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ON RANDOM MULTIPLE DIRICHLET SERIES

GAUTAMI BHOWMIK AND KOHJI MATSUMOTO

1. INTRODUCTION

A sequences of independent random variables, each of which takes only the values ± 1 , is called a Rademacher sequence. Dirichlet series or Taylor series whose coefficients include Rademacher type of variables are studied, for example, in Kahane [5].

In the present paper we study a multiple analogue of Rademacher sequences and associated Dirichlet series. Let $m_1, \dots, m_r \in \mathbb{N}$, and let $\varepsilon_{m_1, \dots, m_r}$ be independent random variables which take only the values ± 1 and are defined on a certain probability space (Ω, P) . Let

$$(1.1) \quad P(\omega \in \Omega \mid \varepsilon_{m_1, \dots, m_r}(\omega) = 1) = P(\omega \in \Omega \mid \varepsilon_{m_1, \dots, m_r}(\omega) = -1) = \frac{1}{2}.$$

The sequence $\{\varepsilon_{m_1, \dots, m_r} \mid m_1, \dots, m_r \in \mathbb{N}\}$ may be called a Rademacher sequence of r -tuple indices. The associated multiple Dirichlet series can be defined as

$$(1.2) \quad F_r(s, \omega) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\varepsilon_{m_1, \dots, m_r}(\omega) X(m_1, \dots, m_r)}{(m_1 + \cdots + m_r)^s},$$

where $s \in \mathbb{C}$ and $X(m_1, \dots, m_r) \in \mathbb{R}$.

As an example, we will treat the case where $X(m_1, \dots, m_r)$ is a product of the von Mangoldt function $\Lambda(\cdot)$, that is

$$(1.3) \quad \Phi_r(s, \omega) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\varepsilon_{m_1, \dots, m_r}(\omega) \Lambda(m_1) \cdots \Lambda(m_r)}{(m_1 + \cdots + m_r)^s}.$$

The Dirichlet series

$$(1.4) \quad \Phi_r(s) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\Lambda(m_1) \cdots \Lambda(m_r)}{(m_1 + \cdots + m_r)^s}$$

is connected with the problem of expressing positive integers as a sum of r prime numbers and the analytic properties of $\Phi_r(s)$ have been studied actively in the recent past (see [1] for references).

In the present paper we discuss the natural boundary of (1.2) and (1.3), and our result for the random series could be compared with those of the corresponding ordinary Dirichlet series.

2. THE NATURAL BOUNDARY

First we write $F_r(s, \omega)$ in a form of single Dirichlet series:

$$(2.1) \quad F_r(s, \omega) = \sum_{n=1}^{\infty} Y_r(n, \omega) n^{-s},$$

where

$$(2.2) \quad Y_r(n, \omega) = \sum_{m_1 + \dots + m_r = n} \varepsilon_{m_1, \dots, m_r}(\omega) X(m_1, \dots, m_r).$$

Since $\varepsilon_{m_1, \dots, m_r}$ are independent variables, we see that $Y_r(n, \omega)$ are also independent of each other.

For each fixed $\omega \in \Omega$, the series (2.1) is a Dirichlet series, so we can find its abscissa of convergence $\sigma_c(F_r, \omega)$. Then, according to the zero-one law, we can find a constant $\sigma_c(F_r)$ such that $\sigma_c(F_r, \omega) = \sigma_c(F_r)$ almost surely (see [5, Section 6 of Chapter 4]). In general $-\infty \leq \sigma_c(F_r) \leq \infty$, but here we assume $-\infty < \sigma_c(F_r) < \infty$. Our main result in this section is as follows.

Theorem 1. *The line $\Re s = \sigma_c(F_r)$ is the natural boundary of $F_r(s, \omega)$ almost surely.*

Proof. We say that $Y_r(n, \omega)$ is symmetric if $Y_r(n, \omega)$ and $-Y_r(n, \omega)$ have the same distribution. We prove that $Y_r(n, \omega)$ is symmetric; and the conclusion then follows immediately from the expression (2.1) and [5, Theorem 4 in Section 6 of Chapter 4].

Let $L(n)$ be the number of tuples (m_1, \dots, m_r) satisfying $m_1 + \dots + m_r = n$. Let \mathbf{b} be any tuple of $L(n)$ elements, each element being 1 or -1 . Write $\mathbf{b} = (b_{m_1, \dots, m_r})_{m_1 + \dots + m_r = n}$, where $b_{m_1, \dots, m_r} \in \{\pm 1\}$. Define

$$Z_r(n, \mathbf{b}) = \sum_{m_1 + \dots + m_r = n} b_{m_1, \dots, m_r} X(m_1, \dots, m_r).$$

Let A be any Borel subset of \mathbb{R} , and let $\mathcal{B}(A)$ be the set of all \mathbf{b} such that $Z_r(n, \mathbf{b}) \in A$. Then

$$(2.3) \quad \begin{aligned} & P(\omega \in \Omega \mid Y_r(n, \omega) \in A) \\ &= \sum_{\mathbf{b} \in \mathcal{B}(A)} P(\omega \in \Omega \mid (\varepsilon_{m_1, \dots, m_r}(\omega))_{m_1 + \dots + m_r = n} = \mathbf{b}). \end{aligned}$$

Since $\varepsilon_{m_1, \dots, m_r}$ are independent, we have

$$(2.4) \quad P(\omega \in \Omega \mid (\varepsilon_{m_1, \dots, m_r}(\omega))_{m_1 + \dots + m_r = n} = \mathbf{b}) = 2^{-L(n)},$$

and the same equality holds if we replace \mathbf{b} by $-\mathbf{b}$. Therefore the right-hand side of (2.3) is equal to

$$\begin{aligned} & \sum_{\mathbf{b} \in \mathcal{B}(A)} P(\omega \in \Omega \mid (\varepsilon_{m_1, \dots, m_r}(\omega))_{m_1 + \dots + m_r = n} = -\mathbf{b}) \\ &= P(\omega \in \Omega \mid Y_r(n, \omega) \in -A). \end{aligned}$$

This implies that two random variables $Y_r(n, \omega)$ and $-Y_r(n, \omega)$ have the same distribution. \square

3. THE CASE OF GOLDBACH GENERATING FUNCTIONS

In this section we consider the special case (1.3). The function $\Phi_r(s)$ defined by (1.4) is the generating function of

$$(3.1) \quad G_r(n) = \sum_{m_1 + \dots + m_r = n} \Lambda(m_1) \cdots \Lambda(m_r).$$

The series $\Phi_r(s)$ was introduced in [3] where it was conjectured that:

(C-r) the line $\Re s = r - 1$ would be the natural boundary of $\Phi_r(s)$.

The conjecture is out of reach for the moment. However one can say more under the RH and other reasonable ones on the zeros of the Riemann zeta function like the following due to Fujii [4] :

(Z) Let \mathcal{I} be the set of all imaginary parts of non-trivial zeros of $\zeta(s)$. If $\gamma_j \in \mathcal{I}$ ($1 \leq j \leq 4$) and $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 \neq 0$, then $\{\gamma_1, \gamma_2\} = \{\gamma_3, \gamma_4\}$.

In [3] and [2], it was shown that (C-2) is indeed true under the RH and a certain quantitative version of (Z) or (Z) itself. The case of (C-r) is shown to be true if and only if (C-2) is.

The reason of introducing the random series (1.3) is to observe the situation from a stochastic viewpoint. Let

$$(3.2) \quad G_r(n, \omega) = \sum_{m_1 + \dots + m_r = n} \varepsilon_{m_1, \dots, m_r}(\omega) \Lambda(m_1) \cdots \Lambda(m_r).$$

Then

$$(3.3) \quad |G_r(n, \omega)| \leq n^{r-1} (\log n)^r,$$

and hence (1.3), which can be written as

$$(3.4) \quad \Phi_r(s, \omega) = \sum_{n=1}^{\infty} G_r(n, \omega) n^{-s},$$

is absolutely convergent for $\Re s > r$. We will show:

Theorem 2. *The random series (3.4) converges for $\Re s > r/2$ almost surely.*

We will prove this theorem in the next section. In particular this theorem implies $\sigma_c(\Phi_r, \omega) \leq r/2$ almost surely, and hence $\sigma_c(\Phi_r) \leq r/2$. The lower-bound of $\sigma_c(\Phi_r)$ will be discussed in Section 5.

4. PROOF OF THEOREM 2

Let $E(\cdot)$ denote the expected value, and $V(\cdot)$ the variance. Write $X_n = G_r(n, \omega) n^{-s}$. From (3.3) we see that

$$(4.1) \quad \int_{\Omega} |X_n|^2 d\omega \leq \left(\frac{n^{r-1} (\log n)^r}{n^{\sigma}} \right)^2 \int_{\Omega} d\omega < +\infty,$$

that is $X_n \in L^2(\Omega)$.

We use Theorem 2 in [5, Section 2 of Chapter 3], which asserts that if $X_n \in L^2(\Omega)$ are independent random variables satisfying $E(X_n) = 0$ and $\sum_{n=1}^{\infty} V(X_n) < +\infty$, then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Since

$$(4.2) \quad E(X_n) = \frac{1}{n^s} \sum_{m_1 + \dots + m_r = n} \Lambda(m_1) \cdots \Lambda(m_r) \int_{\Omega} \varepsilon_{m_1, \dots, m_r}(\omega) d\omega,$$

and $\varepsilon_{m_1, \dots, m_r}$ is an element of a Rademacher sequence, it is obvious that $E(X_n) = 0$ for any n . Next consider $V(X_n) = E(|X_n - E(X_n)|^2) = E(|X_n|^2)$. We see that

$$(4.3) \quad V(X_n) = \frac{1}{n^{2\sigma}} \sum_{\substack{m_1 + \dots + m_r = n \\ m'_1 + \dots + m'_r = n}} \Lambda(m_1) \cdots \Lambda(m_r) \Lambda(m'_1) \cdots \Lambda(m'_r) \\ \times \int_{\Omega} \varepsilon_{m_1, \dots, m_r}(\omega) \varepsilon_{m'_1, \dots, m'_r}(\omega) d\omega.$$

If $(m_1, \dots, m_r) \neq (m'_1, \dots, m'_r)$, then

$$P(\omega \in \Omega \mid \varepsilon_{m_1, \dots, m_r}(\omega) = \pm 1, \varepsilon_{m'_1, \dots, m'_r}(\omega) = \pm 1) = \frac{1}{4}$$

for any choice of double signs, hence

$$\int_{\Omega} \varepsilon_{m_1, \dots, m_r}(\omega) \varepsilon_{m'_1, \dots, m'_r}(\omega) d\omega = 0.$$

Therefore

$$(4.4) \quad \begin{aligned} V(X_n) &= \frac{1}{n^{2\sigma}} \sum_{m_1+\dots+m_r=n} \Lambda(m_1)^2 \cdots \Lambda(m_r)^2 \int_{\Omega} \varepsilon_{m_1, \dots, m_r}(\omega)^2 d\omega \\ &= \frac{1}{n^{2\sigma}} \sum_{m_1+\dots+m_r=n} \Lambda(m_1)^2 \cdots \Lambda(m_r)^2, \end{aligned}$$

and the sum on the right-hand side can be estimated as $\leq n^{r-1}(\log n)^{2r}$. Therefore

$$(4.5) \quad \sum_{n=1}^{\infty} V(X_n) \leq \sum_{n=1}^{\infty} n^{r-1-2\sigma} (\log n)^{2r},$$

which is convergent if $\sigma > r/2$. This implies the assertion of Theorem 2.

5. THE ABCISSA OF CONVERGENCE OF $\Phi_r(s, \omega)$

Theorem 2 gives the upper-bound $\sigma_c(\Phi_r) \leq r/2$. In this section we comment on its lower-bound. Since G_r is symmetric (as we have seen in the proof of Theorem 1), we have the criterion that $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if $\sum_{n=1}^{\infty} V(X'_n)$ converges, where

$$(5.1) \quad X'_n(\omega) = \begin{cases} X_n(\omega) & \text{if } |X_n(\omega)| \leq 1 \\ X_n(\omega)/|X_n(\omega)| & \text{if } |X_n(\omega)| > 1 \end{cases}$$

([5, Theorem 7 in Section 5 of Chapter 3]).

In the case $r = 1$, we have $X_n(\omega) = \varepsilon_n(\omega)\Lambda(n)n^{-s}$. Hence if $\sigma > 0$ then $|X_n(\omega)| \leq 1$ for sufficiently large n , so we may assume $X'_n = X_n$ if $\sigma > 0$. From (4.4) we have

$$(5.2) \quad \sum_{n=1}^{\infty} V(X_n) = \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}},$$

which is convergent if and only if $\sigma > 1/2$. This implies that $\sigma_c(\Phi_1) = 1/2$, and hence, by Theorem 1, the line $\Re s = 1/2$ is the natural boundary of $\Phi_1(s, \omega)$ almost surely.

Next consider the case $r = 2$. Then

$$X_n(\omega) = \frac{1}{n^s} \sum_{m_1+m_2=n} \varepsilon_{m_1, m_2} \Lambda(m_1) \Lambda(m_2).$$

If $\sigma > 1$ then we may assume $X'_n = X_n$ for sufficiently large n . From (4.4) we have

$$(5.3) \quad \sum_{n=1}^{\infty} V(X_n) = \sum_{m_1+m_2=n} \frac{\Lambda(m_1)^2 \Lambda(m_2)^2}{n^{2\sigma}},$$

whose right-hand side is

$$(5.4) \quad \geq \frac{1}{n^{2\sigma}} \sum_{\substack{p_1, p_2: \text{prime} \\ p_1 + p_2 = n}} (\log p_1)^2 (\log p_2)^2 \gg \frac{(\log n)^2}{n^{2\sigma}} \sum_{\substack{p_1, p_2: \text{prime} \\ p_1 + p_2 = n}} 1,$$

because one of p_1 or p_2 is $\geq n/2$. If the Hardy-Littlewood conjectural asymptotic formula for the Goldbach conjecture is true, then the last sum of (5.4) is $\gg n(\log n)^{-2}$. This implies that (5.3) diverges for $\sigma = 1$. Though $\sigma = 1$ is out of the range where $X'_n = X_n$ is valid, this argument suggests that $\sigma_c(\Phi_2) = 1$, and hence $\Re s = 1$ would be the natural boundary of $\Phi_2(s, \omega)$ almost surely.

The above observation in the cases $r = 1$ and $r = 2$ further suggests that, even in the case $r \geq 3$, perhaps $\sigma_c(\Phi_r) = r/2$ would hold. It is to be noted that Theorem 2 says, in the case $r \geq 3$, $\Phi_r(s, \omega)$ is convergent almost surely beyond the (conjectural) natural boundary (C-r) of $\Phi_r(s)$.

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