On Singular Semi-Riemannian Manifolds

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Abstract. On a Riemannian or a semi-Riemannian manifold, the metric determines invariants like the Levi-Civita connection and the Riemann curvature. If the metric becomes degenerate (as in singular semi-Riemannian geometry), these constructions no longer work, because they are based on the inverse of the metric, and on related operations like the contraction between covariant indices.

In this article we develop the geometry of singular semi-Riemannian manifolds. First, we introduce an invariant and canonical contraction between covariant indices, applicable even for degenerate metrics. This contraction applies to a special type of tensor fields, which are radical-annihilator in the contracted indices. Then, we use this contraction and the Koszul form to define the covariant derivative for radical-annihilator indices of covariant tensor fields, on a class of singular semi-Riemannian manifolds named radical-stationary. We use this covariant derivative to construct the Riemann curvature, and show that on a class of singular semi-Riemannian manifolds, named semi-regular, the Riemann curvature is smooth.

We apply these results to construct a version of Einstein's tensor whose density of weight 2 remains smooth even in the presence of semi-regular singularities. We can thus write a densitized version of Einstein's equation, which is smooth, and which is equivalent to the standard Einstein equation if the metric is non-degenerate.

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Contents

1. Introduction

1.1. Motivation and related advances

Let M be a differentiable manifold with a symmetric inner product structure, named metric, on its tangent bundle. If the metric is non-degenerate, we can construct in a canonical way a Levi-Civita connection and the Riemann, Ricci and scalar curvatures. If the metric is allowed to be degenerate (hence M is a singular semi-Riemannian manifold), some obstructions prevented the construction of such invariants.

Degenerate metrics are useful because they can arise in various contexts in which semi-Riemannian manifolds are used. They are encountered even in manifolds with non-degenerate (but indefinite) metric, because the metric induced on a submanifold can be degenerate. The properties of such submanifolds were studied *e.g.* in [35, 37], [5, 13].

In General Relativity, there are models or situations when the metric becomes degenerate or changes its signature. As the Penrose and Hawking *singularity theorems* [42, 19, 23, 22] show, Einstein's equation leads to singularities under very general conditions, apparently similar to the matter distribution in our Universe. Therefore, many attempts were done to deal with such singularities. For example it was suggested that Ashtekar's method of "new variables" [1, 2, 45] can be used to pass beyond the singularities, because the variable $\widetilde{E}^a_i$ – a densitized frame of vector fields – defines the metric, which can be degenerate. Unfortunately, it turned out that in this case the connection variable A^i_a may become singular *cf. e.g.* [52].

In some cosmological models the initial singularity of the Big Bang is eliminated by making the metric Riemannian for the early Universe. The metric changes the signature when traversing a hypersurface, becoming Lorentzian,

so that time emerges from a space dimension. Some particular junction conditions were studied (see [46],[15, 16],[25, 26, 27], [7], [10, 11, 28, 12, 8, 9], [29, 30, 31, 32, 33, 34] *etc.*).

Other situation where the metric can become degenerate was proposed by Einstein and Rosen, as a model of charged particles [14].

All these applications in Geometry and General Relativity demand a generalization of the standard methods of semi-Riemannian Geometry, to cover the degenerate case. A degenerate metric prevents the standard constructions like covariant derivative and curvature. Manifolds endowed with degenerate metrics were studied by Moisil [39], Strubecker [47, 48, 49, 50], Vrănceanu [51]. Notable is the work of Kupeli [35, 36, 37], which is limited to the constant signature case.

1.2. **Presentation of this article**

The purpose of this article twofold:

(1) to provide a toolbox of geometric invariants, which extend the standard constructions from semi-Riemannian geometry to the non-degenerate case, with constant or variable signature,
(2) and to apply these constructions to extend Einstein's equation to a class of singular spacetimes.

The first goal of this article is to construct canonical invariants such as the covariant derivative and Riemann curvature tensor, in the case of singular semi-Riemannian geometry. The main obstruction for this is the fact that when the metric is non-degenerate, it doesn't admit an inverse. This prohibits operations like index raising and contractions between covariant indices. This prevents the definition of a Levi-Civita connection, and by this, the construction of the curvature invariants. This article presents a way to construct such invariants even if the metric is degenerate, for a class of singular semi-Riemannian manifolds which are named *semi-regular*.

The second goal is to apply the tools developed here to write a densitized version of Einstein's tensor which remains smooth in the presence of singularities, if the spacetime is semi-regular. Consequently, we can write a version of Einstein's equation which is equivalent to the standard one if the metric is non-degenerate. This allows us to extend smoothly the equations of General Relativity beyond the apparent limits imposed by the singularity theorems of Penrose and Hawking [42, 19, 23, 22].

Section §2 contains generalities on singular semi-Riemannian manifolds, in particular the radical bundle associated to the metric, made of the degenerate tangent vectors. In section §3 are studied the properties of the radical-annihilator bundle, consisting in the covectors annihilating the degenerate vectors. Tensor fields which are radical-annihilator in some of their covariant indices are introduced. On this bundle we can define a metric which is the next best thing to the inverse of the metric, and which will be used to

perform contractions between covariant indices. Section §4 shows how we can contract covariant indices of tensor fields, so long as these indices are radical-annihilators.

Normally, the Levi-Civita connection is obtained by raising an index of the right member of the Koszul formula (named here Koszul form), operation which is not available when the metric is degenerate. Section §5 studies the properties of the Koszul form, which are similar to those of the Levi-Civita connection. This allows us to construct in section §6 a sort of covariant derivative for vector fields, and in §6.3 a covariant derivative for differential forms.

The notion of semi-regular semi-Riemannian manifold is defined in section §7 as a special type of singular semi-Riemannian manifold with variable signature on which the lower covariant derivative of any vector field, which is a 1-form, admits smooth covariant derivatives.

The Riemann curvature tensor is constructed in §7 with the help of the Koszul form and of the covariant derivative for differential forms introduced in section §6. For semi-regular semi-Riemannian manifolds, the Riemann curvature tensor is shown to be smooth, and to have the same symmetry properties as in the non-degenerate case. In addition, it is radical-annihilator in all of its indices, this allowing the construction of the Ricci and scalar curvatures. Then, in section §8, the Riemann curvature tensor is expressed directly in terms of the Koszul form, obtaining an useful formula. Then the Riemann curvature is compared with a curvature tensor obtained by Kupeli by other means [36].

Section §9 presents two examples of semi-regular semi-Riemannian manifolds. The first is based on diagonal metrics, and the second on degenerate metrics which are conformal to non-degenerate metrics.

The final section, §10, applies the results of this article to General Relativity. This section studies the Einstein's equation on semi-regular semi-Riemannian manifolds. It proposes a densitized version of this equation, which remains smooth on semi-regular spacetimes, and reduces to the standard Einstein equation if the metric is non-degenerate.

2. Singular semi-Riemannian manifolds

2.1. Definition of singular semi-Riemannian manifolds

Definition 2.1. (see *e.g.* [36], [41], p. 265 for comparison) A *singular semi-Riemannian manifold* is a pair (M, g), where M is a differentiable manifold, and $g \in \Gamma(T^*M \odot_M T^*M)$ is a symmetric bilinear form on M, named *metric tensor* or *metric*. If the signature of g is fixed, then (M, g) is said to be with *constant signature*. If the signature of g is allowed to vary from point to point, (M, g) is said to be with *variable signature*. If g is non-degenerate,

then (M, g) is named *semi-Riemannian manifold*. If g is positive definite, (M, g) is named *Riemannian manifold*.

Remark 2.2. Let (M, g) be a singular semi-Riemannian manifold and let $M_\wr \subseteq M$ be the set of the points where the metric changes its signature. The set $M - M_\wr$ is dense in M, and it is a union of singular semi-Riemannian manifolds with constant signature.

Example 2.3 (Singular Semi-Euclidean Spaces $\mathbb{R}^{r,s,t}$, *cf. e.g.* [41], p. 262)**.** Let $r, s, t \in \mathbb{N}$, $n = r + s + t$, We define the singular semi-Euclidean space $\mathbb{R}^{r,s,t}$ by:

$$\mathbb{R}^{r,s,t} := (\mathbb{R}^n, \langle , \rangle), \tag{1}$$

where the metric acts on two vector fields X, Y on $\mathbb{R}^n$ at a point p on the manifold, in the natural chart, by

$$\langle X_p, Y_p \rangle = -\sum_{i=r+1}^{s} X^i Y^i + \sum_{j=r+s+1}^{n} X^j Y^j. \tag{2}$$

If $r = 0$ we fall over the semi-Euclidean space $\mathbb{R}^n_s := \mathbb{R}^{0,s,t}$ (see *e.g.* [40], p. 58). If $s = 0$ we find the degenerate Euclidean space. If $r = s = 0$, then $t = n$ and we recover the Euclidean space $\mathbb{R}^n$ endowed with the natural scalar product.

2.2. **The radical of a singular semi-Riemannian manifold**

Definition 2.4. (*cf. e.g.* [5], p. 1, [38], p. 3 and [40], p. 53) Let (V, g) be a finite dimensional inner product space, where the inner product g may be degenerate. The totally degenerate space $V_\circ := V^\perp$ is named the *radical* of V. An inner product g on a vector space V is non-degenerate if and only if $V_\circ = \{0\}$.

Definition 2.5. (see *e.g.* [36], p. 261, [41], p. 263) We denote by $T_\circ M$ and we call *the radical of* TM the following subset of the tangent bundle: $T_\circ M = \cup_{p\in M}(T_pM)_\circ$. We can define vector fields on M valued in $T_\circ M$, by taking those vector fields $W \in \mathfrak{X}(M)$ for which $W_p \in (T_pM)_\circ$. We denote by $\mathfrak{X}_\circ(M) \subseteq \mathfrak{X}(M)$ the set of these sections – they form a vector space over $\mathbb{R}$ and a module over $\mathscr{F}(M)$. $T_\circ M$ is a vector bundle if and only if the signature of g is constant on all M, and in this case, $T_\circ M$ is a distribution.

Example 2.6. The radical $T_\circ\mathbb{R}^{r,s,t}$ of the singular semi-Euclidean manifold $\mathbb{R}^{r,s,t}$ in the Example 2.3 is spanned at each point p by the tangent vectors ∂_{ap} with $a \leq r$:

$$T_\circ\mathbb{R}^{r,s,t} = \bigcup_{p\in\mathbb{R}^{r,s,t}} \operatorname{span}(\{(p, \partial_{ap}) | \partial_{ap} \in T_p\mathbb{R}^{r,s,t}, a \leq r\}). \tag{3}$$

The sections of $T_\circ\mathbb{R}^{r,s,t}$ are therefore given by

$$\mathfrak{X}_\circ(\mathbb{R}^{r,s,t}) = \{X \in \mathfrak{X}(\mathbb{R}^{r,s,t}) | X = \sum_{a=1}^{r} X^a \partial_a\}. \tag{4}$$

3. The radical-annihilator inner product space

Let (V, g) be an inner product vector space. If the inner product g is non-degenerate, it defines an isomorphism $\flat : V \to V^*$ (see *e.g.* [18], p. 15; [17], p. 72). If g is degenerate, $\flat$ remains a linear morphism, but not an isomorphism. This is why we can no longer define a dual for g on V^* in the usual sense. We will see that we can still define canonically an inner product $g_\bullet \in \flat(V)^* \odot \flat(V)^*$, and use it to define contraction and index raising in a weaker sense than in the non-degenerate case. This rather elementary construction can be immediately extended to singular semi-Riemannian manifolds. It provides a tool to contract covariant indices and construct the invariants we need.

3.1. The radical-annihilator vector space

This section applies well-known elementary properties of linear algebra, with the purpose is to extend fundamental notions related to the non-degenerate inner product g on a vector space V induced on the dual space V^* (*cf. e.g.* [44], p. 59), to the case when g is allowed to be degenerate. Let (V, g) be an inner product space over $\mathbb{R}$.

Definition 3.1. The inner product g defines a vector space morphism, named the *index lowering morphism* $\flat : V \to V^*$, by associating to any $u \in V$ a linear form $\flat(u) : V \to \mathbb{R}$ defined by $\flat(u)v := \langle u, v\rangle$. Alternatively, we use the notation $u^\flat$ for $\flat(u)$. For reasons which will become apparent, we will also use the notation $u^\bullet := u^\flat$.

Remark 3.2. It is easy to see that $V_\circ = \ker\flat$, so $\flat$ is an isomorphism if and only if g is non-degenerate.

Definition 3.3. The *radical-annihilator* vector space $V^\bullet := \operatorname{im}\flat \subseteq V^*$ is the space of 1-forms ω which can be expressed as $\omega = u^\bullet$ for some u, and they act on V by $\omega(v) = \langle u, v\rangle$.

Obviously, in the case when g is non-degenerate, we have the identification $V^\bullet = V^*$.

Remark 3.4. In other words, $V^\bullet$ is the annihilator of $V_\circ$. It follows that $\dim V^\bullet + \dim V_\circ = n$.

Remark 3.5. Any $u' \in V$ satisfying $u'^\bullet = \omega$ differs from u by $u' - u \in V_\circ$. Such 1-forms $\omega \in V^\bullet$ satisfy $\omega|_{V_\circ} = 0$.

Definition 3.6. On the vector space $V^\bullet$ we can define a unique non-degenerate inner product $g_\bullet$ by $g_\bullet(\omega,\tau) := \langle u, v\rangle$, where $u^\bullet = \omega$ and $v^\bullet = \tau$. We alternatively use the notation $\langle\!\langle \omega, \tau \rangle\!\rangle_\bullet = g_\bullet(\omega,\tau)$.

Proposition 3.7. The inner product $g_\bullet$ from above is well-defined, being independent on the vectors u, v chosen to represent the 1-forms ω, τ.

Proof. If $u', v' \in V$ are other vectors satisfying $u'^\bullet = \omega$ and $v'^\bullet = \tau$, then $u' - u \in V_\circ$ and $v' - v \in V_\circ$. $\langle u', v'\rangle = \langle u, v\rangle + \langle u' - u, v\rangle + \langle u, v' - v\rangle + \langle u' - u, v' - v\rangle = \langle u, v\rangle$. □

Proposition 3.8. The inner product $g_\bullet$ from above is non-degenerate, and if g has the signature (r, s, t), then the signature of $g_\bullet$ is $(0, s, t)$.

Proof. Let's take an orthonormal basis $(e_a)_{a=1}^n$ in which the inner product is diagonal, with the first r diagonal elements being 0. We have ${e_a}^\bullet = 0$ for $a \in \{1, \dots, r\}$, and the 1-forms $\omega_a := {e_{r+a}}^\bullet$ for $a \in \{1, \dots, s + t\}$ are the generators of $V^\bullet$. They satisfy $\langle\!\langle \omega_a, \omega_b \rangle\!\rangle_\bullet = \langle e_{r+a}, e_{r+b}\rangle$. Therefore, $(\omega_a)_{a=1}^{s+t}$ are linear independent and the signature of $g_\bullet$ is $(0, s, t)$. □

3.2. **The radical-annihilator vector bundle**

Definition 3.9. We denote by $T^\bullet M$ the subset of the cotangent bundle defined as

$$T^\bullet M = \bigcup_{p\in M} (T_pM)^\bullet \tag{5}$$

where $(T_pM)^\bullet \subseteq T_p^*M$ is the space of covectors at p which can be expressed as $\omega_p(X_p) = \langle Y_p, X_p\rangle$ for some $Y_p \in T_pM$ and any $X_p \in T_pM$. $T^\bullet M$ is a vector bundle if and only if the signature of the metric is constant. We can define sections of $T^\bullet M$ in the general case, by

$$\mathcal{A}^\bullet(M) := \{\omega \in \mathcal{A}^1(M) | \omega_p \in (T_pM)^\bullet \text{ for any } p \in M\}. \tag{6}$$

Remark 3.10. $(T_pM)^\bullet$ is the annihilator space (*cf. e.g.* [44], p. 102) of the radical space $T_{\circ p}M$, that is, it contains the linear forms ω_p which satisfy $\omega_p|_{T_{\circ p}M} = 0$.

Example 3.11. The radical-annihilator $T^\bullet\mathbb{R}^{r,s,t}$ of the singular semi-Euclidean manifold $\mathbb{R}^{r,s,t}$ in the Example 2.3 is:

$$T^\bullet\mathbb{R}^{r,s,t} = \bigcup_{p\in\mathbb{R}^{r,s,t}} \operatorname{span}(\{\mathrm{d}x^a \in T_p^*\mathbb{R}^{r,s,t} | a > r\}). \tag{7}$$

Consequently, the radical-annihilator 1-forms have the general form

$$\omega = \sum_{a=r+1}^{n} \omega_a \mathrm{d}x^a, \tag{8}$$

and

$$\mathcal{A}^\bullet(\mathbb{R}^{r,s,t}) = \{\omega \in \mathcal{A}^1(\mathbb{R}^{r,s,t}) | \omega^i = 0, i \le r\}. \tag{9}$$

3.3. **The radical-annihilator inner product in a basis**

Let us consider an inner product space (V, g), and a basis $(e_a)_{a=1}^n$ of V in which g takes the diagonal form $g = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_a \in \mathbb{R}$ for all $1 \leq a \leq n$. The inner product satisfies:

$$g_{ab} = \langle e_a, e_b \rangle = \alpha_a \delta_{ab}. \tag{10}$$

We also have

$$e_a{}^\bullet(e_b) := \langle e_a, e_b \rangle = \alpha_a \delta_{ab},$$

and, if $(e^{*a})_{a=1}^n$ is the dual basis of $(e_a)_{a=1}^n$,

$$e_a{}^\bullet = \alpha_a e^{*a}. \tag{11}$$

Proposition 3.12. If in a basis the inner product has the form $g_{ab} = \alpha_a \delta_{ab}$, then

$$g_\bullet{}^{ab} = \frac{1}{\alpha_a} \delta^{ab}, \tag{12}$$

for all a so that $\alpha_a \neq 0$.

Proof. Since

$$\langle\!\langle e_a{}^\bullet, e_b{}^\bullet \rangle\!\rangle_\bullet = \langle e_a, e_b \rangle = \alpha_a \delta_{ab},$$

and in the same time

$$\langle\!\langle e_a{}^\bullet, e_b{}^\bullet \rangle\!\rangle_\bullet = \alpha_a \alpha_b \langle\!\langle e^{*a}, e^{*b} \rangle\!\rangle_\bullet = \alpha_a \alpha_b g_\bullet{}^{ab},$$

we have that

$$\alpha_a \alpha_b g_\bullet{}^{ab} = \alpha_a \delta_{ab},$$

This leads, for $\alpha_a \neq 0$, to

$$g_\bullet{}^{ab} = \frac{1}{\alpha_a} \delta_{ab}.$$

The case when $\alpha_a = 0$ doesn't happen, since $g_\bullet$ is defined only on $\text{im}\, \flat$. □

3.4. **Radical and radical-annihilator tensors**

For inner product vector spaces we define tensors that are radical in a contravariant slot, and radical-annihilator in a covariant slot, and give their characterizations.

Definition 3.13. Let T be a tensor of type (r, s). We call it *radical* in the k-th contravariant slot if $T \in \mathcal{T}_0^{k-1} M \otimes_M T_\circ M \otimes_M \mathcal{T}_s^{r-k} M$. We call it *radical-annihilator* in the l-th covariant slot if $T \in \mathcal{T}_{l-1}^r M \otimes_M T^\bullet M \otimes_M \mathcal{T}_{s-l}^0 M$.

Proposition 3.14. A tensor $T \in \mathcal{T}_s^r M$ is radical in the k-th contravariant slot if and only if its contraction $C_{s+1}^k(T \otimes \omega)$ with any radical-annihilator linear 1-form $\omega \in \mathcal{A}^1(M)$ is zero.

Proof. For simplicity, we can work on an inner product space (V, g) and consider $k = r$ (if $k < r$, we can make use of the permutation automorphisms of the tensor space $\mathcal{T}^r_s V$). T can be written as a sum of linear independent terms having the form $\sum_\alpha S_\alpha \otimes v_\alpha$, with $S_\alpha \in \mathcal{T}^{r-1}_s V$ and $v_\alpha \in V$. We keep only the terms with $S_\alpha \neq 0$. The contraction of the r-th contravariant slot with any $\omega \in V^\bullet$ becomes $\sum_\alpha S_\alpha \omega(v_\alpha)$.

If T is radical in the r-th contravariant slot, for all α and any $\omega \in V^\bullet$ we have $\omega(v_\alpha) = 0$, therefore $\sum_\alpha S_\alpha \omega(v_\alpha) = 0$.

Reciprocally, if $\sum_\alpha S_\alpha \omega(v_\alpha) = 0$, it follows that for any α, $S_\alpha \omega(v_\alpha) = 0$. Then, $\omega(v_\alpha) = 0$, because $S_\alpha \neq 0$. It follows that $v_\alpha \in V_\circ$. □

Proposition 3.15. A tensor $T \in \mathcal{T}^r_s M$ is radical-annihilator in the l-th covariant slot if and only if its l-th contraction with any radical vector field is zero.

Proof. The proof goes as in Proposition 3.14. □

Example 3.16. The inner product g is radical-annihilator in both of its slots. This means that $g \in \mathcal{A}^\bullet(M) \odot_M \mathcal{A}^\bullet(M)$.

Proof. Follows directly from the definition of $TM_\circ$ and of radical-annihilator tensor fields. □

Proposition 3.17. The contraction between a radical slot and a radical-annihilator slot of a tensor is zero.

Proof. Follows from the Proposition 3.14 combined with the commutativity between tensor products and linear combinations with contraction. The proof goes similar to that of the Proposition 3.14. □

4. Covariant contraction of tensor fields

We don't need an inner product to define contractions between one covariant and one contravariant indices. We can use the inner product g to contract between two contravariant indices, obtaining the *contravariant contraction operator* C^{kl} (*cf. e.g.* [40], p. 83). On the other hand, the contraction is not always well defined for two covariant indices. We will see that we can use $g_\bullet$ for such contractions, but this works only for vectors or tensors which are radical-annihilator in covariant slots. Fortunately, this kind of tensors turn out to be the relevant ones in the applications to singular semi-Riemannian geometry.

4.1. Covariant contraction on inner product spaces

Definition 4.1. We can define uniquely the *covariant contraction* or *covariant trace* operator by the following steps.

(1) We define it first on tensors $T \in V^\bullet \otimes V^\bullet$, by $C_{12}T = g_\bullet{}^{ab} T_{ab}$. This definition is independent on the basis, because $g_\bullet \in V^{\bullet *} \otimes V^{\bullet *}$.

(2) Let $T \in \mathcal{T}^r_s V$ be a tensor with $r \geq 0$ and $s \geq 2$, which satisfies $T \in V^{\otimes r} \otimes V^{*\otimes s-2} \otimes V^\bullet \otimes V^\bullet$, that is, $T(\omega_1, \ldots, \omega_r, v_1, \ldots, v_s) = 0$ for any $\omega_i \in V^*, i = 1, \ldots, r$, $v_j \in V, j = 1, \ldots, s$ whenever $v_{s-1} \in V_\circ$ or $v_s \in V_\circ$. Then, we define the covariant contraction between the last two covariant slots by the operator

$$C_{s-1\,s} := 1_{\mathcal{T}^r_{s-2}V} \otimes g_\bullet : \mathcal{T}^r_s V \otimes V^\bullet \otimes V^\bullet \to \mathcal{T}^r_{s-2} V,$$

where $1_{\mathcal{T}^r_{s-2}V} : \mathcal{T}^r_{s-2}V \to \mathcal{T}^r_{s-2}V$ is the identity. In a radical basis, the contraction can be expressed by

$$(C_{s-1\,s}T)^{a_1\ldots a_r}{}_{b_1\ldots b_{s-2}} := g_\bullet{}^{b_{s-1}b_s} T^{a_1\ldots a_r}{}_{b_1\ldots\ldots b_{s-2}b_{s-1}b_s}.$$

(3) Let $T \in \mathcal{T}^r_s V$ be a tensor with $r \geq 0$ and $s \geq 2$, which satisfies

$$T \in V^{\otimes r} \otimes V^{*\otimes k-1} \otimes V^\bullet \otimes V^{*\otimes l-k-1} \otimes V^\bullet \otimes V^{*\otimes s-l}, \tag{13}$$

$1 \leq k < l \leq s$, that is, $T(\omega_1, \ldots, \omega_r, v_1, \ldots, v_k, \ldots, v_l, \ldots, v_s) = 0$ for any $\omega_i \in V^*, i = 1, \ldots, r$, $v_j \in V, j = 1, \ldots, s$ whenever $v_k \in V_\circ$ or $v_l \in V_\circ$. We define the contraction

$$C_{kl} : V^{\otimes r} \otimes V^{*\otimes k-1} \otimes V^\bullet \otimes V^{*\otimes l-k-1} \otimes V^\bullet \otimes V^{*\otimes s-l} \to V^{\otimes r} \otimes V^{*\otimes s-2},$$

by $C_{kl} := C_{s-1\,s} \circ P_{k,s-1;l,s}$, where $C_{s-1\,s}$ is the contraction defined above, and $P_{k,s-1;l,s} : T \in \mathcal{T}^r_s V \to T \in \mathcal{T}^r_s V$ is the permutation isomorphisms which moves the k-th and l-th slots in the last two positions. In a basis, the components take the form

$$(C_{kl}T)^{a_1\ldots a_r}{}_{b_1\ldots\widehat{b_k}\ldots\widehat{b_l}\ldots b_s} := g_\bullet{}^{b_k b_l} T^{a_1\ldots a_r}{}_{b_1\ldots b_k\ldots b_l\ldots b_s}. \tag{14}$$

We denote the contraction $C_{kl}T$ of T also by

$$C(T(\omega_1, \ldots, \omega_r, v_1, \ldots, \bullet, \ldots, \bullet, \ldots, v_s))$$

or simply

$$T(\omega_1, \ldots, \omega_r, v_1, \ldots, \bullet, \ldots, \bullet, \ldots, v_s).$$

4.2. **Covariant contraction on singular semi-Riemannian manifolds**

In §4.1 we have seen that we can contract in two covariant slots, so long as they are radical-annihilators. The covariant contraction uses the inner product $g_\bullet \in V^{\bullet *} \odot V^{\bullet *}$. In Section §3.4 we have extended the notion of tensors which are radical-annihilator in some slots to a singular semi-Riemannian manifold (M, g) by imposing the condition that the corresponding factors in the tensor product, at $p \in M$, are from $T^\bullet{}_p M$, which is just a subset of $T^*_p M$. This allows us easily to extend the covariant contraction (*cf. e.g.* [40], p. 40) in radical-annihilator slots to singular semi-Riemannian manifolds.

Definition 4.2. Let $T \in \mathcal{T}^r_s M$, $s \geq 2$, be a tensor field on M, which is radical-annihilator in the k-th and l-th covariant slots, where $1 \leq k < l \leq s$. The *covariant contraction* or *covariant trace* operator is the linear operator

$$C_{kl} : \mathcal{T}^r_{k-1} M \otimes_M \mathcal{A}^\bullet(M) \otimes_M \mathcal{T}^0_{l-k-1} M \otimes_M \mathcal{A}^\bullet(M) \otimes_M \mathcal{T}^0_{s-l} M \to \mathcal{T}^r_{s-2} M$$

by

$$(C_{kl}T)(p) = C_{kl}(T(p))$$

in terms of the covariant contraction defined for inner product vector spaces, as in §4.1. In local coordinates we have

$$(C_{kl}T)^{a_1 \ldots a_r}{}_{b_1 \ldots \widehat{b}_k \ldots \widehat{b}_l \ldots b_s} := g_\bullet{}^{b_k b_l} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_k \ldots b_l \ldots b_s}. \tag{15}$$

We denote the contraction $C_{kl}T$ of T also by

$$C(T(\omega_1, \ldots, \omega_r, X_1, \ldots, \bullet, \ldots, \bullet, \ldots, X_s))$$

or simply

$$T(\omega_1, \ldots, \omega_r, X_1, \ldots, \bullet, \ldots, \bullet, \ldots, X_s).$$

Lemma 4.3. If T is a tensor field $T \in \mathcal{T}^r_s M$ with $r \geq 0$ and $s \geq 1$, which is radical-annihilator in the k-th covariant slot, $1 \leq k \leq s$, then its contraction with the metric tensor gives again T:

$$\begin{aligned} &T(\omega_1, \ldots, \omega_r, X_1, \ldots, \bullet, \ldots, X_s)\langle X_k, \bullet \rangle \\ &\quad = T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_k, \ldots, X_s) \end{aligned} \tag{16}$$

Proof. For simplicity, we can work on an inner product space (V, g). Let's first consider the case when $T \in \mathcal{T}^0_1 V$, in fact, $T = \omega \in V^\bullet$. Then, equation (16) reduces to

$$\omega(\bullet)\langle v, \bullet \rangle = \omega(v). \tag{17}$$

But since $\omega \in V^\bullet$, it takes the form $\omega = u^\bullet$ for $u \in V$, and $\omega(\bullet)\langle v, \bullet \rangle = \langle\!\langle \omega, v^\bullet \rangle\!\rangle_\bullet = \langle u, v \rangle = u^\bullet(v) = \omega(v)$.

The general case is obtained from the linearity of the tensor product in the k-th covariant slot. □

Corollary 4.4. $\langle X, \bullet \rangle \langle Y, \bullet \rangle = \langle X, Y \rangle$.

Proof. Follows from Lemma 4.3 and from $g \in \mathcal{A}^\bullet(M) \odot_M \mathcal{A}^\bullet(M)$. □

Example 4.5. $\langle \bullet, \bullet \rangle = \operatorname{rank} g$.

Proof. For simplicity, we can work on an inner product space (V, g). We recall that $g \in V^\bullet \odot V^\bullet$, $g_\bullet \in V^{\bullet *} \odot V^{\bullet *}$. When restricted to $V^\bullet$ and $V^{\bullet *}$ they are non-degenerate and inverse to one another. Since $\dim V^\bullet = \dim \ker \flat = \operatorname{rank} g$, we obtain $\langle \bullet, \bullet \rangle = \operatorname{rank} g$. □

Theorem 4.6. Let (M, g) be a singular semi-Riemannian manifold with constant signature. Let $T \in \mathcal{T}^r_s M$, $s \geq 2$, be a tensor field which is radical-

annihilator in the k-th and l-th covariant slots ($1 \leq k < l \leq n$). Let $(E_a)_{a=1}^n$ be an orthogonal basis on M, so that $E_1, \dots, E_{n-\text{rank } g} \in \mathfrak{X}_\circ(M)$. Then

(18)
$$T(\omega_1, \dots, \omega_r, X_1, \dots, \bullet, \dots, \bullet, \dots, X_s) = \sum_{a=n-\text{rank } g+1}^{n} \frac{1}{\langle E_a, E_a\rangle} T(\omega_1, \dots, \omega_r, X_1, \dots, E_a, \dots, E_a, \dots, X_s),$$

for any $X_1, \dots, X_s \in \mathfrak{X}(M), \omega_1, \dots, \omega_r \in \mathcal{A}^1(M)$.

Proof. For simplicity, we will work on an inner product space (V, g). From the Proposition 3.12 we recall that $g_\bullet$ is diagonal and $g_\bullet{}^{aa} = \dfrac{1}{g_{aa}}$, for $a > n - \text{rank } g$. Therefore

$$g_\bullet{}^{ab} T(\omega_1, \dots, \omega_r, v_1, \dots, E_a, \dots, E_b, \dots, v_s) = \sum_{a=n-\text{rank } g+1}^{n} \frac{1}{\langle E_a, E_a\rangle} T(\omega_1, \dots, \omega_r, v_1, \dots, E_a, \dots, E_a, \dots, v_s).$$

□

Remark 4.7. Since in fact

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_\bullet = \sum_{a=n-\text{rank } g+1}^{n} \frac{\omega_1(E_a)\omega_2(E_a)}{\langle E_a, E_a\rangle}, \tag{19}$$

for any radical-annihilator 1-forms $\omega_1, \omega_2 \in \mathcal{A}^\bullet(M)$, it follows that if we define the contraction alternatively by the equation (18), the definition is independent on the frame $(E_a)_{a=1}^n$.

Remark 4.8. On regions of constant signature, the covariant contraction of a smooth tensor is smooth. But at the points where the signature changes, the contraction is not necessarily smooth, because the inverse of the metric becomes divergent at the points where the signature changes, as it follows from Proposition 3.12. The fact that $g_{\bullet p} \in (T^\bullet{}_p M)^* \odot (T^\bullet{}_p M)^*$ raises some problems, because the union of $(T^\bullet{}_p M)^*$ does not form a bundle, and for $g_\bullet$ the notions of continuity and smoothness don't even make sense.

Counterexample 4.9. The covariant contraction of the two indices of the metric tensor at a point $p \in M$ is $g_p(\bullet, \bullet) = \text{rank } g(p)$ (see Example 4.5). When rank $g(p)$ is not constant, $g_p(\bullet, \bullet)$ is discontinuous.

On the other hand, the following example shows that it is possible to have smooth contractions even when the signature changes:

Example 4.10. If $X \in \mathfrak{X}(M)$ and $\omega \in \mathcal{A}^\bullet(M)$, $C_{12}(\omega \otimes_M X^\flat) = \langle\!\langle \omega, X^\flat \rangle\!\rangle_\bullet = \omega(X)$ and it is smooth, even if the signature is variable.

Remark 4.11. Since the points where the signature doesn't change form a dense subset of M (Remark 2.2), it makes sense to impose the condition of smoothness of the covariant contraction of a smooth tensor. To check smoothness, we simply check whether the extension by continuity of the contraction is smooth.

5. The Koszul form

For convenience, we name *Koszul form* the right member of the Koszul formula (see *e.g.* [40], p. 61):

Definition 5.1 (The Koszul form, see *e.g.* [36], p. 263)**.** *The Koszul form* is defined as

$$\mathcal{K} : \mathfrak{X}(M)^3 \to \mathbb{R},$$

$$\begin{aligned} \mathcal{K}(X,Y,Z) &:= \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}. \end{aligned} \tag{20}$$

The Koszul formula becomes

$$\langle \nabla_X Y, Z\rangle = \mathcal{K}(X,Y,Z), \tag{21}$$

and for non-degenerate metric, the unique Levi-Civita connection is obtained by raising the 1-form $\mathcal{K}(X,Y,\_)$:

$$\nabla_X Y = \mathcal{K}(X,Y,\_)^\sharp. \tag{22}$$

If the metric is degenerate, then this is not in general possible. We can raise $\mathcal{K}(X,Y,\_)$ on regions of constant signature, and what we obtain is what Kupeli ([36], p. 261–262) called *Koszul derivative* – which is in general not a connection and is not unique. Kupeli's construction is done only for singular semi-Riemannian manifolds with metrics with constant signature, which satisfy the condition of *radical-stationarity* (Definition 6.4). But if the metric changes its signature, the Koszul derivative is discontinuous at the points where the signature changes. In this article we would not need to use the Koszul derivative, because for our purpose it will be enough to work with the Koszul form.

5.1. Basic properties of the Koszul form

Let's recall the Lie derivative of a tensor field $T \in \mathcal{T}^0_2 M$:

Definition 5.2. (see *e.g.* [22], p. 30) Let M be a differentiable manifold. Recall that the *Lie derivative* of a tensor field $T \in \mathcal{T}^0_2 M$ with respect to a vector field $Z \in \mathfrak{X}(M)$ is given by

$$(\mathcal{L}_Z T)(X,Y) := ZT(X,Y) - T([Z,X],Y) - T(X,[Z,Y]) \tag{23}$$

for any $X, Y \in \mathfrak{X}(M)$.

The following properties of the Koszul form correspond directly to standard properties of the Levi-Civita connection of a non-degenerate metric (*cf. e.g.* [40], p. 61). We prove them explicitly here, because in the case of degenerate metric the proofs need to avoid using the Levi-Civita connection and the index raising. These properties will turn out to be important for what it follows.

Theorem 5.3. The Koszul form of a singular semi-Riemannian manifold (M, g) has, for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathscr{F}(M)$, the following properties:

(1) It is additive and $\mathbb{R}$-linear in each of its arguments.
(2) It is $\mathscr{F}(M)$-linear in the first argument:
$\mathcal{K}(fX, Y, Z) = f\mathcal{K}(X, Y, Z)$.
(3) Satisfies the *Leibniz rule*:
$\mathcal{K}(X, fY, Z) = f\mathcal{K}(X, Y, Z) + X(f)\langle Y, Z\rangle$.
(4) It is $\mathscr{F}(M)$-linear in the third argument:
$\mathcal{K}(X, Y, fZ) = f\mathcal{K}(X, Y, Z)$.
(5) It is *metric*:
$\mathcal{K}(X, Y, Z) + \mathcal{K}(X, Z, Y) = X\langle Y, Z\rangle$.
(6) It is *symmetric* or *torsionless*:
$\mathcal{K}(X, Y, Z) - \mathcal{K}(Y, X, Z) = \langle [X, Y], Z\rangle$.
(7) Relation with the Lie derivative of g:
$\mathcal{K}(X, Y, Z) + \mathcal{K}(Z, Y, X) = (\mathcal{L}_Y g)(Z, X)$.
(8) $\mathcal{K}(X, Y, Z) + \mathcal{K}(Y, Z, X) = Y\langle Z, X\rangle + \langle [X, Y], Z\rangle$.

Proof. (1) Follows from Definition 5.1, and from the linearity of g, of the action of vector fields on scalars, and of the Lie brackets.

$$
\begin{aligned}
(2)\quad 2\mathcal{K}(fX, Y, Z) &= fX\langle Y, Z\rangle + Y\langle Z, fX\rangle - Z\langle fX, Y\rangle \\
&\quad -\langle fX, [Y, Z]\rangle + \langle Y, [Z, fX]\rangle + \langle Z, [fX, Y]\rangle \\
&= fX\langle Y, Z\rangle + Y(f\langle Z, X\rangle) - Z(f\langle X, Y\rangle) \\
&\quad -f\langle X, [Y, Z]\rangle + \langle Y, f[Z, X] + Z(f)X\rangle \\
&\quad +\langle Z, f[X, Y] - Y(f)X\rangle \\
&= fX\langle Y, Z\rangle + fY\langle Z, X\rangle \\
&\quad +Y(f)\langle Z, X\rangle - fZ\langle X, Y\rangle \\
&\quad -Z(f)\langle X, Y\rangle - f\langle X, [Y, Z]\rangle + f\langle Y, [Z, X]\rangle \\
&\quad +Z(f)\langle Y, X\rangle + f\langle Z, [X, Y]\rangle - Y(f)\langle Z, X\rangle \\
&= fX\langle Y, Z\rangle + fY\langle Z, X\rangle - fZ\langle X, Y\rangle \\
&\quad -f\langle X, [Y, Z]\rangle + f\langle Y, [Z, X]\rangle + f\langle Z, [X, Y]\rangle \\
&= 2f\mathcal{K}(X, Y, Z)
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad 2\mathcal{K}(X, fY, Z) &= X\langle fY, Z\rangle + fY\langle Z, X\rangle - Z\langle X, fY\rangle \\
&\quad -\langle X, [fY, Z]\rangle + \langle fY, [Z, X]\rangle + \langle Z, [X, fY]\rangle \\
&= X(f)\langle Y, Z\rangle + fX\langle Y, Z\rangle \\
&\quad +fY\langle Z, X\rangle - Z(f)\langle X, Y\rangle \\
&\quad -fZ\langle X, Y\rangle - f\langle X, [Y, Z]\rangle + Z(f)\langle X, Y\rangle \\
&\quad +f\langle Y, [Z, X]\rangle + f\langle Z, [X, Y]\rangle + X(f)\langle Z, Y\rangle \\
&= f(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&\quad -\langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle) \\
&\quad +X(f)\left(\langle Y, Z\rangle + \langle Z, Y\rangle\right) \\
&= 2\left(f\mathcal{K}(X, Y, Z) + X(f)\langle Y, Z\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad 2\mathcal{K}(X,Y,fZ) &= X\langle Y,fZ\rangle + Y\langle fZ,X\rangle - fZ\langle X,Y\rangle \\
&\quad -\langle X,[Y,fZ]\rangle + \langle Y,[fZ,X]\rangle + \langle fZ,[X,Y]\rangle \\
&= fX\langle Y,Z\rangle + X(f)\langle Y,Z\rangle \\
&\quad +fY\langle Z,X\rangle + Y(f)\langle Z,X\rangle \\
&\quad -fZ(\langle X,Y\rangle) - f\langle X,[Y,Z]\rangle - Y(f)\langle X,Z\rangle \\
&\quad +f\langle Y,[Z,X]\rangle - X(f)\langle Y,Z\rangle + f\langle Z,[X,Y]\rangle \\
&= fX\langle Y,Z\rangle + fY\langle Z,X\rangle - fZ(\langle X,Y\rangle) \\
&\quad -f\langle X,[Y,Z]\rangle + f\langle Y,[Z,X]\rangle + f\langle Z,[X,Y]\rangle \\
&= 2f\mathcal{K}(X,Y,Z)
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad 2[\mathcal{K}(X,Y,Z) &+ \mathcal{K}(X,Z,Y)] \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\
&\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \\
&\quad +X\langle Z,Y\rangle + Z\langle Y,X\rangle - Y\langle X,Z\rangle \\
&\quad -\langle X,[Z,Y]\rangle + \langle Z,[Y,X]\rangle + \langle Y,[X,Z]\rangle \\
&= X\langle Y,Z\rangle - \langle X,[Y,Z]\rangle \\
&\quad +\langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle + X\langle Y,Z\rangle \\
&\quad +\langle X,[Y,Z]\rangle - \langle Z,[X,Y]\rangle - \langle Y,[Z,X]\rangle \\
&= 2X\langle Y,Z\rangle
\end{aligned}
$$

$$
\begin{aligned}
(7)\quad 2[\mathcal{K}(X,Y,Z) &+ \mathcal{K}(Z,Y,X)] \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\
&\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \\
&\quad +Z\langle Y,X\rangle + Y\langle X,Z\rangle - X\langle Z,Y\rangle \\
&\quad -\langle Z,[Y,X]\rangle + \langle Y,[X,Z]\rangle + \langle X,[Z,Y]\rangle \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\
&\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \\
&\quad +Z\langle X,Y\rangle + Y\langle Z,X\rangle - X\langle Y,Z\rangle \\
&\quad +\langle Z,[X,Y]\rangle - \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \\
&= 2Y\langle Z,X\rangle - 2\langle X,[Y,Z]\rangle + 2\langle Z,[X,Y]\rangle \\
&= 2(Y\langle Z,X\rangle - \langle X,\mathcal{L}_Y Z\rangle - \langle Z,\mathcal{L}_Y X\rangle) \\
&= 2(\mathcal{L}_Y g)(Z,X)
\end{aligned}
$$

$$
\begin{aligned}
(6)\quad 2[\mathcal{K}(X,Y,Z) &- \mathcal{K}(Y,X,Z)] \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\
&\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \\
&\quad -Y\langle X,Z\rangle - X\langle Z,Y\rangle + Z\langle Y,X\rangle \\
&\quad +\langle Y,[X,Z]\rangle - \langle X,[Z,Y]\rangle - \langle Z,[Y,X]\rangle \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\
&\quad -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \\
&\quad -Y\langle Z,X\rangle - X\langle Y,Z\rangle + Z\langle X,Y\rangle \\
&\quad -\langle Y,[Z,X]\rangle + \langle X,[Y,Z]\rangle + \langle Z,[X,Y]\rangle \\
&= 2\langle Z,[X,Y]\rangle = 2\langle [X,Y],Z\rangle
\end{aligned}
$$

(8) By subtracting (6) from (5), we obtain

$$\mathcal{K}(Y,X,Z)+\mathcal{K}(X,Z,Y)=X\langle Y,Z\rangle-\langle[X,Y],Z\rangle.$$

By applying the permutation $(X,Y,Z)\mapsto(Y,X,Z)$ we get

$$\mathcal{K}(X,Y,Z)+\mathcal{K}(Y,Z,X)=Y\langle Z,X\rangle+\langle[X,Y],Z\rangle.$$

□

Remark 5.4. If $U\subseteq M$ is an open set in M and $(E_a)_{a=1}^n\subset\mathfrak{X}(U)$ are vector fields on U forming a frame of T_pU at each $p\in U$, then

(24)

$$\begin{aligned}\mathcal{K}_{abc} &:= \mathcal{K}(E_a,E_b,E_c)\\ &= \frac{1}{2}\{E_a(g_{bc})+E_b(g_{ca})-E_c(g_{ab})-g_{as}\mathscr{C}^s_{bc}+g_{bs}\mathscr{C}^s_{ca}+g_{cs}\mathscr{C}^s_{ab}\},\end{aligned}$$

where $g_{ab}=\langle E_a,E_b\rangle$ and $\mathscr{C}^c_{ab}$ are the coefficients of the Lie bracket of vector fields (see *e.g.* [6], p. 107), $[E_a,E_b]=\mathscr{C}^c_{ab}E_c$.

The equations (5 – 8) in Theorem 5.3 become in the basis $(E_a)_{a=1}^n$:

$$\begin{aligned}&(5')\quad \mathcal{K}_{abc}+\mathcal{K}_{acb}=E_a(g_{bc}).\\ &(7')\quad \mathcal{K}_{abc}+\mathcal{K}_{cba}=(\mathcal{L}_{E_b}g)_{ca}.\\ &(6')\quad \mathcal{K}_{abc}-\mathcal{K}_{bac}=g_{sc}\mathscr{C}^s_{ab}.\\ &(8')\quad \mathcal{K}_{abc}+\mathcal{K}_{bca}=E_b(g_{ca})+g_{sc}\mathscr{C}^s_{ab}.\end{aligned}$$

If $E_a=\partial_a:=\dfrac{\partial}{\partial x^a}$ for all $a\in\{1,\dots,n\}$ are the partial derivatives in a coordinate system, $[\partial_a,\partial_b]=0$ and the equation (24) reduces to

$$\mathcal{K}_{abc}=\mathcal{K}(\partial_a,\partial_b,\partial_c)=\frac{1}{2}(\partial_a g_{bc}+\partial_b g_{ca}-\partial_c g_{ab}),\tag{25}$$

which are Christoffel's symbols of the first kind (*cf. e.g.* [22], p. 40).

Corollary 5.5. Let $X,Y\in\mathfrak{X}(M)$ two vector fields. The map $\mathcal{K}(X,Y,\_):\mathfrak{X}(M)\to\mathbb{R}$ defined as

$$\mathcal{K}(X,Y,\_)(Z):=\mathcal{K}(X,Y,Z)\tag{26}$$

is a differential 1-form.

Proof. It is a direct consequence of Theorem 5.3, properties (1) and (4). □

Corollary 5.6. If $X,Y\in\mathfrak{X}(M)$ and $W\in\mathfrak{X}_\circ(M)$, then

$$\mathcal{K}(X,Y,W)=\mathcal{K}(Y,X,W)=-\mathcal{K}(X,W,Y)=-\mathcal{K}(Y,W,X).\tag{27}$$

Proof. From Theorem 5.3, property (6),

$$\mathcal{K}(X,Y,W)=\mathcal{K}(Y,X,W)+\langle[X,Y],W\rangle=\mathcal{K}(Y,X,W).\tag{28}$$

From Theorem 5.3, property (5),

$$\mathcal{K}(X,Y,W)=-\mathcal{K}(X,W,Y)+X\langle Y,W\rangle=-\mathcal{K}(X,W,Y)\tag{29}$$

and

$$\mathcal{K}(Y,X,W)=-\mathcal{K}(Y,W,X).\tag{30}$$

□

6. The covariant derivative

6.1. The lower covariant derivative of vector fields

Definition 6.1 (The lower covariant derivative)**.** The *lower covariant derivative* of a vector field Y in the direction of a vector field X is the differential 1-form $\nabla^\flat_X Y \in \mathcal{A}^1(M)$ defined as

$$(\nabla^\flat_X Y)(Z) := \mathcal{K}(X, Y, Z) \tag{31}$$

for any $Z \in \mathfrak{X}(M)$. The *lower covariant derivative operator* is the operator

$$\nabla^\flat : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{A}^1(M) \tag{32}$$

which associates to each $X, Y \in \mathfrak{X}(M)$ the differential 1-form $\nabla^\flat_X Y$.

Remark 6.2. Unlike the case of the covariant derivative defined when the metric is non-degenerate, the result of applying the lower covariant derivative to a vector field is not another vector field, but a differential 1-form. When the metric is non-degenerate the two are equivalent by changing the type of the 1-form $\nabla^\flat_X Y$ into a vector field $\nabla_X Y = (\nabla^\flat_X Y)^\sharp$. Similar objects mapping vector fields to 1-forms were used in *e.g.* [29], p. 464–465. The lower covariant derivative doesn't require a non-degenerate metric, and it will be very useful in what follows.

The following properties correspond to standard properties of the Levi-Civita connection of a non-degenerate metric (*cf. e.g.* [40], p. 61), and are extended here to the case when the metric can be degenerate.

Theorem 6.3. The lower covariant derivative operator $\nabla^\flat$ of vector fields defined on a singular semi-Riemannian manifold (M, g) has the following properties:

(1) It is additive and $\mathbb{R}$-linear in both of its arguments.
(2) It is $\mathscr{F}(M)$-linear in the first argument:
$\nabla^\flat_{fX} Y = f \nabla^\flat_X Y$.
(3) Satisfies the *Leibniz rule*:
$\nabla^\flat_X fY = f \nabla^\flat_X Y + X(f) Y^\flat$.
or, explicitly,
$(\nabla^\flat_X fY)(Z) = f(\nabla^\flat_X Y)(Z) + X(f)\langle Y, Z\rangle$.
(4) It is *metric*:
$(\nabla^\flat_X Y)(Z) + (\nabla^\flat_X Z)(Y) = X\langle Y, Z\rangle$.
(5) It is *symmetric* or *torsionless*:
$\nabla^\flat_X Y - \nabla^\flat_Y X = [X, Y]^\flat$
or, explicitly,
$(\nabla^\flat_X Y)(Z) - (\nabla^\flat_Y X)(Z) = \langle [X, Y], Z\rangle$.
(6) Relation with the Lie derivative of g:
$(\nabla^\flat_X Y)(Z) + (\nabla^\flat_Z Y)(X) = (\mathcal{L}_Y g)(Z, X)$.
(7) $(\nabla^\flat_X Y)(Z) + (\nabla^\flat_Y Z)(X) = Y\langle Z, X\rangle + \langle [X, Y], Z\rangle$.

for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathscr{F}(M)$.

Proof. Follows from the direct application of Theorem 5.3. □

6.2. **Radical-stationary singular semi-Riemannian manifolds**

The radical-stationary singular semi-Riemannian manifolds of constant signature were introduced by Kupeli in [36], p. 259–260, where he called them singular semi-Riemannian manifolds. Later, in [38] Definition 3.1.3, he named them "stationary singular semi-Riemannian manifolds". Here we use the term "radical-stationary singular semi-Riemannian manifolds" to avoid possible confusion, since the word "stationary" is used in general for manifolds admitting a Killing vector field, and in particular for spacetimes invariant at time translation. Kupeli introduced them to ensure the existence of the Koszul derivative. Our need is different, since we don't rely on Kupeli's Koszul derivative.

Definition 6.4 (*cf.* [38] Definition 3.1.3)**.** A singular semi-Riemannian manifold (M, g) is *radical-stationary* if it satisfies the condition

$$\mathcal{K}(X, Y, \_) \in \mathcal{A}^\bullet(M), \tag{33}$$

for any $X, Y \in \mathfrak{X}(M)$.

Remark 6.5. The condition from Definition 6.4 means that $\mathcal{K}(X, Y, W_p) = 0$ for any $X, Y \in \mathfrak{X}(M)$ and $W_p \in \mathfrak{X}_\circ(M_p)$, $p \in M$.

Corollary 6.6. If (M, g) is radical-stationary and $X, Y \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}_\circ(M)$, then

$$\mathcal{K}(X, Y, W) = \mathcal{K}(Y, X, W) = -\mathcal{K}(X, W, Y) = -\mathcal{K}(Y, W, X) = 0. \tag{34}$$

Proof. Follows directly from the Corollary 5.6. □

Remark 6.7. The condition (33) can be expressed in terms of the lower derivative as

$$\nabla^\flat_X Y \in \mathcal{A}^\bullet(M), \tag{35}$$

for any $X, Y \in \mathfrak{X}(M)$.

6.3. **The covariant derivative of differential 1-forms**

For non-degenerate metrics the covariant derivative of a differential 1-form is defined in terms of $\nabla_X Y$ (*cf. e.g.* [17], p. 70) by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \tag{36}$$

In order to generalize this formula to the case of degenerate metrics, we need to express $\omega(\nabla_X Y)$ in terms of $\nabla^\flat_X Y$. We can use the identity

$$\omega(\nabla_X Y) = \langle \nabla_X Y, \omega^\sharp \rangle \tag{37}$$

and rewrite it in a way compatible to the degenerate case as

$$\omega\left(\nabla_X Y\right) = \langle \nabla_X Y, {}_\bullet\rangle\langle \omega^\sharp, {}_\bullet\rangle \tag{37'}$$

Remark 6.8. If the metric is degenerate, we need to be allowed to define the contraction $\mathcal{K}(X, Y, {}_\bullet)\omega({}_\bullet)$. This is possible on radical-stationary singular semi-Riemannian manifolds – since $\nabla^\flat_X Y$ is radical-annihilator – if the differential form ω is radical-annihilator too.

We can therefore give the following definition:

Definition 6.9. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the covariant derivative of a radical-annihilator 1-form $\omega \in \mathcal{A}^\bullet(M)$ in the direction of a vector field $X \in \mathfrak{X}(M)$ by

$$\nabla : \mathfrak{X}(M) \times \mathcal{A}^\bullet(M) \to {A_d}^1(M) \tag{38}$$

$$\left(\nabla_X \omega\right)(Y) := X\left(\omega(Y)\right) - \langle\!\langle \nabla^\flat_X Y, \omega\rangle\!\rangle_\bullet, \tag{39}$$

where ${A_d}^1(M)$ is the set of sections of T^*M smooth at the points of M where the signature is constant.

Proposition 6.10. If (M, g) is radical-stationary and $\omega \in \mathcal{A}^\bullet(M)$ is a radical-annihilator 1-form, then for any $X \in \mathfrak{X}(M)$ and $p \in M - M_\wr$, $\nabla_{X_p}\omega_p \in {T_p}^\bullet M$.

Proof. It follows from the Definition 6.9. Let U be a neighborhood of p where g has constant signature, and let $W \in \mathfrak{X}_\circ(U)$ so that $W_p \in T_{p\circ}M$. Then, on U, $\left(\nabla_X\omega\right)(W) = X\left(\omega(W)\right) - \langle\!\langle \nabla^\flat_X W, \omega\rangle\!\rangle_\bullet = 0$. □

Corollary 6.11. If $\nabla_X\omega$ is smooth, then it is a radical-annihilator differential 1-form, $\nabla_X\omega \in \mathcal{A}^\bullet(M)$.

Proof. Follows from Proposition 6.10 because of continuity. □

Definition 6.12. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the following vector spaces of differential forms having smooth covariant derivatives:

$$\mathscr{A}^{\bullet 1}(M) = \{\omega \in \mathcal{A}^\bullet(M) | (\forall X \in \mathfrak{X}(M))\ \nabla_X\omega \in \mathcal{A}^\bullet(M)\}, \tag{40}$$

$$\mathscr{A}^{\bullet k}(M) := \bigwedge_M^k \mathscr{A}^{\bullet 1}(M). \tag{41}$$

The following theorem extends some properties of the covariant derivative known from the non-degenerate case (*cf. e.g.* [40], p. 59).

Theorem 6.13. The covariant derivative operator ∇ of differential 1-forms defined on a radical-stationary semi-Riemannian manifold (M, g) has the following properties:

(1) It is additive and $\mathbb{R}$-linear in both of its arguments.

(2) It is $\mathscr{F}(M)$-linear in the first argument:
$\nabla_{fX}\omega = f\nabla_X\omega$.
(3) It satisfies the *Leibniz rule*:
$\nabla_X f\omega = f\nabla_X\omega + X(f)\omega$.
(4) It commutes with the lowering operator:
$\nabla_X Y^\flat = \nabla^\flat_X Y$.

for any $X, Y \in \mathfrak{X}(M)$, $\omega \in \mathcal{A}^\bullet(M)$ and $f \in \mathscr{F}(M)$.

Proof. The property (1) follows from the direct application of Theorem 6.3 to the Definition 6.9.

For property (2),

$$(\nabla_{fX}\omega)(Y) = fX\left(\omega(Y)\right) - \langle\!\langle \nabla^\flat_{fX}Y, \omega\rangle\!\rangle_\bullet = f(\nabla_X\omega)(Y). \tag{42}$$

Property (3) results by

$$\begin{aligned} (\nabla_X f\omega)(Y) &= X\left(f\omega(Y)\right) - \langle\!\langle \nabla^\flat_X Y, f\omega\rangle\!\rangle_\bullet \\ &= X(f)\omega(Y) + fX\left(\omega(Y)\right) - f\langle\!\langle \nabla^\flat_X Y, \omega\rangle\!\rangle_\bullet \\ &= f(\nabla_X\omega)(Y) + X(f)\omega(Y). \end{aligned} \tag{43}$$

For property (4), we apply Definition 6.9 to $\omega = Y^\flat$. Let $Z \in \mathfrak{X}(M)$. Then,

$$\begin{aligned} (\nabla_X Y^\flat)(Z) &= X\left(Y^\flat(Z)\right) - \langle\!\langle \nabla^\flat_X Z, Y^\flat\rangle\!\rangle_\bullet \\ &= X\langle Y, Z\rangle - (\nabla^\flat_X Z)(Y) \\ &= (\nabla^\flat_X Y)(Z), \end{aligned} \tag{44}$$

where the last identity follows from Theorem 4 property (4). □

Corollary 6.14. Let (M, g) be a radical-stationary semi-Riemannian manifold, and

$$\mathscr{F}^\bullet(M) = \{f \in \mathscr{F}(M) | \mathrm{d}f \in \mathcal{A}^{\bullet 1}(M)\}. \tag{45}$$

Then, $\mathscr{A}^{\bullet k}(M)$ from Definition 6.12 are $\mathscr{F}^\bullet(M)$-modules of differential forms.

Proof. From Theorem 6.13 property (3) follows that for any $f \in \mathscr{F}^\bullet(M)$ and $\omega \in \mathscr{A}^{\bullet k}(M)$, $f\omega \in \mathscr{A}^{\bullet k}(M)$. □

6.4. **The covariant derivative of differential forms**

We define now the covariant derivative for tensors which are covariant and radical annihilator in all their slots, in particular on differential forms (generalizing the corresponding formulas from the non-degenerate case, see *e.g.* [17], p. 70).

Definition 6.15. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the covariant derivative of tensors of type $(0, s)$ as the operator

$$\nabla : \mathfrak{X}(M) \times \otimes^s_M \mathscr{A}^{\bullet 1}(M) \to \otimes^s_M \mathcal{A}^{\bullet 1}(M) \tag{46}$$

acting by

$$\nabla_X(\omega_1 \otimes \ldots \otimes \omega_s) := \nabla_X(\omega_1) \otimes \ldots \otimes \omega_s + \ldots + \omega_1 \otimes \ldots \otimes \nabla_X(\omega_s) \tag{47}$$

In particular,

Definition 6.16. On a radical-stationary semi-Riemannian manifold (M, g) we define the covariant derivative of k-differential forms by

$$\nabla : \mathfrak{X}(M) \times \mathscr{A}^{\bullet k}(M) \to \mathcal{A}^{\bullet k}(M), \tag{48}$$

acting by

$$\nabla_X(\omega_1 \wedge \ldots \wedge \omega_s) := \nabla_X(\omega_1) \wedge \ldots \wedge \omega_s + \ldots + \omega_1 \wedge \ldots \wedge \nabla_X(\omega_s) \tag{49}$$

Theorem 6.17. The covariant derivative of a tensor $T \in \otimes^k_M \mathscr{A}^{\bullet 1}(M)$ on a radical-stationary semi-Riemannian manifold (M, g) satisfies the formula

$$\begin{array}{rcl} (\nabla_X T)(Y_1, \ldots, Y_k) & = & X\left(T(Y_1, \ldots, Y_k)\right) \\ & & -\sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) T(Y_1, , \ldots, \bullet, \ldots, Y_k) \end{array} \tag{50}$$

Proof. Because of linearity, it is enough to prove it for the case

$$T = \omega_1 \otimes_M \ldots \otimes_M \omega_k. \tag{51}$$

From the Definitions 6.15 and 6.9,

(52)

$$\begin{array}{rcl} (\nabla_X T)(Y_1, \ldots, Y_k) & = & \nabla_X(\omega_1 \otimes_M \ldots \otimes_M \omega_k)(Y_1, \ldots, Y_k) \\ & = & (\nabla_X \omega_1)(Y_1) \cdot \ldots \cdot \omega_k(Y_k) + \ldots \\ & & +\omega_1(Y_1) \cdot \ldots \cdot (\nabla_X \omega_k)(Y_k) \\ & = & (X(\omega_1(Y_1)) - \langle\!\langle \nabla^\flat_X Y_1, \omega_1 \rangle\!\rangle_\bullet) \cdot \ldots \cdot \omega_k(Y_k) + \ldots \\ & & +\omega_1(Y_1) \cdot \ldots \cdot (X(\omega_k(Y_k)) - \langle\!\langle \nabla^\flat_X Y_k, \omega_k \rangle\!\rangle_\bullet) \\ & = & X(\omega_1(Y_1)) \cdot \ldots \cdot \omega_k(Y_k) + \ldots \\ & & +\omega_1(Y_1) \cdot \ldots \cdot X(\omega_k(Y_k)) \\ & & -\langle\!\langle \nabla^\flat_X Y_1, \omega_1 \rangle\!\rangle_\bullet \cdot \ldots \cdot \omega_k(Y_k) \\ & & -\omega_1(Y_1) \cdot \ldots \cdot \langle\!\langle \nabla^\flat_X Y_k, \omega_k \rangle\!\rangle_\bullet \\ & = & X\left(T(Y_1, \ldots, Y_k)\right) \\ & & -\sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) T(Y_1, , \ldots, \bullet, \ldots, Y_k) \end{array}$$

and the desired formula follows. □

Corollary 6.18. Let (M, g) be a radical-stationary semi-Riemannian manifold. The covariant derivative of a k-differential form $\omega \in \mathscr{A}^{\bullet k}(M)$ takes the form

$$\begin{array}{rcl} (\nabla_X \omega)(Y_1, \ldots, Y_k) & := & X\left(\omega(Y_1, \ldots, Y_k)\right) \\ & & -\sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) \omega(Y_1, , \ldots, \bullet, \ldots, Y_k) \end{array} \tag{53}$$

Proof. Follows from Theorem 6.17, by verifying that the antisymmetry property of ω is maintained. □

Corollary 6.19. On a radical-stationary semi-Riemannian manifold (M, g), the metric g is parallel:

$$\nabla_X g = 0. \tag{54}$$

Proof. Follows from Theorems 6.17 and 5.3, property (5):

$$(\nabla_X g)(Y, Z) = X\langle Y, Z\rangle - \mathcal{K}(X, Y, \bullet)g(\bullet, Z) - \mathcal{K}(X, Z, \bullet)g(Y, \bullet) = 0. \tag{55}$$

□

6.5. **Semi-regular semi-Riemannian manifolds**

An important particular type of radical-stationary semi-Riemannian manifold is provided by the semi-regular semi-Riemannian manifolds, introduced below.

Definition 6.20. A *semi-regular semi-Riemannian manifold* is a singular semi-Riemannian manifold (M, g) which satisfies

$$\nabla^\flat_X Y \in \mathscr{A}^{\bullet 1}(M) \tag{56}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

Remark 6.21. By Definition 6.12, this is equivalent to saying that for any $X, Y, Z \in \mathfrak{X}(M)$

$$\nabla_X \nabla^\flat_Y Z \in \mathcal{A}^\bullet(M). \tag{57}$$

Remark 6.22. Recall that $\mathscr{A}^{\bullet 1}(M) \subseteq \mathcal{A}^\bullet(M)$. This means that any semi-regular semi-Riemannian manifold is also radical-stationary (*cf.* Definition 6.4).

Proposition 6.23. Let (M, g) be a radical-stationary semi-Riemannian manifold. Then, the manifold (M, g) is semi-regular if and only if for any $X, Y, Z, T \in \mathfrak{X}(M)$

$$\mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet) \in \mathscr{F}(M). \tag{58}$$

Proof. From the Definition 6.9 of the covariant derivative of 1-forms we obtain that

$$\begin{aligned} (\nabla_X \nabla^\flat_Y Z)(T) &= X\left((\nabla^\flat_Y Z)(T)\right) - \langle\!\langle \nabla^\flat_X T, \nabla^\flat_Y Z\rangle\!\rangle_\bullet \\ &= X\left((\nabla^\flat_Y Z)(T)\right) - \mathcal{K}(X, T, \bullet)\mathcal{K}(Y, Z, \bullet). \end{aligned} \tag{59}$$

It follows that $(\nabla_X \nabla^\flat_Y Z)(T)$ is smooth if and only if $\mathcal{K}(X, T, \bullet)\mathcal{K}(Y, Z, \bullet)$ is. □

7. Curvature of semi-regular semi-Riemannian manifolds

The standard way to define the curvature invariants is to construct the Levi-Civita connection of the metric (*cf. e.g.* [40], p. 59), and from this the curvature operator (*cf. e.g.* [40], p. 74). The Ricci tensor and the scalar curvature (*cf. e.g.* [40], p. 87–88) follow by contraction (*cf. e.g.* [40], p. 83).

Unfortunately, in the case of singular semi-Riemannian manifolds the usual road is not available, because there is no intrinsic Levi-Civita connection. But, as we shall see in this section, the Riemann curvature tensor can be obtained from the lower covariant derivative and the covariant derivative of radical-annihilator differential forms. For radical-stationary manifolds the Riemann curvature tensor thus introduced is guaranteed to be smooth only on the regions of constant signature, but for semi-regular manifolds it is smooth everywhere.

In order to obtain the Ricci curvature tensor, and further the scalar curvature, we need to contract the Riemann curvature tensor in two covariant indices. Because the metric may be degenerate, this covariant contraction can be defined only if the Riemann curvature tensor is radical-annihilator in its slots. We will see that this is the case, and in §7.3 we define the Ricci tensor and the scalar curvature.

7.1. Riemann curvature of semi-regular semi-Riemannian manifolds

Definition 7.1. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the *lower Riemann curvature operator* as

$$\mathcal{R}^\flat : \mathfrak{X}(M)^3 \to {A_d}^1(M) \tag{60}$$

$$\mathcal{R}^\flat_{XY} Z := \nabla_X \nabla^\flat_Y Z - \nabla_Y \nabla^\flat_X Z - \nabla^\flat_{[X,Y]} Z \tag{61}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Definition 7.2. We define the *Riemann curvature tensor* as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}, \tag{62}$$

$$R(X, Y, Z, T) := (\mathcal{R}^\flat_{XY} Z)(T) \tag{63}$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Remark 7.3. The Riemann curvature tensor from Definition 7.2 generalizes the Riemann curvature tensor $R(X, Y, Z, T) := \langle R_{XY} Z, T \rangle$ known from semi-Riemannian geometry (*cf. e.g.* [40], p. 75).

Remark 7.4. It follows from the Definition 7.2 that

$$R(X,Y,Z,T) = (\nabla_X \nabla^\flat_Y Z)(T) - (\nabla_Y \nabla^\flat_X Z)(T) - (\nabla^\flat_{[X,Y]} Z)(T) \tag{64}$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Theorem 7.5. Let (M, g) be a semi-regular semi-Riemannian manifold. The Riemann curvature is a smooth tensor field $R \in \mathcal{T}^0_4 M$.

Proof. Remember from Theorem 6.3, property (1) that the lower covariant derivative for vector fields is additive and $\mathbb{R}$-linear in both of is arguments. From the same Theorem 6.13 property (1), we recall that the covariant derivative for differential 1-forms is additive and $\mathbb{R}$-linear in both of is arguments. By combining the two, it follows *the additivity and $\mathbb{R}$-linearity* of the Riemann curvature R in all of its four arguments.

We will show now that R is $\mathscr{F}(M)$-linear in its four arguments. The proof goes almost similar to the non-degenerate case, but we will give it explicitly, because in our proof we need to avoid any use of the Levi-Civita connection or of the inverse of the metric tensor, for example index raising.

We apply the properties of the lower covariant derivative for vector fields, as exposed in Theorem 6.13 properties (2)-(4), and those of the covariant derivative for differential 1-forms, as known from Theorem 6.13, properties (2)-(4), to verify that for any function $f \in \mathscr{F}(M)$, $R(fX,Y,Z,T) = R(X,fY,Z,T) = R(X,Y,fZ,T) = R(X,Y,Z,fT) = fR(X,Y,Z,T)$.

Since $[fX,Y] = f[X,Y] - Y(f)X$,

$$\begin{aligned}
R(fX,Y,Z,T) &= (\nabla_{fX}\nabla^\flat_Y Z)(T) - (\nabla_Y \nabla^\flat_{fX} Z)(T) - (\nabla^\flat_{[fX,Y]} Z)(T) \\
&= f(\nabla_X \nabla^\flat_Y Z)(T) - (\nabla_Y (f \nabla^\flat_X Z))(T) \\
&\quad -(\nabla^\flat_{f[X,Y]-Y(f)X} Z)(T) \\
&= f(\nabla_X \nabla^\flat_Y Z)(T) - f(\nabla_Y \nabla^\flat_X Z)(T) \\
&\quad -Y(f)(\nabla^\flat_X Z)(T) - f(\nabla^\flat_{[X,Y]} Z)(T) \\
&\quad +Y(f)(\nabla^\flat_X Z)(T) \\
&= fR(X,Y,Z,T).
\end{aligned}$$

The Definition 7.2 implies that $R(X,Y,Z,T) = -R(Y,X,Z,T)$, which leads immediately to

$$R(X,fY,Z,T) = fR(X,Y,Z,T). \tag{65}$$

$$\begin{aligned}
R(X,Y,fZ,T) &= (\nabla_X\nabla^\flat_Y fZ)(T) - (\nabla_Y\nabla^\flat_X fZ)(T) - (\nabla^\flat_{[X,Y]}fZ)(T)\\
&= (\nabla_X(f\nabla^\flat_Y Z + Y(f)Z))(T)\\
&\quad -(\nabla_Y(f\nabla^\flat_X Z + X(f)Z))(T)\\
&\quad -(f\nabla^\flat_{[X,Y]}Z + [X,Y](f)Z^\flat)(T)\\
&= (\nabla_X(f\nabla^\flat_Y Z))(T) + (\nabla_X(Y(f)Z^\flat))(T)\\
&\quad -(\nabla_Y(f\nabla^\flat_X Z))(T) - (\nabla_Y(X(f)Z^\flat))(T)\\
&\quad -f(\nabla^\flat_{[X,Y]}Z)(T) - [X,Y](f)Z^\flat(T)\\
&= f(\nabla_X\nabla^\flat_Y Z)(T) + X(f)(\nabla^\flat_Y Z)(T)\\
&\quad +X(Y(f))(Z^\flat)(T) + Y(f)(\nabla_X Z^\flat)(T)\\
&\quad -f(\nabla_Y\nabla^\flat_X Z)(T) - Y(f)(\nabla^\flat_X Z)(T)\\
&\quad -Y(X(f))(Z^\flat)(T) - X(f)(\nabla_Y Z^\flat)(T)\\
&\quad -f(\nabla^\flat_{[X,Y]}Z)(T) - [X,Y](f)Z^\flat(T)\\
&= fR(X,Y,Z,T).
\end{aligned}$$

The $\mathscr{F}(M)$-linearity in T follows from the definition of R, observing that $\nabla_X\nabla^\flat_Y Z$, $\nabla_Y\nabla^\flat_X Z$ and $\nabla^\flat_{[X,Y]}Z$ are in fact differential 1-forms.

The lower covariant derivative of a smooth vector field is a smooth differential 1-form on M, therefore $\nabla^\flat_X Z$, $\nabla^\flat_Y Z$ and $\nabla^\flat_{[X,Y]}Z$ are smooth on M. It follows that R is also smooth on M. □

Remark 7.6. One can write

$$\mathcal{R}^\flat : \mathfrak{X}(M)^2 \to \mathcal{T}^0_2 M \tag{66}$$

$$\mathcal{R}^\flat_{XY} := \nabla_X\nabla^\flat_Y - \nabla_Y\nabla^\flat_X - \nabla^\flat_{[X,Y]}, \tag{67}$$

with the amendment that

$$\mathcal{R}^\flat_{XY}(Z,T) := (\mathcal{R}^\flat_{XY}Z)(T) \tag{68}$$

for any $Z, T \in \mathfrak{X}(M)$.

7.2. **The symmetries of the Riemann curvature tensor**

The following proposition generalizes well-known symmetry properties of the Riemann curvature tensor of a non-degenerate metric (*cf. e.g.* [40], p. 75) to semi-regular metrics. The proofs are similar to the non-degenerate case, except that they avoid using the covariant derivative and the index raising, so we prefer to give them explicitly.

Proposition 7.7 (The symmetries of the Riemann curvature)**.** Let (M, g) be a semi-regular semi-Riemannian manifold. Then, for any $X, Y, Z, T \in \mathfrak{X}(M)$, the Riemann curvature has the following symmetry properties

(1) $\mathcal{R}^\flat_{XY} = -\mathcal{R}^\flat_{YX}$
(2) $\mathcal{R}^\flat_{XY}(Z,T) = -\mathcal{R}^\flat_{XY}(T,Z)$
(3) $\mathcal{R}^\flat_{YZ}X + \mathcal{R}^\flat_{ZX}Y + \mathcal{R}^\flat_{XY}Z = 0$
(4) $\mathcal{R}^\flat_{XY}(Z,T) = \mathcal{R}^\flat_{ZT}(X,Y)$

Proof. (1) Follows from the Definition 7.1:

$$\begin{aligned}\mathcal{R}^\flat_{XY}Z &= \nabla_X\nabla^\flat_Y Z - \nabla_Y\nabla^\flat_X Z - \nabla^\flat_{[X,Y]}Z\\ &= -\mathcal{R}^\flat_{YX}Z\end{aligned}$$

(2) This is equivalent to

$$\mathcal{R}^\flat_{XY}(V,V) = 0 \tag{69}$$

for any $V \in \mathfrak{X}(M)$. From the property of the lower covariant derivative of being metric (Theorem 6.3, property (4)) it follows that

$$(\nabla^\flat_{[X,Y]}V)(V) = \frac{1}{2}[X,Y]\langle V,V\rangle$$

and

$$X((\nabla^\flat_Y V)(V)) = \frac{1}{2}XY\langle V,V\rangle.$$

From the Definition 6.9 of the covariant derivative of 1-forms we obtain that

$$(\nabla_X\nabla^\flat_Y V)(V) = X\left((\nabla^\flat_Y V)(V)\right) - \langle\!\langle \nabla^\flat_X V, \nabla^\flat_Y V\rangle\!\rangle_\bullet. \tag{70}$$

By combining them we get

$$(\nabla_X\nabla^\flat_Y V)(V) = \frac{1}{2}XY\langle V,V\rangle - \langle\!\langle \nabla^\flat_X V, \nabla^\flat_Y V\rangle\!\rangle_\bullet. \tag{71}$$

Therefore,

$$\begin{aligned}\mathcal{R}^\flat_{XY}(V,V) &= (\nabla_X\nabla^\flat_Y V)(V) - (\nabla_Y\nabla^\flat_X V)(V) - (\nabla^\flat_{[X,Y]}V)(V)\\ &= \frac{1}{2}X\left((\nabla^\flat_Y V)(V)\right) - \langle\!\langle \nabla^\flat_X V, \nabla^\flat_Y V\rangle\!\rangle_\bullet\\ &\quad -\frac{1}{2}Y\left((\nabla^\flat_X V)(V)\right) + \langle\!\langle \nabla^\flat_Y V, \nabla^\flat_X V\rangle\!\rangle_\bullet\\ &\quad -\tfrac{1}{2}[X,Y]\langle V,V\rangle = 0\end{aligned}$$

(3) As the proof of this identity usually goes, we define the cyclic sum for any $F : \mathfrak{X}(M)^3 \to \mathcal{A}^1(M)$ by

$$\textstyle\sum_{\circlearrowleft} F(X,Y,Z) := F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) \tag{72}$$

and observe that it doesn't change at cyclic permutations of X, Y, Z. Then, from the properties of the lower covariant derivative and from Jacobi's identity,

$$\begin{aligned}\textstyle\sum_{\circlearrowleft}\mathcal{R}^\flat_{XY}Z &= \textstyle\sum_{\circlearrowleft}\nabla_X\nabla^\flat_Y Z - \sum_{\circlearrowleft}\nabla_Y\nabla^\flat_X Z - \sum_{\circlearrowleft}\nabla^\flat_{[X,Y]}Z\\ &= \textstyle\sum_{\circlearrowleft}\nabla_X\nabla^\flat_Y Z - \sum_{\circlearrowleft}\nabla_X\nabla^\flat_Z Y - \sum_{\circlearrowleft}\nabla^\flat_{[X,Y]}Z\\ &= \textstyle\sum_{\circlearrowleft}\nabla_X\left(\nabla^\flat_Y Z - \nabla^\flat_Z Y\right) - \sum_{\circlearrowleft}\nabla^\flat_{[X,Y]}Z\\ &= \textstyle\sum_{\circlearrowleft}\nabla_X[Y,Z]^\flat - \sum_{\circlearrowleft}\nabla^\flat_{[X,Y]}Z\\ &= \textstyle\sum_{\circlearrowleft}\nabla^\flat_X[Y,Z] - \sum_{\circlearrowleft}\nabla^\flat_{[Y,Z]}X\\ &= \textstyle\sum_{\circlearrowleft}[X,[Y,Z]]^\flat = 0.\end{aligned}$$

To show (4) we apply (3) four times (as in the usual proof of the properties of the curvature):

$$\begin{array}{rcrcrcl} \mathcal{R}^\flat_{XY}(Z,T) & + & \mathcal{R}^\flat_{YZ}(X,T) & + & \mathcal{R}^\flat_{ZX}(Y,T) & = & 0 \\ \mathcal{R}^\flat_{YZ}(T,X) & + & \mathcal{R}^\flat_{ZT}(Y,X) & + & \mathcal{R}^\flat_{TY}(Z,X) & = & 0 \\ \mathcal{R}^\flat_{ZT}(X,Y) & + & \mathcal{R}^\flat_{TX}(Z,Y) & + & \mathcal{R}^\flat_{XZ}(T,Y) & = & 0 \\ \mathcal{R}^\flat_{TX}(Y,Z) & + & \mathcal{R}^\flat_{XY}(T,Z) & + & \mathcal{R}^\flat_{YT}(X,Z) & = & 0 \end{array}$$

then sum up, divide by 2 and get:

$$\mathcal{R}^\flat_{XY}(Z,T) = \mathcal{R}^\flat_{ZT}(X,Y).$$

□

Corollary 7.8 (see [36], p. 270)**.** For any $X, Y, Z \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}_\circ(M)$, the Riemann curvature tensor R satisfies

(73)
$$R(W,X,Y,Z) = R(X,W,Y,Z) = R(X,Y,W,Z) = R(X,Y,Z,W) = 0.$$

Proof. From the Remark 6.21, $\nabla_X \nabla^\flat_Y Z \in \mathcal{A}^\bullet(M)$, and from the Remark 6.7, $\nabla^\flat_X Y \in \mathcal{A}^\bullet(M)$, for any $X, Y, Z \in \mathfrak{X}(M)$. Therefore, $R(X,Y,Z,W) = 0$. From the symmetry properties (1) and (4) from Theorem 7.7, this property extends to all other slots of the Riemann curvature tensor. □

Corollary 7.9. Let (M, g) be a semi-regular semi-Riemannian manifold. Then, for any $X, Y \in \mathfrak{X}(M)$, $\mathcal{R}^\flat_{XY} \in \mathcal{A}^{\bullet 2}(M)$ ($\mathcal{R}^\flat_{XY}$ is a radical-annihilator).

Proof. Follows from the Corollary 7.8. □

7.3. **Ricci curvature tensor and scalar curvature**

In non-degenerate semi-Riemannian geometry, the Ricci tensor is obtained by tracing the Riemann curvature, and the scalar curvature by tracing the Ricci tensor (*cf. e.g.* [40], p. 87–88). In the degenerate case, an invariant contraction can be performed only on radical-annihilator slots. Fortunately, this is the case of the Riemann tensor even in the case when the metric is degenerate (Corollary 7.8), so it is possible to define the Ricci tensor as:

Definition 7.10. Let (M, g) be a radical-stationary singular semi-Riemannian manifold with constant signature. The *Ricci curvature tensor* is defined as the covariant contraction of the Riemann curvature tensor

$$\mathrm{Ric}(X,Y) := R(X, \bullet, Y, \bullet) \tag{74}$$

for any $X, Y \in \mathfrak{X}(M)$.

The symmetry of the Ricci tensor works just like in the non-degenerate case (*cf. e.g.* [40], p. 87):

Proposition 7.11. The Ricci curvature tensor on a radical-stationary singular semi-Riemannian manifold with constant signature is symmetric:

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X) \tag{75}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. The Proposition 7.7 states that $R(X,Y,Z,T) = R(Z,T,X,Y)$ for any $X, Y, Z, T \in \mathfrak{X}(M)$. Therefore, $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$. □

The scalar curvature is obtained from the Ricci tensor like in the non-degenerate case (*cf. e.g.* [40], p. 88):

Definition 7.12. Let (M,g) be a radical-stationary singular semi-Riemannian manifold with constant signature. The *scalar curvature* is defined as the covariant contraction of the Ricci curvature tensor

$$s := \mathrm{Ric}(\bullet, \bullet). \tag{76}$$

Remark 7.13. The Ricci and the scalar curvatures are smooth for the case of radical-stationary singular semi-Riemannian manifolds having the metric with constant signature. For semi-regular semi-Riemannian manifolds, the Ricci and scalar curvatures are smooth in the regions of constant curvature, and become in general divergent as we approach the points where the signature changes.

8. Curvature of semi-regular semi-Riemannian manifolds II

This section contains some complements on the Riemann curvature tensor of semi-regular semi-Riemannian manifolds. A useful formula of this curvature in terms of the Koszul form is provided in §8.1.

In the subsection §8.2 we recall some results from [36] concerning the (non-unique) Koszul derivative ∇ and the associated curvature function R_∇, and show that $\langle R_\nabla(\_,\_)\_,\_\rangle$ coincides with that of the Riemann curvature tensor given in this article in §7.

8.1. Riemann curvature in terms of the Koszul form

Proposition 8.1. For any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$ on a semi-regular semi-Riemannian manifold (M,g):

$$\begin{aligned} R(X,Y,Z,T) &= X\left((\nabla^\flat_Y Z)(T)\right) - Y\left((\nabla^\flat_X Z)(T)\right) - (\nabla^\flat_{[X,Y]} Z)(T) \\ &\quad + \langle\!\langle \nabla^\flat_X Z, \nabla^\flat_Y T\rangle\!\rangle_\bullet - \langle\!\langle \nabla^\flat_Y Z, \nabla^\flat_X T\rangle\!\rangle_\bullet \end{aligned} \tag{77}$$

and, alternatively,

$$\begin{aligned} R(X,Y,Z,T) &= X\mathcal{K}(Y,Z,T) - Y\mathcal{K}(X,Z,T) - \mathcal{K}([X,Y],Z,T) \\ &\quad + \mathcal{K}(X,Z,\bullet)\mathcal{K}(Y,T,\bullet) - \mathcal{K}(Y,Z,\bullet)\mathcal{K}(X,T,\bullet) \end{aligned} \tag{78}$$

Proof. From the Definition 6.9 of the covariant derivative of 1-forms we obtain that

$$(\nabla_X \nabla^\flat_Y Z)(T) = X\left((\nabla^\flat_Y Z)(T)\right) - \langle\!\langle \nabla^\flat_X T, \nabla^\flat_Y Z\rangle\!\rangle_\bullet, \tag{79}$$

therefore

(80)
$$\begin{aligned} R(X,Y,Z,T) &= (\nabla_X \nabla_Y^\flat Z)(T) - (\nabla_Y \nabla_X^\flat Z)(T) - (\nabla_{[X,Y]}^\flat Z)(T) \\ &= X\left((\nabla_Y^\flat Z)(T)\right) - Y\left((\nabla_X^\flat Z)(T)\right) - (\nabla_{[X,Y]}^\flat Z)(T) \\ &\quad + \langle\!\langle \nabla_X^\flat Z, \nabla_Y^\flat T \rangle\!\rangle_\bullet - \langle\!\langle \nabla_Y^\flat Z, \nabla_X^\flat T \rangle\!\rangle_\bullet \end{aligned}$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$. The second formula (78) follows from the definition of the lower derivative of vector fields. □

Remark 8.2. In a coordinate basis, the components of the Riemann curvature tensor are given by

$$R_{abcd} = \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + g_\bullet{}^{st}(\mathcal{K}_{acs}\mathcal{K}_{bdt} - \mathcal{K}_{bcs}\mathcal{K}_{adt}). \tag{81}$$

Proof.

$$\begin{aligned} R_{abcd} &:= R(\partial_a, \partial_b, \partial_c, \partial_d) \\ &= \partial_a \mathcal{K}(\partial_b, \partial_c, \partial_d) - \partial_b \mathcal{K}(\partial_a, \partial_c, \partial_d) - \mathcal{K}([\partial_a, \partial_b], \partial_c, \partial_d) \\ &\quad + \mathcal{K}(\partial_a, \partial_c, \bullet)\mathcal{K}(\partial_b, \partial_d, \bullet) - \mathcal{K}(\partial_b, \partial_c, \bullet)\mathcal{K}(\partial_a, \partial_d, \bullet) \\ &= \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + g_\bullet{}^{st}(\mathcal{K}_{acs}\mathcal{K}_{bdt} - \mathcal{K}_{bcs}\mathcal{K}_{adt}) \end{aligned} \tag{82}$$

□

8.2. **Relation with Kupeli's curvature function**

Through the work of Demir Kupeli [36] we have seen that for a radical-stationary singular semi-Riemannian manifold (with constant signature) (M, g) there is always a Koszul derivative ∇, from whose curvature function R_∇ we can construct a tensor field $\langle R_\nabla(\_,\_)\_,\_\rangle$. We may wonder how is $\langle R_\nabla(\_,\_)\_,\_\rangle$ related to the Riemann curvature tensor from the Definition 7.2. We will see that they coincide for a radical-stationary singular semi-Riemannian manifold.

Definition 8.3 (Koszul derivative, *cf.* Kupeli [36], p. 261)**.** A *Koszul derivative* on a radical-stationary semi-Riemannian manifold with constant signature is an operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which satisfies the *Koszul formula*

$$\langle \nabla_X Y, Z \rangle = \mathcal{K}(X, Y, Z). \tag{83}$$

Remark 8.4 (*cf.* Kupeli [36], p. 262)**.** The Koszul derivative corresponds, for the non-degenerate case, to the Levi-Civita connection.

Definition 8.5 (Curvature function, *cf.* Kupeli [36], p. 266)**.** The *curvature function* $R_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ of a Koszul derivative ∇ on a singular semi-Riemannian manifold with constant signature (M, g) is defined by

$$R_\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{84}$$

Remark 8.6. In [36], p. 266-268 it is shown that $\langle R_\nabla(\_,\_)\_,\_\rangle \in \mathcal{T}^0_4 M$ and it has the same symmetry properties as the Riemann curvature tensor of a Levi-Civita connection.

Theorem 8.7. Let (M, g) be a radical-stationary singular semi-Riemannian manifold with constant signature, and ∇ a Koszul derivative on M. The Riemann curvature tensor is related to the curvature function by

$$\langle R_\nabla(X,Y)Z,T\rangle = R(X,Y,Z,T) \tag{85}$$

for any $X, Y, Z, T \in \mathfrak{X}(M)$.

Proof. From Theorem 5.3 and Definition 8.5, applying the property of contraction with the metric from Lemma 4.3 and the Koszul formula for the Riemann curvature tensor (78), we obtain

$$\begin{aligned}
\langle R_\nabla(X,Y)Z,T\rangle &= \langle\nabla_X\nabla_Y Z,T\rangle - \langle\nabla_Y\nabla_X Z,T\rangle - \langle\nabla_{[X,Y]}Z,T\rangle \\
&= X\langle\nabla_Y Z,T\rangle - \langle\nabla_Y Z,\nabla_X T\rangle \\
&\quad -Y\langle\nabla_X Z,T\rangle + \langle\nabla_X Z,\nabla_Y T\rangle - \langle\nabla_{[X,Y]}Z,T\rangle \\
&= X\mathcal{K}(Y,Z,T) - \mathcal{K}(Y,Z,\bullet)\mathcal{K}(X,T,\bullet) \\
&\quad -Y\mathcal{K}(X,Z,T) + \mathcal{K}(X,Z,\bullet)\mathcal{K}(Y,T,\bullet) \\
&\quad -\mathcal{K}([X,Y],Z,T) \\
&= R(X,Y,Z,T)
\end{aligned}$$

□

9. Examples of semi-regular semi-Riemannian manifolds

9.1. Diagonal metric

Let (M, g) be a singular semi-Riemannian manifold with variable signature having the property that for each point $p \in M$ there is a local coordinate system around p in which the metric takes a diagonal form $g = \mathrm{diag}(g_{11}, \ldots, g_{nn})$. According to equation (25), $2\mathcal{K}_{abc} = \partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}$, but since g is diagonal, we have only the following possibilities: $\mathcal{K}_{baa} = \mathcal{K}_{aba} = -\mathcal{K}_{aab} = \frac{1}{2}\partial_b g_{aa}$, for $a \neq b$, and $\mathcal{K}_{aaa} = \frac{1}{2}\partial_a g_{aa}$.

The manifold (M, g) is radical-stationary if and only if whenever $g_{aa} = 0$, $\partial_b g_{aa} = \partial_a g_{bb} = 0$.

According to Proposition 6.23, the manifold (M, g) is semi-regular if and only if

$$\sum_{\substack{s\in\{1,\ldots,n\}\\ g_{ss}\neq 0}} \frac{\partial_a g_{ss}\partial_b g_{ss}}{g_{ss}}, \sum_{\substack{s\in\{1,\ldots,n\}\\ g_{ss}\neq 0}} \frac{\partial_s g_{aa}\partial_s g_{bb}}{g_{ss}}, \sum_{\substack{s\in\{1,\ldots,n\}\\ g_{ss}\neq 0}} \frac{\partial_a g_{ss}\partial_s g_{bb}}{g_{ss}} \tag{86}$$

are all smooth.

One way to ensure this is for instance if the functions $u, v : M \to \mathbb{R}$ defined as

$$u(p) := \begin{cases} \dfrac{\partial_b g_{aa}}{\sqrt{|g_{aa}|}} & g_{aa} \neq 0 \\ 0 & g_{aa} = 0 \end{cases} \text{ and } v(p) := \begin{cases} \dfrac{\partial_a g_{bb}}{\sqrt{|g_{aa}|}} & g_{aa} \neq 0 \\ 0 & g_{aa} = 0 \end{cases} \tag{87}$$

and $\sqrt{|g_{aa}|}$ are smooth for all $a, b \in \{1, \ldots, n\}$. In this case it is easy to see that all the terms of the sums in equation (86) are smooth.

9.2. **Conformally-non-degenerate metrics**

Another example of semi-regular metric is given by those that can be obtained by a conformal transformation (*cf. e.g.* [22], p. 42) from non-degenerate metrics.

Definition 9.1. A singular semi-Riemannian manifold (M, g) is said to be *conformally non-degenerate* if there is a non-degenerate semi-Riemannian metric $\tilde{g}$ on M and a smooth function $\Omega \in \mathscr{F}(M)$, $\Omega \geq 0$, so that $g(X, Y) = \Omega^2 \tilde{g}(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. The manifold (M, g) is alternatively denoted by $(M, \tilde{g}, \Omega)$.

The following proposition shows what happens to the Koszul form at a conformal transformation of the metric, similar to the non-degenerate case (*cf. e.g.* [22], p. 42).

Proposition 9.2. Let $(M, \tilde{g}, \Omega)$ be a conformally non-degenerate singular semi-Riemannian manifold. Then, the Koszul form $\mathcal{K}$ of g is related to the Koszul form $\tilde{\mathcal{K}}$ of $\tilde{g}$ by:

$$\mathcal{K}(X, Y, Z) = \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \Omega \left[\tilde{g}(Y, Z)X + \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z\right](\Omega) \tag{88}$$

Proof. From the Koszul formula we obtain

$$\begin{aligned} \mathcal{K}(X, Y, Z) &= \frac{1}{2}\{X(\Omega^2 \tilde{g}(Y, Z)) + Y(\Omega^2 \tilde{g}(Z, X)) - Z(\Omega^2 \tilde{g}(X, Y)) \\ &\quad -\Omega^2 \tilde{g}(X, [Y, Z]) + \Omega^2 \tilde{g}(Y, [Z, X]) + \Omega^2 \tilde{g}(Z, [X, Y])\} \\ &= \frac{1}{2}\{\Omega^2 X(\tilde{g}(Y, Z)) + \tilde{g}(Y, Z)X(\Omega^2) + \Omega^2 Y(\tilde{g}(X, Z)) \\ &\quad +\tilde{g}(X, Z)Y(\Omega^2) - \Omega^2 Z(\tilde{g}(X, Y)) - \tilde{g}(X, Y)Z(\Omega^2) \\ &\quad -\Omega^2 \tilde{g}(X, [Y, Z]) + \Omega^2 \tilde{g}(Y, [Z, X]) + \Omega^2 \tilde{g}(Z, [X, Y])\} \\ &= \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \frac{1}{2}\{\tilde{g}(Y, Z)X(\Omega^2) \\ &\quad +\tilde{g}(X, Z)Y(\Omega^2) - \tilde{g}(X, Y)Z(\Omega^2)\} \\ &= \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \Omega\big[\tilde{g}(Y, Z)X \\ &\quad +\tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z\big](\Omega) \end{aligned}$$

□

Theorem 9.3. Let $(M, \tilde{g}, \Omega)$ be a singular semi-Riemannian manifold which is conformally non-degenerate. Then, $(M, g = \Omega^2 \tilde{g})$ is a semi-regular semi-Riemannian manifold.

Proof. The metric g is either non-degenerate, or it is 0. Therefore, the manifold (M, g) is radical-stationary.

Let $(E_a)_{a=1}^n$ be a local frame of vector fields on an open $U \subseteq M$, which is orthonormal with respect to the non-degenerate metric $\tilde{g}$. Then, the metric g is diagonal in $(E_a)_{a=1}^n$.

Proposition 9.2 implies that the Koszul form has the form $\mathcal{K}(X, Y, Z) = \Omega h(X, Y, Z)$, where

$$h(X, Y, Z) = \Omega \tilde{\mathcal{K}}(X, Y, Z) + [\tilde{g}(Y, Z)X + \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z]\,(\Omega) \tag{89}$$

is a smooth function depending on X, Y, Z. Moreover, if $\Omega = 0$, then $h(X, Y, Z) = 0$ as well, because the first term is multiple of Ω, and the second is a partial derivative of Ω, which reaches its minimum at 0.

Theorem 4.6 saids that, on the regions of constant signature, if $r = n - \operatorname{rank} g + 1$, for any $X, Y, Z, T \in U$ and for any $a \in \{1, \ldots, n\}$,

$$\begin{aligned} \mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet) &= \textstyle\sum_{a=r}^n \dfrac{\mathcal{K}(X, Y, E_a)\mathcal{K}(Z, T, E_a)}{g(E_a, E_a)} \\ &= \textstyle\sum_{a=r}^n \dfrac{\Omega^2 h(X, Y, E_a)h(Z, T, E_a)}{\Omega^2 \tilde{g}(E_a, E_a)} \\ &= \textstyle\sum_{a=1}^n \dfrac{h(X, Y, E_a)h(Z, T, E_a)}{\tilde{g}(E_a, E_a)}. \end{aligned} \tag{90}$$

If $\Omega = 0$, then $h(X, Y, Z) = 0$, therefore the last member does not depend on r. It follows that $\mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet) \in \mathscr{F}(M)$, and according to Proposition 6.23, (M, g) is semi-regular. □

10. Einstein's equation on semi-regular spacetimes

10.1. The problem of singularities

In 1965 Roger Penrose [42], and later he and S. Hawking [19, 23, 22], proved a set of *singularity theorems*. These theorems state that under reasonable conditions the spacetime turns out to be *geodesic incomplete – i.e.* it has *singularities*. Consequently, some researchers proclaimed that General Relativity predicts its own breakdown, by predicting the singularities [23, 21, 2, 24, 3, 4]. Hawking's discovery of the black hole evaporation, leading to his *information loss paradox* [20, 21], made the things even worse. The singularities seem to destroy information, in particular violating the unitary evolution of quantum systems. The reason is that the field equations cannot be continued through singularities.

By applying the results presented in this article we shall see that, at least for semi-regular semi-Riemannian manifolds, we can extend Einstein's

equation through the singularities. Einstein's equation is replaced by a densitized version which is equivalent to the standard version if the metric is non-degenerate. This equation remains smooth at singularities, which now become harmless.

10.2. **Einstein's equation on semi-regular spacetimes**

To define the Einstein tensor on a semi-regular semi-Riemannian manifold, we normally make use of the Ricci tensor and the scalar curvature:

$$G := \mathrm{Ric} - \frac{1}{2} s g \tag{91}$$

These two quantities can be defined even for a degenerate metric, so long as the metric doesn't change its signature (see §7.3), but at the points where the signature changes, they can become infinite.

Definition 10.1. A *semi-regular spacetime* is a four-dimensional semi-regular semi-Riemannian manifold having the signature $(0, 3, 1)$ at the points where it is non-degenerate.

Theorem 10.2. Let (M, g) be a semi-regular spacetime. Then its Einstein density tensor of weight 2, $G \det g$, is smooth.

Proof. At the points p where the metric is non-degenerate, the Einstein tensor (91) can be expressed using the Hodge $*$ operator by:

$$G_{ab} = g^{st}(*R*)_{asbt}, \tag{92}$$

where $(*R*)_{abcd}$ is obtained by taking the Hodge dual of R_{abcd} with respect to the first and the second pairs of indices (*cf. e.g.* [43], p. 234). Explicitly, if we write the components of the volume form associated to the metric as ε_{abcd}, we have

$$(*R*)_{abcd} = \varepsilon_{ab}{}^{st}\varepsilon_{cd}{}^{pq}R_{stpq}. \tag{93}$$

If we employ coordinates, the volume form can be expressed in terms of the Levi-Civita symbol by

$$\varepsilon_{abcd} = \epsilon_{abcd}\sqrt{-\det g}. \tag{94}$$

We can rewrite the Einstein tensor as

$$G^{ab} = \frac{g_{kl}\epsilon^{akst}\epsilon^{blpq}R_{stpq}}{\det g}, \tag{95}$$

If we allow the metric to become degenerate, the Einstein tensor so defined becomes divergent, as it is expected. But the tensor density $G^{ab} \det g$, of weight 2, associated to it remains smooth, and we get

$$G^{ab} \det g = g_{kl}\epsilon^{akst}\epsilon^{blpq}R_{stpq}. \tag{96}$$

Since the spacetime is semi-regular, this quantity is indeed smooth, because it is constructed only from the Riemann curvature tensor, which is

smooth (see Theorem 7.5), and from the Levi-Civita symbol, which is constant in the particular coordinate system. The determinant of the metric converges to 0 so that it cancels the divergence which normally would appear in G^{ab}. The tensor density $G_{ab}\det g$, being obtained by lowering its indices, is also smooth. □

Remark 10.3. Because the densitized Einstein tensor $G_{ab}\det g$ is smooth, it follows that the densitized curvature scalar is smooth

$$s\det g = -g_{ab}G^{ab}\det g, \tag{97}$$

and so is the densitized Ricci tensor

$$R_{ab}\det g = g_{as}g_{bt}G^{st}\det g + \frac{1}{2}sg_{ab}\det g. \tag{98}$$

Remark 10.4. In the context of General Relativity, on a semi-regular spacetime, if T is the stress-energy tensor, we can write the *densitized Einstein equation*:

$$G\det g + \Lambda g\det g = \kappa T\det g, \tag{99}$$

or, in coordinates or local frames,

$$G_{ab}\det g + \Lambda g_{ab}\det g = \kappa T_{ab}\det g, \tag{100}$$

where $\kappa := \dfrac{8\pi\mathcal{G}}{c^4}$, with $\mathcal{G}$ and c being Newton's constant and the speed of light.

References

1. A. Ashtekar, *New Hamiltonian formulation of general relativity*, Phys. Rev. D **36** (1987), no. 6, 1587–1602.
2. A. Ashtekar, *Non-perturbative canonical gravity, Lecture notes in collaboration with R. S. Tate*, World Scientific, Singapore, 1991.
3. A. Ashtekar, *Singularity Resolution in Loop Quantum Cosmology: A Brief Overview*, J. Phys. Conf. Ser. **189** (2009), 012003.
4. A. Ashtekar and E. Wilson-Ewing, *Loop quantum cosmology of Bianchi I models*, Phys. Rev. **D79** (2009), 083535.
5. A. Bejancu and K.L. Duggal, *Lightlike submanifolds of Semi-Riemannian manifolds*, Acta Appl. Math. **38** (1995), no. 2, 197–215.
6. A. Das, *Tensors: the mathematics of relativity theory and continuum mechanics*, Springer Verlag, 2007.
7. T. Dereli and R.W. Tucker, *Signature dynamics in general relativity*, Classical Quantum Gravity **10** (1993), 365.
8. T. Dray, *Einstein's equations in the presence of signature change*, Arxiv preprint gr-qc/9610064 (1996).
9. T. Dray, G. Ellis, and C. Hellaby, *Note on signature change and Colombeau theory*, Gen. Relativity Gravitation **33** (2001), no. 6, 1041–1046.
10. T. Dray, C.A. Manogue, and R.W. Tucker, *Particle production from signature change*, Gen. Relativity Gravitation **23** (1991), no. 8, 967–971.
11. T. Dray, C.A. Manogue, and R.W. Tucker, *Scalar field equation in the presence of signature change*, Phys. Rev. D **48** (1993), no. 6, 2587–2590.

12. T. Dray, C.A. Manogue, and R.W. Tucker, *Boundary conditions for the scalar field in the presence of signature change*, Classical Quantum Gravity **12** (1995), 2767.
13. K.L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, vol. 364, Kluwer Academic, 1996.
14. A. Einstein and N. Rosen, *The particle problem in the general theory of relativity*, Phys. Rev. **48** (1935), no. 1, 73.
15. G. Ellis, A. Sumeruk, D. Coule, and C. Hellaby, *Change of signature in classical relativity*, Classical Quantum Gravity **9** (1992), 1535.
16. G. Ellis, *Covariant change of signature in classical relativity*, Gen. Relativity Gravitation **24** (1992), no. 10, 1047–1068.
17. S. Gallot, D. Hullin, and J. Lafontaine, *Riemannian Geometry*, 3rd ed., Springer-Verlag, Berlin, New York, 2004.
18. G.W. Gibbons, *Part III: Applications of Differential Geometry to Physics*, Cambridge CB3 0WA, UK (2006).
19. S. Hawking, *The occurrence of singularities in cosmology. iii. causality and singularities*, Proc. Roy. Soc. London Ser. A (1967), no. 300, 187–201.
20. S. Hawking, *Particle Creation by Black Holes*, Comm. Math. Phys. (1973), no. 33, 323.
21. S. Hawking, *Breakdown of Predictability in Gravitational Collapse*, Phys. Rev. D (1976), no. 14, 2460.
22. S. Hawking and G. Ellis, *The Large Scale Structure of Space Time*, Cambridge University Press, 1995.
23. S. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. London Ser. A (1970), no. 314, 529–548.
24. S. Hawking and R. Penrose, *The Nature of Space and Time*, Princeton University Press, 1996.
25. S.A. Hayward, *Signature change in general relativity*, Classical Quantum Gravity **9** (1992), 1851.
26. S.A. Hayward, *Junction conditions for signature change*, Arxiv preprint gr-qc/9303034 (1993).
27. S.A. Hayward, *Comment on "Failure of standard conservation laws at a classical change of signature"*, Phys. Rev. D **52** (1995), no. 12, 7331–7332.
28. C. Hellaby and T. Dray, *Failure of standard conservation laws at a classical change of signature*, Phys. Rev. D **49** (1994), no. 10, 5096–5104.
29. M. Kossowski, *Fold singularities in pseudo Riemannian geodesic tubes*, Proc. Amer. Math. Soc. (1985), 463–469.
30. M. Kossowski, *Pseudo-Riemannian metric singularities and the extendability of parallel transport*, Proc. Amer. Math. Soc. **99** (1987), no. 1, 147–154.
31. M. Kossowski and M. Kriele, *Signature type change and absolute time in general relativity*, Classical Quantum Gravity **10** (1993), 1157.
32. M. Kossowski and M. Kriele, *Smooth and discontinuous signature type change in general relativity*, Classical Quantum Gravity **10** (1993), 2363.
33. M. Kossowski and M. Kriele, *The Einstein equation for signature type changing spacetimes*, Proceedings: Math. Phys. Sciences **446** (1994), no. 1926, 115–126.
34. M. Kossowski and M. Kriele, *Transverse, type changing, pseudo riemannian metrics and the extendability of geodesics*, Proceedings: Math. Phys. Sciences (1994), 297–306.
35. D. Kupeli, *On Null Submanifolds in Spacetimes*, Geom. Dedicata **23** (1987), no. 1, 33–51.
36. D. Kupeli, *Degenerate Manifolds*, Geom. Dedicata **23** (1987), no. 3, 259–290.
37. D. Kupeli, *Degenerate submanifolds in Semi-Riemannian geometry*, Geom. Dedicata **24** (1987), no. 3, 337–361.
38. D. Kupeli, *Singular semi-Riemannian geometry*, Kluwer Academic Publishers Group, 1996.

39. G. C. Moisil, *Sur les géodésiques des espaces de Riemann singuliers*, Bull. Math. Soc. Roumaine Sci. (1940), no. 42, 33–52.
40. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure Appl. Math. (1983), no. 103.
41. A. Pambira, *Harmonic morphisms between degenerate semi-Riemannian manifolds*, arXiv:math / 0303275v1 (2003).
42. R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. (1965), no. 14, 57–59.
43. R. Penrose and W. Rindler, *Spinors and space-time: Volume 1, two-spinor calculus and relativistic fields (cambridge monographs on mathematical physics)*, Cambridge University Press, 1987.
44. S. Roman, *Advanced Linear Algebra*, Springer, 2008.
45. Joseph D. Romano, *Geometrodynamics vs. Connection Dynamics*, Gen. Rel. Grav. (1993), no. 25, 759–854.
46. AD Sakharov, *Cosmological transitions with a change in metric signature*, Sov. Phys. JETP **60** (1984), 214.
47. K. Strubecker, *Differentialgeometrie des isotropen Raumes. 1. Theorie der Raumkurven*, Sitzungsber. Akad. Wiss. Wien, Math.-Naturw. Kl., Abt. IIa **150** (1941), 1–53.
48. K. Strubecker, *Differentialgeometrie des isotropen Raumes. II. Die Flächen konstanter RelativkrümmungK= rt- s 2*, Math. Z. **47** (1942), no. 1, 743–777.
49. K. Strubecker, *Differentialgeometrie des isotropen Raumes. III. Flächentheorie*, Math. Z. **48** (1942), no. 1, 369–427.
50. K. Strubecker, *Differentialgeometrie des isotropen Raumes. IV. Theorie der flächentreuen Abbildungen der Ebene*, Math. Z. **50** (1944), no. 1, 1–92.
51. G. Vrănceanu, *Sur les invariants des espaces de Riemann singuliers*, Disqu. Math. Physic. (1942), no. 2, 253–281.
52. G. Yoneda, H. Shinkai, and A. Nakamichi, *Trick for passing degenerate points in the Ashtekar formulation*, Phys. Rev. D **56** (1997), no. 4, 2086–2093.