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Lorentz Transform and Staggered Finite Differences for Advective Acoustics

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Abstract. – We study acoustic wave propagation in a uniform stationary flow. We develop a method founded on the Lorentz transform and a hypothesis of irrotationality of the acoustic perturbation. After a transformation of the space-time and of the unknown fields, we derive a system of partial differential equations that eliminates the external flow and deals with the classical case of non advective acoustics. A sequel of the analysis is a new set of perfectly matched layers equations in the spirit of the work of Berenger and Collino. The numerical implementation of the previous ideas is presented with the finite differences method HaWAY on cartesian staggered grids. Relevant numerical tests are proposed.

Keywords: Perfectly matched layers, finite differences, HaWAY method.

AMS classification: 65M06, 76N15

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1) Introduction

The acoustic dimensioning of a civil aircraft requires the use of numerical models to predict the radiated acoustic field emitted by the engine. We study this problem in the case of advective acoustics. The specific case of classical acoustic wave propagation can be viewed as a scalar version of Maxwell equations. We use our previous experience acquired in the simulation of the propagation of electromagnetic waves using the finite differences method [DDS94] to rapidly develop a three-dimensional software simulating the propagation of acoustic waves.

Our work is structured as follows. In order to take into account the external aerodynamic flow, we first come back to the equations of gas dynamics and consider the acoustic field as a first order linear perturbation of such a flow. Then we use physical ideas based on the Lorentz group invariance in Section 3, in the spirit of [AB86], to deal in Section 4 with the case of advective acoustics in the same way as the non-advective ones. We develop in Section 5 a smart solution of the difficult problem of absorbing layers. The numerical aspect with the use of staggered grids is tackled in Section 6, and relevant physical and numerical tests are proposed in Section 7

2) Non linear acoustics

2-1 Barotrope gas dynamics

We consider the propagation of sound waves in a uniform two-dimensional subsonic flow of a compressible fluid. This phenomenon is described by the nonlinear Euler equations for gas dynamics, see [LL54] for example, as :

$$\left\{ \begin{array}{l} \dfrac{\partial \check{\rho}}{\partial t} + \text{div } (\check{\rho}\check{\mathbf{u}}) = 0 \\ \dfrac{\partial (\check{\rho}\check{u})}{\partial t} + \dfrac{\partial}{\partial x}(\check{\rho}\check{u}^2 + \check{p}) + \dfrac{\partial}{\partial y}(\check{\rho}\check{u}\check{v}) = 0 \\ \dfrac{\partial (\check{\rho}\check{v})}{\partial t} + \dfrac{\partial}{\partial x}(\check{\rho}\check{u}\check{v}) + \dfrac{\partial}{\partial y}(\check{\rho}\check{v}^2 + \check{p}) = 0 \\ \dfrac{\partial \check{s}}{\partial t} + \check{u}\dfrac{\partial \check{s}}{\partial x} + \check{v}\dfrac{\partial \check{s}}{\partial y} = 0 \ , \end{array} \right. \tag{1}$$

where $\check{\mathbf{u}} = (\check{u}, \check{v})$ is the velocity vector, $\check{\rho}$ the density of the fluid, $\check{p}$ the pressure of the fluid and $\check{s}$ the entropy. We also know that :

$$\frac{\check{p}}{\check{p}_0} = \frac{\check{\rho}^\gamma}{\check{\rho}_0{}^\gamma} \exp(\frac{\check{s}}{C_V}) \ , \tag{2}$$

where C_V is the calorific capacity at constant volume and (ρ_0, p_0) a state of reference.

2-2 Linearization around a stationary state

We linearize the system (1) around a constant state W_0 defined by :

$$W_0 = (\rho_0, u_0, v_0, s_0)^t \ ,$$

where $(...)^t$ is the transpose of a vector. The global state $\check{W}$ of the system is defined around the state W_0 thanks to the perturbation $W = (\rho, u, v, s)^t$, as :

$$\check{W} = W_0 + W \ .$$

A first idea of our approach is to use the impulses :

$$\begin{cases} \check{\rho}\check{u} = (\rho_0 + \rho)(u_0 + u) \equiv \rho_0 u_0 + \xi + \rho u \\ \check{\rho}\check{v} = (\rho_0 + \rho)(v_0 + v) \equiv \rho_0 v_0 + \zeta + \rho v \ , \end{cases}$$

and to linearize them considering the variables ρ, u, v, s as first order infinitesimal quantities. We then introduce the **linearized impulses** :

$$\begin{cases} \xi = \rho_0 u + \rho u_0 \\ \zeta = \rho_0 v + \rho v_0 \ . \end{cases}$$

We have the following classical hypothesis, see [LL54] :

Hypothesis 1 *Isentropy of the flow.*
The linearization of the fourth equation of the system (1) gives :

$$\frac{\partial s}{\partial t} + u_0 \frac{\partial s}{\partial x} + v_0 \frac{\partial s}{\partial y} \equiv \frac{\mathrm{ds}}{\mathrm{dt}} = 0 \ .$$

If we consider the perturbation of entropy at the initial time to be null, that is to say $s(x, y, t = 0) \equiv 0$, we deduce that $s(x, y, t) \equiv 0$ during the time evolution.

Then the system (1) can be shared, first into a **stationary aerodynamic** system :

$$\begin{cases} \mathrm{div}\ (\rho_0 \mathbf{u}_0) = 0 \\ \rho_0 \mathbf{u}_0 \bullet \nabla \mathbf{u}_0 + \nabla p_0 = \mathbf{0} \ , \end{cases}$$

and then into an **isentropic acoustic system** :

$$(3) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \xi}{\partial x} + \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \dfrac{\partial}{\partial x}\left(2u_0\xi + \dfrac{c_0^2 - u_0^2}{c_0^2} p\right) + \dfrac{\partial}{\partial y}(u_0\zeta + v_0\xi - \rho u_0 v_0) = 0 \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial}{\partial x}(u_0\zeta + v_0\xi - \rho u_0 v_0) + \dfrac{\partial}{\partial y}\left(2v_0\zeta + \dfrac{c_0^2 - v_0^2}{c_0^2} p\right) = 0 \ , \end{cases}$$

with $p = c_0^2 \rho$ and c_0 the speed of sound, deduced from the linearization of (2).

Proposition 1 (Advection of the acoustic vorticity)**.**

If the external flow $W_0 = (\rho_0, u_0, v_0, s_0)^t$ is stationary and uniform, the acoustic vorticity $\omega = \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}$ is advected by the flow, *i.e.* $\dfrac{\mathrm{d}\omega}{\mathrm{dt}} = 0$.

Proof. This property is classical, see [LL54] for example. We give the proof for completeness. We have from the system (3) :

$$\begin{aligned}
\frac{\partial \xi}{\partial t} &+ \frac{\partial}{\partial x}\left(2u_0\xi + \frac{c_0^2 - u_0^2}{c_0^2}p\right) + \frac{\partial}{\partial y}(u_0\zeta + v_0\xi - \rho u_0 v_0) \\
&= \frac{\partial \xi}{\partial t} + u_0\frac{\partial}{\partial x}(\xi - \rho u_0) + v_0\frac{\partial}{\partial y}(\xi - \rho u_0) + u_0\frac{\partial \xi}{\partial x} + u_0\frac{\partial \zeta}{\partial y} + \frac{\partial p}{\partial x} \\
&= (\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y})(\xi - \rho u_0) + \frac{\partial p}{\partial x} = 0 \ , \\
\frac{\partial \zeta}{\partial t} &+ \frac{\partial}{\partial x}(u_0\zeta + v_0\xi - \rho u_0 v_0) + \frac{\partial}{\partial y}\left(2v_0\zeta + \frac{c_0^2 - v_0^2}{c_0^2}p\right) \\
&= \frac{\partial \zeta}{\partial t} + u_0\frac{\partial}{\partial x}(\zeta - \rho v_0) + v_0\frac{\partial}{\partial y}(\zeta - \rho v_0) + v_0\frac{\partial \xi}{\partial x} + v_0\frac{\partial \zeta}{\partial y} + \frac{\partial p}{\partial y} \\
&= (\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y})(\zeta - \rho v_0) + \frac{\partial p}{\partial y} = 0 \ .
\end{aligned}$$

We differentiate the first set of equations by y and the second by x, we eliminate the pressure field and obtain :

$$\frac{1}{\rho_0}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial}{\partial y}(\xi - \rho u_0) - \frac{\partial}{\partial x}(\zeta - \rho v_0)\right) = \frac{\mathrm{d}}{\mathrm{dt}}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) = \frac{\mathrm{d}\omega}{\mathrm{dt}} = 0 \ . \tag{4}$$

□

Hypothesis 2 *Irrotationality of the acoustic vorticity.*
If we consider the acoustic perturbation at the initial time to be **irrotational**, *i.e.* $\mathbf{rot}\,\mathbf{u}\,(x, y, t = 0) \equiv \omega\,(x, y, t = 0) = \mathbf{0}$, we then deduce with equation (4) that $\mathbf{rot}\,\mathbf{u}\,(x, y, t) = \mathbf{0}$ during the time evolution.

3) Lorentz Transform

We consider the two-dimensional equations of advective acoustics when the velocity of the fluid is parallel to a particular direction; we suppose specifically :

$$\mathbf{u} = u_0\ \mathbf{e_x} \ . \tag{5}$$

We search a space-time transform $(x, t) \longmapsto (x', t')$ so that in the new space-time (x', t'), the pressure field is the solution of the wave equation. We find that this space-time transform is a **Lorentz transform**. With it, we derive a new set of equations and

prove that the corresponding system can be reduced to the classical case of non advective acoustics.

3-1 Change of space-time

Considering a flow of velocity given by equation (5), the system (3) is written as :

$$\left\{\begin{array}{l} \dfrac{\partial p}{\partial t} + c_0^2 \dfrac{\partial \xi}{\partial x} + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \dfrac{\partial}{\partial x}\left(2u_0\xi + \dfrac{(c_0^2 - u_0^2)}{c_0^2} p\right) + \dfrac{\partial}{\partial y}(u_0\zeta) = 0 \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial}{\partial x}(u_0\zeta) + \dfrac{\partial p}{\partial y} = 0 \ , \end{array}\right. \tag{6}$$

which is a pleasant conservative form. We easily deduce that the pressure field $p\,(x, y, t)$ is solution in the (initial) space-time (x, y, t) of a wave equation :

$$\frac{\partial^2 p}{\partial t^2} + 2u_0 \frac{\partial^2 p}{\partial x \partial t} + u_0^2 \frac{\partial^2 p}{\partial x^2} - c_0^2 \Delta p = 0 \ , \tag{7}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the laplacian in two dimension space.

Proposition 2 (Lorentz transform and equation of pressure)**.**

We suppose that the advective velocity satisfies equation (5). We define the Mach number as $M_0 = \frac{u_0}{c_0}$ and the Lorentz space-time transform as :

$$\left\{\begin{array}{l} x' = \dfrac{1}{\sqrt{1 - M_0^2}}\, x \\ y' = y \\ t' = t + \dfrac{M_0}{c_0(1 - M_0^2)}\, x \ . \end{array}\right. \tag{8}$$

In this new space-time, the pressure field is considered as a function of the new set of space-time coordinates (x', y', t'), *i.e.* :

$$p'(x', y', t') \equiv p\,(x, y, t) \ , \tag{9}$$

and is the solution of the wave equation with a **modified celerity** :

$$\frac{\partial^2 p'}{\partial t'^2} - c_0^2(1 - M_0^2)\left(\frac{\partial^2 p'}{\partial x'^2} + \frac{\partial^2 p'}{\partial y'^2}\right) = 0 \ . \tag{10}$$

reduced from the "pure" sound celerity by a similarity factor $\sqrt{1 - M_0^2}$.

Proof. We first explain the way we derive the Lorentz transform (8) to remove the advective contribution $2u_0 \frac{\partial^2 p}{\partial x \partial t}$ in equation (7). In the new space-time (x', y', t'), we want the pressure field to be solution of the wave equation. We search the new space-time coordinates (x', y', t') as :

$$\left\{ \begin{array}{l} x' = \alpha x \\ y' = y \\ t' = t + \beta x \ . \end{array} \right. \tag{11}$$

The transformed equation (7) takes the form :

$$\begin{array}{rcl} \dfrac{\partial^2 p}{\partial t^2} & + & 2u_0 \dfrac{\partial^2 p}{\partial x \partial t} + u_0^2 \dfrac{\partial^2 p}{\partial x^2} - c_0^2 \Delta p \\ & = & \left[\left(\dfrac{\partial}{\partial t} + u_0 \dfrac{\partial}{\partial x} \right)^2 - c_0^2 \Delta \right] p\,(x, y, t) \\ & = & \left[\left((1 + u_0 \beta)^2 - c_0^2 \beta^2 \right) \dfrac{\partial^2}{\partial t'^2} + 2\alpha \left(u_0 (1 + u_0 \beta) - \beta c_0^2 \right) \dfrac{\partial^2}{\partial t' \partial x'} \right. \\ & & \left. - \alpha^2 (c_0^2 - u_0^2) \dfrac{\partial^2}{\partial x'^2} - c_0^2 \dfrac{\partial^2}{\partial y'^2} \right] p'(x', y', t') \ . \end{array}$$

Then we obtain :

$$\begin{array}{c} \left[\left((1 + u_0 \beta)^2 - c_0^2 \beta^2 \right) \dfrac{\partial^2}{\partial t'^2} + 2\alpha \left(u_0 (1 + u_0 \beta) - \beta c_0^2 \right) \dfrac{\partial^2}{\partial t' \partial x'} \right. \\ \left. - \alpha^2 (c_0^2 - u_0^2) \dfrac{\partial^2}{\partial x'^2} - c_0^2 \dfrac{\partial^2}{\partial y'^2} \right] p'(x', y', t') = 0 \ . \end{array} \tag{12}$$

The conditions upon α and β to find the wave equation are clear from equation (12); on the first hand no further crossed partial derivation between space and time, that is :

$$2\alpha \left(u_0 (1 + u_0 \beta) - \beta c_0^2 \right) = 0 \ , \tag{13}$$

and on the other hand equality of the coefficients of double derivations in space to have a laplacian operator invariant by rotation :

$$\alpha^2 (c_0^2 - u_0^2) = c_0^2 \ . \tag{14}$$

The unique solution (α, β) of the previous 2×2 linear system (13)-(14) is :

$$\left\{ \begin{array}{l} \alpha = \dfrac{c_0}{\sqrt{c_0^2 - u_0^2}} \\ \beta = \dfrac{u_0}{c_0^2 - u_0^2} \ , \end{array} \right.$$

and with this set of coefficients, the space-time transform (11) is exactly equal to the system (8). Moreover, we remark that the coefficient of $\dfrac{\partial^2}{\partial t^2}$ in equation (12) is now equal to :

$$\left((1+u_0\beta)^2 - c_0^2\beta^2\right) = \frac{c_0^2}{c_0^2 - u_0^2} = \frac{1}{1-M_0^2} \ ,$$

and, in our transformed space-time (x', y', t'), the pressure field defined by the condition (9) is the solution of the wave equation (10) . □

3-2 Change of unknown functions

Let us apply the Lorentz transform (8) to the acoustic system (6). We have the following proposition :

Proposition 3 (New unknown functions for advective acoustic)**.**
We assume that the hypothesis 2 of irrotationality of the acoustic vorticity is satisfied and that the advective velocity field is defined by equation (5). After applying the Lorentz transform (8) and the following change of pressure and impulse functions :

$$\left\{\begin{array}{l} \widetilde{p} = p' + \dfrac{u_0}{(1-M_0^2)}\,\xi' \\ \widetilde{\xi} = \dfrac{1}{\sqrt{1-M_0^2}}\,\xi' \\ \widetilde{\zeta} = \zeta' \ , \end{array}\right. \tag{15}$$

the advective acoustic system (6) can be written as :

$$\left\{\begin{array}{l} \dfrac{\partial \widetilde{p}}{\partial t'} + c_0^2\dfrac{\partial \widetilde{\xi}}{\partial x'} + c_0^2\dfrac{\partial \widetilde{\zeta}}{\partial y'} = 0 \\ \dfrac{\partial \widetilde{\xi}}{\partial t'} + (1-M_0^2)\dfrac{\partial \widetilde{p}}{\partial x'} = 0 \\ \dfrac{\partial \widetilde{\zeta}}{\partial t'} + (1-M_0^2)\dfrac{\partial \widetilde{p}}{\partial y'} = 0 \ . \end{array}\right. \tag{16}$$

Proof. We first use the hypothesis of irrotationality of the acoustic vorticity in the third equation of the system (6) and obtain :

$$\frac{\partial \zeta}{\partial x} = \frac{\partial(\xi - \rho u_0)}{\partial y} = \frac{\partial \xi}{\partial y} - \frac{u_0}{c_0^2}\frac{\partial p}{\partial y} \ .$$

Secondly we introduce the Lorentz transform (8) into the system (6). We have the following transform of partial derivations :

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial x} = \dfrac{1}{\sqrt{1-M_0^2}}\dfrac{\partial}{\partial x'} + \dfrac{u_0}{c_0^2(1-M_0^2)}\dfrac{\partial}{\partial t'} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial y'} \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t'} \ . \end{array}\right.$$

We then substract the first equation of the system (6) multiplied by $\frac{u_0}{c_0^2}$ from the second one and, using the following notations :

$$\left\{ \begin{array}{l} p'(x',y',t') \equiv p\,(x,y,t) \\ \xi'(x',y',t') \equiv \xi(x,y,t) \\ \zeta'(x',y',t') \equiv \zeta(x,y,t) \ , \end{array} \right. \tag{17}$$

we find :

$$\left\{ \begin{array}{l} \dfrac{\partial p'}{\partial t'} + \dfrac{c_0^2}{\sqrt{1-M_0^2}} \dfrac{\partial \xi'}{\partial x'} + \dfrac{M_0 c_0}{(1-M_0^2)} \dfrac{\partial \xi'}{\partial t'} + c_0^2 \dfrac{\partial \zeta'}{\partial y'} = 0 \\ \dfrac{\partial \xi'}{\partial t'} + \dfrac{u_0}{\sqrt{1-M_0^2}} \dfrac{\partial \xi'}{\partial x'} + \dfrac{M_0^2}{1-M_0^2} \dfrac{\partial \xi'}{\partial t'} + \dfrac{c_0^2 - u_0^2}{c_0^2 \sqrt{1-M_0^2}} \dfrac{\partial p'}{\partial x'} = 0 \\ \dfrac{\partial \zeta'}{\partial t'} + u_0 \dfrac{\partial \xi'}{\partial y'} + \dfrac{(c_0^2 - u_0^2)}{c_0^2} \dfrac{\partial p'}{\partial y'} = 0 \ . \end{array} \right.$$

We gather the terms associated with the same operator of derivation :

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t'} \left[p' + \dfrac{M_0 c_0}{(1-M_0^2)} \xi' \right] + c_0^2 \dfrac{\partial}{\partial x'} (\dfrac{\xi'}{\sqrt{1-M_0^2}}) + c_0^2 \dfrac{\partial \zeta'}{\partial y'} = 0 \\ \dfrac{\partial}{\partial t'} (\dfrac{\xi'}{\sqrt{1-M_0^2}}) + (1-M_0^2) \dfrac{\partial}{\partial x'} \left[p' + \dfrac{M_0 c_0}{(1-M_0^2)} \xi' \right] = 0 \\ \dfrac{\partial \zeta'}{\partial t'} + (1-M_0^2) \dfrac{\partial}{\partial y'} \left[p' + \dfrac{M_0 c_0}{(1-M_0^2)} \xi' \right] = 0 \ , \end{array} \right.$$

and we substitute into the previous system the new unknown functions $(\widetilde{p}\,, \widetilde{\xi}\,, \widetilde{\zeta})$ introduced in the system (15). Then the system of equations (16) is satisfied. □

Remark 4.
The major consequence of propositions 2 and 3 is the following (operational!) remark. The resolution of the advective acoustic system is absolutly **identical** to the one obtained **without** advective flow, but with a propagation celerity **scaled** by a factor $\sqrt{1-M_0^2}$.

4) Lorentz transform for multi-dimensional flows

In the previous section, we dealt with the case of a velocity field described by the equation (5). In [AGH99], Abarbanel *et* al consider a multi-dimensional flow as a one-dimensional flow, after a correct rotation of the studied medium by an angle $\theta = \tan^{-1}(\frac{v_0}{u_0})$ and considering the new velocity to be $u_{new} = \sqrt{u_0^2 + v_0^2}$. We observe that in order to study numerically the influence of the flow for acoustic propagation near objects, such an idea imposes a remeshing of the geometry for each change of the advective flow. In our opinion, this process is not compatible with the use of finite differences and with operational industrial constraints. We propose in this section to generalize the Lorentz space-time transform to a multi-dimensional flow and to extend our approach with the help of **space**

affinities to a multi-dimensional flow under the same hypotheses as before. In the next two paragraphs, we present the generalization of the Lorentz transform respectively to the two and three-dimensional cases. Only the two-dimensional case is proven in the present document. The proof of the three-dimensional case can be found in [Ma2k].

4-1 The two-dimensional case

We now consider a subsonic uniform flow described by a velocity vector :

$$\mathbf{u} = (u_0, v_0) \ . \tag{18}$$

With such an external flow, the linearized isentropic Euler equations for advective acoustics are :

$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + c_0^2 \dfrac{\partial \xi}{\partial x} + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \dfrac{\partial}{\partial x}\left(2u_0\xi + \dfrac{(c_0^2 - u_0^2)}{c_0^2} p\right) + \dfrac{\partial}{\partial y}(u_0\zeta + v_0\xi - \dfrac{u_0 v_0}{c_0^2} p) = 0 \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial}{\partial x}(u_0\zeta + v_0\xi - \dfrac{u_0 v_0}{c_0^2} p) + \dfrac{\partial}{\partial y}\left(2v_0\zeta + \dfrac{(c_0^2 - v_0^2)}{c_0^2} p\right) = 0 \ . \end{array} \right. \tag{19}$$

Proposition 4 (Simplification of the acoustic system)**.**
We generalize the Lorentz space-time transform when the external flow verifies (18). We introduce a new set of space-time coordinates as :

$$\left\{ \begin{array}{l} x' = \dfrac{1}{\sqrt{1 - \frac{u_0^2}{c_0^2}}} \, x \\ y' = \dfrac{1}{\sqrt{1 - \frac{v_0^2}{c_0^2}}} \, y \\ t' = t + \dfrac{u_0}{c_0^2(1 - M_0^2)} \, x + \dfrac{v_0}{c_0^2(1 - M_0^2)} \, y \ , \end{array} \right. \tag{20}$$

the Mach number as $M_0 = \frac{\sqrt{u_0^2 + v_0^2}}{c_0}$, a coupling coefficient α between the two cartesian coordinates as :

$$\alpha = \frac{u_0 v_0}{c_0^2 \sqrt{1 - \frac{u_0^2}{c_0^2}} \sqrt{1 - \frac{v_0^2}{c_0^2}}} \ , \tag{21}$$

and the new unknown functions $\widetilde{p}$, $\widetilde{\xi}$ et $\widetilde{\zeta}$ defined by :

$$\left\{ \begin{array}{l} \widetilde{p} = p' + \dfrac{1}{1 - M_0^2} \, (u_0 \xi' + v_0 \zeta') \\ \widetilde{\xi} = \sqrt{1 - \dfrac{u_0^2}{c_0^2}} \left((1 - \dfrac{v_0^2}{c_0^2}) \dfrac{\xi'}{1 - M_0^2} + \dfrac{u_0 v_0}{c_0^2} \dfrac{\zeta'}{1 - M_0^2} \right) \\ \widetilde{\zeta} = \sqrt{1 - \dfrac{v_0^2}{c_0^2}} \left(\dfrac{u_0 v_0}{c_0^2} \dfrac{\xi'}{1 - M_0^2} + (1 - \dfrac{u_0^2}{c_0^2}) \dfrac{\zeta'}{1 - M_0^2} \right) \ . \end{array} \right. \tag{22}$$

Under the hypothesis 2 of irrotationality of the acoustic vorticity, the new form of the acoustic system (19) is the following :

$$\left\{ \begin{array}{l} \dfrac{\partial \widetilde{p}}{\partial t'} + c_0^2 \dfrac{\partial}{\partial x'}(\widetilde{\xi} - \alpha \widetilde{\zeta}) + c_0^2 \dfrac{\partial}{\partial y'}(\widetilde{\zeta} - \alpha \widetilde{\xi}) = 0 \\ \dfrac{\partial \widetilde{\xi}}{\partial t'} + (1 - M_0^2) \dfrac{\partial \widetilde{p}}{\partial x'} = 0 \\ \dfrac{\partial \widetilde{\zeta}}{\partial t'} + (1 - M_0^2) \dfrac{\partial \widetilde{p}}{\partial y'} = 0 \ . \end{array} \right. \tag{23}$$

Proof. First, by substracting with a correct coefficient the first equation of the system (19) from the two others, we find :

$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + c_0^2 \dfrac{\partial \xi}{\partial x} + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial}{\partial t}(\xi - \dfrac{u_0}{c_0^2} p) + \dfrac{\partial}{\partial x} \left(u_0 \xi + \dfrac{(c_0^2 - u_0^2)}{c_0^2} p \right) + \dfrac{\partial}{\partial y}(v_0 \xi - \dfrac{u_0 v_0}{c_0^2} p) = 0 \\ \dfrac{\partial}{\partial t}(\zeta - \dfrac{v_0}{c_0^2} p) + \dfrac{\partial}{\partial x}(u_0 \zeta - \dfrac{u_0 v_0}{c_0^2} p) + \dfrac{\partial}{\partial y} \left(v_0 \zeta + \dfrac{(c_0^2 - v_0^2)}{c_0^2} p \right) = 0 \ . \end{array} \right.$$

Using the hypothesis of irrotationality of the acoustic vorticity, we have :

$$\begin{array}{rcl} \dfrac{\partial}{\partial y}(v_0 \xi - \dfrac{u_0 v_0}{c_0^2} p) & = & \dfrac{\partial}{\partial y} \left[v_0(\rho_0 u + \rho u_0) - \dfrac{u_0 v_0}{c_0^2} p \right] = \dfrac{\partial}{\partial y}(\rho_0 v_0 u) \\ & = & \dfrac{\partial}{\partial x}(\rho_0 v_0 v) = \dfrac{\partial}{\partial x} v_0 (\zeta - \rho v_0) \\ & = & \dfrac{\partial}{\partial x}(v_0 \zeta - \dfrac{v_0^2}{c_0^2} p) \ . \end{array}$$

The previous calculation gives the new system :

$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + c_0^2 \dfrac{\partial \xi}{\partial x} + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial}{\partial t}(\xi - \dfrac{u_0}{c_0^2} p) + \dfrac{\partial}{\partial x} \left[u_0 \xi + v_0 \zeta + (1 - M_0^2) p \right] = 0 \\ \dfrac{\partial}{\partial t}(\zeta - \dfrac{v_0}{c_0^2} p) + \dfrac{\partial}{\partial y} \left[u_0 \xi + v_0 \zeta + (1 - M_0^2) p \right] = 0 \ . \end{array} \right.$$

In the new space-time defined by the change of space-time (20), we have :

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial x} = \dfrac{1}{\sqrt{1 - \frac{u_0^2}{c_0^2}}} \dfrac{\partial}{\partial x'} + \dfrac{u_0}{c_0^2 (1 - M_0^2)} \dfrac{\partial}{\partial t'} \\ \dfrac{\partial}{\partial y} = \dfrac{1}{\sqrt{1 - \frac{v_0^2}{c_0^2}}} \dfrac{\partial}{\partial y'} + \dfrac{v_0}{c_0^2 (1 - M_0^2)} \dfrac{\partial}{\partial t'} \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t'} \ . \end{array} \right.$$

With the notation introduced in the system (17), the transformed equations take the algrebraic form :

$$\begin{cases} \dfrac{\partial}{\partial t'}\left(p' + \dfrac{u_0\xi' + v_0\zeta'}{1-M_0^2}\right) + c_0^2\dfrac{\partial}{\partial x'}\left(\dfrac{\xi'}{\sqrt{1-\frac{u_0^2}{c_0^2}}}\right) + c_0^2\dfrac{\partial}{\partial y'}\left(\dfrac{\zeta'}{\sqrt{1-\frac{v_0^2}{c_0^2}}}\right) = 0 \\ \dfrac{\partial}{\partial t'}\left[\dfrac{\sqrt{1-\frac{u_0^2}{c_0^2}}}{1-M_0^2}\left((1-\dfrac{v_0^2}{c_0^2})\xi' + \dfrac{u_0v_0}{c_0^2}\zeta'\right)\right] + (1-M_0^2)\dfrac{\partial}{\partial x'}\left(p' + \dfrac{u_0\xi' + v_0\zeta'}{1-M_0^2}\right) = 0 \\ \dfrac{\partial}{\partial t'}\left[\dfrac{\sqrt{1-\frac{v_0^2}{c_0^2}}}{1-M_0^2}\left(\dfrac{u_0v_0}{c_0^2}\xi' + (1-\dfrac{u_0^2}{c_0^2})\zeta'\right)\right] + (1-M_0^2)\dfrac{\partial}{\partial y'}\left(p' + \dfrac{u_0\xi' + v_0\zeta'}{1-M_0^2}\right) = 0 \ . \end{cases}$$

By the change of unknown functions (22), the previous system becomes :

$$\begin{cases} \dfrac{\partial \widetilde{p}}{\partial t'} + c_0^2\dfrac{\partial}{\partial x'}\left(\dfrac{\xi'}{\sqrt{1-\frac{u_0^2}{c_0^2}}}\right) + c_0^2\dfrac{\partial}{\partial y'}\left(\dfrac{\zeta'}{\sqrt{1-\frac{v_0^2}{c_0^2}}}\right) = 0 \\ \dfrac{\partial \widetilde{\xi}}{\partial t'} + (1-M_0^2)\dfrac{\partial \widetilde{p}}{\partial x'} = 0 \\ \dfrac{\partial \widetilde{\zeta}}{\partial t'} + (1-M_0^2)\dfrac{\partial \widetilde{p}}{\partial y'} = 0 \ . \end{cases} \tag{24}$$

We focus here on the fact that the pair (ξ', ζ') is present in the first equation of (24) whereas the new unknown functions are $\widetilde{\xi}$ and $\widetilde{\zeta}$. Nevertheless with the last two equations of the system (22), we have the following calculation :

$$\begin{cases} \xi' = \sqrt{1-\dfrac{u_0^2}{c_0^2}}\ (\widetilde{\xi} - \alpha\widetilde{\zeta}) \\ \zeta' = \sqrt{1-\dfrac{v_0^2}{c_0^2}}\ (\widetilde{\zeta} - \alpha\widetilde{\xi}) \ , \end{cases}$$

where α is defined by equation (21). We then find the final form (23) of the system of advective acoustics in the new space-time (x', y', t'). □

4-2 The three-dimensional case

The generalization to **three dimension space** can be done without any major difficulty. We have the following proposition proven in [Ma2k] :

Proposition 5.

We assume that the velocity of the external flow is given by :

$$\mathbf{u_0} = (u_0, v_0, w_0) \ , \tag{25}$$

and that the Hypothesis 2 of irrotationality of the acoustic vorticity is verified. We define the Mach number as $M_0 = \frac{\sqrt{u_0^2+v_0^2+w_0^2}}{c_0}$ and a change of space-time as :

$$
\left\{
\begin{array}{l}
x' = \dfrac{1}{\sqrt{1-\frac{u_0^2}{c_0^2}}}\, x \\
y' = \dfrac{1}{\sqrt{1-\frac{v_0^2}{c_0^2}}}\, y \\
z' = \dfrac{1}{\sqrt{1-\frac{w_0^2}{c_0^2}}}\, z \\
t' = t + \dfrac{u_0}{c_0^2(1-M_0^2)}\, x + \dfrac{v_0}{c_0^2(1-M_0^2)}\, y + \dfrac{w_0}{c_0^2(1-M_0^2)}\, z \ .
\end{array}
\right.
\tag{26}
$$

We introduce **three** coupling coefficients as :

$$
\left\{
\begin{array}{lcl}
\alpha & = & \dfrac{u_0 v_0}{c_0^2\sqrt{1-\frac{u_0^2}{c_0^2}}\sqrt{1-\frac{v_0^2}{c_0^2}}} \\
\beta & = & \dfrac{u_0 w_0}{c_0^2\sqrt{1-\frac{u_0^2}{c_0^2}}\sqrt{1-\frac{w_0^2}{c_0^2}}} \\
\gamma & = & \dfrac{v_0 w_0}{c_0^2\sqrt{1-\frac{v_0^2}{c_0^2}}\sqrt{1-\frac{w_0^2}{c_0^2}}} \ ,
\end{array}
\right.
$$

and a change of unknown functions as :

$$
\left\{
\begin{array}{l}
\widetilde{p} = p' + \dfrac{1}{1-M_0^2}\,(u_0\xi' + v_0\zeta' + w_0\chi') \\
\widetilde{\xi} = \sqrt{1-\dfrac{u_0^2}{c_0^2}}\left((1-\dfrac{v_0^2+w_0^2}{c_0^2})\dfrac{\xi'}{1-M_0^2} + \dfrac{u_0v_0}{c_0^2}\dfrac{\zeta'}{1-M_0^2} + \dfrac{u_0w_0}{c_0^2}\dfrac{\chi'}{1-M_0^2}\right) \\
\widetilde{\zeta} = \sqrt{1-\dfrac{v_0^2}{c_0^2}}\left(\dfrac{u_0v_0}{c_0^2}\dfrac{\xi'}{1-M_0^2} + (1-\dfrac{u_0^2+w_0^2}{c_0^2})\dfrac{\zeta'}{1-M_0^2} + \dfrac{v_0w_0}{c_0^2}\dfrac{\chi'}{1-M_0^2}\right) \\
\widetilde{\chi} = \sqrt{1-\dfrac{w_0^2}{c_0^2}}\left(\dfrac{u_0w_0}{c_0^2}\dfrac{\xi'}{1-M_0^2} + \dfrac{v_0w_0}{c_0^2}\dfrac{\zeta'}{1-M_0^2} + (1-\dfrac{u_0^2+v_0^2}{c_0^2})\dfrac{\chi'}{1-M_0^2}\right) \ .
\end{array}
\right.
$$

The acoustic system takes the new form :

$$\begin{cases} \dfrac{\partial \widetilde{p}}{\partial t'} + c_0^2 \dfrac{\partial}{\partial x'}(\widetilde{\xi} - \alpha\widetilde{\zeta} - \beta\widetilde{\chi}) \\ \qquad\quad + c_0^2 \dfrac{\partial}{\partial y'}(\widetilde{\zeta} - \alpha\widetilde{\xi} - \gamma\widetilde{\chi}) + c_0^2 \dfrac{\partial}{\partial z'}(\widetilde{\chi} - \beta\widetilde{\xi} - \gamma\widetilde{\zeta}) = 0 \\ \dfrac{\partial \widetilde{\xi}}{\partial t'} + (1 - M_0^2)\dfrac{\partial \widetilde{p}}{\partial x'} = 0 \\ \dfrac{\partial \widetilde{\zeta}}{\partial t'} + (1 - M_0^2)\dfrac{\partial \widetilde{p}}{\partial y'} = 0 \\ \dfrac{\partial \widetilde{\chi}}{\partial t'} + (1 - M_0^2)\dfrac{\partial \widetilde{p}}{\partial z'} = 0 \ . \end{cases}$$

5) Acoustic Absorbing Layers

Physical wave phenomena modelling takes often place in the infinite two or three dimensional space. Due to finite computing resources, the numerical simulations of such phenomena must be truncated to confined domains, then numerical artifical boundaries must be considered. Generally, numerical reflections of outgoing waves from the boundaries of the numerical domain reenter the computational domain and falsify the results. Various methods have been proposed to reduce the influence of reentering waves in the computational domain.

For many years, the numerical physicists, Israeli-Orsag [IO81], have developed the idea of layers of absorbing materials. Then the mathematical study of non reflecting boundary conditions has been developed after the pionnering work of Engquist-Majda [EM77]. The discrete studies of such absorbing conditions have been realized for scalar waves by Bayliss-Turkel [BT80], for electromagnetic waves by Joly-Mercier [JM89], Taflove [Ta98] and for sismic waves by Halpern-Trefethen [HT86] among others.

The current perfectly matched layers approach has been introduced by Bérenger [Be94] in the context of computational electromagnetics; a mathematical interpretation of this model has been made by Collino [Co85]. In [Hu96], Hu proposes an adaptation of Berenger's model for advective acoustics. Nevertheless, Abarbanel *et* al [AGH99] and Rahmouni [Rah01] and [Rah99] have demonstrated that this model is mathematically ill-posed, *i.e.* that, if truncated to the first order terms, there exists a perturbation as small as we wish that can make the model unstable. These authors also propose well-posed models.

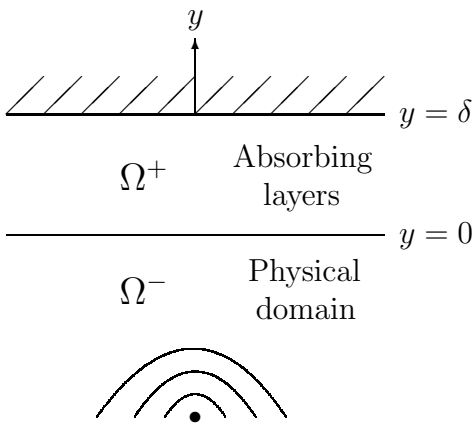
Our approach uses a model of the type introduced by Hu. We focus in our study on the practical disadvantages to deal with a mathematical model whose principal symbol corresponds to a ill-posed problem.

5-1 Acoustic absorbing layers without external flow

We propose in the following a precise description of acoustic absorbing layers, and we follow the ideas developed by Collino [Co85]. We consider a semi-infinite medium in the x-direction defined by $\Omega = \Omega^+ \cup \Omega^-$, where :

$$\left\{ \begin{array}{l} \Omega^- = \{(x,y),\ y < 0\} \\ \Omega^+ = \{(x,y),\ 0 \leq y \leq \delta\} \ , \end{array} \right. \tag{27}$$

Ω^+ representing the absorbing layers domain.



Proposition 6 (System of acoustic absorbing layers)**.**

A system of partial differential equations that models absorbing layers of acoustic waves in the domain Ω^+ introduced in (27) can be given as :

$$\left\{ \begin{array}{lll} \dfrac{\partial p_x}{\partial t} & + & c_0^2 \dfrac{\partial \xi}{\partial x} = 0 \\ \dfrac{\partial p_y}{\partial t} & + & \sigma^*(\eta)\, p_y + c_0^2 \dfrac{\partial \zeta}{\partial \eta} = 0 \\ \dfrac{\partial \xi}{\partial t} & + & \dfrac{\partial}{\partial x}(p_x + p_y) = 0 \\ \dfrac{\partial \zeta}{\partial t} & + & \sigma^*(\eta)\, \zeta + \dfrac{\partial}{\partial \eta}(p_x + p_y) = 0 \ , \end{array} \right. \tag{28}$$

where the absorbing coefficient $\sigma^*(y)$ satisfies :

$$[0, \delta] \ni y \longmapsto \sigma^*(y) \in \mathbf{R}_+ \ , \ \sigma^*(y) > 0 \ \text{ if } \ y > 0 \ \text{ and } \ \sigma^*(0) = 0 \ . \tag{29}$$

The pressure p is defined by $p \equiv p_x + p_y$.

Proof. We follow essentially the idea of Collino [Co85]. The main idea to establish the acoustic system inside the absorbing layers is to introduce the Fourier-Laplace transform and to write the system (19) in the complex plan.

We define the Fourier-Laplace transform by :

$$\hat{v}(k_x, y, \omega) = \int\int v(x, y, t)\, e^{-i(\omega t + k_x x)}\, \mathrm{d}x\, \mathrm{d}t \ .$$

Then in the domain $\Omega^- = \{(x, y),\ y < 0\}$, the system (19) takes the form :

$$\text{(30)} \qquad \begin{cases} i\omega\hat{p} = -ik_x c_0^2 \hat{\xi} - c_0^2 \dfrac{\partial \hat{\zeta}}{\partial y} \\ i\omega\hat{\xi} = -ik_x \hat{p} \\ i\omega\hat{\zeta} = -\dfrac{\partial \hat{p}}{\partial y} \ , \end{cases}$$

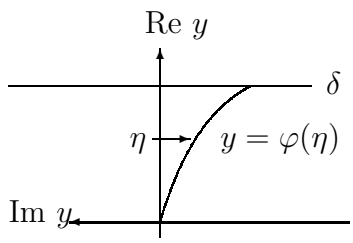
and the solution of the system (30) for a propagation in the growing y-direction is obtained after the integration of an ordinary differential equation of degree 2 :

$$\text{(31)} \qquad \begin{cases} \hat{p} = p_0 e^{-ik_y y} \\ \hat{\xi} = \dfrac{p_0}{\omega} k_x e^{-ik_y y} \\ \hat{\zeta} = -\dfrac{p_0}{\omega} k_y e^{-ik_y y} \end{cases} \qquad \text{with} \quad k_y^2 = \frac{\omega^2}{c_0^2} - k_x^2 \ .$$

We establish a modified form of the system (30) in Ω^+ that ensures that waves leaving the domain are not reflected back. Let us extend the variable y in the complex plan by adding an imaginary part depending on the function σ^* defined by equation (29), and that equals zero for $\sigma^* \equiv 0$. We write precisely in Ω^+ the complex variable y parameterized by a real variable η, $0 \leq \eta \leq \delta$, as :

$$y = \varphi(\eta) = \eta + \frac{1}{i\omega} \int_0^\eta \sigma^*(u)\, \mathrm{d}u \ .$$

We can draw the function $y = \varphi(\eta)$ in the complex plan as :



For v equal to one of the variables of pressure or momentum, we introduce the function $\hat{v}(y)$, with $y = \varphi(\eta)$, as a function $\hat{v}^*$ of the real variable η :

$$\hat{v}^*(\eta) \equiv \hat{v}(\varphi(\eta)) \ , \ v \in \{p\,, \xi\,, \zeta\} \ .$$

An elementary calculation gives us :

$$\begin{aligned}
\hat{v}^*(\eta) &= \hat{v}\left(\eta + \frac{1}{i\omega}\int_0^\eta \sigma^*(u)\,\mathrm{d}u\right) \\
&= v_0\ \exp\left(-ik_y\left[\eta + \frac{1}{i\omega}\int_0^\eta \sigma^*(u)\,\mathrm{d}u\right]\right) \\
&= \hat{v}\,(\eta)\ \exp\left(-\frac{k_y}{\omega}\int_0^\eta \sigma^*(u)\,\mathrm{d}u\right) .
\end{aligned}$$

We then deduce the following important property :

$$|\hat{v}^*(\eta)| < |\hat{v}\,(\eta)| \quad , \quad 0 < \eta \leq \delta \quad , \quad v \in \{p\,,\xi\,,\zeta\} \ . \tag{32}$$

The property (32) is a consequence of an exponential decay of all the variables inside the absorbing layers. It is possible to derive the system of partial differential equations satisfied by those fields. A first algebraic calculation gives us :

$$\frac{\partial \hat{v}}{\partial y} = \frac{\partial \hat{v}^*}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}y} = \frac{i\omega}{i\omega + \sigma^*(\eta)}\frac{\partial \hat{v}^*}{\partial \eta} \quad , \quad v \in \{p\,,\xi\,,\zeta\} \ ,$$

then we obtain with this new set of unknown functions a system in the (x, η) domain issued from (30) :

$$\left\{\begin{array}{l}
i\omega\hat{p}^* = -ik_x c_0^2\hat{\xi}^* - c_0^2\dfrac{i\omega}{i\omega + \sigma^*(\eta)}\dfrac{\partial\hat{\zeta}^*}{\partial\eta} \\
i\omega\hat{\xi}^* = -ik_x\hat{p}^* \\
i\omega\hat{\zeta}^* = -\dfrac{i\omega}{i\omega + \sigma^*(\eta)}\dfrac{\partial\hat{p}^*}{\partial\eta} \ .
\end{array}\right. \tag{33}$$

The first equation can be rewritten while splitting the pressure field into two sub-pressure fields as :

$$\hat{p}^* = \hat{p}_x^* + \hat{p}_y^* \ ,$$

with $\hat{p}_x^*$ and $\hat{p}_y^*$ solutions of :

$$\left\{\begin{array}{l}
i\omega\hat{p}_x^* = -ik_x c_0^2\hat{\xi}^* \\
i\omega\hat{p}_y^* = \dfrac{i\omega}{i\omega + \sigma^*(\eta)}c_0^2\dfrac{\partial\hat{\zeta}^*}{\partial\eta} \ .
\end{array}\right.$$

Taking the inverse Fourier-Laplace transform of the new system, we obtain (28). □

Considering a square domain $[\,0\,,\,L\,]^2$, we define the thickness of the absorbing layers by δ_x in the x-direction and δ_y in the y-direction. The interesting studying medium is then $[\,\delta_x, L - \delta_x\,] \times [\,\delta_y, L - \delta_y\,]$. We have the following proposition that generalizes the proposition 6 :

Proposition 7 (General acoustic system to solve)**.**
We consider two smoothing functions σ_x^* and σ_y^* defined by :

$$[0,\, L] \ni x \longmapsto \sigma_x^*(x) \in \mathbf{R}_+ \quad \begin{array}{l} \sigma_x^*(x) > 0 \,\text{if}\, x \in [\,0\,, \delta_x\,[\,\times\,]\,L - \delta_x,\, L\,] \\ \sigma_x^*(x) = 0 \;\text{if}\, x \in [\delta_x, L - \delta_x\,], \end{array} \tag{34}$$

$$[0,\, L] \ni\ni y \longmapsto \sigma_y^*(y) \in \mathbf{R}_+ \quad \begin{array}{l} \sigma_y^*(y) > 0 \,\text{if}\, y \in [\,0\,, \delta_y\,[\,\times\,]\,L - \delta_y,\, L\,] \\ \sigma_y^*(y) = 0 \,\text{if}\, y \in [\delta_y, L - \delta_y\,]. \end{array} \tag{35}$$

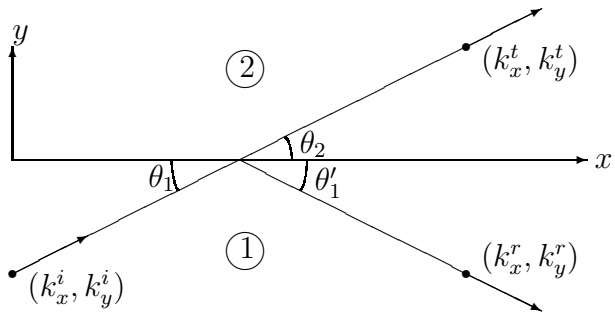
The acoustic system in the studying medium and in the absorbing layers can be written as :

$$\left\{ \begin{array}{lllllll} \dfrac{\partial p_x}{\partial t} & + & \sigma_x^*(x)\, p_x & + & c_0^2 \dfrac{\partial \xi}{\partial x} & = & 0 \\ \dfrac{\partial p_y}{\partial t} & + & \sigma_y^*(y)\, p_y & + & c_0^2 \dfrac{\partial \zeta}{\partial y} & = & 0 \\ \dfrac{\partial \xi}{\partial t} & + & \sigma_x^*(x)\, \xi & + & \dfrac{\partial}{\partial x}(p_x + p_y) & = & 0 \\ \dfrac{\partial \zeta}{\partial t} & + & \sigma_y^*(y)\, \zeta & + & \dfrac{\partial}{\partial y}(p_x + p_y) & = & 0\ . \end{array} \right. \tag{36}$$

Remark 9.
This set of equations is the same as obtained by Hu in [Hu96] using velocity fields rather than impulses.

Proof. The proof is similar to the one of proposition 6 □

5-2 Plane wave analysis



The acoustic system, inside the absorbing layers in the y-direction is, after a Fourier transform :

$$\left\{ \begin{array}{l} i\omega p_x - ik_x c_0^2 \xi = 0 \\ i\omega p_y + \sigma^*(y) p_y - ik_y c_0^2 \zeta = 0 \\ i\omega \xi - ik_x p = 0 \\ i\omega \zeta + \sigma^*(y)\zeta - ik_y p = 0\ . \end{array} \right.$$

The incident wave is a solution of the system :

$$\left\{ \begin{array}{l} i\omega p - ik_x^i c_0^2 \xi - \dfrac{i\omega}{i\omega + \sigma_1^*} ik_y^i c_0^2 \zeta = 0 \\ i\omega \xi - ik_x^i p = 0 \\ i\omega \zeta - \dfrac{i\omega}{i\omega + \sigma_1^*} ik_y^i p = 0 \ . \end{array} \right. \tag{37}$$

The reflected wave is a solution of the system :

$$\left\{ \begin{array}{l} i\omega p_r - ik_x^r c_0^2 \xi_r - \dfrac{i\omega}{i\omega + \sigma_1^*} ik_y^r c_0^2 \zeta_r = 0 \\ i\omega \xi_r - ik_x^r p_r = 0 \\ i\omega \zeta_r - \dfrac{i\omega}{i\omega + \sigma_1^*} ik_y^r p_r = 0 \ . \end{array} \right. \tag{38}$$

The transmitted wave is a solution of the system :

$$\left\{ \begin{array}{l} i\omega p_t - ik_x^t c_0^2 \xi_t - \dfrac{i\omega}{i\omega + \sigma_2^*} ik_y^t c_0^2 \zeta_t = 0 \\ i\omega \xi_t - ik_x^r p_t = 0 \\ i\omega \zeta_t - \dfrac{i\omega}{i\omega + \sigma_2^*} ik_y^r p_t = 0 \ . \end{array} \right. \tag{39}$$

At $y = 0$, we write the continuity of the pressure field. We have : $p + p_r = p_t$ then $p_r = R\,p$ and $p_t = Tp$ with $1 + R = T$. We then have :

$$\left\{ \begin{array}{l} \xi_r = R\,\xi \ \text{ and } \ \xi_t = T\xi \\ \zeta_r = R\,\zeta \ \text{ and } \ \zeta_t = T\zeta \end{array} \right.$$

We deduce from the system (37) :

$$\xi = \frac{k_x^i}{\omega} p \quad \text{and} \quad \zeta = \frac{ik_y^i}{i\omega + \sigma_1^*} p \ ,$$

and we know that $k_x^r = k_x^i$ and $k_y^r = -k_y^i$. We then have :

$$\left\{ \begin{array}{l} \xi + \xi_r = \dfrac{k_x^i}{\omega} p + \dfrac{k_x^i}{\omega} p_r = \dfrac{k_x^i}{\omega}(1 + R)p \\ \zeta + \zeta_r = \dfrac{ik_y^i}{i\omega + \sigma_1^*}(1 - R)p \end{array} \right.$$

The system (39) gives us :

$$\left\{ \begin{array}{l} \xi_t = \dfrac{k_x^t}{\omega}(1 + R)p = \dfrac{k_x^t}{\omega} Tp \\ \xi_t = \dfrac{ik_y^t}{i\omega + \sigma_2^*}(1 + R)p = \dfrac{ik_y^t}{i\omega + \sigma_2^*} Tp \end{array} \right.$$

At the interface $y = 0$, we write the continuity of ζ. We then have $\zeta_t = \zeta + \zeta_r$, that we write as :

$$\frac{ik_y^t}{i\omega + \sigma_2^*(0)}(1+R)p = \frac{ik_y^i}{i\omega + \sigma_1^*(0)}(1-R)p \tag{40}$$

5-3 Mathematical property of the absorbing layers system

The system of acoustic absorbing layers (36) can be written as :

$$\frac{\partial W}{\partial t} + A\frac{\partial W}{\partial x} + B\frac{\partial W}{\partial y} + C\,W = 0\ , \tag{41}$$

where $W = (p_x,\, p_y,\, \xi,\, \zeta)^t$,

$$A = \begin{pmatrix} 0 & 0 & c_0^2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_0^2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} \sigma_x^*(x) & 0 & 0 & 0 \\ 0 & \sigma_y^*(y) & 0 & 0 \\ 0 & 0 & \sigma_x^*(x) & 0 \\ 0 & 0 & 0 & \sigma_y^*(y) \end{pmatrix}.$$

The principal symbol, $M = ik_xA + ik_yB$, of the system (41) is given by :

$$M = \begin{pmatrix} 0 & 0 & ik_xc_0^2 & 0 \\ 0 & 0 & 0 & ik_yc_0^2 \\ ik_x & ik_x & 0 & 0 \\ ik_y & ik_y & 0 & 0 \end{pmatrix}. \tag{42}$$

The eigenvalues and eigenvectors of the principal symbol are re-evaluated in table 1.

Eigenvalue	Eigenvector
0 (double)	$(-1, 1, 0, 0)^t$
$c_0\sqrt{-k_x^2 - k_y^2}$	$(-\frac{ik_xc_0\sqrt{-k_x^2-k_y^2}}{k_x^2+k_y^2}, -\frac{ik_y^2c_0\sqrt{-k_x^2-k_y^2}}{kx(k_x^2+k_y^2)}, 1, \frac{k_y}{k_x})^t$
$-c_0\sqrt{-k_x^2 - k_y^2}$	$(\frac{ik_xc_0\sqrt{-k_x^2-k_y^2}}{k_x^2+k_y^2}, \frac{ik_y^2c_0\sqrt{-k_x^2-k_y^2}}{k_x(k_x^2+k_y^2)}, 1, \frac{k_y}{k_x})^t$

Table 1. Eigenvalues and eigenvectors of matrix (42)

We notice that 0 is an eigenvalue of multiplicity order equal to 2 associated with a one-dimensional eigensubspace. The system (41) is not hyperbolic and classical results relative to well-posedness of such systems (see [Rau91]) can not be applied to the absorbing layers. There exists an arbitrarily small perturbation of the Cauchy problem for the system (36) with $\sigma_x^*(x) = \sigma_y^*(y) = 0$ that makes the system (41) ill-posed for L^2 or Sobolev norms of order 1. Nevertheless, our choice of the system (41) does not produce unstable numerical results.

Proposition 8.
If we look for a solution of the form $W = \varphi(t)\, e^{-ik_x x} e^{-ik_y y}\, V_{M^2}$ of the system (41), where $V_{M^2} = (0, 0, k_y, -k_x)^t$, the scalar function $\varphi(t)$ is an exponential decay in time.

Proof. To establish this result, we determine the characteristic subspace for the eigenvalue $\lambda = 0$, *i.e.* we calculate $ker(M^2)$. We have :

$$(43) \quad -M^2 = \begin{pmatrix} {k_x}^2 & {k_x}^2 & 0 & 0 \\ {k_y}^2 & {k_y}^2 & 0 & 0 \\ 0 & 0 & {k_x}^2 & k_x k_y \\ 0 & 0 & k_x k_y & {k_y}^2 \end{pmatrix} \text{ and } \ker(M^2) = \left[\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ k_y \\ -k_x \end{pmatrix} \right] .$$

We analyse the stability of the system (41) under a perturbation following the direction of the eigenvector of M^2 that is not eigenvector of M. We note $V_{M^2} = (0, 0, k_y, -k_x)^t$ this vector that is a simple impulse pertubation in the direction orthogonal to the wave vector. We choose a state vector W of the form :

$$W = \varphi(t)\, e^{-ik_x x} e^{-ik_y y}\, V_{M^2} \ .$$

In this case, the system (36) is written as :

$$(44) \qquad \frac{\partial \varphi}{\partial t} V_{M^2} + \varphi M\, V_{M^2} + \varphi C\, V_{M^2} = 0 \ ,$$

with :

$$M V_{M^2} = \begin{pmatrix} i k_x k_y \\ -i k_x k_y \\ 0 \\ 0 \end{pmatrix} \ , \ C V_{M^2} = \begin{pmatrix} 0 \\ 0 \\ \sigma_x^*(x) k_y \\ -\sigma_y^*(y) k_x \end{pmatrix} .$$

We then deduce that the first two equations of (44) impose that $k_x k_y = 0$. Therefore, if $k_x = 0$ and $k_y \neq 0$, the third equation of (44) gives us :

$$(45) \qquad \frac{\partial \varphi}{\partial t} k_y + \sigma_x^*(x) k_y \varphi = 0 \quad i.e. \quad \frac{\partial \varphi}{\partial t} + \sigma_x^*(x) \varphi = 0 \ .$$

We assume that $\sigma_x^*(x) > 0$ inside the absorbing layers, the solution of the equation (45) is an exponential decay in time of the function φ. $\square$

We obtain the same result considering $k_y = 0$ and $k_x \neq 0$. Even if the principal symbol of the system (41) is associated to a "ill-posed mathematical problem", the form of the zero order terms shows that even exciting the absorbing layers system in the direction of the characteristic vector, the perturbation is dissipated.

We would like to predict the behavior of our absorbing layers model. Thus, we simplify our set of equations to the simplest model and study it. We establish the following proposition :

Proposition 9.

We consider the simplest 1-D non-hyperbolic model inside the absorbing layers, excited with a source term $\psi(t)$ centered at (x_a, y_a), in the direction of the eigenvector V_{M^2}. We note $W = (u, v)^t$ the state vector, δ_{x_a,y_a} the Dirac mass at the position (x_a, y_a) and we assume that the coefficients σ_1 and σ_2 are strictly positive. The problem :

$$\left\{ \begin{array}{l} \dfrac{\partial W}{\partial t} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dfrac{\partial W}{\partial x} + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} W = \begin{pmatrix} 0 \\ \psi(t)\, \delta_{x_a,y_a} \end{pmatrix} \\ W(0) = 0 \ . \end{array} \right. \tag{46}$$

is stable as long as $\psi(t)$ is bounded.

Proof. The system (46) is written as :

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} + \sigma_1 u = 0 \\ \dfrac{\partial v}{\partial t} + \sigma_2 v = \psi(t)\, \delta_{x_a,y_a} \\ u(0) = v(0) = 0 \ . \end{array} \right. \tag{47}$$

The solution in v is $v(t) = \left(\int_0^t \psi(\theta) e^{-\sigma_2 (t-\theta)} \mathrm{d}\theta \right) \delta_{x_a,y_a}$, we then deduce that u is solution of :

$$\frac{\partial u}{\partial t} + \sigma_1 u = \left(\int_0^t \psi(\theta) e^{-\sigma_2 (t-\theta)} \mathrm{d}\theta \right) \delta'_{x_a,y_a} \ .$$

Then, the solution in u is $u(t) = \mu(t)\, \delta'_{x_a,y_a}$, where $\lim_{t\to\infty} \mu(t) = 0$. Then, for a function $\psi(t)$ bounded, $u \to 0$ and v is bounded. The system (47) is stable as long as $\sigma_1 > 0$ and $\sigma_2 > 0$. $\square$

This simple model makes us think that our absorbing layers model is stable even if we are in the worst situation, *i.e.* there is a source in the direction of the eigenvector V_{M^2}. Numerical simulations proposed in Section 7 confirm this result.

One challenge in the future is to theorically understand the behavior of the solution of the true problem in the absorbing layers defined by the following system :

$$\left\{ \begin{array}{l} \dfrac{\partial W}{\partial t} + A \dfrac{\partial W}{\partial x} + B \dfrac{\partial W}{\partial y} + C\, W = \psi(t)\, \delta_{x_a, y_a}\, V_{M^2} \\ W(0) = 0 \ . \end{array} \right. \tag{48}$$

Nevertheless, the qualitative behavior proposed for the system (46) gives a good idea of the behavior of the system (48), see Section 7.

5-4 Acoustic absorbing layers with external subsonic flow

Proposition 10 (General absorbing layers in two dimensions)**.**

We assume that the velocity for the external subsonic flow is $\mathbf{u} = (u_0, v_0)$. A system of partial differential equations that models absorbing layers of acoustic waves is given by :

$$\left\{ \begin{array}{l} \dfrac{\partial p_x}{\partial t} + c_0 \sqrt{1 - M_0^2}\, \sigma^*(x)\, p_x + \dfrac{c_0 u_0}{\sqrt{1 - M_0^2}} \sigma^*(x)\, \xi + c_0^2 \dfrac{\partial \xi}{\partial x} = 0 \\ \dfrac{\partial p_y}{\partial t} + c_0 \sqrt{1 - M_0^2}\, \sigma^*(y)\, p_y + \dfrac{c_0 v_0}{\sqrt{1 - M_0^2}} \sigma^*(y)\, \zeta + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} \ + \ \dfrac{\partial}{\partial x}(2 u_0 \xi + v_0 \zeta) + (1 - M_0^2) \dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial y}(u_0 \zeta) + \dfrac{c_0 (1 + \frac{u_0^2 - v_0^2}{c_0^2})}{\sqrt{1 - M_0^2}} \sigma^*(x) \xi \\ \qquad + \ \dfrac{u_0 \sqrt{1 - M_0^2}}{c_0} \left(\sigma^*(x) p_x + \sigma^*(y) p_y \right) + \dfrac{u_0 v_0 (\sigma^*(x) + \sigma^*(y))}{c_0 \sqrt{1 - M_0^2}} \zeta = 0 \\ \dfrac{\partial \zeta}{\partial t} \ + \ \dfrac{\partial}{\partial x}(v_0 \xi) + \dfrac{\partial}{\partial y}(u_0 \xi + 2 v_0 \zeta) + (1 - M_0^2) \dfrac{\partial p}{\partial y} + \dfrac{c_0 (1 + \frac{v_0^2 - u_0^2}{c_0^2})}{\sqrt{1 - M_0^2}} \sigma^*(y) \zeta \\ \qquad + \ \dfrac{v_0 \sqrt{1 - M_0^2}}{c_0} \left(\sigma^*(x)\, p_x + \sigma^*(y)\, p_y \right) + \dfrac{u_0 v_0 (\sigma^*(x) + \sigma^*(y))}{c_0 \sqrt{1 - M_0^2}} \xi = 0 \ , \end{array} \right. \tag{49}$$

where $\sigma^*(x)$ et $\sigma^*(y)$ are smoothing functions defined by (35).

Proof. Here are the main ideas of the proof; the details of all the calculus can be found in [Ma2k]. If we consider a two-dimensional flow, we have shown that the acoustic system in the new space-time defined by (20) is given by (23) after the change of unknown functions (22). Using the method described in the section, we easily show that a general system

of dimensionless partial differential equations for the acoustic absorbing layers can be written in (x', y', t') using the functions $\sigma^*(x)$ and $\sigma^*(y)$ introduced by equation (35) as :

$$
(50) \qquad \left\{ \begin{array}{lllll}
\dfrac{\partial \widetilde{p_x}}{\partial t'} & + & c_0\sqrt{1-M_0^2}\,\sigma^*(x)\widetilde{p_x} & + & c_0^2\dfrac{\partial}{\partial x'}(\widetilde{\xi}-\alpha\widetilde{\zeta}) = 0 \\
\dfrac{\partial \widetilde{p_y}}{\partial t'} & + & c_0\sqrt{1-M_0^2}\,\sigma^*(y)\widetilde{p_y} & + & c_0^2\dfrac{\partial}{\partial y'}(\widetilde{\zeta}-\alpha\widetilde{\xi}) = 0 \\
\dfrac{\partial \widetilde{\xi}}{\partial t'} & + & c_0\sqrt{1-M_0^2}\,\sigma^*(x)\widetilde{\xi} & + & \dfrac{\partial}{\partial x'}(\widetilde{p_x}+\widetilde{p_y}) = 0 \\
\dfrac{\partial \widetilde{\zeta}}{\partial t'} & + & c_0\sqrt{1-M_0^2}\,\sigma^*(y)\widetilde{\zeta} & + & \dfrac{\partial}{\partial y'}(\widetilde{p_x}+\widetilde{p_y}) = 0 \ ,
\end{array} \right.
$$

where α is a coupling coefficient given by :

$$
\alpha = \frac{u_0 v_0}{c_0^2\sqrt{1-\frac{u_0^2}{c_0^2}}\sqrt{1-\frac{v_0^2}{c_0^2}}} \ .
$$

We now wish to write the system (50) in the initial space-time (x, y, t), using the initial unknown functions p, ξ, ζ. We have :

$$
(51) \qquad \widetilde{\xi}-\alpha\widetilde{\zeta} = \frac{1}{\sqrt{1-\frac{u_0^2}{c_0^2}}}\,\xi' \quad , \quad \widetilde{\zeta}-\alpha\widetilde{\xi} = \frac{1}{\sqrt{1-\frac{v_0^2}{c_0^2}}}\,\zeta' \ ,
$$

and we easily calculate :

$$
(52) \qquad \left\{ \begin{array}{l}
\dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t} \\
\dfrac{\partial}{\partial x'} = \sqrt{1-\dfrac{u_0^2}{c_0^2}}\dfrac{\partial}{\partial x} - \dfrac{u_0\sqrt{1-\frac{u_0^2}{c_0^2}}}{c_0^2(1-M_0^2)}\dfrac{\partial}{\partial t} \\
\dfrac{\partial}{\partial y'} = \sqrt{1-\dfrac{v_0^2}{c_0^2}}\dfrac{\partial}{\partial y} - \dfrac{v_0\sqrt{1-\frac{v_0^2}{c_0^2}}}{c_0^2(1-M_0^2)}\dfrac{\partial}{\partial t} \ .
\end{array} \right.
$$

We substitute (51) and (52) in the system (50) :

$$
\left\{ \begin{array}{l}
\dfrac{\partial \widetilde{p_x}}{\partial t} + c_0\sqrt{1-M_0^2}\sigma^*(x)\widetilde{p_x} + c_0^2\sqrt{1-\dfrac{u_0^2}{c_0^2}}\dfrac{\partial}{\partial x}(\widetilde{\xi}-\alpha\widetilde{\zeta}) - \dfrac{u_0\sqrt{1-\frac{u_0^2}{c_0^2}}}{1-M_0^2}\dfrac{\partial}{\partial t}(\widetilde{\xi}-\alpha\widetilde{\zeta}) = 0 \\
\dfrac{\partial \widetilde{p_y}}{\partial t} + c_0\sqrt{1-M_0^2}\sigma^*(y)\widetilde{p_y} + c_0^2\sqrt{1-\dfrac{v_0^2}{c_0^2}}\dfrac{\partial}{\partial y}(\widetilde{\zeta}-\alpha\widetilde{\xi}) - \dfrac{v_0\sqrt{1-\frac{v_0^2}{c_0^2}}}{1-M_0^2}\dfrac{\partial}{\partial t}(\widetilde{\zeta}-\alpha\widetilde{\xi}) = 0 \\
\dfrac{\partial \widetilde{\xi}}{\partial t} + c_0\sqrt{1-M_0^2}\sigma^*(x)\widetilde{\xi} + (1-M_0^2)\sqrt{1-\dfrac{u_0^2}{c_0^2}}\dfrac{\partial \widetilde{p}}{\partial x} - \dfrac{u_0\sqrt{1-\frac{u_0^2}{c_0^2}}}{c_0^2}\dfrac{\partial \widetilde{p}}{\partial t} = 0 \\
\dfrac{\partial \widetilde{\zeta}}{\partial t} + c_0\sqrt{1-M_0^2}\sigma^*(y)\widetilde{\zeta} + (1-M_0^2)\sqrt{1-\dfrac{v_0^2}{c_0^2}}\dfrac{\partial \widetilde{p}}{\partial y} - \dfrac{v_0\sqrt{1-\frac{v_0^2}{c_0^2}}}{c_0^2}\dfrac{\partial \widetilde{p}}{\partial t} = 0 \ .
\end{array} \right.
$$

Using the change of variables (22), we deduce :

$$
(53) \qquad \left\{ \begin{array}{l}
\widetilde{p} = p' + \dfrac{1}{1 - M_0^2} \, (u_0 \xi' + v_0 \zeta') \\
\widetilde{\xi} = \sqrt{1 - \dfrac{u_0^2}{c_0^2}} \left((1 - \dfrac{v_0^2}{c_0^2}) \dfrac{\xi'}{1 - M_0^2} + \dfrac{u_0 v_0}{c_0^2} \dfrac{\zeta'}{1 - M_0^2} \right) \\
\widetilde{\zeta} = \sqrt{1 - \dfrac{v_0^2}{c_0^2}} \left(\dfrac{u_0 v_0}{c_0^2} \dfrac{\xi'}{1 - M_0^2} + (1 - \dfrac{u_0^2}{c_0^2}) \dfrac{\zeta'}{1 - M_0^2} \right) .
\end{array} \right.
$$

We substitute the change of variables (53) into the previous system, we then obtain :

$$
(54) \qquad \left\{ \begin{array}{l}
\dfrac{\partial \widetilde{p_x}}{\partial t} - \dfrac{u_0}{1 - M_0^2} \dfrac{\partial \xi}{\partial t} + c_0 \sqrt{1 - M_0^2} \sigma^*(x) \widetilde{p_x} + c_0^2 \dfrac{\partial \xi}{\partial x} = 0 \\
\dfrac{\partial \widetilde{p_y}}{\partial t} - \dfrac{v_0}{1 - M_0^2} \dfrac{\partial \zeta}{\partial t} + c_0 \sqrt{1 - M_0^2} \sigma^*(y) \widetilde{p_y} + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\
\dfrac{\partial \xi}{\partial t} - \dfrac{u_0}{c_0^2} \dfrac{\partial p}{\partial t} \; + \; c_0 \sqrt{1 - M_0^2} \sigma^*(x) \left((1 - \dfrac{v_0^2}{c_0^2}) \dfrac{\xi}{1 - M_0^2} + \dfrac{u_0 v_0}{c_0^2} \dfrac{\zeta}{1 - M_0^2} \right) \\
\qquad\qquad + \; (1 - M_0^2) \dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial x} (u_0 \xi + v_0 \zeta) = 0 \\
\dfrac{\partial \zeta}{\partial t} - \dfrac{v_0}{c_0^2} \dfrac{\partial p}{\partial t} \; + \; c_0 \sqrt{1 - M_0^2} \sigma^*(y) \left(\dfrac{u_0 v_0}{c_0^2} \dfrac{\xi}{1 - M_0^2} + (1 - \dfrac{u_0^2}{c_0^2}) \dfrac{\zeta}{1 - M_0^2} \right) \\
\qquad\qquad + \; \dfrac{\partial}{\partial y} (u_0 \xi + v_0 \zeta) + (1 - M_0^2) \dfrac{\partial p}{\partial y} = 0 \; .
\end{array} \right.
$$

We use as new unknowns :

$$
\widetilde{p_x} = p_x + \frac{u_0}{1 - M_0^2} \xi \quad , \quad \widetilde{p_y} = p_y + \frac{v_0}{1 - M_0^2} \zeta \; ,
$$

we then have :

$$
\widetilde{p} = \widetilde{p_x} + \widetilde{p_y} \; , \qquad p = p_x + p_y \; ,
$$

and we finally obtain :

$$
(55) \qquad \left\{ \begin{array}{l}
\dfrac{\partial p_x}{\partial t} + c_0 \sqrt{1 - M_0^2} \sigma^*(x) p_x + c_0^2 \dfrac{\partial \xi}{\partial x} = 0 \\
\dfrac{\partial p_y}{\partial t} + c_0 \sqrt{1 - M_0^2} \sigma^*(y) p_y + \dfrac{c_0 v_0}{\sqrt{1 - M_0^2}} \sigma^*(y) \zeta + c_0^2 \dfrac{\partial \zeta}{\partial y} = 0 \\
\dfrac{\partial \xi}{\partial t} - \dfrac{u_0}{c_0^2} \dfrac{\partial p}{\partial t} + (1 - M_0^2) \dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial x} (u_0 \xi + v_0 \zeta) = 0 \\
\dfrac{\partial \zeta}{\partial t} - \dfrac{v_0}{c_0^2} \dfrac{\partial p}{\partial t} \; + \; c_0 \sqrt{1 - M_0^2} \sigma^*(y) \left(\dfrac{u_0 v_0}{c_0^2} \dfrac{\xi}{1 - M_0^2} + (1 - \dfrac{u_0^2}{c_0^2}) \dfrac{\zeta}{1 - M_0^2} \right) \\
\qquad\qquad + \; \dfrac{\partial}{\partial y} (u_0 \xi + v_0 \zeta) + (1 - M_0^2) \dfrac{\partial p}{\partial y} = 0 \; .
\end{array} \right.
$$

We wish to find a dynamic system, then we eliminate the $\frac{\partial p}{\partial t}$ terms in the last two equations of the system (55). Using the equality $p = p_x + p_y$, and adding the first two equations of the system (55), we deduce :

$$\frac{\partial p}{\partial t} + c_0^2 \frac{\partial \xi}{\partial x} + c_0^2 \frac{\partial \zeta}{\partial y} + c_0 \sqrt{1 - M_0^2}\sigma^*(y) p_y + \frac{c_0 v_0}{\sqrt{1 - M_0^2}} \sigma^*(y)\zeta = 0 \ , \tag{56}$$

and we substitute the equation (56) in the last two equations of the system (55). Hence, we find the result (49) that ends the proof. □

Remark 13.
Various authors propose a system of partial differential equation for absorbing layers for advective acoustic (see [AGH99], [Hu96], [Rah99], [Rah01] for example). Each of them have to solve 6 equations, whereas we propose a system composed by only 4 equations. We see this property as a consequence of our precise physical analysis based on the Lorentz transform and our change of unknown functions.

6) Discretization with the "HaWAY" method

This section deals with the numerical resolution of the equation of acoustic **without** an external flow. We use finite differences with staggered grids as introduced by Harlow-Welsch (MAC method) [HW65], Arakawa (C grids) [Ar66] and Yee [Yee66] for electromagnetism. HaWAY comes from **Ha**rlow-**W**elsch, **A**rakawa, **Y**ee. The acoustic system can be written as :

$$\begin{cases} \dfrac{\partial p'}{\partial t'} + c_0^2 \dfrac{\partial \xi'}{\partial x'} + c_0^2 \dfrac{\partial \zeta'}{\partial y'} = 0 \\ \dfrac{\partial \xi'}{\partial t'} + \dfrac{\partial p'}{\partial x'} = 0 \\ \dfrac{\partial \zeta'}{\partial t'} + \dfrac{\partial p'}{\partial y'} = 0 \ . \end{cases} \tag{57}$$

We first propose to nondimensionalize the previous system. We then explain the numerical scheme chosen in the free space and in the acoustic absorbing layers.

6-1 Dimensionlessness of the acoustic system

This section is introduced for the completeness of our meaning. We refer to [Se75] for this kind of purpose. We nondimensionalize the set of equations (57) by writing each variable as : $X' = X^* X$, for X' pressure, impulse, time and space variables and X^* a reference dimension for each variable: a reference pressure p^*, reference impulses ξ^* and ζ^*, a time reference t^* and reference lengths x^* and y^*. The new form of the system (57) is :

$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + c_0^2 \dfrac{t^*}{p^*} \dfrac{\xi^*}{x^*} \dfrac{\partial \xi}{\partial x} + c_0^2 \dfrac{t^*}{p^*} \dfrac{\zeta^*}{y^*} \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \dfrac{t^*}{\xi^*} \dfrac{p^*}{x^*} \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{t^*}{\zeta^*} \dfrac{p^*}{y^*} \dfrac{\partial p}{\partial y} = 0 \ . \end{array} \right.$$

We decide here to **choose** the following coefficients $c_0^2 \dfrac{t^*}{p^*} \dfrac{\xi^*}{x^*}$, $c_0^2 \dfrac{t^*}{p^*} \dfrac{\zeta^*}{y^*}$, $\dfrac{t^*}{\xi^*} \dfrac{p^*}{x^*}$ and $\dfrac{t^*}{\zeta^*} \dfrac{p^*}{y^*}$ equal to 1. We then deduce that we have :

$$\xi^* = \frac{1}{c_0} p^* \ , \ \zeta^* = \frac{1}{c_0} p^* \ , \ \frac{x^*}{t^*} = c_0 \ , \ \frac{y^*}{t^*} = c_0 \ ,$$

and the resulting set of dimensionless equations is :

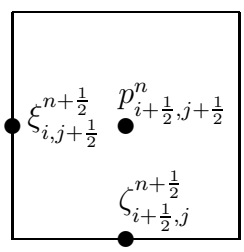
$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + \dfrac{\partial \xi}{\partial x} + \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial p}{\partial y} = 0 \ . \end{array} \right. \tag{58}$$

6-2 Staggered grids for acoustics

By analogy with electromagnetism (see [DDS94]), we decide to use the cartesian staggered finite differences method to solve the system (58). We decompose a model domain $\Omega = [\,0, L\,]^2$ into finite elements with an isotropic meshing of space step $\Delta x = \Delta y = \frac{L}{J}$ ($J \in \mathbf{N}^*$ is the number of cells in each direction). The cell $K_{i+\frac{1}{2}, j+\frac{1}{2}}$ is defined as :

$$K_{i+\frac{1}{2}, j+\frac{1}{2}} =]\, i\Delta x \,,\, (i+1)\Delta x \,[\, \times \,]\, j\Delta y \,,\, (j+1)\Delta y \,[$$

We share the time with the help of a time step Δt and introduce the nth "entire time" $t^n = n\Delta t$. By convention, we know that the pressure is defined at entire times t^n in the center of the mesh $K_{i+\frac{1}{2}, j+\frac{1}{2}}$ and that the impulses are defined at semi-entire times $t^{n+\frac{1}{2}}$ on the edge of the mesh. The variables in a mesh are defined as below :



The numerical scheme used in the free space is in two-dimensional space :

- Discretization of the pressure equation :

$$p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = p^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - \sigma \left[(\xi^{n+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - \xi^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}) + (\zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j+1} - \zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j}) \right] ,$$

- Discretization of the impulse equations :

$$\begin{cases} \xi^{n+\frac{3}{2}}_{i,j+\frac{1}{2}} = \xi^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} - \sigma(p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}}) \\ \zeta^{n+\frac{3}{2}}_{i+\frac{1}{2},j} = \zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - \sigma(p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i+\frac{1}{2},j-\frac{1}{2}}) \end{cases} ,$$

where we have $\sigma = \dfrac{\Delta t}{\Delta x} = \dfrac{\Delta t}{\Delta y}$.

This numerical scheme is an explicit second order in time and space scheme, stable under the Courant-Friedrichs-Lewy condition :

$$\text{CFL} \quad \begin{cases} \Delta t \leq \dfrac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}} & \text{in two dimension space,} \\ \Delta t \leq \dfrac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}} & \text{in three dimension space.} \end{cases} \tag{59}$$

The boundary condition is supposed to be on the edge of the mesh. The Dirichlet boundary condition is written as : $\mathbf{u}.\mathbf{n} = 0$.

6-3 Numerical acoustic absorbing layers

We deduce from section 4 the set of equations to solve in the absorbing layers :

$$\begin{cases} \dfrac{\partial p_x}{\partial t} + \sigma^*(x)\, p_x + \dfrac{\partial \xi}{\partial x} = 0 \\ \dfrac{\partial p_y}{\partial t} + \sigma^*(y)\, p_y + \dfrac{\partial \zeta}{\partial y} = 0 \\ \dfrac{\partial \xi}{\partial t} + \sigma^*(x)\, \xi + \dfrac{\partial}{\partial x}(p_x + p_y) = 0 \\ \dfrac{\partial \zeta}{\partial t} + \sigma^*(y)\, \zeta + \dfrac{\partial}{\partial y}(p_x + p_y) = 0 \end{cases} ,$$

where $\sigma^*(x)$ and $\sigma^*(y)$ are the smoothing functions strictly positive inside the absorbing layers. The set of equations is ended by a Dirichlet condition on the edge of the whole studied domain : $\mathbf{u}.\mathbf{n} = 0$, where $\mathbf{n}$ is the external normal to the domain. The discretization of such a boundary condition for the whole studied domain is :

$$\left\{\begin{array}{ll}
\xi^{n+\frac{1}{2}}_{0,j+\frac{1}{2}} = 0, & 0 \leq j \leq J-1, \quad n \geq 0 \\
\xi^{n+\frac{1}{2}}_{J,j+\frac{1}{2}} = 0, & 0 \leq j \leq J-1, \quad n \geq 0 \\
\zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},0} = 0, & 0 \leq i \leq J-1, \quad n \geq 0 \\
\zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},J} = 0, & 0 \leq i \leq J-1, \quad n \geq 0
\end{array}\right.$$

We propose to use the same discretisation as before. In the absorbing layers, we have to know p , ξ and ζ respectively at times $n+\frac{1}{2}$, $n+1$ et $n+1$. We decide to center those values in time, we write :

$$\left\{\begin{array}{lcl}
p^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} & = & \dfrac{1}{2}\left(p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + p^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\
\xi^{n+1}_{i,j+\frac{1}{2}} & = & \dfrac{1}{2}\left(\xi^{n+\frac{3}{2}}_{i,j+\frac{1}{2}} + \xi^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}\right) \\
\zeta^{n+1}_{i+\frac{1}{2},j} & = & \dfrac{1}{2}\left(\zeta^{n+\frac{3}{2}}_{i+\frac{1}{2},j} + \zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j}\right) .
\end{array}\right.$$

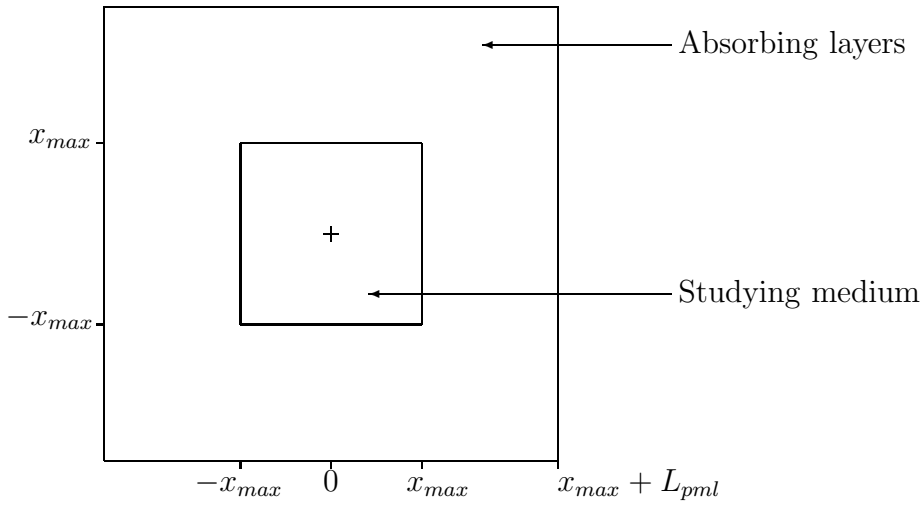
The numerical scheme, while noting $\sigma = \dfrac{\Delta t}{\Delta x} = \dfrac{\Delta t}{\Delta y}$ (isotrop meshing), can be written as :

$$\left\{\begin{array}{l}
{p_x}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \dfrac{2-\sigma^*_x(i+\frac{1}{2})\ \Delta t}{2+\sigma^*_x(i+\frac{1}{2})\ \Delta t}\ {p_x}^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - \dfrac{2\sigma\left(\xi^{n+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - \xi^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}\right)}{2+\sigma^*_x(i+\frac{1}{2})\ \Delta t} \\
{p_y}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \dfrac{2-\sigma^*_y(j+\frac{1}{2})\ \Delta t}{2+\sigma^*_y(j+\frac{1}{2})\ \Delta t}\ {p_y}^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - \dfrac{2\sigma\left(\zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j+1} - \zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j}\right)}{2+\sigma^*_y(j+\frac{1}{2})\ \Delta t} \\
\xi^{n+\frac{3}{2}}_{i,j+\frac{1}{2}} = \dfrac{2-\sigma^*_x(i)\ \Delta t}{2+\sigma^*_x(i)\ \Delta t}\ \xi^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} - \dfrac{2\sigma}{2+\sigma^*_x(i)\ \Delta t}\ \left(p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}}\right) \\
\zeta^{n+\frac{3}{2}}_{i+\frac{1}{2},j} = \dfrac{2-\sigma^*_y(j)\ \Delta t}{2+\sigma^*_y(j)\ \Delta t}\ \zeta^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - \dfrac{2\sigma}{2+\sigma^*_y(j)\ \Delta t}\ \left(p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i+\frac{1}{2},j-\frac{1}{2}}\right) ,
\end{array}\right.$$

with $p \equiv p_x + p_y$.

7) Numerical tests

The computational domain is shared into two areas : the studying medium and the absorbing layers. Those two areas are defined as shown below :



7-1 Mathematical experiments

We first propose to numerically validate our mathematical analysis of the absorbing layers acoustic system (41) with a test case proposed by O. Pironneau [Pi99]. We decide to place an acoustic pulse **inside** the absorbing layers, in the direction of the eigenvector associated to the eigenvalue 0 in order to enforce the unstability due to the lack of hyperbolicity as shown before. For this test, the computational domain, symetric in the x-direction and the y-direction, is defined by $x_{max} = 5$ and $L_{pml} = 45$. We consider two different tests. For the first one, we solve :

$$\left\{ \begin{array}{l} \dfrac{\partial W}{\partial t} + A\dfrac{\partial W}{\partial x} + B\dfrac{\partial W}{\partial y} + C\,W = \psi(t)\,\delta_{x_a,y_a}\,V_{M^2} \\ W(0) = 0\ . \end{array} \right. \tag{60}$$

For the second one, we solve :

$$\left\{ \begin{array}{l} \dfrac{\partial W}{\partial t} + A\dfrac{\partial W}{\partial x} + B\dfrac{\partial W}{\partial y} + C\,W = 0\ , \\ W(0) = \begin{pmatrix} 0 \\ 0 \\ \exp\left(-(\ln 2)\frac{(x-x_a)^2+(y-y_a)^2}{9}\right) \\ \exp\left(-(\ln 2)\frac{(x-x_a)^2+(y-y_a)^2}{9}\right) \end{pmatrix}\ . \end{array} \right. \tag{61}$$

For the first test case, the excitation $\psi(t)\,V_{M^2}$ is centered **inside** the absorbing layers at $(x_a, y_a) = (25, 0)$. The absorbing coefficients in the absorbing layers are constant in the x and in the y-direction. The reference excitation is :

$$\psi_{xy} = \exp\left(-(\ln 2)\frac{(x-x_a)^2+(y-y_a)^2}{9}\right) .$$

We solve the problem (60) with the particular data given by :

$$\left\{\begin{array}{l} \check{p}(x_a, y_a, t) = 0 \\ \check{\xi}(x_a, y_a, t) = \frac{\partial \psi_{xy}}{\partial y} \\ \check{\zeta}(x_a, y_a, t) = -\frac{\partial \psi_{xy}}{\partial x} \end{array}\right. .$$

The observing points are centered at $(x_1, y_1) = (45, 0)$, $(x_2, y_2) = (25, 0)$, $(x_3, y_3) = (0, 0)$, $(x_4, y_4) = (-45, 0)$, $(x_5, y_5) = (0, 25)$ and $(x_6, y_6) = (0, -25)$. We observe the results at $(x_2, y_2) = (25, 0)$. We obtain the graph presented of Figure 1.

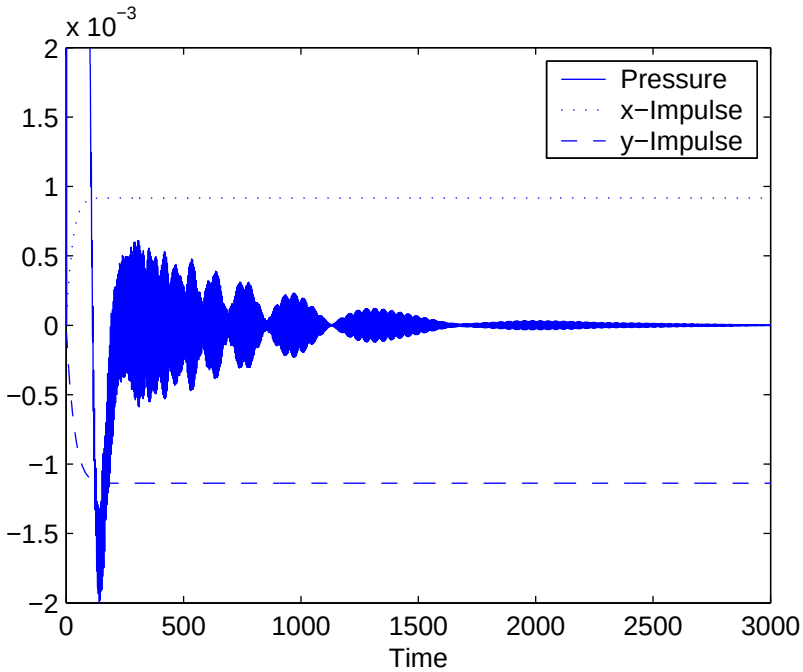


Figure 1. Pressure and impulse fields at $(x, y) = (25, 0)$.

We remark that, for long times, the pressure field converges to 0 and the impulse fields to a non-zero value. We obtain the results predicted by our simple 1-D model (see proposition 9) noting u the pressure field and v the impulse fields ξ or ζ. We notice that the sign is changing if we consider a positive or a negative source. We observe approximately the same results for the other observing points. The pressure field converges to 0 except at (x_5, y_5) and (x_6, y_6), where there is a slight residual r_p at (x_5, y_5) and $-r_p$ at (x_6, y_6). The ξ impulse field converges to 0 at (x_1, y_1), (x_3, y_3) and (x_4, y_4). The residual at (x_5, y_5) is r_ξ and $-r_\xi$ at (x_6, y_6). The ζ impulse field always converges to a constant value as predicted by the 1-D model.

For the second test, we solve the problem (61) with an excitation centered **inside** the absorbing layers at $(x_a, y_a) = (25, 0)$. As before, we analyse the results at $(x_2, y_2) = (25, 0)$. The evolution of the pressure and impulse fields are drawn in the following figure :

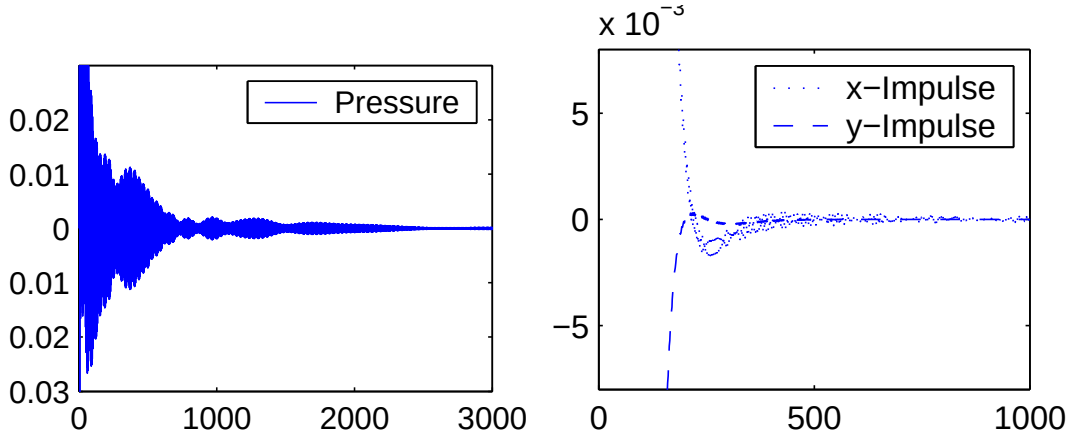


Figure 2. Pressure and impulse fields at $(x, y) = (25, 0)$.

All the fields converge to 0 for all the observing points. We obtain the same results as predicted by the proposition 9 with an excitation function $\psi(t)$ such as $\int_0^\infty \psi(t)\mathrm{d}t$ is bounded.

The conclusion of these mathematical experiments is that even if we excite this non hyperbolic system in the direction of the non caracteristic vector, the zero order damping terms insure that there is no numerical explosion of our results.

7-2 Physical experiments

We have proven the stability of our absorbing layers. We now want to study numerical reflections of outgoing waves from the boundaries of the computational domain for various speeds of the external flow. We then consider the problem (6) in the computational domain defined by $x_{max} = 25$ with the absorbing layers outside (L_{pml} is now a parameter), an acoustic pulse centered at $(x_a, y_a) = (0, 0)$ and an excitation in the right hand side $(\check{p}, \check{\xi}, \check{\zeta})^t$ given by :

$$\left\{\begin{array}{l} \check{p}\,(x, y, t) = \exp\left(-(\ln 2)\dfrac{(x - x_a)^2 + (y - y_a)^2}{9}\right)\sin(\pi t) \\ \check{\xi}(x, y, t) = 0 \\ \check{\zeta}(x, y, t) = 0\ . \end{array}\right. \tag{62}$$

We compare the calculated solution, denoted by p, to the numerical solution obtained in the domain defined by $x_{max} = 150$. We take $\Delta x = 1$ and Δt following the CFL (59). For $t < 300\,\Delta t$, it is easy to see that no reflection from the boundaries can interact with the solution within the small domain $[-25, 25]^2$ and such a solution is the numerical solution

in an infinite domain for $t < 300\,\Delta t$. This solution is considered as a reference, noted p_{ref}, and the computation of the error $|p - p_{ref}|$ for each time step indicates the efficiency of the absorbing layers. As explained in Section 5, the boundary condition is imposed on the edge of a cell and the pressure is calculated in the middle of the cell. The observing point is then taken only half a cell near the absorbing layers at $(25, 0)$.

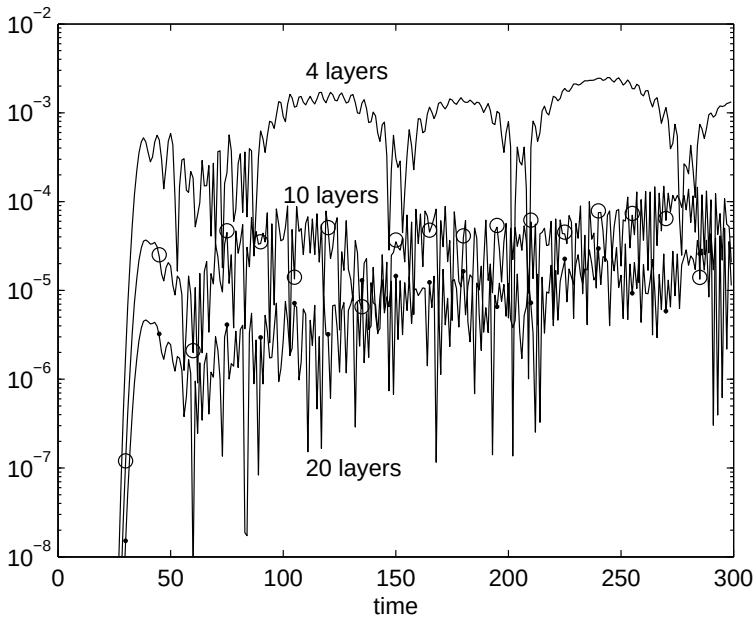


Figure 3. L_2-error of the pressure for 4, 10 and 20 absorbing layers, $\frac{\mathbf{u}}{c_0} = (0.5, 0)$.

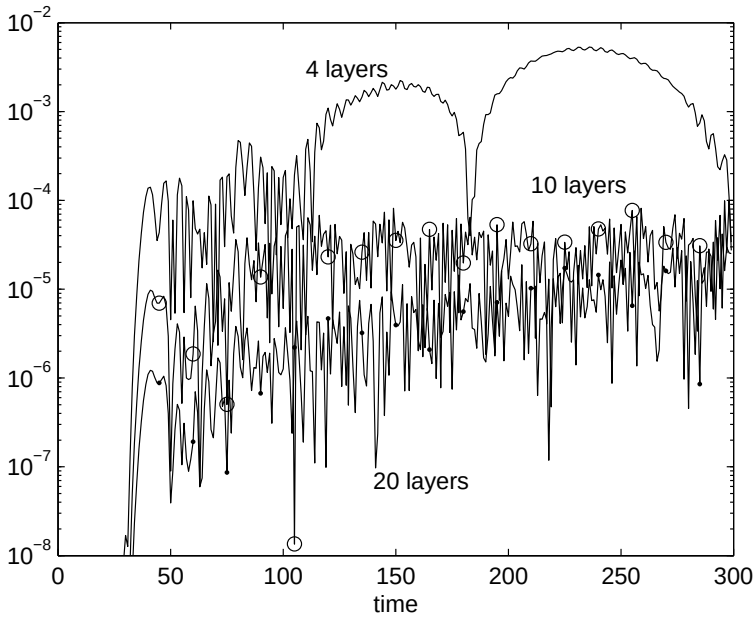


Figure 4. L_2-error of the pressure for 4, 10 and 20 absorbing layers, $\frac{\mathbf{u}}{c_0} = (0.5, 0)$.

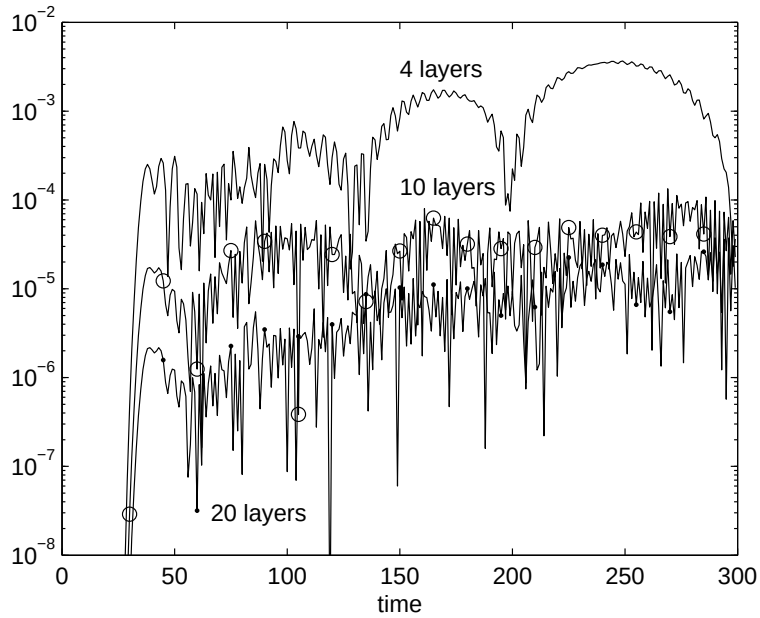


Figure 5. L_2-error of the pressure for 4, 10 and 20 absorbing layers, $\frac{\mathbf{u}}{c_0} = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$.

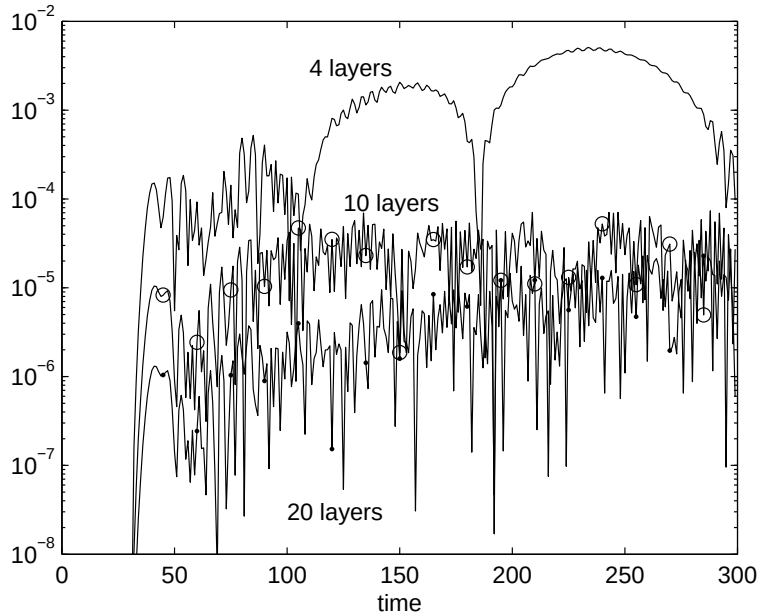


Figure 6. L_2-error of the pressure for 4, 10 and 20 absorbing layers, $\frac{\mathbf{u}}{c_0} = (\frac{2}{\sqrt{17}}, \frac{1}{2\sqrt{17}})$.

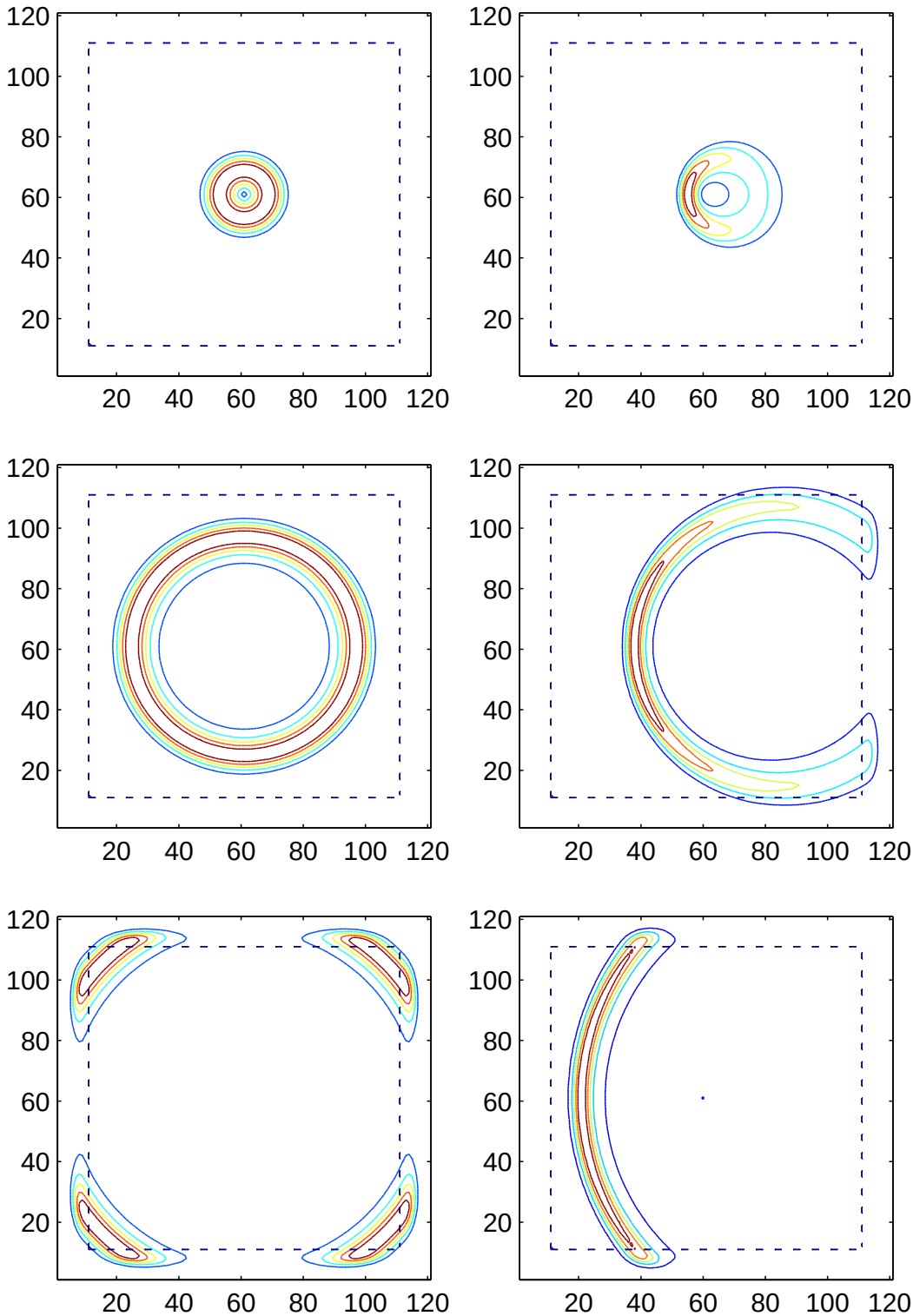


Figure 7. Iso-lines of the pressure field for $t = 40\,\Delta t$, $t = 80\,\Delta t$ and $t = 120\,\Delta t$.

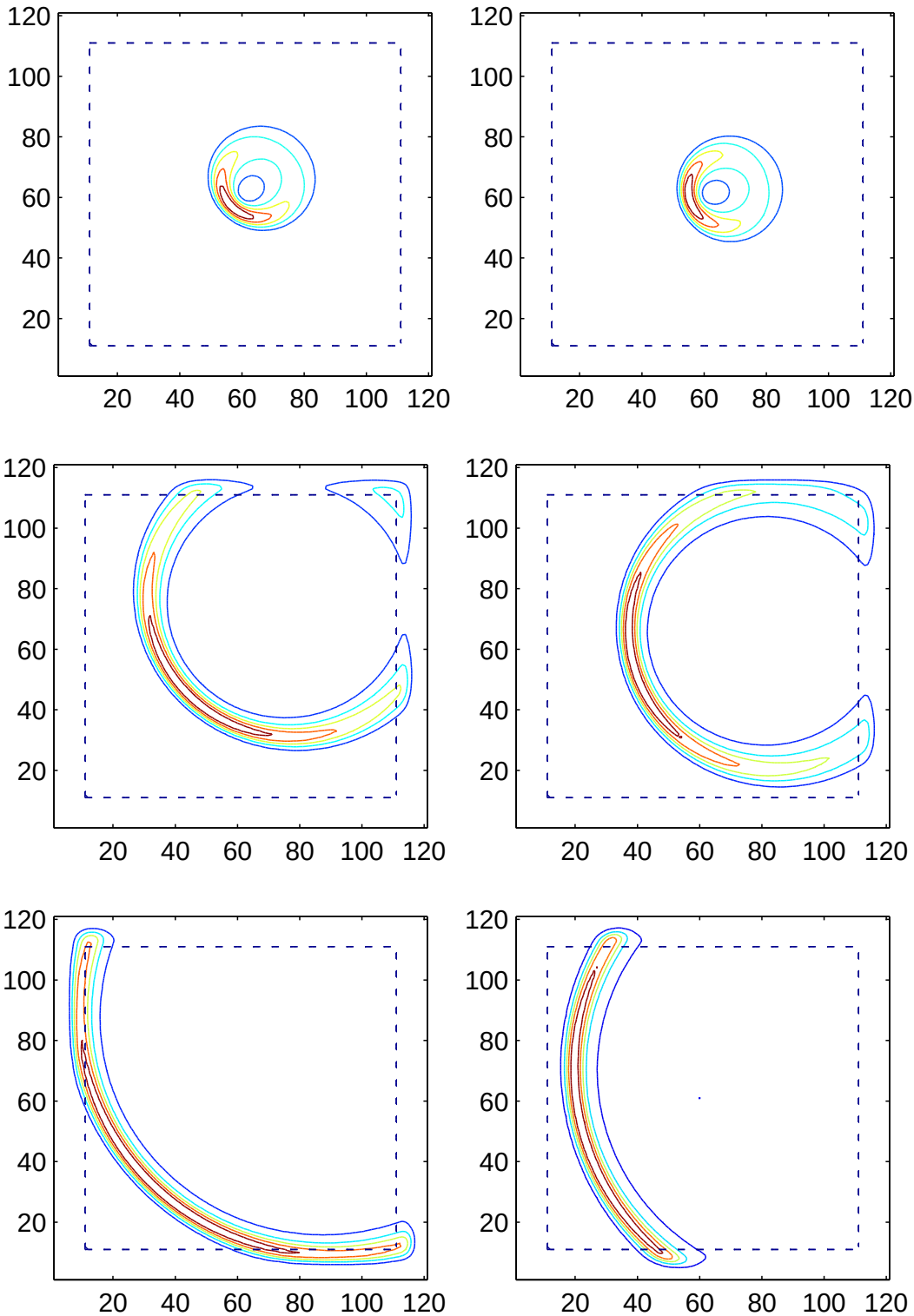


Figure 8. Iso-lines of the pressure field for $t = 40\,\Delta t$, $t = 80\,\Delta t$ and $t = 120\,\Delta t$.

We first consider a domain without external flow. We notice that the L_2-error of the pressure field p computed for each time step at the observing point $(25, 0)$ for various thickness for the absorbing layers. We notice that for small absorbing layers (4 cells), we have a good accuracy for the results, and that increasing the thickness from 4 cells to 20 cells improves the accuracy by 2 orders of magnitude.

We also compare, for various thickness of the absorbing layers, the "exact" and the numerical solution considering the same observing point and the same acoustic source, for three velocity vectors defined by $\mathbf{u} = (0.5\, c_0, 0)$, $\mathbf{u} = (\frac{1}{2\sqrt{2}}\, c_0, \frac{1}{2\sqrt{2}}\, c_0)$ and $\mathbf{u} = (\frac{2}{\sqrt{17}}\, c_0, \frac{1}{2\sqrt{17}}\, c_0)$. We notice an improvement of the accuracy by 2 orders of magnitude for an absorbing layers growing from 4 cells to a 20 cells.

The results are satisfying. Nevertheless, when the number of cells in the absorbing layers is increasing, the error is small but remains measurable, even for the long times. We think that this behavior could be improved in future work.

Conclusion

We have explored a new method for solving the equations of advective acoustics based on a change a space-time variables (Lorentz transform) and a change of unknown variables. We have also derived a system of equations (49) to modelize the absorbing layers for the acoustic model. The system of partial differential equations established in the absorbing layers is well-posed due to the zero order term. The staggered grid "HaWAY" method has been used for the numerical implementation and experiments have proven the efficiency of such a method. When we force a punctual acoustic source inside this numerical domain, our experiments show that the results remain bounded and our method is stable from a practical point of view. Notice that we explain our method only in two-dimensional space, but we extend it easily to three-dimensional space.

Acknowledgments

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