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Entropy Analysis of Integer and Fractional Dynamical Systems

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Abstract: This paper investigates the adoption of entropy for analyzing the dynamics of a multiple independent particles system. Several entropy definitions and particle dynamics with integer and fractional behavior are studied. The results reveal the adequacy of the entropy concept in the analysis of complex dynamical systems.

Keywords: Fractional calculus, Entropy, Dynamics, Complex systems.

1. Introduction

In the last two decades we witnessed an increasing interest in the generalization of the classical concepts of differential calculus and of entropy. The notion of 'Fractional calculus' (FC) stem from Leibniz [1-5], but only recently relevant applications emerged in the areas of physics and engineering [6-18]. The concept of entropy was introduced in the field of thermodynamics by Clausius (1862) and Boltzmann (1896) and was later applied by Shannon (1948) and Jaynes (1957) to information theory [19-21]. However, recently more general entropy measures have being proposed, allowing the relaxation of the additivity axiom for application in several types of complex systems [22-29].
The novel ideas are presently under a large development and open up ambitious perspectives. Bearing these facts in mind, the present study combines both concepts in the analysis of dynamical systems and is organized as follows. Section 2 introduces a brief description of the fractional calculus and the entropy. Section 3 formulates the conditions underlying the integer and fractional order dynamical system and develops their analysis through several entropy measures. Finally, section 4 outlines the main conclusions.

2. Fundamental Concepts

This section presents the main mathematical tools adopted in this study, namely the fractional calculus and the entropy.

2.1. Fractional calculus

Fractional calculus (FC) is a generalization of the ordinary integer differentiation and integration to an arbitrary order. The subject was initiated in 1695 by Leibniz that sent a

letter to L'Hospital with the question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". FC was an ongoing topic in the last three centuries and many mathematicians, such as Liouville, Riemann and Weyl, contributed to its development.

There are several definitions of fractional derivatives, being three of the most important the Riemann - Liouville, the Grunwald - Letnikov, and the Caputo given by:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \; n-1 < \alpha < n \tag{1} $$

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) \tag{2} $$

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \; n-1 < \alpha < n \tag{3} $$

where $\Gamma(\;)$ is the Euler's gamma function, $[x]$ means the integer part of *x*, and *h* is the step time increment.

It is also possible to generalize several results based on transforms, yielding expressions such as the Laplace expression:

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} L\{f(t)\} - \sum_{k=0}^{n-1} s^k \, {}_0D_t^{\alpha-k-1} f(0^+) \tag{4} $$

where *s* and *L* represent the Laplace variable and operator, respectively.

These definitions demonstrate that fractional derivatives capture the history of the variable, or, by other words, have memory, contrary to integer derivatives, that are local operators.

The Mittag-Leffler function $E_{\alpha}(x)$ arises in the solution of fractional integral equations and interpolates between a purely exponential law and a power-like behavior for phenomena governed by ordinary equations and their fractional counterparts. The function $E_{\alpha}(x)$ is defined by:

$$ E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \tag{5} $$

In particular, when $\alpha = 1$ we have $E_1(x) = e^x$. An important characteristic of the Mittag-Leffler function is its asymptotic behavior. In the case where the argument $x \le 0$, the Mittag-Leffler function decreases monotonically and, for large values of *x*, we can write:

$$ E_{\alpha}(-x) \approx \frac{1}{\Gamma(1-\alpha)} \frac{1}{x}, \; \alpha \neq 1 \tag{6} $$

The Laplace transform yields:

$$ L\left\{ E_{\alpha}\left(\pm a t^{\alpha} \right) \right\} = \frac{s^{\alpha-1}}{s^{\alpha} \mp a} \tag{7} $$

Therefore, we see a natural extension of the Laplace transform pairs for the exponential function in terms of integer powers of *s* to the Mittag-Leffler function in terms of fractional powers of the transform parameter *s*.

The Grunwald - Letnikov formulation inspires the numerical calculation of the fractional derivative based on the approximation of the time increment *h* through the sampling

period T and the series truncation at the rth term. This method is often denoted as Power Series Expansion (PSE) yielding the equation in the z – domain:

$$Z\{D^{\alpha}x(t)\} \approx \left[\frac{1}{T^{\alpha}}\sum_{k=0}^{r}\frac{(-1)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}z^{-k}\right]X(z) \tag{8}$$

where $X(z)=Z\{x(t)\}$ and z and Z represent the z-transform variable and operator, respectively.

In fact, expression (2) represents the Euler (or first backward difference) approximation in the $s \to z$ conversion scheme, being the Tustin approximation another possibility. The Euler and Tustin rational expressions, $G_0(z^{-1})=\frac{1}{T}(1-z^{-1})$ and $G_1(z^{-1})=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ respectively, are often called generating approximants of zero and first order, respectively. Therefore, the generalization of these conversion methods leads to the non-integer order α results:

$$s^{\alpha} \approx \left[\frac{1}{T}(1-z^{-1})\right]^{\alpha} = G_0^{\alpha}(z^{-1}) \tag{9.a}$$

$$s^{\alpha} \approx \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha} = G_1^{\alpha}(z^{-1}) \tag{9.b}$$

We can obtain a family of fractional differentiators generated by $G_0^{\alpha}(z^{-1})$ and $G_1^{\alpha}(z^{-1})$ weighted by the factors p and $1-p$, yielding:

$$s^{\alpha} \approx pG_0^{\alpha}(z^{-1})+(1-p)G_1^{\alpha}(z^{-1}) \tag{10}$$

For example, the Al-Alaoui operator corresponds to an interpolation of the Euler and Tustin rules with weighting factor $p=3/4$ [30].

In order to get a rational expression, the final approximation corresponds to a PSE or a rational fraction expansion. This approach is often denoted by Continued Fraction Expansion (CFE) of order $k \in \aleph$, based on a Padé expansion in the neighborhood of $z^{-1}=0$, yielding:

$$G_k(z^{-1})=\frac{\sum_{i=0}^{k}a_i z^{-i}}{\sum_{i=0}^{k}b_i z^{-i}},\ a_i, b_i \in \Re \tag{11}$$

Since one parameter is linearly dependent, usually it is established $b_0=1$.

2.2. Entropy

Khinchin formulated four axioms for a 'classical' (i.e., yielding an ordinary Boltzmann-Gibbs statistical mechanics) measure H, namely:

$$\text{Axiom 1: } H=H(p_1,\cdots,p_W) \tag{12}$$

$$\text{Axiom 2: } H(W^{-1},\cdots,W^{-1}) \geq H(p_1,\cdots,p_W) \tag{13}$$

$$\text{Axiom 3: } H(p_1,\cdots,p_W)=H(p_1,\cdots,p_W,0) \tag{14}$$

$$\text{Axiom 4: } H(\{p_{ij}^{A,B}\})=H(\{p_i^{A}\})+\sum_i p_i^{A}H(\{p_i^{B}(i\mid j)\}) \tag{15}$$

where W represents the number of possible events and p_i is the probability that event i occurs, so that $\sum_{i=1}^{W} p_i = 1$.

Axiom 1 means that H only depends on the probabilities p_i, $i = 1, \cdots, W$. Axiom 2 states that H takes a maximum for the uniform probability distribution (i.e., all probabilities are equal to $p_i = W^{-1}$). Axiom 3 says that H does not change if the sample set is enlarged by another event with zero probability. Axiom 4 postulates that given two systems A and B, not necessarily independent, H should be independent of the way information is collected. When systems A and B are independent, it results $p_{ij}^{A,B} = p_i^A p_j^B$ and axiom 4 reduces to the rule of additivity:

$$H\left(\left\{p_{ij}^{A,B}\right\}\right) = H\left(\left\{p_i^A\right\}\right) + H\left(\left\{p_j^B\right\}\right) \tag{16}$$

This condition is less stringent than axiom (15) and states that the entropy of independent systems should be additive.

The most celebrated entropy is the so-called Shannon entropy S defined by:

$$S = -\sum_{i=1}^{W} p_i \, ln(p_i) \tag{17}$$

that satisfies the four Shannon-Khinchin axioms (12)-(14).

The Shannon entropy represents the expected value of the information $-\ln(p_i)$. Therefore, for the uniform probability distribution we have $p_i = W^{-1}$ and the Shannon entropy takes its maximum value $S = \ln(W)$, yielding the Boltzmann's famous formula, up to a multiplicative factor k denoting the Boltzmann constant. Therefore, in thermodynamic equilibrium, the Shannon entropy can be identified as the 'physical entropy' of the system.

Two of the most studied generalizations of the entropy are the Rényie and Tsallis entropies given by:

$$S_q^{(R)} = \frac{1}{1-q} ln\left(\sum_{i=1}^{W} p_i^{\,q}\right), q > 0 \tag{18}$$

$$S_q^{(T)} = \frac{1}{q-1}\left(1 - \sum_{i=1}^{W} p_i^{\,q}\right) \tag{19}$$

that reduce to the Shannon entropy when $q \to 1$.

Recently M. Ubriaco [31] proposed the fractional entropy:

$$S_q^{(U)} = \sum_{i=1}^{W} \left(-ln\, p_i\right)^q p_i \tag{20}$$

that has the same properties as the Shannon entropy except additivity.

Several other measures were proposed such as the Landsberg-Vedral, Abel, Kaniagakis and Sharma-Mital entropies.

3. Entropy Analysis of Integer and Fractional Dynamical Systems

In this section we analyze integer and fractional order dynamical systems through the entropy measure.

We consider an isolated system consisting of n independent particles, each one having a one-dimensional trajectory evolution $x_i(t), i = 1,\cdots,n$. There is neither dynamical interaction between particles (i.e., the probability of collisions between particles is zero) nor impacts with some container or walls. Therefore, the system dynamical state is characterized by the phase space $\{X,\dot{X}\} \equiv \{(x_i,\dot{x}_i), i = 1,\cdots,n\}$ and each particle has a specific dynamics.

In this paper the particle dynamics is considered to be one of the following integer and fractional order test cases:

$$x_i^{(1)} + ax_i = 0, \quad i = 1,\cdots,n \tag{21}$$

$$x_i^{(2)} + ax_i^{(1)} + bx_i = 0, \quad i = 1,\cdots,n \tag{22}$$

$$x_i^{(\alpha)} + ax_i = 0, 0 < \alpha < 1, \quad i = 1,\cdots,n \tag{23}$$

$$x_i^{(1+\alpha)} + ax^{(1)}{}_i + bx_i^{(\alpha)} + cx_i = 0, \quad 0 < \alpha < 1, \quad i = 1,\cdots,n \tag{24}$$

It is adopted $n = 10^4$ and the initial conditions are generated by a normalized Gaussian distribution $N(0,1)$ with zero average and unit standard deviation. For the cases (23)-(24) the fractional derivates are evaluated numerically through the PSE method (8), with $r = 10$, and the initialization is completed numerically by a preliminary simulation of r iterations. Moreover, in all cases it is adopted an integration step time of $dt = 10^{-2}$ sec.

The system global state is measured through the entropies S, $S_q^{(R)}$, $S_q^{(T)}$, $S_q^{(U)}$ and, for the Rényie, Tsallis and Ubriaco entropy measures, are evaluated the cases $q = \{1/2,2\}$. The theoretical probabilities are approximated through the relative frequencies of occurrence. For this purpose, in each time step dt, is constructed a two-dimensional histogram characterizing the phase space $\{X,\dot{X}\}$ with $N_1 \times N_2 = 10^2 \times 10^2$ bins in the range $N_1 : -max(x_i) < x_i < max(x_i)$ and $N_2 : -max\left(x_i^{(1)}\right) < x_i^{(1)} < max\left(x_i^{(1)}\right)$. Therefore, by applying to the phase-space dynamics a time-sliding window with duration dt, a sequence of time-stamped values $S(t)$, $S_q^{(R)}(t)$, $S_q^{(T)}(t)$, $S_q^{(U)}(t)$ is obtained, producing a particular entropy curve that depends on the system, the entropy formula and the time. Furthermore, in order to simplify the comparison the different measures are rescaled so that that we get an evolution between one and zero.

Figures 1-4 show the seven cases of entropy measuring when applied to systems with particle dynamics described by (21)-(24), respectively. The Shannon entropy S is represented with a thicker trace. As expected we verify that, in all cases, the entropy decreases monotonically and that the $S(t)$ plot is in the 'middle' of the traces corresponding to $S_q^{(R)}(t)$, $S_q^{(T)}(t)$, $S_q^{(U)}(t)$ for $q = \{1/2,2\}$. In what concerns the global system dynamics we observe that the first order equation (21) leads to a single phase evolution while the second order equation (22) leads to a more 'sophisticated' behavior. The fractional dynamics (23) and (24) somehow interpolate the integer order cases.

Figures 5-6 show the variation of $S(t)$, $S_q^{(R)}(t)$, $S_q^{(T)}(t)$, $S_q^{(U)}(t)$ for different eigenvalues in (22) and distinct fractional orders α in (24).

In the simulations the initial state was totally disordered and, therefore, it has maximal entropy normalized to one. For time increasing the systems tend to a final state of static or dynamic equilibrium with entropy zero.

We conclude that the entropy characterizes adequately the dynamical evolution of the multiple particle system. The tested cases did not show any pros or cons for a particular entropy measure. Since the benchmarks were developed for an isolated system, without any type of further dynamical interaction, it remains to be investigated the behavior of more complex systems.

The time evolution of the entropy plotted in the charts can be approximated easily and with good accuracy by polynomials or rational fractions; nevertheless, under the process of numerical fitting, the dynamic behavior will be hidden. Bearing this idea in mind, it was devised an alternative strategy and the asymptotic behavior, that is, the evolution at the beginning or at the end of the time evolution, was approximated numerically by independent functions. In order to avoid the perturbation introduced by the bin counting for the relative frequency calculation when the entropy is close to zero, the final values were disregarded. For the Shannon entropy the approximations of the initial $S_{initial}(t)$ and final $S_{final}(t)$ evolutions, yielded the results:

- Figure 1: $S_{initial}(t) \approx 1 - 0.25t$, $S_{final}(t) \sim -0.62 \ln t$;
- Figure 2: $S_{initial}(t) \approx 1 - 0.36t$, $S_{final}(t) \sim -0.43 \ln t$;
- Figure 3: $S_{initial}(t) \approx 1 - 5.4t$, $S_{final}(t) \sim -0.62 \ln t$;
- Figure 4: $S_{initial}(t) \approx 1 - 0.35t$, $S_{final}(t) \sim -0.05 \ln t$;
- Figure 5: $(a,b) = (2,1)$, $S_{initial}(t) \approx 1 - 0.24t$, $S_{final}(t) \sim -0.40 \ln t$,
  $(a,b) = (2,1)$, $S_{initial}(t) \approx 1 - 0.31t$, $S_{final}(t) \sim -0.38 \ln t$,
  $(a,b) = (2,1)$, $S_{initial}(t) \approx 1 - 0.47t$, $S_{final}(t) \sim -0.38 \ln t$,
  $(a,b) = (2,1)$, $S_{initial}(t) \approx 1 - 0.59t$, $S_{final}(t) \sim -0.40 \ln t$;
- Figure 6: $\alpha = 0.1$, $S_{initial}(t) \approx 1 - 0.47t$, $S_{final}(t) \sim -0.61 \ln t$,
  $\alpha = 0.3$, $S_{initial}(t) \approx 1 - 0.41t$, $S_{final}(t) \sim -0.61 \ln t$,
  $\alpha = 0.5$, $S_{initial}(t) \approx 1 - 0.35t$, $S_{final}(t) \sim -0.61 \ln t$,
  $\alpha = 0.7$, $S_{initial}(t) \approx 1 - 0.32t$, $S_{final}(t) \sim -0.59 \ln t$,
  $\alpha = 0.9$, $S_{initial}(t) \approx 1 - 0.25t$, $S_{final}(t) \sim -0.63 \ln t$.

We verify that while the initial transient has a fast linear time variation, the final dynamics is much slower revealing logarithmic time dependence.

While the focus of the present study was mainly the comparison of different entropy formulations for integer and fractional dynamical systems, another important aspect deserves future attention. In fact, it was assumed that the effect of interaction terms between the subsystems is negligible in comparison with the effects of their own equations of motion. Nevertheless, in complex systems interaction effects modify the dynamics [32] and it remains to be investigated the resulting decrease of entropy which will be pursuit in future research.

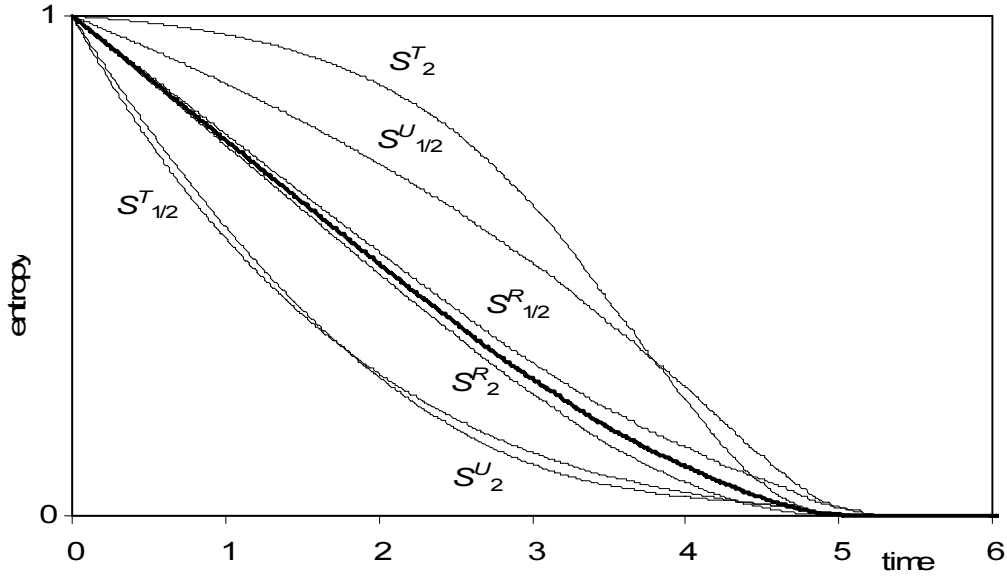

Fig. 1. Evolution of $S(t)$ (thicker trace), $S_q^{(R)}(t)$, $S_q^{(T)}(t)$ and $S_q^{(U)}(t)$ for $q = \{1/2, 2\}$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (21) with $a = 1.0$.

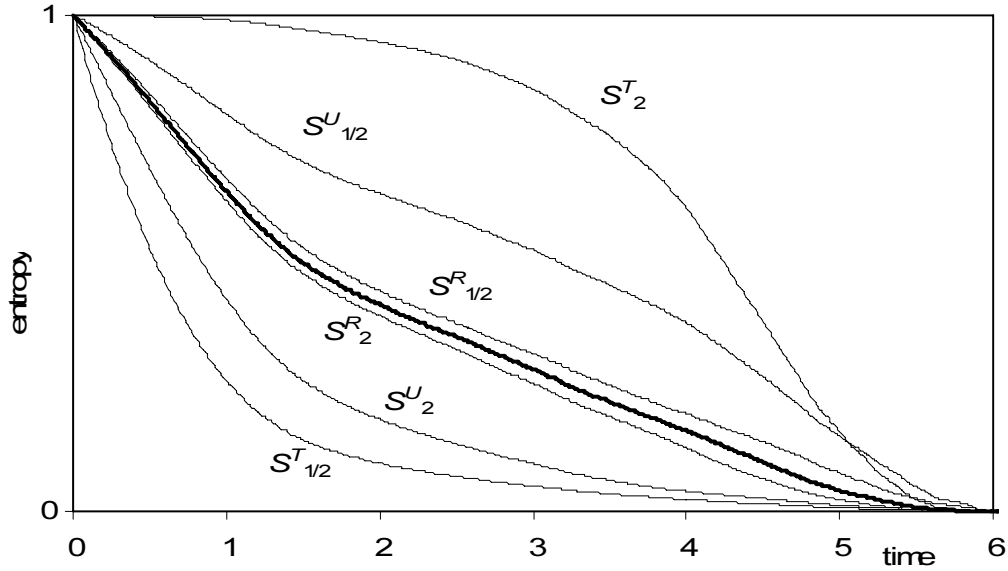

Fig. 2. Evolution of $S(t)$ (thicker trace), $S_q^{(R)}(t)$, $S_q^{(T)}(t)$ and $S_q^{(U)}(t)$ for $q = \{1/2, 2\}$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (22) with $a = 3.0$, $b = 2.0$.

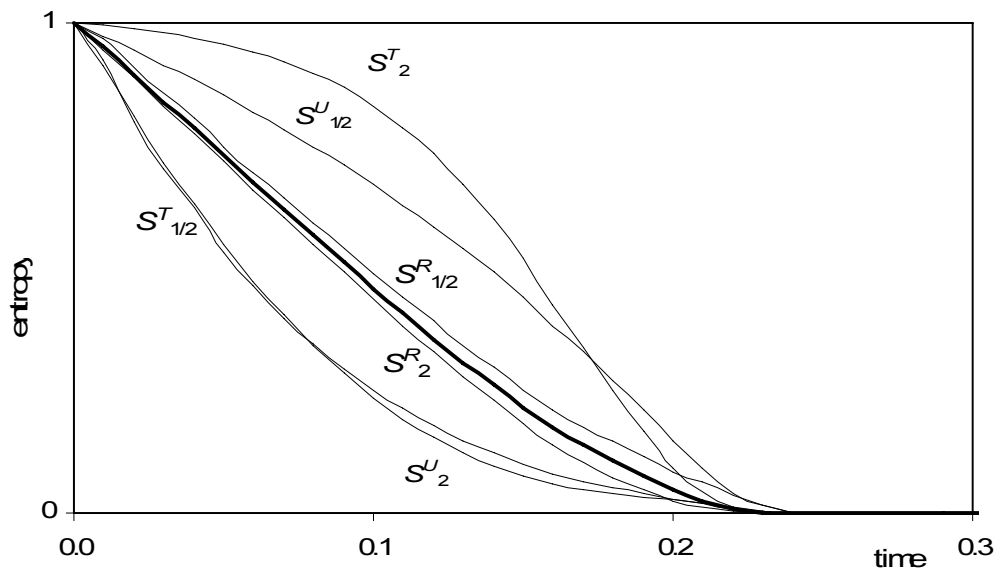

Fig. 3. Evolution of $S(t)$ (thicker trace), $S_q^{(R)}(t)$, $S_q^{(T)}(t)$ and $S_q^{(U)}(t)$ for $q = \{1/2, 2\}$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (23) with $\alpha = 0.5$, $a = 1.0$.

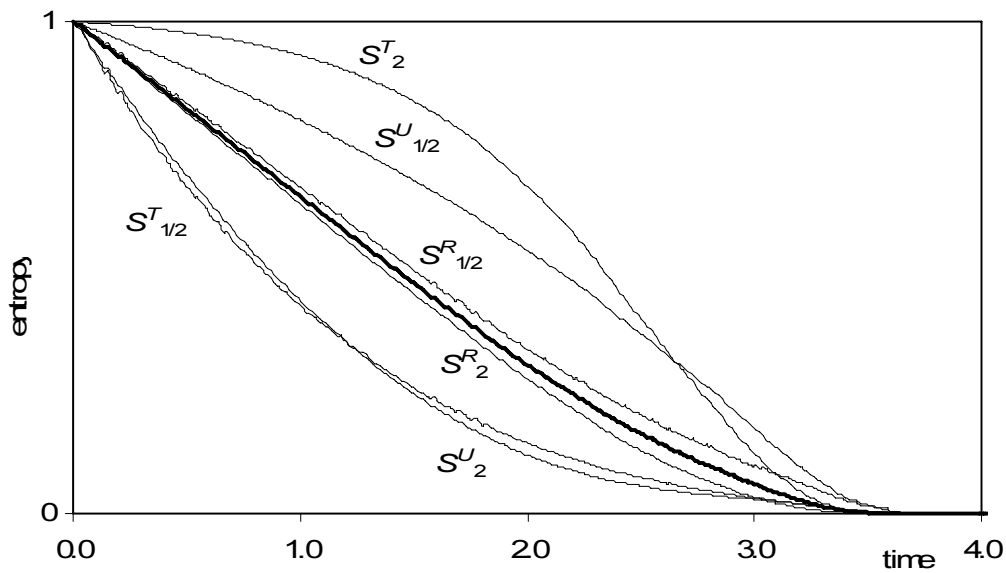

Fig. 4. Evolution of $S(t)$ (thicker trace), $S_q^{(R)}(t)$, $S_q^{(T)}(t)$ and $S_q^{(U)}(t)$ for $q = \{1/2, 2\}$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (24) with $\alpha = 0.5$, $a = 1.0$, $b = 2.0$ and $c = 2.0$.

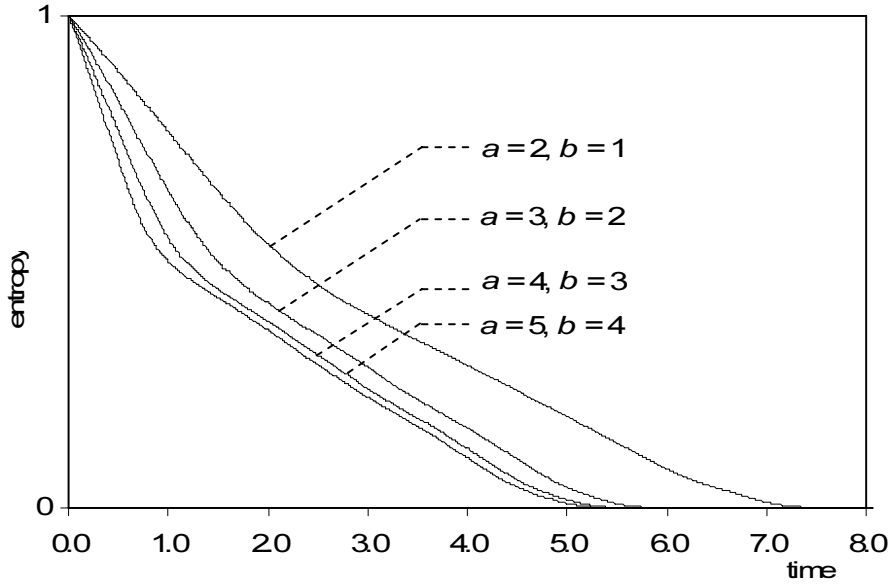

Fig. 5. Evolution of $S(t)$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (22) with $(a,b) = \{(2,1), (3,2), (4,3), (5,4)\}$.

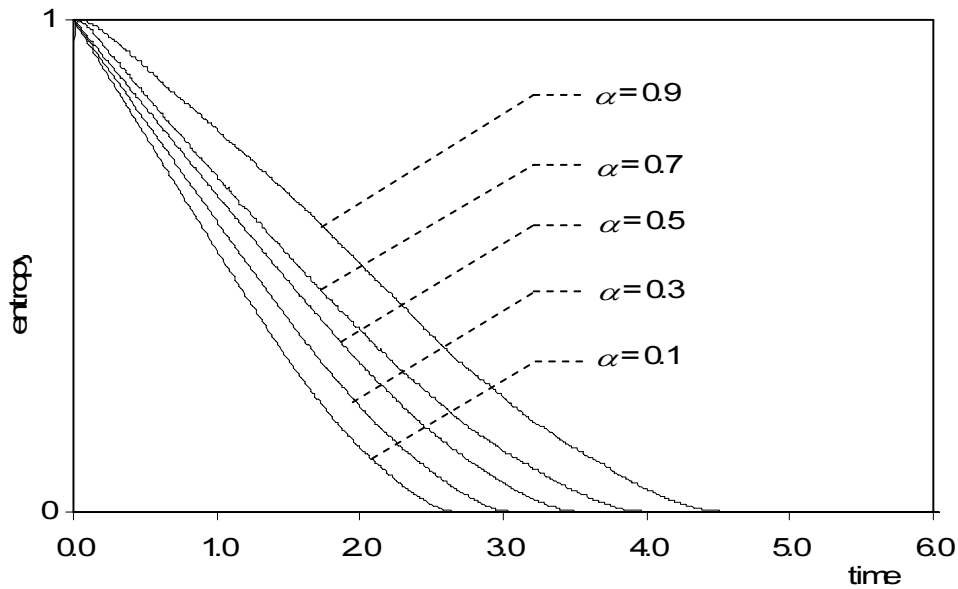

Fig. 6. Evolution of $S(t)$ for a system composed by $n = 10^4$ independent particles having dynamics given by expression (24) with $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$, $a = 1.0$, $b = 2.0$ and $c = 2.0$.

4. Conclusions

This paper reviewed two important mathematical tools, namely the fractional calculus and the entropy. These concepts allow a fruitful interplay in the analysis of system dynamics. Nevertheless, the synergies of applying both tools has been, somehow, neglected in engineering and applied sciences. The paper analyzed multi-particle systems with integer and fractional order behavior and demonstrated that the concepts are simple and straightforward to apply. In this line of thought, future research will address the analysis of more complex systems.

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