcal{O}\} \tag{3.4}$$

The aim of this paragraph is to study the penalisation by the functional

$$\Gamma_t = 1_{\{T_{\mathcal{C}} > t\}} \exp\Big(\frac{\gamma}{2} H_t + \rho R_t\Big) \qquad (\gamma \in \mathbb{R},\ \rho \ge 0)$$

Of course, the particular case : $\gamma = \rho = 0$ amounts to study Brownian motion $(X_t,\, t \geq 0)$ conditioned to stay in the cone $\mathcal{C}$. We shall prove the following :

Theorem 3.1. *Let $x \in \mathcal{C}\,(x \neq 0)$, and let $T_{\mathcal{C}}$ denote the exit time of $\mathcal{C}$, as defined by (3.3). Let $\rho \geq 0$, and $\gamma \in \mathbb{R}$ such that : $\mu^2 \geq \gamma - \lambda_1^2$, where : $\mu = \frac{d}{2} - 1$.*
Then :
1) For every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$, the limit as $t \to \infty$ of :

$$\frac{E_x\left[1_{\Lambda_s} 1_{\{T_{\mathcal{C}}>t\}} \exp\left(\frac{\gamma}{2} H_t + \rho R_t\right)\right]}{E_x\left[1_{\{T_{\mathcal{C}}>t\}} \exp\left(\frac{\gamma}{2} H_t + \rho R_t\right)\right]} \qquad \textit{exists.} \tag{3.5}$$

2) This limit equals :

$$E_x\left[1_{\Lambda_s} 1_{\{T_{\mathcal{C}}>s\}} M_s\right] \tag{3.6}$$

where :

$$M_s := k \exp\left(-\frac{\rho^2}{2} s + \frac{\gamma}{2} H_s\right) \varphi_1(\Theta_{H_s}) R_s^{-\mu} I_\nu(\rho R_s) \tag{3.7}$$

with :

$$\nu = \sqrt{\mu^2 + \lambda_1^2 - \gamma} \quad \textit{and} \quad k = \left(\varphi_1\left(\frac{x}{|x|}\right) |x|^{-\mu} I_\nu(\rho |x|)\right)^{-1} \tag{3.8}$$

where I_ν denotes the modified Bessel function with index ν. (cf. [4], p.108).

3) Formula (3.6) induces a probability Q_x on $(\Omega, \mathcal{F}_\infty)$. Under this probability Q_x, the process $(X_t, t \geq 0)$ satisfies :

i) $\quad Q_x\,(T_{\mathcal{C}} = \infty) = 1 \qquad (3.9)$

ii) $\quad (X_t, t \geq 0)$ *admits the skew-product decomposition :*

$$X_t = R_t \Theta_{H_t} \tag{3.10}$$

where :
a) $(R_t, t \geq 0)$ is the "Bessel process with drift", whose generator is given by :

$$\mathcal{L}^R : f \to \mathcal{L}^R f(r) = \frac{1}{2} f''(r) + \left(\frac{1+2\nu}{2r} + \frac{\rho I_{\nu+1}(\rho r)}{I_\nu(\rho r)}\right) f'(r), \tag{3.11}$$

see [21].
b) $(\Theta_u, u \geq 0)$ is a diffusion taking values in $\mathcal{O}$, with generator :

$$\mathcal{L}^\Theta : f \to \mathcal{L}^\Theta f(\theta) = \frac{1}{2} \widetilde{\Delta} f(\theta) + \frac{\nabla \varphi_1}{\varphi_1}(\theta) \cdot \nabla f(\theta) \tag{3.12}$$

where the above scalar product and the gradient are taken in the sense of the Riemannian metric on S_{d-1};

c) The processes $(R_t, t \geq 0)$ and $(\Theta_u, u \geq 0)$ are independent. (3.13)

Remark 3.2. i) $\rho = 0$ is allowed in Theorem 3.1. In this case, the process $(R_t, t \geq 0)$ is, under Q_x, a Bessel process with index ν (ν depends on γ via formula (3.8)).
i') $\rho = \gamma = 0$ is allowed in Theorem 3.1. In this case, $(R_t, t \geq 0)$ is, under Q_x, a Bessel process with index $\nu = \sqrt{\mu^2 + \lambda^2}$.
ii) Note that, when $\gamma > 0$, with respect to the penalisation with $\Gamma_t = 1_{(T_\mathcal{C}>t)} \exp\left(\frac{\gamma}{2}H_t + \rho R_t\right)$, the terms $\exp\left(\frac{\gamma}{2}H_t\right)$ and $\exp(\rho R_t)$ play conflicting roles : the term $\exp\left(\frac{\gamma}{2}H_t\right)$ favors the trajectories for which R is small, whereas the term $\exp(\rho R_t)$ favors those for which R is large.
This explains, intuitively, that the process $(R_t, t \geq 0)$ may have, for $\rho = 0$, and $\gamma > 0$, a smaller "dimension" than the process $(R_t, t \geq 0)$ under P_x. Note that this situation never happens when one penalizes with $1_{(T_\mathcal{C}>t)}$, i.e : when one considers the Brownian motion in $\mathbb{R}^d$, conditioned never to leave the cone $\mathcal{C}$.
iii) We shall show, in the course of the proof, that :

$$E_x\left[\exp\left(\frac{\gamma}{2}H_t + \rho R_t\right)\right] < \infty, \text{ for all } t \geq 0,$$

as soon as : $\mu^2 + \lambda_1^2 - \gamma \geq 0$.
iv) If $\rho > 0$, and $\gamma = 0$, the limit process is "very transient", since :

$$\frac{\rho I_{\nu+1}(\rho r)}{I_\nu(\rho r)} \underset{r\to\infty}{\sim} \rho. \tag{3.14}$$

Thus, in this case, the process $(R_t, t \geq 0)$ behaves, as $t \to \infty$, as a one-dimensional Brownian motion with drift ρ.
We also remark that, if we take $\rho < 0$ in Theorem 3.1, the limiting probability Q_x is the same as for $\rho = 0$.

□

Proof of Theorem 3.1.
1) We begin with the

Lemma 3.3. *Let $T_\mathcal{O}^\Theta = \inf\{u \geq 0 : \Theta_u \notin \mathcal{O}\}$, and $a \in \mathcal{O}$. Then :*

$$P_a\left(T_\mathcal{O}^\Theta > t\right) = \sum_{n\geq 1} \exp\left(-\frac{\lambda_n^2 t}{2}\right)\varphi_n(a)\int_\mathcal{O}\varphi_n(b)db \tag{3.15}$$

$$\underset{t\to\infty}{\sim} \exp\left(-\frac{\lambda_1^2 t}{2}\right)\varphi_1(a)k', \tag{3.16}$$

with : $k' = \int_{\mathcal{O}} \varphi_1(b)db > 0$.

Proof of lemma 3.3

This lemma is classical. Note $\tilde{p}_u(a,b)$ the density, with respect to the Riemannian measure (db), of the semi-group of the process $(\widetilde{\Theta}_u, u \geq 0)$, i.e : the process $(\Theta_u, u \geq 0)$ killed as it exits from $\mathcal{O}$. Then (see [1]) :

$$\tilde{p}_u(a,b) = \sum_{n=1}^{\infty} \exp\left(-\frac{\lambda_n^2 u}{2}\right) \varphi_n(a)\varphi_n(b), \tag{3.17}$$

hence, for every $a \in \mathcal{O}$:

$$P_a\left(T_{\mathcal{O}}^{\Theta} > t\right) = E_a\left[1_{\mathcal{O}}(\widetilde{\Theta}_t)\right] = \sum_{n\geq 1} \exp\left(-\frac{\lambda_n^2 t}{2}\right) \varphi_n(a) \int_{\mathcal{O}} \varphi_n(b)db.$$

■

2) For every $x \in \mathbb{R}^d$, $x \neq 0$, we denote by (r, θ) its polar coordinates, with :

$$x = (r, \theta)\ ,\ r = |x|\ ,\ \theta \in S_{d-1} \tag{3.18}$$

Lemma 3.4. *For every $x = (r, \theta)$ in $\mathcal{O}$, we have :*

$$E_{r,\theta}\left[1_{\{T_{\mathcal{C}}>t\}} \exp\left(\frac{\gamma}{2}H_t + \rho R_t\right)\right] \underset{t\to\infty}{\sim} \left(k'\varphi_1(\theta)\sqrt{2\pi}\rho^{1+\mu}r^{-\mu}I_\nu(\rho r)\right)t^{\mu+\frac{1}{2}}e^{\frac{\rho^2}{2}t}$$

with $\mu = \frac{d}{2} - 1$, $\nu^2 = \mu^2 - \gamma + \lambda_1^2$, *and* $k' = \int_{\mathcal{O}} \varphi_1(b)db$.

Proof of lemma 3.4

Conditioning with respect to $\mathcal{R}_t = \sigma\{R_s, s \leq t\}$, we get :

$$E_{r,\theta}\Big[1_{\{T_{\mathcal{C}}>t\}} \exp\left(\frac{\gamma}{2}H_t + \rho R_t\right)\Big] = E_{r,\theta}\Big[\exp\left(\frac{\gamma}{2}H_t + \rho R_t\right) E_{r,\theta}\left[1_{\{T_{\mathcal{C}}>t\}}|\mathcal{R}_t\right]\Big].$$

It is clear that (3.1) implies :

$$H_{T_{\mathcal{C}}} = T_{\mathcal{O}}^{\Theta}. \tag{3.19}$$

Consequently, applying Lemma 3.3, we obtain :

$$E_{r,\theta}\left[1_{\{T_{\mathcal{C}}>t\}}|\mathcal{R}_t\right] = E_{r,\theta}\left[1_{\{H_{T_{\mathcal{C}}}>H_t\}}|\mathcal{R}_t\right] \underset{t\to\infty}{\sim} k'\varphi_1(\theta)\exp\big(-\frac{\lambda_1^2}{2}H_t\big).$$

As a result :

$$E_{r,\theta}\Big[1_{\{T_{\mathcal{C}}>t\}} \exp\left(\frac{\gamma}{2}H_t + \rho R_t\right)\Big] \underset{t\to\infty}{\sim} k'\varphi_1(\theta)E_r^{(\mu)}\Big[\exp\big(\rho R_t + \frac{\gamma}{2}H_t - \frac{\lambda_1^2}{2}H_t\big)\Big].$$

Choosing $\nu^2 = \mu^2 - \gamma + \lambda_1^2$ and $\xi_t = \exp\big(\rho R_t\big)$ in (2.20), we have :

$$E_r^{(\mu)}\Big[\exp\big(\rho R_t + \frac{\gamma}{2}H_t - \frac{\lambda_1^2}{2}H_t\big)\Big] = E_r^{(\nu)}\Big[\Big(\frac{r}{R_t}\Big)^{\nu-\mu}\exp\big(\rho R_t\big)\Big], \tag{3.20}$$

with $\nu^2 = \mu^2 - \gamma + \lambda_1^2$.
Hence :

$$E_{r,\theta}\Big[1_{\{T_C>t\}}\exp\Big(\frac{\gamma}{2}H_t + \rho R_t\Big)\Big] \underset{t\to\infty}{\sim} k'\varphi_1(\theta)E_r^{(\nu)}\Big[\Big(\frac{r}{R_t}\Big)^{\nu-\mu}\exp\big(\rho R_t\big)\Big]. \tag{3.21}$$

But, the second term in (3.21) may be computed explicitly :

$$\begin{aligned}
& E_r^{(\nu)}\Big[\Big(\frac{r}{R_t}\Big)^{\nu-\mu}\exp\rho R_t\Big] \\
= \quad & r^{\nu-\mu}\int_0^\infty \frac{e^{\rho y}}{y^{\nu-\mu}}\frac{1}{t}\Big(\frac{y}{r}\Big)^{\nu} y I_\nu\Big(\frac{ry}{t}\Big)\exp-\Big(\frac{r^2+y^2}{2t}\Big)dy \\
= \quad & \frac{e^{-\frac{r^2}{2t}}}{tr^\mu}\int_0^\infty y^{\mu+1}I_\nu\Big(\frac{ry}{t}\Big)e^{(\rho y-\frac{y^2}{2t})}dy \\
= \quad & \frac{e^{(\frac{\rho^2}{2}t-\frac{r^2}{2t})}}{tr^\mu}\int_0^\infty y^{\mu+1}I_\nu\Big(\frac{ry}{t}\Big)e^{-\frac{1}{2t}(y-\rho t)^2}dy \\
= \quad & \frac{e^{(\frac{\rho^2}{2}t-\frac{r^2}{2t})}}{\sqrt{t}r^\mu}\int_{-\rho\sqrt{t}}^\infty (z\sqrt{t}+\rho t)^{\mu+1}I_\nu\Big(\frac{r}{t}(z\sqrt{t}+\rho t)\Big)e^{-\frac{z^2}{2}}dz \\
\underset{t\to\infty}{\sim} \quad & \Big(\sqrt{2\pi}\rho^{1+\mu}r^{-\mu}I_\nu(\rho r)\Big)t^{\mu+\frac{1}{2}}e^{\frac{\rho^2}{2}t}.
\end{aligned}$$

■

<u>3) We now prove points 1) and 2) of Theorem 3.1.</u>
Conditioning with respect to $\mathcal{F}_s$, we get :

$$\begin{aligned}
& \frac{E_{r,\theta}\big[1_{\Lambda_s}1_{\{T_C>t\}}\exp\big(\frac{\gamma}{2}H_t+\rho R_t\big)\big]}{E_{r,\theta}\big[1_{\{T_C>t\}}\exp\big(\frac{\gamma}{2}H_t+\rho R_t\big)\big]} \\
= \quad & \frac{E_{r,\theta}\Big[1_{\Lambda_s}1_{\{T_C>s\}}e^{\frac{\gamma}{2}H_s}E_{r',\theta'}\big[1_{\{T_C>t-s\}}e^{\frac{\gamma}{2}H_{t-s}+\rho R_{t-s}}\big]\Big]}{E_{r,\theta}\big[1_{\{T_C>t\}}\exp\big(\frac{\gamma}{2}H_t+\rho R_t\big)\big]}
\end{aligned}$$

with $r' = |X_s| = R_s$ and $\theta' = \Theta_{H_s}$.

Hence, from Lemma 3.4 :

$$\begin{aligned}
&\frac{E_{r,\theta}\left[1_{\Lambda_s}1_{\{T_C>t\}}\exp\left(\frac{\gamma}{2}H_t+\rho R_t\right)\right]}{E_{r,\theta}\left[1_{\{T_C>t\}}\exp\left(\frac{\gamma}{2}H_t+\rho R_t\right)\right]}\\
\underset{t\to\infty}{\sim}\quad &\frac{E_{r,\theta}\left[1_{\Lambda_s}1_{\{T_C>s\}}(t-s)^{\mu+\frac{1}{2}}e^{\frac{\gamma}{2}H_s}\varphi_1(\Theta_{H_s})R_s^{-\mu}I_\nu(\rho R_s)e^{\frac{\rho^2}{2}(t-s)}\right]}{\varphi_1(\theta)r^{-\mu}I_\nu(\rho r)e^{\frac{\rho^2 t}{2}}t^{\mu+\frac{1}{2}}}\\
\underset{t\to\infty}{\to}\quad & k\,E_{r,\theta}\left[1_{\Lambda_s}1_{\{T_C>s\}}\varphi_1(\Theta_{H_s})R_s^{-\mu}I_\nu(\rho R_s)e^{-\frac{\rho^2}{2}s+\frac{\gamma}{2}H_s}\right],
\end{aligned}$$

with $k:=\Big(\varphi_1(\theta)r^{-\mu}I_\nu(\rho r)\Big)^{-1}$.

■

4) We prove that $M_s1_{\{T_C>s\}}$ is a positive martingale

Since $(\Theta_u,u\geq 0)$ is the diffusion associated with $\frac{1}{2}\widetilde{\Delta}$, and (3.2) holds, we get :

$$\begin{aligned}
d\varphi_1(\Theta_{H_t}) &= dM_t^{(1)}+\frac{1}{2}\widetilde{\Delta}\varphi_1(\Theta_{H_t})dH_t\\
&= dM_t^{(1)}-\frac{\lambda_1^2}{2}\varphi_1(\Theta_{H_t})\frac{dt}{R_t^2} \qquad (3.22)
\end{aligned}$$

where $(M_t^{(1)},t\geq 0)$ is a local martingale.

On the other hand, denoting by $\mathcal{L}^{(\mu)}$ the infinitesimal generator of the Bessel semigroup, with index μ :

$$\mathcal{L}^{(\mu)}f(r)=\frac{1}{2}f''(r)+\frac{1+2\mu}{2r}f'(r) \qquad (3.23)$$

an elementary computation, which follows from the classical identity (see [4], p. 110)

$$I_\nu''(r)+\frac{1}{r}I_\nu'(r)=\left(1+\frac{\nu^2}{r^2}\right)I_\nu(r)$$

shows that, with :

$$\Psi(r):=r^{-\mu}I_\nu(\rho r) \qquad (r\geq 0) \qquad (3.24)$$

we get :

$$\mathcal{L}^{(\mu)}\Psi(r)=\Psi(r)\left[\frac{\rho^2}{2}+\frac{\nu^2-\mu^2}{2r^2}\right]. \qquad (3.25)$$

Thus :

$$d(R_t^{-\mu}I_\nu(\rho R_t))=dM_t^{(2)}+\left(\frac{\rho^2}{2}+\frac{\nu^2-\mu^2}{2R_t^2}\right)R_t^{-\mu}I_\nu(\rho R_t)dt \qquad (3.26)$$

where $(M_t^{(2)}, t \geq 0)$ is a local martingale. We then apply Itô's formula (Notation : given our aim in this point 4), we now prefer to use s for the time variable, instead of t):

$$
\begin{aligned}
d\frac{1}{k}M_s &= d\left(e^{-\frac{\rho^2}{2}s+\frac{\gamma}{2}H_s}\varphi_1(\Theta_{H_s})R_s^{-\mu}I_\nu(\rho R_s)\right)\\
&= \left(-\frac{\rho^2}{2}+\frac{\gamma}{2}\frac{1}{R_s^2}\right)\frac{M_s}{k}\,ds\\
&\quad + e^{-\frac{\rho^2}{2}s+\frac{\gamma}{2}H_s}R_s^{-\mu}I_\nu(\rho R_s)\left(dM_s^{(1)}-\frac{\lambda_1^2}{2}\varphi_1(\Theta_{H_s})\frac{ds}{R_s^2}\right)\\
&\quad + e^{-\frac{\rho^2}{2}s+\frac{\gamma}{2}H_s}\varphi_1(\Theta_{H_s})\left[dM_s^{(2)}+\left\{\left(\frac{\rho^2}{2}+\frac{\nu^2-\mu^2}{2R_s^2}\right)R_s^{-\mu}I_\nu(\rho R_s)\right\}ds\right]\\
&= e^{-\frac{\rho^2}{2}s+\frac{\gamma}{2}H_s}\left[R_s^{-\mu}I_\nu(\rho R_s)dM_s^{(1)}+\varphi_1(\Theta_{H_s})dM_s^{(2)}\right] \qquad (3.27)
\end{aligned}
$$

since $\nu^2 = \mu^2 + \lambda_1^2 - \gamma$.

■

Recall that $\varphi_1(x) = 0$ when $x \in \partial\mathcal{O}$; hence $M_{T_\mathcal{C}} = 0$. This proves that $(M_s 1_{\{T_\mathcal{C}>s\}}, s \geq 0)$ is a local martingale. Since it is positive, it is a supermartingale. Hence, to prove that $(M_s 1_{\{T_\mathcal{C}>s\}}, s \geq 0)$ is a martingale, it suffices to prove that $E_{r,\theta}\big[M_s 1_{\{T_\mathcal{C}>s\}}\big] = 1$.
Due to (3.17) and (3.2) iii) we have :

$$
\begin{aligned}
E_{r,\theta}\big[1_{\{T_\mathcal{O}^\Theta>t\}}\varphi_1(\Theta_t)\big] &= E_\theta\big[\varphi_1(\widetilde{\Theta}_t)\big]\\
&= \sum_{n\geq 1} e^{-\frac{\lambda_n^2}{2}t}\varphi_n(\theta)\int_\mathcal{O}\varphi_n(b)\varphi_1(b)db\\
&= e^{-\frac{\lambda_1^2}{2}t}\varphi_1(\theta). \qquad (3.28)
\end{aligned}
$$

We proceed as in the proof of Lemma 3.4, taking the conditional expectation with respect to $\mathcal{R}_t$ and using the previous result we get :

$$
\begin{aligned}
E_{r,\theta}\big[M_s 1_{\{T_\mathcal{C}>s\}}\big] &= k\,E_{r,\theta}\left[1_{\{T_\mathcal{C}>s\}}\,\varphi_1(\Theta_{H_s})e^{\frac{\gamma}{2}H_s-\frac{\rho^2}{2}s}R_s^{-\mu}I_\nu(\rho R_s)\right]\\
&= k\,E_{r,\theta}\left[e^{\frac{\gamma}{2}H_s-\frac{\rho^2}{2}s}R_s^{-\mu}I_\nu(\rho R_s)\,e^{-\frac{\lambda_1^2}{2}H_s}\varphi_1(\theta)\right]\\
&= k\varphi_1(\theta)\,E_r^{(\mu)}\left[R_s^{-\mu}I_\nu(\rho R_s)\exp\left\{(\gamma-\lambda_1^2)\frac{H_s}{2}-\frac{\rho^2}{2}s\right\}\right]
\end{aligned}
$$

According to the absolute continuity formula (2.20), with $\nu^2 = \mu^2 - \gamma + \lambda_1^2$, we have :

$$
\begin{aligned}
E_{r,\theta}\big[M_s 1_{\{T_C>s\}}\big] &= k\varphi_1(\theta)e^{-\frac{\rho^2}{2}s}\, E_r^{(\nu)}\Big[R_s^{-\mu} I_\nu(\rho R_s)\left(\frac{r}{R_s}\right)^{\nu-\mu}\Big] \\
&= k\varphi_1(\theta)r^{\nu-\mu}e^{-\frac{\rho^2}{2}s}\, E_r^{(\nu)}\Big[R_s^{-\nu} I_\nu(\rho R_s)\Big].
\end{aligned}
$$

But $\mathcal{L}^{(\nu)}(\widetilde{\Psi})(r) = \frac{\rho^2}{2}\widetilde{\Psi}(r)$, with $\widetilde{\Psi}(r) = r^{-\nu} I_\nu(\rho r)$, then :

$$
\big(R_s^{-\nu} I_\nu(\rho R_s)e^{-\frac{\rho^2}{2}s},\ s \geq 0\big) \text{ is a martingale under } P_r^{(\nu)}. \tag{3.29}
$$

Therefore (3.8) implies

$$
E_{r,\theta}\big[M_s 1_{\{T_C>s\}}\big] = k\varphi_1(\theta)r^{\nu-\mu}r^{-\nu}I_\nu(\rho r) = 1,
$$

from the definition of k, at the end of point 3) above.

5) Description of the process $(R_t, t \geq 0)$ under Q_x

For every positive functional F, and every $x \in \mathcal{C}, x \neq 0$, we write :

$$
\begin{aligned}
& E_{Q_x}(F(R_s, s \leq t)) \\
= \quad & kE_{r,\theta}\Big[F(R_s, s \leq t)\varphi_1(\Theta_{H_t})1_{\{T_C>t\}}\, e^{-\frac{\rho^2}{2}t + \frac{\gamma}{2}H_t} R_t^{-\mu} I_\nu(\rho R_t)\Big]
\end{aligned} \tag{3.30}
$$

Then, conditioning with respect to $\mathcal{R}_t = \sigma\{R_s, s \leq t\}$ and using (3.28) we get :

$$
E_{Q_x}\big[F(R_s, s \leq t)\big] = kE_r^{(\mu)}\Big[F(R_s, s \leq t)R_t^{-\mu} I_\nu(\rho R_t)e^{-\frac{\rho^2}{2}t - (\frac{\lambda_1^2}{2} - \frac{\gamma}{2})H_t}\Big] \tag{3.31}
$$

Relation (3.25) implies that :

$$
\frac{\mathcal{L}^{(\mu)}\Psi(r)}{\Psi(r)} = \frac{\rho^2}{2} + \frac{\nu^2 - \mu^2}{2r^2} = \frac{\rho^2}{2} + \frac{\lambda_1^2 - \gamma}{2r^2}, \tag{3.32}
$$

the function Ψ being defined by (3.24).
Consequently,

$$
\Big(R_t^{-\mu} I_\nu(\rho R_t)e^{-\frac{\rho^2}{2}t - (\frac{\lambda_1^2}{2} - \frac{\gamma}{2})H_t}, t \geq 0\Big) \text{ is a martingale under } P_r^{(\mu)}, \tag{3.33}
$$

since it is of the form :

$$
\Psi(R_t)\exp\Big(-\int_0^t \frac{\mathcal{L}^{(\mu)}\Psi}{\Psi}(R_s)ds\Big).
$$

Thus, the function $\tilde{h}(t,a,r) = \exp\left(-\frac{\rho^2 t}{2} - \left(\frac{\lambda_1^2}{2} - \frac{\gamma}{2}\right) a\right)\Psi(r)$ is a harmonic function for the Markov process $((t,H_t,R_t), t \geq 0)$. The formula (3.31) then indicates that the process $(R_t, t \geq 0)$ is under Q_x the $\tilde{h}$-Doob transform of the process $(R_t, t \geq 0)$ under $P_r^{(\mu)}$. Thus, it is a Markov process, with infinitesimal generator $\mathcal{L}^{\mathcal{R}}$:

$$\mathcal{L}^{\mathcal{R}} f(r) = \frac{1}{\tilde{h}} \widetilde{\mathcal{L}}^{\mu}(f\tilde{h})$$

where $\widetilde{\mathcal{L}}^{\mu}$ is the infinitesimal generator of the process $((t,H_t,R_t), t \geq 0)$.
Hence :

$$\begin{aligned}
\mathcal{L}^{\mathcal{R}} f &= \frac{1}{2} f''(r) + \left(\frac{\partial}{\partial r}(\log \tilde{h}) + \frac{1+2\mu}{2r}\right) f'(r) \\
&= \frac{1}{2} f''(r) + \left(\frac{1}{2r} + \frac{\rho I'_{\nu}(\rho r)}{I_\nu(\rho r)}\right) f'(r) \\
&= \frac{1}{2} f''(r) + \left(\frac{1+2\nu}{2r} + \frac{\rho I_{\nu+1}(\rho r)}{I_\nu(\rho r)}\right) f'(r) \qquad (3.34)
\end{aligned}$$

since from ([4], p. 110) :

$$\frac{d}{dz}\big(z^{-\nu} I_\nu(z)\big) = z^{-\nu} I_{\nu+1}(z). \qquad (3.35)$$

Note that, since :

$$I_\nu(z) \underset{z\to 0}{\sim} \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}, \qquad (3.36)$$

then :

$$\frac{\rho I_{\nu+1}(\rho r)}{I_\nu(\rho r)} \underset{r\to 0}{\sim} \frac{\rho^2}{2\nu} r, \qquad (3.37)$$

the process $(R_t, t \geq 0)$ under Q_x behaves, near 0, as a Bessel process with index $\nu = \sqrt{\mu^2 + \lambda_1^2 - \gamma}$. In particular, when $\rho = 0$, this process is then a Bessel process whose index equals $\sqrt{\mu^2 + \lambda_1^2 - \gamma}$. Thus, the dimension of this Bessel process may be smaller than the original dimension d; this happens if $\lambda_1^2 < \gamma$.

6) Description of the process $(\Theta_u, u \geq 0)$ under Q_x

i) Let $f : \mathcal{O} \to \mathbb{R}$ be regular. Since, under $P_x^{(\mu)}, (\Theta_u, u \geq 0)$ is a spherical

Brownian motion associated with $\frac{1}{2}\widetilde{\Delta}$, then :

$$M_t^f \;:=\; f\big(\Theta_{t\wedge T_{\mathcal{O}}^{\Theta}}\big) - \frac{1}{2}\int_0^{t\wedge T_{\mathcal{O}}^{\Theta}} \widetilde{\Delta} f(\Theta_s)ds \tag{3.38}$$

$$= \; f(\theta) + \int_0^{t\wedge T_{\mathcal{O}}^{\Theta}} \nabla f(\Theta_s)\cdot d\Theta_s \tag{3.39}$$

is a P_x-martingale whose bracket equals $\displaystyle\int_0^{t\wedge T_{\mathcal{O}}^{\Theta}} |\nabla f|^2(\Theta_s)ds$ (the gradient and its norm being taken in the sense of the Riemannian structure on S^{d-1}). Hence, since R and Θ are independent, under P_x :

$$\widetilde{M}_t^f \;:=\; f\big(\Theta_{H_{t\wedge T_{\mathcal{C}}}}\big) - \frac{1}{2}\int_0^{t\wedge T_{\mathcal{C}}} \widetilde{\Delta} f(\Theta_{H_s})dH_s \tag{3.40}$$

$$= \; f(\theta) + \int_0^{t\wedge T_{\mathcal{C}}} \nabla f(\Theta_{H_s})\cdot d\Theta_{H_s} \tag{3.41}$$

is a P_x-martingale whose bracket is equal to $\displaystyle\int_0^{t\wedge T_{\mathcal{C}}} |\nabla f|^2(\Theta_{H_s})dH_s$.
In the same way :

$$M_t^{(1)} = \varphi_1(\theta) + \int_0^t \nabla\varphi_1(\Theta_{H_s})\cdot d\Theta_{H_s}, \tag{3.42}$$

where $M_t^{(1)}$ has been introduced in (3.22).
ii) $\big(M_{t\wedge T_{\mathcal{C}}}, t\geq 0\big)$ is a P_x positive martingale and, from Girsanov's theorem

$$\widetilde{M}_t^f - \int_0^{t\wedge T_{\mathcal{C}}} \frac{1}{M_s} d < \widetilde{M}^f, M >_s \;\text{ is a }\; Q_x \;\text{ local martingale.} \tag{3.43}$$

iii) We now determine the bracket $< \widetilde{M}^f, M >$. Since the bracket of $\widetilde{M}^f$ and of $M^{(2)}$(which was introduced in (3.26)) is equal to 0, as R and Θ are independent, we deduce from (3.27), (3.41) and (3.42) :

$$\begin{aligned} d < \widetilde{M}^f, M >_t \;&=\; k e^{-\frac{\rho^2}{2}t + \frac{\gamma}{2}H_t} R_t^{-\mu} I_\nu(\rho R_t)\big(\nabla f\cdot\nabla\varphi_1\big)\big(\Theta_{H_t}\big)dH_t \\ &=\; M_t\big(\nabla f\cdot\frac{\nabla\varphi_1}{\varphi_1}\big)\big(\Theta_{H_t}\big)dH_t, \end{aligned}$$

for any $t\leq T_{\mathcal{C}}$.

Relations (3.40) and (3.43) imply :

$$f(\Theta_{H_t})1_{\{T_{\mathcal{C}}>t\}} - \frac{1}{2}\int_0^{t\wedge T_{\mathcal{C}}} \widetilde{\Delta} f(\Theta_{H_s})dH_s - \int_0^{t\wedge T_{\mathcal{C}}} \left(\nabla f \cdot \frac{\nabla \varphi_1}{\varphi_1}\right)(\Theta_{H_s})dH_s \tag{3.44}$$

is a Q_x-martingale.
Performing the time change $H_t = u$ in (3.44), we deduce :

$$f(\Theta_u)1_{\{T_{\mathcal{O}}^{\Theta}>u\}} - \frac{1}{2}\int_0^{u\wedge T_{\mathcal{O}}^{\Theta}} \widetilde{\Delta} f(\Theta_s)ds - \int_0^{u\wedge T_{\mathcal{O}}^{\Theta}} \left(\frac{\nabla f . \nabla \varphi_1}{\varphi_1}\right)(\Theta_s)ds$$

is a martingale.
Thus, from Stroock and Varadhan [20], $(\Theta_u, u \geq 0)$ is a diffusion process, with infinitesimal generator : $\frac{1}{2}\widetilde{\Delta} + \frac{\nabla \varphi_1}{\varphi_1}.\nabla$.

7) We prove that, under Q_x, $T_{\mathcal{C}} = \infty$., a.s.

This follows from the fact that the normal derivative of φ_1 on the boundary of $\mathcal{O}$ does not vanish. Thus :

$$\frac{\nabla \varphi_1}{\varphi_1}(\theta) \underset{\theta \to \partial\mathcal{O}}{\sim} \frac{\overrightarrow{n}}{d(\theta, \partial\mathcal{O})}$$

where $d(\theta, \partial\mathcal{O})$ denotes the distance of θ to the boundary of $\mathcal{O}$, and where $\overrightarrow{n}$ is the inward normal vector. This implies that the process $(\Theta_u, u \geq 0)$ under Q_x has, in the neighborhood of the boundary of $\mathcal{O}$, "a radial part which behaves like a BES (3) process", hence which does not reach the boundary.

8) We prove the independence, under Q_x, of $(R_t, t \geq 0)$ and $(\Theta_u, u \geq 0)$

For the sake of simplicity, we shall only give the proof for dimension $d = 2$. Under P_x, we write the complex-valued Brownian motion :

$$X_t := x_t + iy_t = |X_t| \exp\left(i\beta^{(1)}_{H_t}\right) ,$$

where

$$\beta^{(1)}_{H_t} = \mathrm{Im}\left(\int_0^t \frac{dX_s}{X_s}\right) = \int_0^t \frac{x_s\, dy_s - y_s\, dx_s}{|X_s|^2}. \tag{3.45}$$

$(|X_t|,\, t \geq 0)$ decomposes as a semi-martingale :

$$\begin{aligned} &\bullet \text{ under } P_x : \quad |X_t| = \beta^{(2)}_t + \frac{1}{2}\int_0^t \frac{ds}{|X_s|} \\ &\bullet \text{ under } Q_x : \quad |X_t| = \widetilde{\beta}^{(2)}_t + \int_0^t h(|X_s|)ds \quad , \end{aligned} \tag{3.46}$$

for a certain h, where $\beta^{(2)}$, resp : $\widetilde{\beta}^{(2)}$, is a P_x, resp : Q_x Brownian motion. Moreover :

$$d\beta_t^{(2)} = \frac{x_t\, dx_t + y_t\, dy_t}{|X_t|} \quad ,$$

which implies :

$$d < \beta^{(2)} , \ \beta^{(1)}_{H_\bullet} >_t = 0 \quad ;$$

hence, from Knight's representation theorem of continuous orthogonal martingales, $(\beta_t^{(2)})$ and $(\beta_u^{(1)})$ are two independent real-valued Brownian motions. After applying Girsanov's theorem to go from P_x to Q_x, we obtain likewise that $(\widetilde{\beta}_t^{(2)}, \, t \geq 0)$ and $(\widetilde{\beta}_u^{(1)}, \, u \geq 0)$, which are respectively the martingale parts of $(\beta_u^{(2)})$ and $(\beta_u^{(1)})$ under Q_x are two independent Q_x Brownian motions.

Moreover, from (3.46), $(|X_t|, \, t \geq 0)$ is the solution, under Q_x, of an SDE with driving Brownian motion $(\widetilde{\beta}_t^{(2)})$; likewise, from point 6) of the proof, or even more directly in dimension 2, $(\beta_u^{(1)})_{u \geq 0}$ solves an SDE directed by $(\widetilde{\beta}_u^{(1)})$. Consequently, $(|X_t|, \, t \geq 0)$ and $(\beta_u^{(1)}, \, u \geq 0)$ are independent under Q_x.

For dimensions $d > 2$, we leave the variant of this proof to the reader.

■

We shall now end this Section 3 by giving, for $d = 2$, a slightly different form of Theorem 3.1, where we make $\rho = \gamma = 0$, to simplify matters. This time, we shall use the skew-product decomposition given by (2.1),$\cdots$,(2.5) :

$$X_t = R_t \exp(i\beta_{H_t}) \tag{3.47}$$

where $(R_t, \, t \geq 0)$ denotes a Bessel process with dimension 2 (or index 0). We denote, for $\theta_t = \beta_{H_t}$:

$$\overline{\theta}_t = \sup_{s \leq t} \theta_s = S^{\beta}_{H_t} = \sup_{u \leq H_t} \beta_u \tag{3.48}$$

and

$$\underline{\theta}_t = \inf_{s \leq t} \theta_s = I^{\beta}_{H_t} = \inf_{u \leq H_t} \beta_u . \tag{3.49}$$

On the other hand, θ_- and θ_+ denote two reals such that :

$$\theta_- < 0 < \theta_+ \tag{3.50}$$

and we now propose to study the penalisations with $\Gamma_t := 1_{\{\overline{\theta}_t < \theta_+, \, \underline{\theta}_t > \theta_-\}}$. When $\theta_- > -\pi$ and $\theta_+ < \pi$, this study is a particular case of Theorem 3.1, with $\rho = \gamma = 0$.

In what follows, x is a point of $\mathbb{R}^2$ whose first coordinate is strictly positive, while the second one is 0, and we shall write x for $(x, 0)$.

Theorem 3.5.
Let x be as just assumed.
1) *For every $s \geq 0$, and every $\Lambda_s \in \mathcal{F}_s$, the limit :*

$$Q_x(\Lambda_s) := \lim_{t\to\infty} \frac{E_x\left[1_{\Lambda_s} 1_{\{\overline{\theta}_t < \theta_+ ,\, \underline{\theta}_t > \theta_-\}}\right]}{E_x\left[1_{\{\overline{\theta}_t < \theta_+ ,\, \underline{\theta}_t > \theta_-\}}\right]} \qquad \textit{exists.} \tag{3.51}$$

This limit equals

$$Q_x(\Lambda_s) = E_x(1_{\Lambda_s}\, M_s) \tag{3.52}$$

with

$$M_s := k' R_s^\lambda \sin\big(\lambda(\theta_+ - \theta_s)\big)\, 1_{\{\overline{\theta}_s < \theta_+ ,\, \underline{\theta}_s > \theta_-\}} \tag{3.53}$$

and

$$\lambda = \frac{\pi}{\theta_+ - \theta_-}\ , \quad k' = \frac{1}{x^\lambda}\, \frac{1}{\sin(\lambda\theta_+)}\ . \tag{3.54}$$

Moreover, $(M_s,\, s \geq 0)$ is a positive martingale such that $M_0 = 1$.

2) *Formula (3.52) induces a probability on $(\Omega,\, \mathcal{F}_\infty)$, and under Q_x, the process $(X_t,\, t \geq 0)$ writes :*

$$X_t = R_t\, e^{i\beta_{H_t}} \tag{3.55}$$

where :

a) $(R_t,\, t \geq 0)$ and $(\beta_u,\, u \geq 0)$ are independent

b) $(R_t,\, t \geq 0)$ is a Bessel process with dimension $2(1+\lambda)$, and

$$H_t = \int_0^t \frac{ds}{R_s^2}. \tag{3.56}$$

c) $(\beta_u,\, u \geq 0)$ is distributed as the solution of the SDE :

$$Z_u = \widehat{\beta}_u - \lambda \int_0^u \operatorname{cotg}\big(\lambda(\theta_+ - Z_s)\big)\, ds \tag{3.57}$$

where $(\widehat{\beta}_u,\, u \geq 0)$ is a Brownian motion.
In particular, the process $(\beta_t,\, t \geq 0)$ never reaches the levels θ_- and θ_+, although :

$$\sup_{s\leq t} \beta_s \underset{t\to\infty}{\longrightarrow} \theta_+ \quad a.s., \quad \inf_{s\leq t} \beta_s \underset{t\to\infty}{\longrightarrow} \theta_- \quad a.s.,$$

Proof of Theorem 3.5

It is essentially the same as that of Theorem 3.1. We briefly indicate the main lines.

1) When written in our present context, formula (3.17) yields the density of the process $(\beta_t,\, t \geq 0)$ killed when it exits the interval $[\theta_-,\, \theta_+]$:

$$\begin{aligned} & P(S_u^\beta < \theta_+,\, I_u^\beta > \theta_-,\, \beta_u \in dx) \\ = & \sum_{k\geq 1} \left\{ \cos\Big(\frac{k\pi x}{\theta_+ - \theta_-}\Big) - \cos\Big(\frac{k\pi(2\theta_+ - x)}{\theta_+ - \theta_-}\Big)\right\} e^{-\frac{k^2\pi^2 u}{2(\theta_+ - \theta_-)^2}} \\ & \qquad\qquad \times \frac{1}{\theta_+ - \theta_-} 1_{\{\theta_- < x < \theta_+\}}\, dx. \end{aligned} \tag{3.58}$$

Consequently :

$$P(S_u^\beta < \theta_+,\, I_u^\beta > \theta_-) = P(u < T_{\theta_+} \wedge T_{\theta_-}) \underset{u\to\infty}{\sim} C e^{-\frac{\lambda^2 u}{2}}, \tag{3.59}$$

where :

$$C = \frac{1}{\theta_+ - \theta_-} \int_{\theta_-}^{\theta_+} \Big(\cos(\lambda x) - \cos\big(\lambda(2\theta_+ - x)\big)\Big) dx.$$

We have :

$$C = \frac{2\sin(\lambda\theta_+)}{\theta_+ - \theta_-} \int_{\theta_-}^{\theta_+} \sin\big(\lambda(\theta_+ - x)\big) dx = \frac{4\sin(\lambda\theta_+)}{\pi}. \tag{3.60}$$

From formulae (3.48), (3.49), (3.59), and the independence of H_t from (β_u), we deduce, for every starting point $(r, 0)$:

$$P_r(\overline{\theta}_t < \theta_+,\, \underline{\theta}_t > \theta_-) \underset{t\to\infty}{\sim} C\, E_r\left[e^{-\frac{\lambda^2 H_t}{2}}\right]. \tag{3.61}$$

Applying (2.20) with $\xi_t = 1, \mu = 0$ and $\nu = \lambda$, we get :

$$E_r\left[e^{-\frac{\lambda^2 H_t}{2}}\right] = E_r^{(\lambda)}\left[\Big(\frac{r}{R_t}\Big)^\lambda\right]. \tag{3.62}$$

Reasoning as in the proof of Lemma 2.3, we obtain :

$$E_r^{(\lambda)}\left[\Big(\frac{r}{R_t}\Big)^\lambda\right] \underset{t\to\infty}{\sim} \Big(\frac{r}{\sqrt{t}}\Big)^\lambda \frac{1}{2^{\lambda/2}} \frac{\Gamma(1+\lambda/2)}{\Gamma(1+\lambda)}. \tag{3.63}$$

Finally, we get :

$$P_r(\overline{\theta}_t < \theta_+,\, \underline{\theta}_t > \theta_-) \underset{t\to\infty}{\sim} \frac{4\sin(\lambda\theta_+)}{\pi 2^{\lambda/2}} \frac{\Gamma(1+\lambda/2)}{\Gamma(1+\lambda)} \Big(\frac{r}{\sqrt{t}}\Big)^\lambda. \tag{3.64}$$

Observe that the Markov property implies :

$$E_x\big[1_{\{\overline{\theta}_t<\theta_+,\,\underline{\theta}_t>\theta_-\}}\big|\mathcal{F}_s\big] = 1_{\{\overline{\theta}_s<\theta_+,\,\underline{\theta}_s>\theta_-\}}g(R_s,\theta_+-\theta_s,\theta_--\theta_s,t-s)$$

with

$$g(r,\theta_+,\theta_-,u) = P_r\big(\overline{\theta}_u<\theta_+,\,\underline{\theta}_u>\theta_-\big).$$

Consequently (3.64) implies :

$$\lim_{t\to\infty}\frac{E_x\big[1_{\{\overline{\theta}_t<\theta_+,\,\underline{\theta}_t>\theta_-\}}\big|\mathcal{F}_s\big]}{P_x\big(\overline{\theta}_t<\theta_+,\,\underline{\theta}_t>\theta_-\big)} = 1_{\{\overline{\theta}_s<\theta_+,\,\underline{\theta}_s>\theta_-\}}\frac{\sin\big(\lambda(\theta_+-\theta_s)\big)}{\sin(\lambda\theta_+)}\Big(\frac{R_s}{x}\Big)^{\lambda}.$$

This proves the first part of Theorem 3.5, if we admit for a while that $E_x[M_t]=1$. This equality is actually a direct consequence of the next step 2).

2) <u>We now verify that $\Big(R_s^{\lambda}\sin\big(\lambda(\theta_+-\theta_s)\big),\,s\geq 0\Big)$ is a martingale under P_x and $M_0=1$.</u>

Indeed, $\big(R_s^{\lambda}\sin\big(\lambda(\theta_+-\theta_s)\big),s\geq 0\big)$ is the imaginary part of the conformal martingale $\big(R_s^{\lambda}\exp\big(i\lambda(\theta_+-\theta_s)\big),s\geq 0\big)$.
Moreover, we have, by Itô's formula :

$$\begin{aligned}
R_t^{\lambda}\sin\big(\lambda(\theta_+-\theta_t)\big) &= x^{\lambda}\sin(\lambda\theta_+)+\lambda\int_0^t R_s^{\lambda-1}\sin\big(\lambda(\theta_+-\theta_s)\big)dB_s\\
&+\frac{\lambda^2}{2}\int_0^t R_s^{\lambda-2}\sin\big(\lambda(\theta_+-\theta_s)\big)ds-\lambda\int_0^t R_s^{\lambda}\cos\big(\lambda(\theta_+-\theta_s)\big)d\theta_s\\
&-\frac{\lambda^2}{2}\int_0^t R_s^{\lambda}\sin\big(\lambda(\theta_+-\theta_s)\big)\frac{ds}{R_s^2}\\
&= x^{\lambda}\sin(\lambda\theta_+)+\lambda\int_0^t R_s^{\lambda-1}\sin\big(\lambda(\theta_+-\theta_s)\big)dB_s\\
&-\lambda\int_0^t R_s^{\lambda}\cos\big(\lambda(\theta_+-\theta_s)\big)d\theta_s \qquad (3.65)
\end{aligned}$$

where $(B_s,s\geq 0)$ is the driving Brownian motion of $(R_s,s\geq 0)$.

■

3) <u>We now compute the law of $(R_t,\,t\geq 0)$ under Q_x.</u>

We have, for every functional $F\geq 0$:

$$\begin{aligned}
E_{Q_x}\big[F(R_s,\,s\leq t)\big] &= k'\,E_x\Big[F(R_s,\,s\leq t)R_t^{\lambda}\sin\big(\lambda(\theta_+-\theta_t)\big)1_{\{\overline{\theta}_t<\theta_+,\underline{\theta}_t>\theta_-\}}\Big]\\
&= k'\,E_x\Big[F(R_s,\,s\leq t)R_t^{\lambda}\chi(H_t)\Big] \qquad (3.66)
\end{aligned}$$

where

$$\chi(u) = E\Big[\sin\big(\lambda(\theta_+ - \beta_u)\big)1_{\{S_u^\beta < \theta_+,\, I_u^\beta > \theta_-\}}\Big] \tag{3.67}$$

$$= \sin(\lambda\theta_+)e^{-\frac{\lambda^2 u}{2}} \tag{3.68}$$

by an easy martingale argument.
Plugging (3.68) in (3.66), we get :

$$E_{Q_x}\big[F(R_s,\, s\le t)\big] = k' \sin(\lambda\theta_+)\, E_x\Big[F(R_s,\, s\le t)R_t^\lambda e^{-\frac{\lambda^2 H_t}{2}}\Big].$$

Using (2.20) with $\mu = 0, \xi_t = R_t^\lambda$ and $\nu = \lambda$ and the definition of k' in (3.54), we obtain :

$$E_{Q_x}\big[F(R_s,\, s\le t)\big] = E_x^{(\lambda)}\big[F(R_s,\, s\le t)\big].$$

This proves that, under Q_x, $(R_s,\, s\ge 0)$ is a Bessel process with index λ, i.e with dimension $2(1+\lambda)$. In particular, this process is transient.

4) Computation of the law of β under Q_x.
Relation (3.65) implies that :

$$M_t = 1 + \lambda k' \int_0^t R_s^{\lambda-1}\sin\big(\lambda(\theta_+ - \theta_s)\big)dB_s - \lambda k' \int_0^t R_s^\lambda \cos\big(\lambda(\theta_+ - \theta_s)\big)d\beta_{H_s}. \tag{3.69}$$

$(B_s,\, s\ge 0)$ being the driving Brownian motion of $(R_s,\, s\ge 0)$ is independent from $(\beta_u,\, u\ge 0)$. Since (M_t) is a positive P_x-martingale then Girsanov's theorem provides us with :

$$\beta_{H_t} = \widehat{\gamma}_t - \lambda \int_0^t \frac{R_s^\lambda \cos\big(\lambda(\theta_+ - \theta_s)\big)}{R_s^\lambda \sin\big(\lambda(\theta_+ - \theta_s)\big)}\, d < \beta_H,\, \beta_H >_s$$

$$= \widehat{\gamma}_t - \lambda \int_0^t \mathrm{cotg}\big(\lambda(\theta_+ - \beta_{H_s})\big)dH_s.$$

Performing the time change $u = H_t$, yields :

$$\beta_u = \widehat{\beta}_u - \lambda \int_0^u \mathrm{cotg}\big(\lambda(\theta_+ - \beta_s)\big)ds$$

where $(\widehat{\beta}_u,\, u\ge 0)$ is a Q_x-Brownian motion.

5) The last point 2) c) of Theorem 3.5 is now classical : in order to prove that the hitting time of the interval $[\theta_-,\, \theta_+]$ by the process β is a.s. infinite, it suffices to apply Feller's test. We also note that, under Q_x :

$$H_\infty = \int_0^\infty \frac{ds}{R_s^2} = \infty \quad \text{a.s.}$$

■

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