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Asymptotic Arbitrage in Large Financial Markets With Friction

Emmanuel Denis · Lavinia Ostafe

Abstract In the modern version of Arbitrage Pricing Theory suggested by Kabanov and Kramkov the fundamental financially meaningful concept is an asymptotic arbitrage. The "real world" large market is represented by a sequence of "models" and, though each of them is arbitrage free, investors may obtain non-risky profits in the limit. Mathematically, absence of the asymptotic arbitrage is expressed as contiguity of envelopes of the sets of equivalent martingale measures and objective probabilities. The classical theory deals with frictionless markets. In the present paper we extend it to markets with transaction costs. Assuming that each model admits consistent price systems, we relate them with families of probability measures and consider their upper and lower envelopes. The main result concerns the necessary and sufficient conditions for absence of asymptotic arbitrage opportunities of the first and second kinds expressed in terms of contiguity. We provide also more specific conditions involving Hellinger processes and give applications to particular models of large financial markets.

Key words Large financial market, asymptotic arbitrage, transaction costs, contiguity, hedging theorem.

Mathematics Subject Classification (2000) 60G44 **JEL Classification** G11-G13

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1 Introduction

The idea to describe a financial market by a sequence of market models with a finite number of securities can be traced back to the paper [7] by Huberman who formalized intuitive arguments of Arbitrage Pricing Theory initiated by Ross, [20]. The famous conclusion of this theory is: under the absence of arbitrage, appropriately defined, the expected returns on assets are approximately linearly related to the factor loadings, "betas", proportional to the return covariances with the factors. In economic literature, the APT is considered as a substitute for the Capital Asset Pricing Model (CAPM) by Lintner and Sharp. The Ross–Huberman theory is single-period and uses a definition of arbitrage different from the one that is now standard. Its generalization to the standard continuous-time framework of modern mathematical finance was considered for a long time as a challenging problem of large importance.

This problem was solved in 1994 by Kabanov and Kramkov, [12], who suggested a concept of large financial market described by a sequence of "standard" financial market models with finite number of securities whose price processes admit martingale measures. They introduced new notions of Asymptotic Arbitrage of the First and Second Kind and, assuming that martingale measures are unique for each model, they established necessary and sufficient conditions for the absence of asymptotic arbitrage in terms of contiguity of the sequences of objective probabilities and martingale measures. As a particular example of application of their general approach, Kabanov and Kramkov considered a large Black–Scholes market where the stock prices are given by correlated geometric Brownian motions. For this case their general criteria give a result of the same type as the Ross–Huberman condition but involving instantaneous returns and covariances.

Significant progress in the theory was achieved in the paper by Klein and Schachermayer, [9], where the geometric functional analysis was used to obtain criteria of absence of asymptotic arbitrage for the case of incomplete market models when the martingale measures are not unique. The next step in the development of the general theory as well as in the understanding of financial framework was again done by Kabanov and Kramkov, [13]. They added several new criteria of absence of asymptotic arbitrage in terms of contiguity of sequences of upper and lower envelopes of martingale measures and objective probabilities. The technique of the proofs was based on the optional decomposition theorem. The criteria of Klein and Schachermayer was also obtained by an elegant use of the minimax theorem. Kabanov and Kramkov related their criteria with an extension of the Liptser–Shiryaev theory of contiguity of sequences of probability measures on filtered spaces in terms of the Hellinger processes. One should emphasize that Kabanov–Kramkov framework is very general and flexible. It covers discrete and continuous-time models, models with time horizons tending to infinity, etc. For the further development of the theory of large financial markets we send the reader to the articles [5], [17] but also [10] and [11].

In the present paper we extend the framework of large financial markets to the case of a market with friction. It is well known, in the theory of markets with proportional transaction costs the concept of martingale measures is not natural and is replaced by the notion of consistent price systems, i.e. the martingales evolving in the duals to the solvency cones expressed in physical units, [14]. The consistent price systems are vector objects. Nevertheless, the criteria of absence of asymp-

otic arbitrage can be formulated in terms of contiguity of objective probabilities and envelopes of measures naturally arising from consistent price systems. These are our principal results. We follow the lines of [13] but do not use the optional decomposition theorem (it has no analogue for models with transaction costs) but the hedging theorem. We use the abstract setting of the recent paper by Denis and Kabanov [3], which allows us to avoid detailed discussions on the structure of continuous-time models and cover both major approaches to the definition of the value processes, those of Kabanov and of Campi–Schachermayer [2].

Some examples are given. The first one is a large financial market in a two-dimensional setting. We also extend the results of [13] to models with transaction costs: the one-stage APM by Ross, the large Black–Scholes market, and a two-asset model with infinite horizon.

2 The Model: Definitions and Assumptions

2.1 Example

Before introducing our general model, we recall the simplest discrete-time model of financial market with proportional transaction costs following the book [14]. The investor portfolio is now vector-valued and its evolution, in units of the *numéraire*, is given by the following controlled difference equation:

$$\Delta V_t = \text{diag } V_{t-1} \Delta R_t + \Delta B_t, \quad V_{-1} = v,$$

where $\Delta R_t^i = \Delta S_t^i / S_{t-1}^i$, $i \leq d$, is the relative price increment of the i th security, ΔB_t is the control, and $\text{diag } x$ denotes the diagonal operator generated by the vector x . The first term in the rhs of the dynamics means that the portfolio, before an action of the agent, evolves according to the price movements. The second one corresponds to transfers decided by the agent. In the model where one can exchange any asset to any other with losses,

$$\Delta B_t^i := \sum_{j=1}^d \Delta L_t^{ji} - \sum_{j=1}^d (1 + \lambda_t^{ij}) \Delta L_t^{ij}$$

where ΔL_t^{ji} represents the net amount transferred from the position j to the position i at date t and λ_t^{ij} are the transaction costs rates. The investor action ΔB_t is a \mathcal{F}_t -measurable random variable taking values in a cone $-K_t$ where the so-called solvency cone K_t is defined by the matrix of transaction costs coefficients $\Lambda_t = (\lambda_t^{ij})$:

$$K_t := \text{cone} \{(1 + \lambda_t^{ij})e_i - e_j, e_i, 1 \leq i, j \leq d\}. \quad (2.1)$$

In the theory, as in practice, the coefficients $\lambda_t^{ij} \geq 0$ are adapted random processes. The above dynamics naturally falls into a scope of linear difference equations with control constraints to be taken from cones which are, in general, random.

One can express the portfolio dynamics also in “physical units”. It is much simpler. Assuming that $S_{-1} = S_0 = (1, \dots, 1)$ and introducing the diagonal operator

$$\phi_t : (x^1, \dots, x^d) \mapsto (x^1 / S_t^1, \dots, x^d / S_t^d),$$

we have:

$$\Delta\widehat{V}_t = \widehat{\Delta B}_t, \quad \widehat{V}_{-1} = v,$$

where $\widehat{V}_t := \phi_t V_t$, $\widehat{\Delta B}_t \in L^0(-\widehat{K}_t, \mathcal{F}_t)$, $\widehat{K}_t := \phi_t K_t$. Note that, in contrast to K_t , the cones \widehat{K}_t are always random, even in the model with constant transaction costs. So, (\widehat{K}_t) is an adapted cone-valued process. Though in financial models the cones $\widehat{K}_t(\omega)$ are polyhedral, for the control theory this looks too restrictive and the question about possible extensions to “general” models, with (\widehat{K}_t) replaced by an arbitrary adapted cone-valued process (G_t) , arises naturally.

As pointed out in the book [14], one can find variants of this model which can be imbedded into the former by choosing sufficiently large transaction costs coefficients. The procedure leads to a larger set of portfolio value processes but has no effect on the arbitrage properties. The elements of $\mathcal{M}_0^T(K^* \setminus \{0\})$ and $\mathcal{M}_0^T(\text{int } K^*)$, i.e. the martingales evolving in the positive dual K^* of K , referred to as *consistent price systems* and *strictly consistent price systems*, play a fundamental role in the arbitrage theory for models with transaction costs. We send the reader to Chapter 3, [14], for more details.

2.2 General Model

The framework setting we present in this section is assumed to be satisfied by a sequence of markets of horizon dates T (for the sake of simplicity, we omit the index n). We consider the general model of the paper [3] including the Kabanov and Campi–Schachermayer models with transaction costs.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \leq T}, P)$ be a continuous-time stochastic basis verifying the usual conditions. We are given a pair of set-valued adapted processes $G = (G_t)_{t \leq T}$ and its positive dual $G^* = (G_t^*)_{t \leq T}$ whose values are closed cones in \mathbf{R}^d , i.e.

$$G_t^*(\omega) = \{y : yx \geq 0 \ \forall x \in G_t(\omega)\}.$$

“Adapted” means that the graphs

$$\left\{ (\omega, x) \in \Omega \times \mathbf{R}^d : x \in G_t(\omega) \right\}$$

are $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$ -measurable.

We assume that all the cones G_t are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\text{int } G_t^* \neq \emptyset$. In a financial context it means that the *efficient friction* condition (EF) is fulfilled. We assume also that G_t dominates \mathbf{R}_+^d , i.e. $G_t^* \setminus \{0\} \subset \text{int } \mathbf{R}_+^d$.

In a more specific financial setting (see [14]), the cones G_t are the solvency cones \widehat{K}_t when the portfolio positions are expressed in physical units.

We are given a convex cone \mathcal{Y}_0^T of optional \mathbf{R}^d -valued processes $Y = (Y_t)_{t \leq T}$ with $Y_0 = 0$. We may interpret these processes as portfolios expressed in physical units.

Notations. We denote by $L^0(G_t, \mathcal{F}_t)$ the set of all G_t -valued \mathcal{F}_t -measurable random variables. A cone G induces a natural order among \mathbf{R}^d -valued random variables. More precisely, for two d -dimensional random variables Y and Y' , we write $Y \geq_G Y'$ if $Y - Y' \in G$. The notation $\mathbf{1}$ stands for the vector $(1, \dots, 1) \in \mathbf{R}_+^d$. Denote by $\mathcal{Y}_{0,b}^T$ the subset of \mathcal{Y}_0^T formed by the processes Y dominated from below

in the sense of the partial orders generated by $(G_t)_{t \leq T}$, i.e. there is a constant κ such that the process $Y + \kappa \mathbf{1}$ evolves in G . We also write $\mathcal{Y}_{0,b}^T(T)$ for the set of random variables Y_T where Y belongs to $\mathcal{Y}_{0,b}^T$. The set $\overline{\mathcal{Y}_{0,b}^{T,\infty}(T)}$ is the closure of $\mathcal{Y}_{0,b}^{T,\infty}(T) := \mathcal{Y}_{0,b}^T(T) \cap L^\infty$ in $\sigma\{L^\infty, L^1\}$. We denote by $\mathcal{M}_0^T(G^*)$ the set of all d -dimensional martingales $Z = (Z_t)_{t \leq T}$ with trajectories evolving in G^* , i.e. such that $Z_t \in G_t^*$ a.s. In the literature, such martingales are commonly called *consistent price systems* and *strictly consistent price systems* if they evolve in the interior of G^* .

Assumptions. Throughout the note we assume the following **standing hypotheses** on the sets $\mathcal{Y}_{0,b}^T(T)$:

$$\mathbf{S1} : E\xi Z_T \leq 0, \forall \xi \in \mathcal{Y}_{0,b}^T(T), Z \in \mathcal{M}_0^T(G^*).$$

$$\mathbf{S2} : \bigcup_{t \leq T} L^\infty(-G_t, \mathcal{F}_t) \subseteq \mathcal{Y}_{0,b}^T(T).$$

The hypotheses **S1** and **S2** adopted in this note allow us to avoid the unnecessary repetitions and do not provide the full description of continuous-time models with transaction costs. It is important to know only that these conditions are fulfilled for the known models, see [15], [2], [4].

Recall that in these financial models **S1** holds because, if one calculates the current portfolio value using a price system Z (that is a process from $\mathcal{M}_0^T(G^*)$), the resulting scalar process is a supermartingale. In a discrete-time model, a portfolio process $(V_t)_{t \leq T}$ is such that $V_t \in \sum_{u=0}^t L^0(-G_u, \mathcal{F}_u)$ for all $t \leq T$. If $V \in \mathcal{Y}_{0,b}^T$ then $E(Z_T V_T)^- < \infty$. The process $V_t = \sum_{u=0}^t \Delta V_u$ vérifies $\Delta V_u \in -G_u$. Applying Proposition 3.3.2, [14], we get that

$$\begin{aligned} E(Z_T V_T | \mathcal{F}_t) &= V_t E(Z_T | \mathcal{F}_t) + E(Z_T \sum_{u=t+1}^T \Delta V_u | \mathcal{F}_t), \\ &= V_t Z_t + \sum_{u=t+1}^T E(Z_u \Delta V_u | \mathcal{F}_t). \end{aligned}$$

Since $Z_u \Delta V_u \leq 0$, we deduce that $E(Z_T V_T | \mathcal{F}_t) \leq V_t Z_t$. Condition **S2** naturally holds in the financial models with transaction costs. Indeed, if $\xi_t \in L^\infty(-G_t, \mathcal{F}_t)$ then $V_u = \xi_t I_{u>t}$ is a portfolio process whose only jump is $\Delta V_t = \xi_t \in -G_t$ and we have $\xi_t = V_T$.

For a given payoff $\xi \in L_b^0(\mathbf{R}^d)$ (i.e. bounded from below with respect to the partial ordering induced by G_T), we consider the convex set

$$\Gamma_\xi := \left\{ x \in \mathbf{R}^d : \exists Y_T \in \mathcal{Y}_{0,b}^T(T) \text{ s.t. } x + Y_T \geq_{G_T} \xi \right\} \quad (2.2)$$

and the closed convex set

$$D_\xi := \left\{ x \in \mathbf{R}^d : Z_0 x \geq E Z_T \xi, \forall Z \in \mathcal{M}_0^T(G^*) \right\}. \quad (2.3)$$

We assume given a dual characterization of Γ_ξ in section 3:

$$\mathbf{S3} : \Gamma_\xi = D_\xi.$$

This property is usually an important result, referred to as the “hedging theorem”. It generally holds under some no-arbitrage conditions (see, e.g., [2], [1] and [4]).

3 Asymptotic Arbitrage via Consistent Price Systems

We fix a sequence $(\Omega^n, \mathcal{F}^n, \mathbf{F}^n = (\mathcal{F}_t^n)_{t \leq T}, P^n)$ of continuous-time stochastic basis verifying the usual conditions with $\mathcal{F}^n = \mathcal{F}_T^n$. The positive number T is interpreted as a time horizon and may depend on n . We are given a pair of set-valued adapted processes $G^n = (G_t^n)_{t \leq T}$ and $G^{n*} = (G_t^{n*})_{t \leq T}$ whose values are closed cones in \mathbf{R}^d which are dual and define the corresponding models of Subsection 2.2. Recall that we assume that conditions **S1** and **S2** hold. For the sake of simplicity, we often omit the index n .

Definition 1 A sequence of portfolios (\widehat{V}^n) realizes an **asymptotic arbitrage opportunity of the first kind** if there exists a sequence (x^n) such that the following holds for a subsequence:

- 1.a) $\widehat{V}_T^n \in x^n + \mathcal{Y}_{0,b}^T(T)$,
- 1.b) $\widehat{V}_T^n \in G_T$,
- 1.c) $x^n \rightarrow 0$,
- 1.d) $\lim_n P(\widehat{V}_T^n \geq_{G_T} \mathbf{1}) > 0$.

We associate with every $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ the equivalent probability measure $dQ^Z := (1/Z_0 \mathbf{1}) Z_T \mathbf{1} dP$ and we define the convex set

$$\mathcal{Q}^n = \left\{ Q^Z : Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 \mathbf{1} = 1 \right\}.$$

We assume that \mathcal{Q}^n is not empty meaning that the No Free Lunch (NFL) condition holds, [3], for each model.

We then define the upper and lower envelopes of the measures of \mathcal{Q}^n as follows:

$$\overline{\mathcal{Q}}^n(A) := \sup_{Q \in \mathcal{Q}^n} Q(A), \quad \underline{\mathcal{Q}}^n(A) := \inf_{Q \in \mathcal{Q}^n} Q(A).$$

Definition 2 The sequence (P^n) is **contiguous** with respect to $(\overline{\mathcal{Q}}^n)$ (in symbols: $(P^n) \triangleleft (\overline{\mathcal{Q}}^n)$) when the implication

$$\lim_n \overline{\mathcal{Q}}^n(A^n) = 0 \Rightarrow \lim_n P^n(A^n) = 0$$

holds for any sequence $A^n \in \mathcal{F}^n$, $n \geq 1$.

Now, we give the first result of this section:

Proposition 3.1 *Assume that Assumption **S3** holds. Then the following conditions are equivalent:*

- (a) *there is no asymptotic arbitrage opportunity of the first kind (NAA1);*
- (b) $(P^n) \triangleleft (\overline{\mathcal{Q}}^n)$.

Proof.

• (a) \Rightarrow (b). Suppose that there exists a sequence $A^n \in \mathcal{F}^n$ such that $\overline{Q}^n(A^n)$ tends to 0 and $P(A^n) \rightarrow \alpha > 0$. We consider $F^n = I_{A^n} \mathbf{1}$ as a contingent claim and $x^n = \overline{Q}^n(A^n) \mathbf{1}$ as an initial endowment. For any $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$, we have immediately by definition

$$Z_0 x^n \geq EZ_T F^n.$$

By virtue of Assumption **S3**, we deduce that $F^n \in x^n + \mathcal{Y}_{0,b}^T(T)$ so that the sequence (F^n) realizes an asymptotic arbitrage opportunity of the first kind.

• (b) \Rightarrow (a). Suppose that there exists a sequence (\widehat{V}^n) realizing an asymptotic arbitrage opportunity of the first kind. Consider $Q \in \mathcal{Q}^n$ defined by $dQ = Z_T \mathbf{1} dP$. Then, according to Condition **S1**,

$$0 \leq EZ_T \widehat{V}_T^n \leq Z_0 x^n \leq |x^n|$$

since $x^n \leq_{G_0} |x^n| \mathbf{1}$. Moreover,

$$EZ_T \widehat{V}_T^n \geq EZ_T \widehat{V}_T^n I_{\widehat{V}_T^n \geq_{G_T} \mathbf{1}} \geq EZ_T \mathbf{1} I_{\widehat{V}_T^n \geq_{G_T} \mathbf{1}} = Q(\widehat{V}_T^n \geq_{G_T} \mathbf{1}).$$

It follows that

$$\overline{Q}^n(\widehat{V}_T^n \geq_{G_T} \mathbf{1}) \leq |x^n|$$

and $\overline{Q}^n(\widehat{V}_T^n \geq_{G_T} \mathbf{1}) \rightarrow 0$ which implies $P(\widehat{V}_T^n \geq_{G_T} \mathbf{1}) \rightarrow 0$ in contradiction with 1.d). ■

Remark 3.2 As shown in [13], the condition (b) is equivalent to the following:

(c) there exists a sequence $(R^n) \in \mathcal{Q}^n$ such that $(P^n) \triangleleft (R^n)$.

Let us recall the financial meaning of the following definition. There is an asymptotic arbitrage of the second kind if the agent, selling short his portfolio, achieves almost a non risky positive profit.

Definition 3 A sequence of portfolios (\widehat{V}^n) realizes an **asymptotic arbitrage opportunity of the second kind** if there exists a subsequence satisfying:

3.a) $\widehat{V}_T^n \geq_{G_T} -\mathbf{1}$,

3.b) $\lim_n P(\widehat{V}_T^n \not\geq_{G_T} -\varepsilon \mathbf{1}) = 0, \forall \varepsilon \in]0, 1[$,

3.c) there exists a bounded sequence of initial endowments $(x^n \mathbf{1})$, with $x^n \in \mathbf{R}$, satisfying $\widehat{V}_T^n \in x^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ and $x^\infty := \liminf_n x^n < 0$.

Remark 3.3 It is an easy exercise to notice that the definition of the asymptotic arbitrage of the second kind can be written equivalently if one considers in 3.c) a bounded sequence of initial endowments $x^n \in \mathbf{R}^d$ satisfying $\widehat{V}_T^n \in x^n + \mathcal{Y}_{0,b}^T(T)$, and such that $\liminf_n \max_{i \leq d} x^{n,i} < 0$. Indeed, if x^n is an initial endowment of \widehat{V}_T^n , then $y^n := (\max_{i \leq d} x^{n,i}) \mathbf{1}$ is still an initial endowment for \widehat{V}_T^n .

In the same manner, we can equivalently define the asymptotic arbitrage of the first kind using a sequence of initial endowments of the form $(x^n \mathbf{1})$, with $x^n \in \mathbf{R}$, but for our purposes it is more convenient to consider the definition with an initial endowment $x^n \in \mathbf{R}^d$.

The next condition is only introduced to give an equivalent characterization of the asymptotic arbitrage of the second kind:

Assumption (B₀) If ξ is a \mathcal{F}_0 -measurable \mathbf{R}^d -valued random variable such that $Z_0\xi \geq 0$ for any $Z \in \mathcal{M}_0^T(G^*)$, then $\xi \in G_0$ (a.s.).

Remark 3.4 Assumption (B₀) appears as a weaker form of the No Arbitrage Condition of the second kind introduced by Denis and Kabanov in their recent work, [3], (it was introduced the first time by Rásonyi for discrete time models, [19]). The so-called condition (B), [14], is the following:

(B) If ξ is a \mathcal{F}_t -measurable \mathbf{R}^d -valued random variable such that $Z_t\xi \geq 0$ for any $Z \in \mathcal{M}_0^T(G^*)$, then $\xi \in G_t$ (a.s.).

Condition (B) is stronger than (B₀) and, as noticed in [14], it is fulfilled for the models with constant transaction costs admitting an equivalent martingale measure.

Remark 3.5 In the case where we interpret the first component of the price process as the numéraire, we may give a more economical sense to the last condition. Indeed, under Assumption (B₀), it also means that the agent sells short his portfolio in the numéraire but achieves almost a non risky positive profit as proven in the following. We denote $e_1 := (1, 0, \dots, 0) \in \mathbf{R}^d$. Let us introduce the following statement:

3.c') There exists a bounded sequence of initial endowments $(x^n e_1)$, $x^n \in \mathbf{R}$, satisfying $\widehat{V}_T^n \in x^n e_1 + \mathcal{Y}_{0,b}^T(T)$ and $x^\infty := \liminf_n x^n < 0$.

Lemma 3.6 *Suppose that Assumption (B₀) holds. Then, 3.c) \Rightarrow 3.c'). Moreover, if there exists $k > 0$ such that*

$$\min_{Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 \mathbf{1} = 1} Z_0 e_1 \geq k,$$

then 3.c') \Rightarrow 3.c).

Proof.

• 3.c) \Rightarrow 3.c') Let $x^n \in \mathbf{R}$ be such that $\widehat{V}_T^n \in x^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ and satisfying 3.c). The first step is to find a number $\tilde{x}^n \in \mathbf{R}$ such that $\tilde{x}^n e_1 \geq_{G_0} x^n \mathbf{1}$. This is equivalent to say that $Z_0 \tilde{x}^n e_1 \geq Z_0 x^n \mathbf{1}$ whatever $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$. Assuming, without loss of generality, that $Z_0 e_1 = 1$, the above inequality holds iff $\tilde{x}^n \geq (Z_0 \mathbf{1}) x^n$. Choosing

$$\tilde{x}^n = x^n \max_{Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 e_1 = 1} Z_0 \mathbf{1},$$

the above requirement is fulfilled. It is not difficult to see that \tilde{x}^n is finite. Indeed, suppose $Z_0^k \mathbf{1} \rightarrow \infty$ as $k \rightarrow \infty$ with $Z_0^k e_1 = 1$. Then, $z_k^* := Z_0^k / (Z_0^k \mathbf{1}) \in G_0^*$ is a bounded sequence. We may assume by compactity that $z_k^* \rightarrow z_\infty^* \in G_0^*$. Since $z_k^* \mathbf{1} = 1$, we get $z_\infty^* \mathbf{1} = 1$. On the other hand, $z_k^* e_1 \rightarrow 0$ hence $z_\infty^* e_1 = 0$ which leads to $z_\infty^* = 0$ since $G_0^* \subseteq \text{int } \mathbf{R}_+^d$, hence a contradiction.

Using the hypothesis and $\tilde{x}^n e_1 \geq_{G_0} x^n \mathbf{1}$, we get that $\widehat{V}_T^n \in \tilde{x}^n e_1 + \mathcal{Y}_{0,b}^T(T)$. Indeed, $\widehat{V}_T^n \in \tilde{x}^n e_1 + (x^n \mathbf{1} - \tilde{x}^n e_1) + \mathcal{Y}_{0,b}^T(T)$ where $(x^n \mathbf{1} - \tilde{x}^n e_1) \in L^0(-G_0, \mathcal{F}_0)$. Now applying 3.c), we obtain that $\liminf_n \tilde{x}^n < 0$, i.e. 3.c') holds.

• 3.c') \Rightarrow 3.c) Let $x^n \in \mathbf{R}$ be such that $\widehat{V}_T^n \in x^n e_1 + \mathcal{Y}_{0,b}^T(T)$ and let 3.c') holds. Writing

$$\widehat{V}_T^n \in x^n \mathbf{1} + (x^n e_1 - x^n \mathbf{1}) + \mathcal{Y}_{0,b}^T(T),$$

there are two cases:

1. If $x^n \in \mathbf{R}_+$, then $(x^n e_1 - x^n \mathbf{1}) = (0, -x^n, \dots, -x^n) \in L^0(-G_0, \mathcal{F}_0) \subseteq \mathcal{Y}_{0,b}^T(T)$. Therefore, $\widehat{V}_T^n \in x^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ hence 3.c) holds.
2. Consider $x^n \in \mathbf{R}_-$. Following the same procedure as in the first implication, we can find a finite number

$$\tilde{x}^n = x^n \min_{Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 \mathbf{1} = 1} Z_0 e_1$$

such that $\tilde{x}^n \mathbf{1} \geq_{G_0} x^n e_1$. It follows that $(x^n e_1 - \tilde{x}^n \mathbf{1}) \in L^0(-G_0, \mathcal{F}_0) \subseteq \mathcal{Y}_{0,b}^T(T)$ and from here we have that $\widehat{V}_T^n \in \tilde{x}^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$. Now, knowing 3.c') and the additional hypothesis, we get that $\liminf_n \tilde{x}^n < 0$, i.e. 3.c) holds. ■

To formulate the next result, we give the following definition:

Definition 4 The sequence of sets of probability measures (\mathcal{Q}^n) is said to be **weakly contiguous** with respect to (P^n) and we denote $(\mathcal{Q}^n) \triangleleft_w (P^n)$ if whatever $\varepsilon > 0$, there is $\delta > 0$ and a sequence of measures $Q^n \in \mathcal{Q}^n$ such that for any sequence $A^n \in \mathcal{F}^n$ with the property $\limsup_n P^n(A^n) < \delta$, we have $\limsup_n Q^n(A^n) < \varepsilon$.

Proposition 3.7 Assume that Assumption **S3** holds. Then the following conditions are equivalent:

- (a) there is no asymptotic arbitrage opportunity of the second kind (NAA2);
- (b) $(\underline{\mathcal{Q}}^n) \triangleleft (P^n)$.

Proof.

• (a) \Rightarrow (b). Suppose that there exists a sequence $A^n \in \mathcal{F}^n$ such that $P^n(A^n)$ tends to 0 and $\underline{\mathcal{Q}}^n(A^n) \rightarrow \alpha > 0$. We define the contingent claim $F^n = -I_{A^n} \mathbf{1}$ we may interpret as the terminal value of a portfolio since it is replicable (e.g. by 0). Consider the bounded sequence $y^n := x^n \mathbf{1} := -\underline{\mathcal{Q}}^n(A^n) \mathbf{1}$ of super-hedging prices for F^n , i.e. $y^n \in \Gamma_{F^n}$. Indeed, for any $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ with $Z_0 \mathbf{1} = 1$, $Z_0 y^n = -\underline{\mathcal{Q}}^n(A^n) \geq -Q^Z(A^n) = EZ_T F^n$ and we conclude using Assumption **S3**. Since $\liminf_n x^n = x^\infty < 0$, the sequence (F^n) is an asymptotic arbitrage opportunity of the second kind.

• (b) \Rightarrow (a). Suppose that there exists a sequence of portfolios (\widehat{V}^n) realizing an asymptotic arbitrage opportunity of the second kind. Let us consider a sequence (x^n) such that $\widehat{V}_T^n \in x^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ with $x^\infty := \liminf_n x^n < 0$. Under Assumption **S3**, for any $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ with $Z_0 \mathbf{1} = 1$, we have that

$$\begin{aligned} x^n &= Z_0 x^n \mathbf{1} \geq EZ_T \widehat{V}_T^n = EZ_T \widehat{V}_T^n I_{\{\widehat{V}_T^n \geq -\varepsilon \mathbf{1}\}} + EZ_T \widehat{V}_T^n I_{\{\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}\}}, \\ x^n &\geq -\varepsilon Q^Z(\widehat{V}_T^n \geq -\varepsilon \mathbf{1}) - Q^Z(\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}), \\ &\geq -\varepsilon + (\varepsilon - 1)Q^Z(\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}) \end{aligned}$$

where $\varepsilon \in (0, 1)$ is arbitrarily chosen. Since the property 3.b) holds, $(\underline{Q}^n) \triangleleft (P^n)$ implies that $\underline{Q}^n(\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}) \rightarrow 0$. We choose, for each n , $Z^n \in \mathcal{M}_0^T(G^* \setminus \{0\})$ such that $Q^{Z^n}(\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}) \leq \underline{Q}^n(\widehat{V}_T^n \not\geq -\varepsilon \mathbf{1}) + n^{-1}$. From above, we deduce that $\liminf_n x^n \geq -\varepsilon$ whatever $\varepsilon \in (0, 1)$ which yields a contradiction. ■

Remark 3.8 As shown in [13], (b) \Leftrightarrow (c) \Leftrightarrow (d) where:

$$(c) (\underline{Q}^n) \triangleleft_w (P^n);$$

$$(d) \lim_{K \rightarrow \infty} \limsup_n \inf_{Q \in \underline{Q}^n} Q \left(\frac{dQ}{dP^n} \geq K \right) = 0.$$

Definition 5 A sequence of portfolios (\widehat{V}^n) realizes a **strong asymptotic arbitrage opportunity of the first kind** if there exists a sequence (x^n) such that the following holds for a subsequence:

$$5.a) \widehat{V}_T^n \in x^n + \mathcal{Y}_{0,b}^T(T),$$

$$5.b) \widehat{V}_T^n \geq_{G_T} 0,$$

$$5.c) x^n \rightarrow 0,$$

$$5.d) \lim_n P \left(\widehat{V}_T^n \geq_{G_T} \mathbf{1} \right) = 1.$$

Definition 6 A sequence of portfolios (\widehat{V}^n) realizes a **strong asymptotic arbitrage opportunity of the second kind** if for a subsequence :

$$6.a) \widehat{V}_T^n \geq_{G_T} -\mathbf{1},$$

$$6.b) \lim_n P \left(\widehat{V}_T^n \not\geq_{G_T} -\varepsilon \mathbf{1} \right) = 0, \forall \varepsilon \in]0, 1[.$$

6.c) there exists a bounded sequence of initial endowments $(x^n \mathbf{1})$, with $x^n \in \mathbf{R}$, satisfying $\widehat{V}_T^n \in x^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ and $\liminf_n x^n = -1$.

Lemma 3.9 *There exists a strong asymptotic arbitrage of the first kind if and only if there is a strong asymptotic arbitrage of the second kind.*

Proof.

- Take any sequence (\widehat{V}^n) realizing a strong asymptotic arbitrage opportunity of the first kind. We want to construct a sequence realizing a strong asymptotic arbitrage of the second kind. Define the sequence $\widehat{U}^n = -\mathbf{1} + \widehat{V}^n$. Using 5.b), we obtain that $\widehat{U}_T^n \geq_{G_T} -\mathbf{1}$, which is exactly the condition 6.a) of the definition of the asymptotic arbitrage opportunity of the second kind.

We have

$$P(\widehat{U}_T^n \not\geq_{G_T} -\varepsilon \mathbf{1}) = 1 - P(\widehat{U}_T^n \geq_{G_T} -\varepsilon \mathbf{1}) \leq 1 - P(\widehat{V}_T^n \geq_{G_T} \mathbf{1}) \rightarrow 0, n \rightarrow \infty$$

which shows the condition 6.b).

We only have to prove the condition 6.c). The condition 5.a) holds so that $\widehat{V}_T^n \in y^n + \mathcal{Y}_{0,b}^T(T)$ where $y^n \rightarrow 0$. We deduce that $\widehat{V}_T^n \in \alpha^n \mathbf{1} + \mathcal{Y}_{0,b}^T(T)$ where $\alpha^n := \max_{i \leq d} y^{ni}$ and $\alpha^n \rightarrow 0$. It suffices to consider $x^n := \alpha^n - 1$ to conclude.

- Take any sequence (\widehat{U}^n) realizing a strong asymptotic arbitrage opportunity of the second kind. We define a sequence realizing a strong asymptotic arbitrage

opportunity of the first kind choosing the sequence $\widehat{V}^n = \widehat{U}^n + \mathbf{1}$. We only prove condition 5.c). It suffices to observe that

$$P\left(\widehat{V}_T^n \not\geq_{G_T} \mathbf{1}\right) = P\left(\widehat{U}_T^n \not\geq_{G_T} 0\right) \leq \liminf_{\varepsilon \rightarrow 0, \varepsilon \in \mathbf{Q}^+} P\left(\widehat{U}_T^n \not\geq_{G_T} -\varepsilon \mathbf{1}\right)$$

where \mathbf{Q}^+ is the set of all strictly positive rational numbers. Taking any arbitrary $\delta > 0$, we get that $P\left(\widehat{V}_T^n \not\geq_{G_T} \mathbf{1}\right) \leq \delta + P\left(\widehat{U}_T^n \not\geq_{G_T} -\varepsilon \mathbf{1}\right)$ for some $\varepsilon = \varepsilon(\delta)$. Using 6.b), we obtain $\lim_n P\left(\widehat{V}_T^n \not\geq_{G_T} \mathbf{1}\right) \leq \delta$ and then $\lim_n P\left(\widehat{V}_T^n \not\geq_{G_T} \mathbf{1}\right) = 0$ as $\delta \rightarrow 0$. ■

Definition 7 A sequence (P^n) is **(entirely) asymptotically separable** from $(\overline{\mathcal{Q}}^n)$, notation $(P^n)\Delta(\overline{\mathcal{Q}}^n)$, if there exists a subsequence (m) with sets $A^m \in \mathcal{F}^m$ such that

$$\lim_m \overline{\mathcal{Q}}^m(A^m) = 0, \quad \lim_m P^m(A^m) = 1.$$

Proposition 3.10 Assume that **S3** holds. Then the following conditions are equivalent:

- (a) there is a strong asymptotic arbitrage opportunity of the first kind (SAA1);
- (b) $(P^n)\Delta(\overline{\mathcal{Q}}^n)$;
- (c) $(\underline{\mathcal{Q}}^n)\Delta(P^n)$.

Proof.

• (a) \Rightarrow (b) Assume there exists a sequence of portfolios (\widehat{V}^n) realizing a strong asymptotic arbitrage opportunity of the first kind. This means that there exists a subsequence (m) such that

$$\lim_m P^m(\widehat{V}_T^m \geq_{G_T} \mathbf{1}) = 1, \quad \lim_m \widehat{V}_0^m = 0.$$

Following the arguments of the proof of Proposition 3.1, the implication (b) \Rightarrow (a), we obtain that $\lim_m \overline{\mathcal{Q}}^m(\widehat{V}_T^m \geq_{G_T} \mathbf{1}) = 0$. We take the sets $A^m := \{\widehat{V}_T^m \geq_{G_T} \mathbf{1}\}$ for the separating sequence.

• (b) \Rightarrow (a) Assume $(P^n)\Delta(\overline{\mathcal{Q}}^n)$. Then, there exists a sequence (m) with sets $A^m \in \mathcal{F}^m$ such that

$$\lim_m \overline{\mathcal{Q}}^m(A^m) = 0, \quad \lim_m P^m(A^m) = 1.$$

Using the arguments in the proof of Proposition 3.1, the implication (a) \Rightarrow (b), but with $\alpha = 1$, we obtain a sequence of portfolios realizing a strong asymptotic arbitrage opportunity. ■

4 Variant for markets with a numéraire

We consider markets whose first component of the price process S is a numéraire (the cash B) in which the portfolios are liquidated. The asymptotic arbitrage opportunity concepts are defined similarly as in Section 3 but here we are concerned by the portfolios starting with an initial endowment expressed in cash and which are liquidated at the horizon date. Moreover, it is possible to avoid **Assumption S3**

if we focus on asymptotic arbitrage in the spirit of the Kreps–Yann arbitrage theory, i.e. by extending the set of all portfolio processes to its weak closure $\overline{\mathcal{Y}_{0,b}^{T,\infty}}^w$ in L^∞ . In this case, we use the dual characterization of Lemma 6.1 which holds only under the conditions **(S1)** and **(S2)**.

Definition 8 A sequence of portfolios (\widehat{V}^n) realizes an asymptotic arbitrage of the first kind if for a subsequence there exists a sequence $(x^n) \in \mathbf{R}_+$ such that:

- 8.a) $\widehat{V}_T^n \in x^n e_1 + \overline{\mathcal{Y}_{0,b}^{T,\infty}}^w$,
- 8.b) $\widehat{V}_T^n \in G_T$ a.s.
- 8.c) $x^n \rightarrow 0$,
- 8.d) $\lim_n P\left(\widehat{V}_T^n \geq_{G_T} e_1\right) > 0$.

Definition 9 A sequence of portfolios (\widehat{V}^n) realizes an **asymptotic arbitrage opportunity of the second kind** if there exists a subsequence satisfying:

- 9.a) $\widehat{V}_T^n \geq_{G_T} -e_1$,
- 9.b) $\lim_n P\left(\widehat{V}_T^n \not\geq_{G_T} -\varepsilon e_1\right) = 0, \forall \varepsilon \in]0, 1[$,
- 9.c) There exists a bounded sequence of initial endowments $(x^n e_1)$, with $x^n \in \mathbf{R}$, satisfying $\widehat{V}_T^n \in x^n e_1 + \mathcal{Y}_{0,b}^T(T)$ and $x^\infty := \liminf_n x^n < 0$.

In this setting, we define for each $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$, $Q^Z \sim P$ such that

$$\frac{dQ^Z}{dP} = \frac{Z_T e_1}{Z_0 e_1}$$

and we define the convex set:

$$\mathcal{Q}^n = \left\{ Q^Z, Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 e_1 = 1 \right\}.$$

Notice that in the frictionless case, a consistent price system is a process having the form $Z_t = \rho_t S_t$, $\rho_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. If the first component $S^{(1)} = 1$, i.e. the interest rate of the bond $r = 0$, then $Z_0 e_1 = 1$ means that ρ is a density process or equivalently $dQ = Z_T e_1 dP$ defines an equivalent martingale measure under which S is a martingale. We may interpret our definition as an extension of that of [13]. Consider the upper and lower envelopes of the measures of \mathcal{Q}^n as previously. We then obtain similar results.

Actually, the two approaches turn out to be equivalent under the condition **(B₀)** we introduced above and the additional hypothesis that the sequence (depending on n via the cone and the horizon date)

$$n \mapsto \min_{Z \in \mathcal{M}_0^T(G^* \setminus \{0\}), Z_0 \mathbf{1} = 1} Z_0 e_1$$

is bounded from below by a strictly positive constant (independent of n). Indeed, in this case, we can find $\alpha, \beta > 0$ such that $\beta e_1 \geq_{G_0} \mathbf{1} \geq_{G_0} \alpha e_1$. It is then easy to construct an asymptotic arbitrage opportunity of the first kind (respectively of the second kind) following the former definition from an asymptotic arbitrage opportunity of the first kind (respectively of the second kind) according to the variant approach and vice-versa.

5 Examples

Throughout this section, we consider a continuous-time financial model with transaction costs defined as in [4], i.e. in the setting of the Kabanov and Campi-Schachermayer models.

5.1 Example of an asymptotic arbitrage of the first kind in a two-dimensional setting

We consider a sequence of markets whose horizon dates are $T^n = 1$ for all $n \geq 1$. We assume that the transaction costs coefficients belong to $(0, 1]$. The dynamics of the price processes $(S_t^n)_{t \leq 1}$ are given under a given probability measure P by

$$\begin{aligned} \frac{dS_t^{n1}}{S_t^{n1}} &= dW_t - \frac{1}{(1+n^{-1}-t)^{3/2}} dt, \\ S_t^{n2} &= n = S_0^{n1}, \quad \forall t \leq 1. \end{aligned}$$

We denote $\xi^n := e^{2n^{1/2}}(3 + e^{W_1^+})^{-1}$ and

$$\widetilde{W}_t := W_t - \int_0^t \frac{du}{(1+n^{-1}-u)^{3/2}}, \quad t \leq 1.$$

so that $\widetilde{W}_1 = W_1 - 2n^{1/2} + 2(1+n^{-1})^{-1/2}$. Observe that the Novikov condition holds. It follows that \widetilde{W} is a Brownian motion under an equivalent probability measure $P^n \sim P$ by virtue of the Girsanov theorem. We deduce that the price process $S^n = (S^{n1}, S^{n2})$ is a P^n -strictly consistent price system following the terminology of [4], i.e. $S_\tau^n \in \text{int } G_\tau^*$ whatever the stopping time $\tau \leq 1$ and $S_{\tau-}^n \in \text{int } G_{\tau-}^*$ if $\tau \in [0, 1]$ is a predictable stopping time. It is then straightforward that the sequence of market models we consider, endowed with P^n , satisfy Condition **S3** by virtue of [4]. Each market of this sequence satisfies the No Arbitrage condition $\mathcal{Y}_{0,b}^1(1) \cap L^0(\mathbf{R}_+^2) = \{0\}$. Indeed, if $\widehat{V}_1 \in \mathcal{Y}_{0,b}^1(1) \cap L^0(\mathbf{R}_+^2)$, then $E_{P^n} S_1^n \widehat{V}_1$ is both negative and positive under **(S3)**, i.e. $S_1^n \widehat{V}_1 = 0$, and therefore $\widehat{V}_1 = 0$. However, we can construct an asymptotic arbitrage opportunity as follows. First notice that

$$S_1^{n1} = n \exp \left\{ W_1 - 2n^{1/2} + 2(1+n^{-1})^{-1/2} - 1/2 \right\}$$

and $S_1^{n1} \rightarrow 0$ a.s. as $n \rightarrow \infty$ under P . We consider the events

$$\Gamma_n := \left\{ \widetilde{W}_1 \in [-1/2, 0] \right\}$$

such that $P^n(\Gamma_n) = P(W_1 \in [-1/2, 0]) > 0$. Let us define the sequence of terminal wealths $\widehat{V}_1^n := x^n + \Delta \widehat{V}_1^n I_{\Gamma_n}$ as the terminal values of the portfolio processes equal to x^n on $[0, 1[$ and jumping at date $t = 1$ by $\Delta \widehat{V}_1^n I_{\Gamma_n} \in -G_1$. Recall that $G_1 := \widehat{K}_1$ following [14], i.e. is the set of all vectors $(X^1(S_1^{n1})^{-1}, X^2(S_1^{n2})^{-1})$, where $X = (X^1, X^2) \in K_1$ and

$$K_1 := \text{cone} \left\{ -e_1 + (1 + \lambda_1^{21})e_2; (1 + \lambda_1^{12})e_1 - e_2 \right\}.$$

It is clear that \widehat{V}_1^n is an element of $x^n + \mathcal{Y}_{0,b}^1(1)$, hence Condition 1.a) of Definition 1 is satisfied. Let the constant x^n be defined as

$$x^n := -\frac{1}{S_1^{n2}} \left(\alpha_2 - \alpha_1(1 + \lambda_1^{21}) \right) e_2 = -\frac{1}{n} \left(\alpha_2 - \alpha_1(1 + \lambda_1^{21}) \right) e_2$$

where $\alpha_1, \alpha_2 \in L^0([0, \infty), \mathcal{F}_1)$ are the coefficients defining $\Delta \widehat{V}_1^n$ as an element of $-G_1^n$, i.e. such that

$$\Delta \widehat{V}_1^n := \left(\frac{1}{S_1^{n1}} (\alpha_1 - \alpha_2(1 + \lambda_1^{12})), \frac{1}{S_1^{n2}} (\alpha_2 - \alpha_1(1 + \lambda_1^{21})) \right).$$

Let us choose $\alpha_1 := (\alpha_2 + 1)/(1 + \lambda_1^{21})$. This implies that

$$\alpha_1 - \alpha_2(1 + \lambda_1^{12}) = \frac{1 - \alpha_2 \lambda_1^{21} - \alpha_2 \lambda_1^{12}(1 + \lambda_1^{21})}{1 + \lambda_1^{21}}.$$

Then

$$\Delta \widehat{V}_1^n = \left(\frac{1}{S_1^{n1}} \frac{1 - \alpha_2 \lambda_1^{21} - \alpha_2 \lambda_1^{12}(1 + \lambda_1^{21})}{1 + \lambda_1^{21}}, -n^{-1} \right), \quad x^n = n^{-1} e_2.$$

In order for \widehat{V}_1^n to satisfy condition 1.b), i.e. $\widehat{V}_1^n \geq_{G_1} 0$, we have to impose that

$$0 \leq \alpha_2 \leq (\lambda_1^{21} + \lambda_1^{12}(1 + \lambda_1^{21}))^{-1}.$$

More precisely, we choose $\alpha_2 := (\lambda_1^{21} + \lambda_1^{12}(1 + \lambda_1^{21}) + e^{W_1^+})^{-1}$. Since the transaction costs coefficients belong to $(0, 1]$, we have that

$$1 - \alpha_2 \lambda_1^{21} - \alpha_2 \lambda_1^{12}(1 + \lambda_1^{21}) = \alpha_2 e^{W_1^+} \geq \frac{e^{W_1^+}}{3 + e^{W_1^+}} \geq C, \quad C := \inf_{x \geq 1} \frac{x}{3 + x}.$$

If $W_1 \leq 0$ then $S_1^{n1} \leq c e^{-2n^{1/2}}$ and the first component $(\Delta \widehat{V}_1^n)^{(1)} \geq \tilde{c} e^{2n^{1/2}}$, where c and \tilde{c} are some constants. If $W_1 \geq 0$, we obtain that

$$(\Delta \widehat{V}_1^n)^{(1)} \geq c \frac{e^{2n^{1/2}}}{3 + e^{W_1}} = c \xi^n,$$

for some constant c . Therefore, in both cases, there exists a constant c such that

$$\Delta \widehat{V}_1^n \geq_{\mathbf{R}_+^2} (c \xi^n, -n^{-1}).$$

We only have to prove that $\widehat{V}_1^n = x^n + \Delta \widehat{V}_1^n \geq_{G_1^n} \mathbf{1}$ a.s. on the events Γ_n and this will give us condition 1.d) of Definition 1. To do so, it suffices to find a.s. an element $g_1 \in G_1^n$ such that $\widehat{V}_1^n - g_1 \geq_{\mathbf{R}_+^2} \mathbf{1}$. Without loss of generality, it is sufficient to find $g_1 \in G_1^n$ such that $M \xi^n e_1 - g_1 \geq_{\mathbf{R}_+^2} \mathbf{1}$ where M is a constant independent of n we choose large enough (if needed we renormalize \widehat{V}_1^n). For this, we solve the following problem. Find $\beta_1, \beta_2 \geq 0$ such that the following inequality holds componentwise :

$$X_n := (M \xi^n, 0) + \left(\frac{1}{S_1^{n1}} (\beta_1 - \beta_2(1 + \lambda_1^{12})), \frac{1}{S_1^{n2}} (\beta_2 - \beta_1(1 + \lambda_1^{21})) \right) \geq (1, 1).$$

It is sufficient to take $\beta_1 := \beta_1^n = \sqrt{n}$ and $\beta_2 := \beta_2^n = 2n$. Since

$$\frac{1}{n} \left(\beta_2 - \beta_1(1 + \lambda_1^{21}) \right) \rightarrow 2, \quad n \rightarrow \infty,$$

the second component of X_n is greater than 1 for n large enough. Note that on the set Γ_n , we have $e^{-1}n \leq S_1^{n1} \leq n$. Moreover

$$\xi^n \geq \frac{e^{2n^{1/2}}}{3 + e^{2n^{1/2} - 2(1+n^{-1})^{-1/2}}} \geq c_0$$

provided that n is large enough, c_0 being a constant independent of n . It follows that

$$M\xi^n + \frac{1}{S_1^{n1}} \left(\beta_1 - \beta_2(1 + \lambda_1^{12}) \right) \geq Mc_0 + \frac{\sqrt{n}}{n} - 2e^1(1 + \lambda_1^{12}).$$

Choosing the constant M independently of n such that

$$Mc_0 \geq 1 + 4e^1,$$

we then conclude that the first component is also greater than 1 provided that n is large enough.

We have built in this example a sequence (\widehat{V}^n) which realizes an asymptotic arbitrage opportunity of the first kind even if each market satisfies a No Arbitrage condition. ■

Throughout the sequence, we assume that for each model the exchanges between assets are executed like in a “real world” where we go through the numéraire. To exchange some amount of the i th-asset into the j th-asset, sell the i th-assets, get the money in cash (i.e. the bond) and buy j th-assets with this cash. We model this assumption by the following:

$$\mathbf{RW}: (1 + \lambda^{i,b})(1 + \lambda^{b,j}) = 1 + \lambda^{i,j} \text{ for every } i, j = 0, 1, \dots, n \text{ and } i \neq j.$$

5.2 One-stage APM by Ross

We study the example of [13] under the variant approach and under the **RW** condition. Recall that we are given a sequence of independent random variables $(\epsilon_i)_{i \geq 0}$ on a probability space (Ω, \mathcal{F}, P) taking values in a finite interval $[-N, N]$. We suppose that $E\epsilon_i = 0$, $E\epsilon_i^2 = 1$. At time zero, asset prices are positive numbers X_0^i , $i \geq 0$. After a certain period (at time $T = 1$), their positive discounted values are given by the following relations:

$$\begin{aligned} X_1^0 &= X_0^0(1 + \mu_0 + \sigma_0\epsilon_0), \\ X_1^i &= X_0^i(1 + \mu_i + \sigma_i(\gamma_i\epsilon_0 + \bar{\gamma}_i\epsilon_i)), \quad i \geq 1. \end{aligned}$$

The coefficients are here deterministic, $\sigma_i > 0$, $\bar{\gamma}_i > 0$ and $\gamma_i^2 + \bar{\gamma}_i^2 = 1$, $\gamma_0 = 1$. The asset with number zero is interpreted as the market portfolio, γ_i is the correlation coefficient between the rate of return for the market portfolio and the rate of return for the asset with number i . For $n \geq 1$, we consider the stochastic basis $\mathcal{B}^n := (\Omega, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}_t^n)_{t=0,1}, P^n)$ with the $(n+1)$ -dimensional random process

$S^n = (X_t^0, \dots, X_t^n)_{t=0,1}$ where \mathcal{F}_0^n is the trivial algebra, $\mathcal{F}_1^n = \mathcal{F}^n = \sigma(\epsilon_0, \dots, \epsilon_n)$, and $P^n = P|\mathcal{F}^n$. We assume that the transaction costs coefficients of each model are constant and equal to λ^i , $i \geq 1$. They correspond to the exchanges from the risky assets number i , $i \geq 1$, to the bond (assumed to be constant and equal to 1), as well as from the bond to the risky assets. Moreover, we assume that there are no transaction costs regarding the exchanges between the bond and the portfolio market X^0 , i.e. $\lambda^0 := 0$. We suppose that there exists a constant k such that

$$\frac{1}{1 - \lambda^i} \leq \frac{k}{1 + \lambda^i}. \quad (5.4)$$

This assumption is not too restrictive from a practical point of view. For instance, if $\lambda^i \leq 0.5$ for all i , then $k = 3$. More generally, the assumption means that there exists $\lambda^* \in (0, 1)$ such that $\lambda^i < \lambda^*$, $\forall i$. The sequence $\mathcal{M} = \{(B^n, (1, S^n), 1)\}$ is a large security market by our definition. We may rewrite the dynamics as in [13]:

$$\begin{aligned} X_1^0 &= X_0^0(1 + \sigma_0(\epsilon_0 - b_0)), \\ X_1^i &= X_0^i(1 + \sigma_i\gamma_i(\epsilon_0 - b_0) + \sigma_i\bar{\gamma}_i(\epsilon_i - b_i)), \quad i \geq 1 \end{aligned}$$

where

$$b_0 := -\frac{\mu_0}{\sigma_0}, \quad b_i := \frac{\mu_0\beta_i - \mu_i}{\sigma_i\bar{\gamma}_i}, \quad \beta_i := \gamma_i\sigma_i/\sigma_0, \quad i \geq 1.$$

Let F_i be the distribution function of ϵ_i . Put

$$\underline{s}_i := \inf\{t : F_i(t) > 0\}, \quad \bar{s}_i := \inf\{t : F_i(t) = 1\},$$

$\underline{d}_i := b_i - \underline{s}_i$, $\bar{d}_i := \bar{s}_i - b_i$, and $d_i^0 := \underline{d}_i \wedge \bar{d}_i$. As in [13], we suppose that $d_i^0 \geq 0$. Moreover, let us define:

$$d_i := d_i^0 + \frac{4\lambda^i}{(1 + \lambda^i)\sigma_i\bar{\gamma}_i} := d_i^0 + f_i, \quad i \geq 1, \quad d_0 := d_0^0 := d_0^0 + f_0.$$

As in [13], we suppose that each model has an equivalent probability measure so that there exists also a strictly consistent price system. In particular, we have $|b_i| < N$ and, without loss of generality, we assume that $N > 1$.

Let us consider the following conditions:

$$\mathbf{C2}: \limsup_i \sqrt{b_i^2 + \frac{2\lambda^i}{(1 - \lambda^i)\sigma_i\bar{\gamma}_i}} = 0.$$

$$\mathbf{P2}: \limsup_i \left(|b_i| - 2\frac{\lambda^i}{(1 - \lambda^i)\sigma_i\bar{\gamma}_i} \right) \leq 0 \text{ and } \limsup_i \frac{\lambda^i}{(1 - \lambda^i)\sigma_i\bar{\gamma}_i} \in (0, \infty).$$

Proposition 5.1 *The following statements hold:*

- (a) $\inf_i d_i = 0 \Leftrightarrow \text{SAA1}$;
- (b) $\inf_i d_i > 0 \Leftrightarrow \text{NAA1}$;
- (c) $\mathbf{C2}$ or $\mathbf{P2} \Leftrightarrow \text{NAA2}$.

Proof. Under Condition **RW**, we may assume without loss of generality that the only exchanges occur between the bond and the risky assets, i.e. there is no exchange between two risky assets. Recall that, in this model, there are no transaction costs between the bond and the portfolio market. Then, the terminal value of a portfolio, once liquidated, can be expressed as follows:

$$V_1^n = x^n + \sum_{i=0}^n \phi_i (X_1^i - X_0^i) - \sum_{i=1}^n \lambda^i |\phi_i| (X_0^i + X_1^i)$$

where $(\phi_i)_{i=0,\dots,n}$ is the composition of the portfolio at date zero in the risky assets and x^n is the initial endowment expressed in the bond. The first two terms of V_1^n represent respectively the initial endowment and the variations of the portfolio due to the price movements. The last one corresponds respectively to the transaction costs that have to be paid due to the passage from x^n to ϕ and to the liquidation of the portfolio at date 1. We use the notations of [13]:

$$a_0 := \sum_{i=0}^n \phi_i X_0^i \sigma_i \gamma_i, \quad a_i := \phi_i X_0^i \sigma_i \bar{\gamma}_i, \quad i \geq 1.$$

The terminal value of the portfolio can be rewritten as:

$$V_1^n = x^n + \sum_{i=0}^n a_i (\epsilon_i - b_i) - \sum_{i=1}^n |\phi_i| \lambda^i (X_0^i + X_1^i) \quad (5.5)$$

$$= x^n + \sum_{i=0}^n a_i (\epsilon_i - b_i) - 2 \sum_{i=1}^n \lambda^i |\phi_i| X_0^i - \sum_{i=1}^n \lambda^i |\phi_i| (X_1^i - X_0^i) \quad (5.6)$$

$$= x^n + \sum_{i=0}^n \alpha_i (\epsilon_i - b_i) - 2 \sum_{i=1}^n \lambda^i \frac{|a_i|}{\sigma_i \bar{\gamma}_i} \quad (5.7)$$

where

$$\alpha_0 := a_0 - \sum_{i=1}^n \lambda^i |a_i| \frac{\gamma_i}{\bar{\gamma}_i}, \quad \alpha_i := a_i - \lambda^i |a_i|, \quad i \geq 1.$$

Note that, for $i \geq 1$, $a_i = \alpha_i / (1 - \lambda^i)$ if $\alpha_i \geq 0$ and $a_i = \alpha_i / (1 + \lambda^i)$ if $\alpha_i \leq 0$ so that $(\alpha_i)_{i=0,\dots,n}$ are uniquely determined and vice-versa.

• Assume that $\inf_i d_i = 0$. Then, there exists a subsequence (i_k) such that $d_{i_k}^0 + f_{i_k} < 2^{-i_k}$. We then construct a strong asymptotic arbitrage opportunity only using the risky assets corresponding to this subsequence. We follow the proof of [13]. We set $\alpha_i^{2n} := 1_{\bar{\Gamma} \cap \{i \geq n+1\}} - 1_{\Gamma \cap \{i \geq n+1\}}$, $i \geq n+1$, where $\Gamma := \{i : \bar{d}_i < \underline{d}_i\}$ and $\bar{\Gamma}$ is the complementary of Γ . Note that there is an abuse of notation as in [13]. The number $2n$ means that we work with the model in which we consider the $2n$ assets whose indices belong to the subsequence (i_k) . In other words we only trade the assets having the same indices than the subsequence. As in [13] but taking $x^{2n} := 2^{-n}(1+k)$ we deduce that

$$\begin{aligned} V_1^{2n} &\geq \sum_{i=n+1}^{2n} ((\bar{s}_i - \epsilon_i)1_{\Gamma} + (\epsilon_i - \underline{s}_i)1_{\bar{\Gamma}}) + 2^{-n} - \sum_{i=n+1}^{2n} (\bar{d}_i 1_{\Gamma} + \underline{d}_i 1_{\bar{\Gamma}}) \\ &\quad + k2^{-n} - 2 \sum_{i=n+1}^{2n} \lambda^i \frac{1}{(1 - \lambda^i \text{sign}(\alpha_i)) \sigma_i \bar{\gamma}_i}. \end{aligned}$$

Observe that

$$\lambda^i \frac{1}{(1 - \lambda^i \text{sign}(\alpha_i)) \sigma_i \bar{\gamma}_i} \leq \lambda^i \frac{1}{(1 - \lambda^i) \sigma_i \bar{\gamma}_i} \leq \lambda^i \frac{k}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \leq k f_i / 4 \leq k 2^{-i} / 4.$$

It follows that

$$V_1^{2n} \geq \sum_{i=n+1}^{2n} ((\bar{s}_i - \epsilon_i) 1_{\Gamma} + (\epsilon_i - \underline{s}_i) 1_{\bar{\Gamma}})$$

and we conclude like in [13] that V_1^{2n} converges a.s. to ∞ as $n \rightarrow \infty$, i.e. there is a strong asymptotic arbitrage opportunity of the first kind.

• Assume that $\inf_i d_i = \delta > 0$. Then using a similar argument like in [13], we have the following inequalities on a non-null set:

$$\begin{aligned} V_1^n &\leq x^n - \sum_{i=0}^n |\alpha_i| \frac{d_i^0}{2} - 2 \sum_{i=1}^n \lambda^i \frac{|a_i|}{\sigma_i \bar{\gamma}_i} \\ &\leq x^n - \sum_{i=0}^n |\alpha_i| \frac{d_i}{2} + \sum_{i=1}^n |\alpha_i| \frac{f_i}{2} - 2 \sum_{i=1}^n \lambda^i \frac{|\alpha_i|}{(1 - \lambda^i \text{sign}(\alpha_i)) \sigma_i \bar{\gamma}_i} \\ &\leq x^n - \sum_{i=0}^n |\alpha_i| \frac{d_i}{2} + \sum_{i=1}^n |\alpha_i| \frac{f_i}{2} - 2 \sum_{i=1}^n \lambda^i \frac{|\alpha_i|}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \\ &\leq x^n - \sum_{i=0}^n |\alpha_i| \frac{d_i}{2} + \sum_{i=1}^n \frac{|\alpha_i|}{2} \left(f_i - 4 \lambda^i \frac{|\alpha_i|}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \right) \\ &\leq x^n - \sum_{i=0}^n |\alpha_i| \frac{d_i}{2} \leq x^n - \frac{\delta}{2} \sum_{i=0}^n |\alpha_i|. \end{aligned}$$

With $V_1^n \geq 0$ and $x^n \rightarrow 0$, it follows that $\sum_{i=0}^n |\alpha_i| \rightarrow 0$ as $n \rightarrow \infty$. From the inequality

$$0 \leq V_1^n \leq x^n + 2N \sum_{i=0}^n |\alpha_i|,$$

we deduce that $V_1^n \rightarrow 0$ as $n \rightarrow \infty$. Hence, there is no strong asymptotic arbitrage opportunity of the first kind. We then conclude about (a) and also about (b) as a consequence.

Let us now prove Statement (c).

• Let us first assume that (NAA2) holds and $\limsup_i \tilde{b}_i > 0$ where

$$\tilde{b}_i := \sqrt{b_i^2 + \frac{2\lambda^i}{(1 - \lambda^i) \sigma_i \bar{\gamma}_i}}.$$

Let us also suppose that Condition **P2** does not hold. Under the conditions above, we show that it is possible to construct an asymptotic arbitrage opportunity of

the second kind hence a contradiction. We may assume without loss of generality that $\nu := \inf_i \tilde{b}_i > 0$. Since $N|b_i| \geq b_i^2$ we get that

$$N|b_i| + N \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i} \geq b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i}.$$

From there, we may assume that we also have

$$|b_i| + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i} > \tilde{\nu}$$

where $\tilde{\nu} > 0$ is a constant. Let us denote $D_n^2 := \sum_{i=0}^n \tilde{b}_i^2$ and consider a terminal portfolio value:

$$V_1^n := x^n + \sum_{i=0}^n \alpha_i(\epsilon_i - b_i) - 2 \sum_{i=1}^n \lambda^i \frac{|\alpha_i|}{(1 - \text{sign}(\alpha_i)\lambda^i)\sigma_i\tilde{\gamma}_i}.$$

The idea is to choose the coefficients $\alpha_i = \alpha_i^n$ so that $V_1^n \rightarrow 0$ a.s. and

$$x^n = \sum_{i=0}^n \alpha_i b_i + 2 \sum_{i=1}^n \lambda^i \frac{|\alpha_i|}{(1 - \text{sign}(\alpha_i)\lambda^i)\sigma_i\tilde{\gamma}_i}.$$

It follows that

$$V_1^n = \sum_{i=0}^n \alpha_i \epsilon_i.$$

Renormalizing the sequence (V_1^n) if necessary, we deduce that $|V_1^n| \leq 1$ and applying the strong law of large numbers, we shall conclude that $V_1^n \rightarrow 0$ a.s. It remains to construct the coefficients (α_i) and to show that $\liminf x^n < 0$. We put

$$e_i := b_i - 2 \frac{\lambda^i}{(1+\lambda^i)\sigma_i\tilde{\gamma}_i}.$$

•*First Case.* We suppose there exists $c > 0$ and a subsequence such that

$$e_i \geq c\tilde{b}_i.$$

We choose $\alpha_i := -\frac{\nu^2 \tilde{b}_i}{N^2 D_n^2}$ so that $|V_1^n| \leq 1$. Moreover,

$$x_n = -\frac{\nu^2}{N^2 D_n^2} \sum_{i=0}^n \tilde{b}_i e_i \leq -\frac{c\nu^2}{N^2}$$

implies that $\liminf x_n \in (-\infty, 0)$. Since $D_n \geq Cn$ where $C > 0$, we deduce that $V_1^n \rightarrow 0$ by virtue of the strong law of large numbers.

•*Second Case.* We suppose that $e_i \leq 0$. Since Condition **P2** does not hold, either $\limsup_i \left(|b_i| - 2 \frac{\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i} \right) > 0$ or $\limsup_i \frac{\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i} = 0$. In the second case, we then deduce that $\limsup_i b_i = 0$ if the condition of the first case is not satisfied hence a contradiction. Then, we may assume there exists a constant $c \in (0, 1)$ such that

$$\left(|b_i| - 2 \frac{\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i} \right) \geq c \sqrt{b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\tilde{\gamma}_i}}.$$

Indeed, the second term in the rhs of the inequality above turns out to be bounded (for a subsequence). From now on, consider the terminal value portfolio:

$$V_1^n := -\frac{\nu}{ND_n} \sum_{i=1}^n \text{sign}(b_i) \sqrt{b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\bar{\gamma}_i}} \epsilon_i - \frac{\nu}{ND_n} \epsilon_0.$$

It satisfies $|V_1^n| \leq 1$ and by virtue of the Bienaymé–Tchebychev inequality,

$$P(|V_1^n| \geq \varepsilon) \leq \frac{\nu^2}{N^2 D_n \varepsilon^2} \rightarrow 0, \quad n \rightarrow \infty$$

since $D_n \geq \nu n$. At last, recall that the random variables $(\epsilon_i)_{i \geq 0}$ are independent and identically distributed under the initial probability measure. We deduce that V_1^n is the terminal value of a portfolio of the form (5.7) if and only if

$$\alpha_i = -\text{sign}(b_i) \frac{\nu}{ND_n} \sqrt{b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\bar{\gamma}_i}}, \quad i \geq 1, \quad \alpha_0 = -\frac{\nu}{ND_n} b_0.$$

We deduce that

$$x^n = -\frac{\nu}{ND_n} b_0^2 - \frac{\nu}{ND_n} \sum_{i=1}^n \sqrt{b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\bar{\gamma}_i}} \left(|b_i| - \frac{2\lambda^i}{(1-\lambda^i)\text{sign}(\alpha_i)\sigma_i\bar{\gamma}_i} \right).$$

We then deduce that $x^n \leq -\frac{c\nu}{N}$ and we conclude that (V_1^n) realizes an asymptotic arbitrage opportunity of the second kind.

- Let us suppose that

$$\limsup_i \sqrt{b_i^2 + \frac{2\lambda^i}{(1-\lambda^i)\sigma_i\bar{\gamma}_i}} = 0.$$

It follows that $\limsup_i |b_i| = 0$. Following the reasoning of [13], we deduce that $\limsup_i d_i^0 \geq C$, where C is a strictly positive constant such that $\underline{s}_i \leq -C$ and $\bar{s}_i \geq C$, and $\delta := \inf_i d_i^0 > 0$. We also deduce that

$$\limsup_i \frac{2\lambda^i}{(1+\lambda^i)\sigma_i\bar{\gamma}_i} = 0.$$

We may assume without loss of generality that

$$\sup_i \left(\frac{2\lambda^i}{(1+\lambda^i)\sigma_i\bar{\gamma}_i} \right) \leq \frac{\delta}{4}.$$

We deduce the existence of $\tilde{\delta} > 0$ such that

$$\inf_i \left(\frac{d_i^0}{2} + \frac{2\lambda^i}{(1+\lambda^i)\sigma_i\bar{\gamma}_i} \right) > \tilde{\delta}. \quad (5.8)$$

Let (x^n, α^n) be a sequence such that the properties (3.a) and (3.c) of a strategy realizing (AA2) are fulfilled, i.e. $x^n \rightarrow -x < 0$ and

$$-V_1^n = -x^n - \sum_{i=0}^n \alpha_i (\epsilon_i - b_i) + 2 \sum_{i=1}^n \lambda^i \frac{|a_i|}{\sigma_i \bar{\gamma}_i} \leq 1.$$

Then, on a non-null set, we deduce that

$$\begin{aligned} -x^n + \sum_{i=0}^n |\alpha_i| \frac{d_i^0}{2} + 2 \sum_{i=1}^n \lambda^i \frac{|\alpha_i|}{(1 - \lambda^i \text{sign}(\alpha_i)) \sigma_i \bar{\gamma}_i} &\leq 1 \\ -x^n + \sum_{i=0}^n |\alpha_i| \left(\frac{d_i^0}{2} + \frac{2\lambda^i}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \right) &\leq 1 \end{aligned}$$

Then, with n large enough and $\gamma := x/2$, we have $\gamma + \tilde{\delta} \sum_{i=0}^n |\alpha_i| \leq 1$ and $\sum_{i=0}^n |\alpha_i| \leq (1 - \gamma)/\tilde{\delta}$. Observe that we can also choose $\tilde{\delta}$ smaller so that the last inequality holds for all n . Since

$$\limsup_i \left(|b_i| + \frac{2\lambda^i}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \right) = 0,$$

we also may assume that

$$\sup_i \left(|b_i| - \frac{2\lambda^i}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \right) \leq \frac{\tilde{\delta}\gamma}{2(1 - \gamma)}. \quad (5.9)$$

We deduce that, with n large enough,

$$\begin{aligned} -V_1^n &\geq \gamma + \sum_{i=0}^n \alpha_i \epsilon_i - \sum_{i=0}^n |\alpha_i| \left(|b_i| - \frac{2\lambda^i}{(1 + \lambda^i) \sigma_i \bar{\gamma}_i} \right) \\ &\geq \frac{\gamma}{2} + \sum_{i=0}^n \alpha_i \epsilon_i. \end{aligned}$$

We conclude that for n large enough,

$$\begin{aligned} P(V_1^n \leq -\gamma/4) &= P(-V_1^n \geq \gamma/4) \geq E(-V_1^n - \gamma/4)^+ \wedge 1 \\ &\geq E(-V_1^n - \gamma/4) \wedge 1 \\ &\geq E(-V_1^n - \gamma/4) \geq \gamma/4 \end{aligned}$$

hence (NAA2) holds. Under the condition **P2**, we do the same reasoning since the inequalities (5.8) and (5.9) remains valid. ■

5.3 The large Black–Scholes market

We consider the large Black and Scholes market example of Kabanov and Kramkov [13]. We are given a sequence of markets whose horizon dates are $T^n = T$ for all $n \geq 1$. Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \leq T}, P)$ be a stochastic basis with a countable set of independent one-dimensional standard Brownian motions $(W^i)_{i \geq 0}$. We define $B^n = (W^0, \dots, W^n)$, and let $\mathbf{G}^n = (\mathcal{G}_t^n)$ be a subfiltration of \mathbf{F} such that (B^n, \mathbf{G}^n) is a $(n + 1)$ -dimensional standard Wiener process. Contrarily to [13], we consider here complete markets, i.e. $\mathbf{G}^n = \mathbf{F}$ is the completed natural filtration of the Brownian motions $(W^i)_{i \geq 0}$. The incomplete case remains an open problem. The behaviour of the stock prices is described as follows:

$$dX_t^0 = \mu_t^0 X_t^0 dt + \sigma_t^0 X_t^0 dW_t^0,$$

$$dX_t^i = \mu_t^i X_t^i dt + \sigma_t^i X_t^i (\gamma_t^i dW_t^0 + \bar{\gamma}_t^i dW_t^i), \quad i \in N$$

with deterministic (strictly positive) initial points. The coefficients are \mathbf{G}^i -predictable processes verifying

$$\int_0^T |\mu_s^i|^2 ds < \infty, \quad \int_0^T |\sigma_s^i|^2 ds < \infty$$

and $|\gamma_t^i|^2 + |\bar{\gamma}_t^i|^2 = 1$. To avoid degeneracy we shall assume that $\sigma^i > 0$ and $\bar{\gamma}^i > 0$. Moreover, we assume that there exists a bond $B_t = 1$ for all $t \geq 0$.

We shall study the absence of asymptotic arbitrage opportunities of the first kind according to the variant definition of Section 4. Observe that in our example $\overline{\mathcal{Y}_{0,b}^{T,\infty}(T)}^w = \mathcal{Y}_{0,b}^{T,\infty}(T)$ is Fatou-closed, [4], since the price process provides a strictly consistent price system. We want to characterize probability measures $Q^n \in \mathcal{Q}^n$, i.e. probability measures $Q \sim P$ such that $\frac{dQ}{dP} = Z_T e_1$ where $Z_T \in \mathcal{M}_0^T(G^* \setminus \{0\}, P)$ and $Z_0 e_1 = 1$. To do so, we shall characterize the consistent price systems. Let us denote by $\lambda_t^{i,b}$, $\lambda_t^{b,i}$, for $t \geq 0$, and $i = 0, \dots, n$, the transaction costs coefficients characterizing the exchange between the risky assets and the bond. We assume that $\lambda_t^{i,b} > 0$ and $\lambda_t^{b,i} > 0$ for all $i = 0, \dots, n$.

Definition 10 We say that the process $Y \in \mathbf{R}_+^{n+1}$ is a λ -consistent price system for the prices $(X^i)_{i \leq n}$ if there exists $Q \sim P$ such that Y is a Q -martingale and

$$\frac{X_t^i}{1 + \lambda_t^{i,b}} \leq Y_t^i \leq (1 + \lambda_t^{b,i}) X_t^i, \quad i = 0, \dots, n. \quad (5.10)$$

Lemma 5.2 Assume that Assumption **RW** holds. Then, there exists a consistent price system $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ if and only if there is a λ -consistent price system for the prices $(X^i)_{i \leq n}$.

Proof. • “ \Rightarrow ” Assume that there exists a consistent price system $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$, i.e. Z is a martingale and $Z_t \in G_t^* \setminus \{0\}$, for all $t \leq T$. Recall that G^* is the $(n+2)$ -dimensional cone defined by the transaction costs $\lambda^{i,b}$ and $\lambda^{b,i}$ for $i \leq n$. Denoting $Z = (Z^b, Z^0, \dots, Z^n)$, interpret Z^b as a numéraire and take Y defined as follows:

$$Y_t := \left(\frac{Z_t^0}{Z_t^b}, \dots, \frac{Z_t^n}{Z_t^b} \right).$$

Define Q such that $dQ/dP = Z_T^b/Z_0^b$. Since $(Z_t)_{t \leq T}$ is a martingale, it is clear that Y is a Q -martingale. In order for Y to be a λ -consistent price system, we only have to prove (5.10) but these inequalities follow immediately from the fact that $Z_t \in G_t^* \setminus \{0\}$, for all $t \leq T$.

• “ \Leftarrow ” Assume that Y is a λ -consistent price system, i.e. there exists a probability measure $Q \sim P$ such that Y is a Q -martingale and the inequalities (5.10) hold. Then we define ρ_t^0 by $\rho_t^0 := E[dQ/dP | \mathcal{F}_t]$ and Z_t^j by $Z_t^j := Y_t^j \rho_t^0$ for every $j = 0, \dots, n$, $Z_t^b := \rho_t^0$. Now it is easily seen that, since Y is a Q -martingale, $Z = (Z^b, Z^0, \dots, Z^n)$ is a P -martingale. The proof is now completed because the inequalities (5.10) imply the fact that Z lies in $G^* \setminus \{0\}$ under Assumption **RW**. ■

From there, we deduce that for each model,

$$\bar{Q}^n(A) = \sup_{Q \in \mathcal{Q}^n} Q(A) = \tilde{Q}^n(A) := \sup_{Q \in \tilde{\mathcal{Q}}^n} Q(A)$$

where

$$\tilde{\mathcal{Q}}^n := \{Q : dQ = \rho_T dP, \quad \rho \in \tilde{M}_e\}$$

and \tilde{M}_e is the set of all density processes such that there exists a λ -consistent price system for the prices $(X^i)_{i \leq n}$ under the probability measure defined by $dQ = \rho_T dP$.

Let us now focus on a consistent price system $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$. By virtue of Corollary 2 page 189 in [16], this is a continuous martingale. We deduce the following martingale representation

$$\begin{aligned} (Z_t^i)_{0 \leq i \leq n} &= (Z_0^i)_{0 \leq i \leq n} + \int_0^t H_s dB_s^n = (Z_0^i)_{0 \leq i \leq n} + \int_0^t \text{diag}((Z_s^i)_{0 \leq i \leq n}) K_s dB_s^n \\ Z_t^b &= Z_0^b + \int_0^t H_s^{b \cdot} dB_s^n = Z_0^b + \int_0^t Z_s^b K_s^{b \cdot} dB_s^n \end{aligned}$$

where $K_s^{b \cdot} := (Z_s^b)^{-1} H_s^{b \cdot}$, $K_s := \text{diag}((Z_s^i)_{0 \leq i \leq n})^{-1} H_s$ and H is a matrix-valued predictable process, $H^{b \cdot}$ is a vector-valued predictable process. Writing the components of the process Z in a Doléans-Dade form, we then deduce that a λ -consistent price system for the prices $(X^i)_{i \leq n}$ has the following form:

$$dY_t^i = Y_t^i \Lambda_t^{i \cdot} dB_t^n + Y_t^i \alpha_t^i dt, \quad i \leq n$$

where $\Lambda \in \mathbf{R}^{n+1, n+1}$ and α are respectively matrix and vector-valued predictable processes. Since $Z^b Y^i$, $i \geq 0$ and Z^b are martingales, the integration by parts formula

$$Z_t^b Y_t^i = \int_0^t Z_u^b dY_u^i + \int_0^t Y_u^i dZ_u^b + \langle Z^b, Y^i \rangle_t$$

implies that $H_s^{b \cdot} \Lambda_s^{i \cdot} + Z_s^b \alpha_s^i = 0$ and $\alpha_s^i = -H_s^{b \cdot} \Lambda_s^{i \cdot} / (Z_s^b)$. We then deduce that a λ -consistent price system for the price $(X^i)_{i \leq n}$ has finally the following form:

$$dY_t^i = Y_t^i \Lambda_t^{i \cdot} (dB_t^n - \alpha_t dt), \quad i \leq n \quad (5.11)$$

where the vector-valued predictable process α does not depend on i .

Let us introduce

$$\tilde{B}_t^n := B_t^n - \int_0^t \alpha_u du$$

and the process L^α satisfying the SDE $dL_t^\alpha = L_t^\alpha \alpha_t dB_t^n$, $L_0 = 1$. We can easily observe that $L = Z^b$ so that L is a martingale and \tilde{B}^n is a standard brownian motion under Q by virtue of Girsanov's theorem.

Reciprocally, consider any process Y whose dynamics has the form (5.11) such that the associated process L^α satisfying the SDE $dL_t^\alpha = L_t^\alpha \alpha_t dB_t^n$, $L_0 = 1$, is a martingale and such that the inequalities (5.10) hold. Applying again Girsanov's theorem, we deduce that Y is a λ -consistent price system for the price $(X^i)_{i \leq n}$. We then have:

Lemma 5.3 *The λ -consistent price systems for the price $(X^i)_{i \leq n}$ are the processes Y verifying Inequalities (5.10) and having a dynamics of the form (5.11) where the associated process L^α is a martingale.*

From now on, let us denote for a given λ -consistent price system Y^n of the n -th model,

$$\widetilde{Q}(Y^n) := \{Q : dQ = \rho_T dP, \quad \rho \in \widetilde{M}_e(Y^n)\}$$

and $\widetilde{M}_e(Y^n)$ is the set of all density processes such that the λ -consistent price system Y^n is a martingale under the probability measure defined by $dQ = \rho_T dP$. Notice that \widetilde{M}_e is the union of all $\widetilde{M}_e(Y^n)$. We then denote by $\overline{\widetilde{Q}(Y^n)}$ the upper envelope of the probability measures of $\widetilde{Q}(Y^n)$.

For our next purpose we remind Proposition 3.1 above in its variant version.

Proposition 5.4 *The following conditions are equivalent:*

- (a) *there is no asymptotic arbitrage of the first kind (NAA1),*
- (b) $(P^n) \triangleleft (\overline{Q^n})$,
- (c) *there exists a sequence $(R^n) \in \mathcal{Q}^n$ such that $(P^n) \triangleleft (R^n)$.*

We then apply Proposition 8 of [13] to a λ -consistent price system we interpret as a price process. Precisely, we consider the price process $S := Y$. The process S satisfies the SDE

$$dS_t^i = S_t^i \Lambda_t^i (dB_t^n + \alpha_t dt), \quad i \leq n. \quad (5.12)$$

and S is a martingale under $dQ^\alpha = L_T^\alpha dP$.

Proposition 5.5 *Assume that each model is defined by the matrix-valued transaction costs process $(\lambda^{i,j})_{i,j \in \{b,0,\dots,n\}}$ verifying Condition **RW**. Then, the following conditions are equivalent:*

- (a) *there is no asymptotic arbitrage of the first kind (NAA1),*
- (b) $(P^n) \triangleleft (\overline{Q^n})$,
- (c) *there exists a sequence (Y^n) of λ -consistent price systems such that*

$$(P^n) \triangleleft (\overline{\widetilde{Q}^n(Y^n)}),$$
- (d) *there exists a sequence of predictable processes $\Lambda^n \in \mathbf{R}^{n,n+1}$, $\alpha^n \in \mathbf{R}^n$ such that:*
 - (d₁) *the process L^{α^n} is a martingale,*
 - (d₂) *The process Y^n defined by (5.11) verifies Inequalities (5.10),*
 - (d₃) $\int_0^T \sup_n \sum_{i=0}^n (\alpha_s^{n,i})^2 ds < \infty$ *a.s.*

Proof. The statements (a), (b) and (c) are equivalent by virtue of Proposition 3.1. Let us show the implication (c) \Rightarrow (d). The statements (d₁) and (d₂) are obvious. The third one is a direct consequence of Proposition 8 of [13]. The reverse implication is based on the same proposition and the construction of a λ -consistent price system from Properties (d₁) and (d₂). ■

Remark 5.6 This result is an extension of Proposition 8 of [13]. Taking $\lambda = 0$, a 0-consistent price system is just X so that the process α^n of Statement (d) is known and (d_3) can be rewritten as in Proposition 8 of [13].

We conclude this example with the following proposition which gives a necessary condition, in terms of the coefficients defining the prices for the large security market, to satisfy the NAA1 condition. We put

$$b_i^2 := \min(e^{2\mu_i T}(1 + \lambda^{i,b})^{-2}, (e^{\sigma_i^2 T} - 1)^{-1}).$$

Proposition 5.7 *Assume that the transaction costs coefficients are constant in time and uniformly bounded. Suppose that the coefficients $\mu_i, \sigma_i, \gamma_i, \bar{\gamma}_i$ are deterministic.*

(a) *If $\sum_{i=1}^{\infty} b_i^2 = \infty$, then SAA2 holds.*

(b) *NAA1 $\Rightarrow \sum_{i=1}^{\infty} b_i^2 < \infty$.*

Proof. We define $\alpha_i := -|\tilde{b}_i|/D_n^2$, $i \geq 0$, where $D_n^2 := \sum_{i=0}^n b_i^2$ and \tilde{b}_i is defined by the equality $|\tilde{b}_i|(1 + \lambda^{i,b})^{-1}e^{\mu_i T} = b_i^2$. Corresponding to the bond, we define

$$\alpha_b := 1 = \sum_{i=0}^n (1 + \lambda^{i,b})^{-1} |\alpha_i| e^{\mu_i T}.$$

We denote $\alpha := (\alpha_b, \alpha_0, \dots, \alpha_n) \in \mathbf{R}^{n+2}$. We consider the terminal liquidated portfolio value

$$W_T := \sum_{i=0}^n (1 + \lambda^{i,b})^{-1} |\alpha_i| X_T^i - \alpha_b.$$

Consider an arbitrary λ -consistent price system. Recall the inequalities

$$(1 + \lambda^{i,b})^{-1} X_T^i \leq Y_T^i \leq (1 + \lambda^{b,i}) X_T^i, \quad i \leq n.$$

Multiplying each side by $L_T^\alpha := Z_T^b$, we deduce that for every consistent price system Z ,

$$(1 + \lambda^{i,b})^{-1} X_T^i Z_T^b \leq Z_T^i \leq (1 + \lambda^{b,i}) X_T^i Z_T^b, \quad i \leq n. \quad (5.13)$$

It follows that $EZ_T(-\alpha) \geq EZ_T W_T e_b$ where $e_b \in \mathbf{R}^{n+2}$ is the vector only the first component of which is different from zero and equal to 1. Then, let us define $V_T := W_T e_b \in -\alpha + \mathcal{Y}_{0,b}^T(T)$. Observe that

$$\begin{aligned} \sum_{i=0}^n \alpha_i^2 &= \frac{1}{D_n^4} \sum_{i=0}^n \tilde{b}_i^2 = \frac{1}{D_n^4} \sum_{i=0}^n b_i^4 (1 + \lambda^{i,b})^2 e^{-2\mu_i T} \\ &\leq \frac{1}{D_n^4} \sum_{i=0}^n b_i^2 \leq D_n^{-2} \rightarrow 0. \end{aligned}$$

Let us put

$$x^n := -\alpha_b - \sum_{i=0}^n (1 + \lambda^{b,i}) \alpha_i.$$

Then, $x^n e_b \geq_{G_0} -\alpha$ by definition. It follows that we may replace $-\alpha$ by $x^n e_b$ where $x^n \rightarrow -1$ as $n \rightarrow \infty$. On the other hand, we have $W_T \geq -1$, $EW_T = 0$ by definition and

$$EW_T^2 = \frac{1}{D_n^4} \sum_{i=0}^n b_i^4 (e^{\sigma_i^2 T} - 1) \leq D_n^{-2} \rightarrow 0.$$

We then deduce that the sequence (V_T) realizes a strong asymptotic arbitrage opportunity of the second kind. ■

5.4 Two asset model with infinite horizon

Under the variant approach, we consider the example of [13], i.e. the discrete-time model with only two assets, one of which is taken as a numéraire and its price equals 1 over time. The price dynamics of the strictly positive second asset is given by the following relation

$$X_i = X_{i-1}(1 + \mu_i + \sigma_i \epsilon_i), \quad i \geq 1$$

where $X_0 > 0$, $(\epsilon_i)_{i \geq 1}$ is a sequence of independent random variables on a probability space (Ω, \mathcal{F}, P) and taking values in a finite interval $[-N, N]$, $E\epsilon_i = 0$, $E\epsilon_i^2 = 1$. The coefficients here are deterministic and $\sigma_i \neq 0$ for all i . The support of ϵ_i is $[\underline{s}_i, \bar{s}_i]$ where $\underline{s}_i < 0 < \bar{s}_i$ and we suppose that $\mu_i + \sigma_i \bar{s}_i > 0$ and $\mu_i + \sigma_i \underline{s}_i < 0$.

For $n \geq 1$, we consider the stochastic basis $\mathcal{B}^n = (\Omega, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}_i^n)_{i \leq n}, P^n)$ with the 2-dimensional random process $S^n = (1, X_i)_{i \leq n}$ where $\mathcal{F}_0^n = \mathcal{F}_0$ is the trivial σ -algebra, $\mathcal{F}_i^n = \mathcal{F}_i := \sigma(\epsilon_1, \dots, \epsilon_i)$, and $P^n = P|_{\mathcal{F}_n^n}$. We consider the sequence $\mathcal{M} = \{(\mathcal{B}^n, S^n, n)\}$ of large security markets associated to the deterministic transaction costs coefficients $(\lambda_i^{0,1} = 0, \lambda_i^{1,0})_{i \leq n}$ for the exchanges between the bond and the risky assets X_i . In a bid-ask model, that means that X_i is the ask price at time i and $X_i(1 - \lambda_i^{1,0})$ is the bid-price. As in [13], we suppose that each model has an equivalent probability measure Q with $b_i := E_Q \epsilon_i$ so that there exists also a strictly consistent price system. In particular, we have $|b_i| < N$.

Before presenting our main result, let us observe that we may rewrite the model under an other probability P^n so that we may assume that $\mu_i \mu_{i+1} < 0$ and $\mu_1 > 0$. Indeed, let us choose $\alpha_i \in (b_i, \bar{s}_i)$ if i is odd and $\alpha_i \in (\underline{s}_i, b_i)$ otherwise. As $P(\epsilon_i - \alpha_i > 0) > 0$ and $P(\epsilon_i - \alpha_i < 0) > 0$ for all i , there exists $P^n \sim P$ with $dP^n := \prod_{i=1}^n f_i(\epsilon_i - \alpha_i) dP$ and $E_P f_i(\epsilon_i - \alpha_i) = 1$ such that $E_P f_i(\epsilon_i - \alpha_i) \epsilon_i = \alpha_i$ (see [13]). We then deduce that

$$\frac{X_i}{X_{i-1}} = 1 + \tilde{\sigma}_i \tilde{\epsilon}_i + \tilde{\mu}_i$$

where

$$\begin{aligned} \tilde{\sigma}_i \tilde{\epsilon}_i &:= \sigma_i \epsilon_i + \mu_i - E_{P^n}(\sigma_i \epsilon_i + \mu_i), \\ \tilde{\mu}_i &:= E_{P^n}(\sigma_i \epsilon_i + \mu_i) = \sigma_i \alpha_i + \mu_i, \\ E_{P^n} \tilde{\epsilon}_i^2 &:= 1. \end{aligned}$$

Since $\mu_i + \sigma_i \bar{s}_i > 0$ and $\mu_i + \sigma_i \underline{s}_i < 0$, we can choose $|\alpha_i|$ large enough such that $\tilde{\mu}_i > 0$ if i is odd and $\tilde{\mu}_i \leq 0$ otherwise. Observe that the random variables $(\epsilon_i)_{i \leq n}$ are still independent under P^n and so do $(\tilde{\epsilon}_i)_{i \leq n}$. We denote by $\tilde{\mu}_i$, $\tilde{\sigma}_i$ and \tilde{b}_i the coefficients of the model when we write it under P^n .

Let

$$b_i := -\frac{\mu_i}{\sigma_i}, \quad D_{0,n}^2 := \sum_{i=1}^n b_i^2, \quad D_n^2 := \sum_{i=1}^n \widehat{b}_i^2$$

where $\widehat{b}_i := b_i - \Delta_i$ and $\Delta_i := 0$ if $b_i = 0$, otherwise:

$$\begin{aligned} \Delta_i &:= \mu_i^{-1} \left(A_i^l - 1 \right) b_i, & b_i > 0, \\ \Delta_i &:= \mu_i^{-1} \left(A_i^r - 1 \right) b_i, & b_i < 0 \end{aligned}$$

with

$$A_i^r := A_i := (1 + \lambda_i^{10}), \quad A_i^l := A_i := (1 + \lambda_{i-1}^{10})^{-1}, \quad \lambda_0^{10} := 0.$$

We also define the analogous coefficients (\widehat{b}_i) we deduce from (\tilde{b}_i) and $(\tilde{\mu}_i)$. Then,

$$\begin{aligned} X_i &= X_{i-1} (1 + \sigma_i (\epsilon_i - b_i)), & i \geq 1, \\ X_i &= X_{i-1} (1 + \tilde{\sigma}_i (\tilde{\epsilon}_i - \tilde{b}_i)), & i \geq 1. \end{aligned}$$

At last, we suppose that $b_i \widehat{b}_i \geq 0$ and so $-\underline{s}_i < \widehat{b}_i < \bar{s}_i$ meaning that the transaction costs coefficients are small enough.

Lemma 5.8

- (a) If $D_\infty^2 < \infty$, then $(P^n) \triangleleft (\overline{Q}^n)$ (equivalently NAA1 holds);
- (b) If $D_\infty^2 = \infty$, then $(P^n) \triangle (\overline{Q}^n)$ (equivalently SAA1 holds).

Proof.

(a) Notice that in the case where $D_{0,n}^2 < \infty$, i.e. when the model without friction of [13] does not admit any asymptotic arbitrage opportunity, it is straightforward to conclude using the results of [13] since (X_i) is a strictly consistent price system. The case $D_{0,n}^2 = \infty$ is the most interesting case; indeed the natural question is how to increase the transaction costs coefficients in order to eliminate an arbitrage opportunity of the frictionless model.

Recall that $\tilde{\mu}_1 > 0$. For each n , we construct a λ -consistent price system (Y_i) such that $Y_0 = X_0$ and $Y_i/Y_{i-1} = (X_i/X_{i-1})k_i$ where $k_i > 0$ is defined by the relation

$$k_i := (1 - \tilde{\sigma}_i \tilde{\Delta}_i)^{-1}$$

i.e. $k_i = (A_i^r)^{-1}$ or $k_i = (A_i^l)^{-1}$. We have $Y_i/Y_{i-1} = 1 + \tilde{\sigma}_i k_i (\tilde{\epsilon}_i - \widehat{b}_i)$ but also

$$Y_i/Y_{i-1} = 1 + k_i \sigma_i (\epsilon_i - \widehat{b}_i). \quad (5.14)$$

Recall that $-\underline{s}_i < \widehat{b}_i < \bar{s}_i$. Then, $P(\epsilon_i - \widehat{b}_i > 0) > 0$ and $P(\epsilon_i - \widehat{b}_i < 0) > 0$ for all i . It follows that there exists $Q \sim P$ such that Y is a Q -martingale.

Since $\tilde{b}_1 < 0$, it follows that

$$\prod_{j=1}^i k_j = (1 + \lambda_1^{10})^{-1} (1 + \lambda_1^{10}) (1 + \lambda_2^{10})^{-1} (1 + \lambda_2^{10}) \cdots$$

and we get that $\prod_{j=1}^i = 1$ or $\prod_{j=1}^i = (1 + \lambda_i^{10})^{-1}$. It follows that

$$(1 + \lambda_i^{10})^{-1} X_i \leq Y_i \leq X_i$$

and (Y_i) is a λ -consistent price system. We then consider the frictionless model of [13] defined by the prices (Y_i) with the coefficients (\hat{b}_i) in (5.14). Since $D_\infty^2 < \infty$, Proposition 11 (a) of [13] and Proposition 5 of [13] implies the NAA1 condition for our large market defined by (X_i) .

(b) Let us consider an arbitrary sequence of measures $Q^n \in \mathcal{Q}^n$ associated to consistent price systems $(Z_i^n)_{i \leq n}$ such that $dQ^n = Z_n^{0n} dP^n$. Then the real valued process $Y^n := Z^{1n}/Z^{0n}$ is a Q^n -martingale verifying the inequality:

$$\frac{1}{1 + \lambda_i^{10}} X_i \leq Y_i^n \leq X_i.$$

It follows that

$$Y_i^n \leq X_i \leq (1 + \lambda_i^{10}) Y_i^n$$

and

$$\frac{1}{(1 + \lambda_{i-1}^{10})} \frac{Y_i^n}{Y_{i-1}^n} \leq \frac{X_i}{X_{i-1}} \leq (1 + \lambda_i^{10}) \frac{Y_i^n}{Y_{i-1}^n}.$$

We deduce that

$$\frac{1}{\sigma_i} \left(\frac{1}{(1 + \lambda_{i-1}^{10})} - 1 \right) \leq E_{Q^n}(\epsilon_i - b_i | \mathcal{F}_{i-1}) \leq \frac{(1 + \lambda_i^{10}) - 1}{\sigma_i}.$$

Consider the case where $b_i < 0$. Using the definition $\sigma_i b_i := -\mu_i$ and $\Delta_i := b_i - \hat{b}_i$, we get the inequalities

$$b_i \left(A_i^r - 1 - \frac{\mu_i \Delta_i}{b_i} \right) \leq -\mu_i E_{Q^n}(\epsilon_i - \hat{b}_i | \mathcal{F}_{i-1}) \leq b_i \left(A_i^l - 1 - \frac{\mu_i \Delta_i}{b_i} \right).$$

Since $\Delta_i = \mu_i^{-1} (A_i^r - 1) b_i$, we deduce that $E_{Q^n}(\epsilon_i - \hat{b}_i | \mathcal{F}_{i-1}) \leq 0$ and

$$\hat{b}_i E_{Q^n}(\epsilon_i - \hat{b}_i | \mathcal{F}_{i-1}) \geq 0. \quad (5.15)$$

The case $b_i > 0$ also yields Inequality (5.15). We then deduce that

$$\text{essinf}_{Q^n \in \mathcal{Q}^n} E_{Q^n}(\hat{b}_i(\epsilon_i - \hat{b}_i) | \mathcal{F}_{i-1}) \geq 0. \quad (5.16)$$

Let us define the Q^n -martingale $M^n(Q^n)$ by

$$M_k^n(Q^n) := \sum_{i=1}^k \left[\hat{b}_i(\epsilon_i - \hat{b}_i) - E_{Q^n}(\hat{b}_i(\epsilon_i - \hat{b}_i) | \mathcal{F}_{i-1}) \right].$$

It verifies

$$\begin{aligned} E_{Q^n}(M_n^n(Q^n))^2 &= \sum_{i=1}^n E_{Q^n} \left[\widehat{b}_i(\epsilon_i - \widehat{b}_i) - E_{Q^n}(\widehat{b}_i(\epsilon_i - \widehat{b}_i) | \mathcal{F}_{i-1}) \right]^2 \\ &\leq C D_n^2 \end{aligned}$$

where C is a constant. Let us define M^n by

$$M^n := \sum_{i=1}^n \left[\widehat{b}_i(\epsilon_i - \widehat{b}_i) - \text{essinf}_{Q^n \in \mathcal{Q}^n} E_{Q^n}(\widehat{b}_i(\epsilon_i - \widehat{b}_i) | \mathcal{F}_{i-1}) \right].$$

Then, let us define the sets $A^n := \{-D_n^{-3/2} M_n > 1\} \in \mathcal{F}^n$. Observe that $M^n \geq M_n^n(Q^n)$ for any $Q^n \in \mathcal{Q}^n$. By the Tchebychev inequality, we get that

$$Q^n(A^n) \leq Q^n(\{-D_n^{-3/2} M_n(Q^n) > 1\}) \leq D_n^{-3} E_{Q^n}(M_n^n(Q^n))^2 \leq 4N^2 D_n^{-1} \rightarrow 0, \quad n \rightarrow \infty.$$

On the other hand, since Inequality (5.16) holds, the complement \overline{A}^n of A^n verifies

$$P^n(\overline{A}^n) \leq P^n \left(\sum_{i=1}^n \widehat{b}_i \epsilon_i \geq (D_n^2 - D_n^{3/2}) \right) \leq \frac{4N^2 D_n^2}{(D_n^2 - D_n^{3/2})^2} \rightarrow 0, \quad n \rightarrow \infty.$$

Using Proposition 7 [13], we deduce that $(P^n) \Delta(\overline{Q}^n)$. ■

Corollary 5.9

- (a) $D_\infty^2 < \infty \Leftrightarrow NAA1$;
- (b) $D_\infty^2 = \infty \Leftrightarrow SAA1$.

Remark 5.10 Consider a model where $\mu_i \mu_{i+1} \leq 0$ for all i and such that $\mu_i > 0$ and $\mu_{i+1} < 0$ implies that $(1 + \mu_{i+1})(1 + \mu_i) = 1$, i.e. $E_P(X_{i+1}/X_{i-1}) = 1$. Assume that $\sum_{i=1}^\infty b_i^2 = \infty$, i.e. there is a strong asymptotic arbitrage opportunity in the model without transaction costs. Let us define for $b_i > 0$, $\lambda_i^{10} = \mu_i$. We then get the equality $(1 + \lambda_i^{10})^{-1} = 1 + \mu_{i+1}$ and $b_{i+1} < 0$, i.e. $\Delta_i = b_i$ and $\widehat{b}_i = 0$ for all i . We then deduce that there is no more asymptotic arbitrage opportunity.

6 Appendix

For a given $\zeta \in L^\infty(\mathbb{R}_+^d)$, we define the convex set

$$\overline{\Gamma}_\zeta := \left\{ x \in \mathbb{R}^d : \zeta - x \in \overline{\mathcal{Y}_{0,b}^{T,\infty}(T)}^w \right\}$$

and the closed convex set

$$D_\zeta := \{x \in \mathbb{R}^d : Z_0 x \geq EZ_T \zeta \quad \forall Z \in \mathcal{M}_0^T(G^*)\}.$$

Let us recall a result from [3] which we use in Section 4:

Lemma 6.1 *Under Conditions S1 and S2, $\overline{\Gamma}_\zeta = D_\zeta$.*

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