Crossings of smooth Shot Noise Processes

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Abstract. In this paper, we consider smooth shot noise processes and their expected number of level crossings. When the kernel response function is sufficiently smooth, the mean number of crossings function is obtained through an integral formula. Moreover, as the intensity increases, or equivalently as the number of shots becomes larger, a normal convergence to the classical Rice's formula for Gaussian processes is obtained. The Gaussian kernel function, that corresponds to many applications in Physics, is studied in detail and two different regimes are exhibited.

1. Introduction

In this paper, we will consider a *shot noise process* which is a real-valued random process given by

$$X(t) = \sum_i \beta_i g(t-\tau_i),\ t \in \mathbb{R} \tag{1}$$

where g is a given (deterministic) measurable function (it will be called the *kernel function* of the shot noise process), the $\{\tau_i\}$ are the points of a Poisson point process on the line of intensity $\lambda\nu(ds)$, where $\lambda > 0$ and ν is a positive σ-finite measure on $\mathbb{R}$, and the $\{\beta_i\}$ are independent copies of a random variable β (called the *impulse*), independent of $\{\tau_i\}$.

Shot noise processes are related to many problems in Physics as they result from the superposition of "shot effects" which occur at random. Fundamental results were obtained by Rice in [23]. Daley in [10] gave sufficient conditions on the kernel function to ensure the convergence of the formal series in a preliminary work. General results, including sample paths properties, were given by Rosiński [24] in a more general setting. In most of the literature the measure ν is the Lebesgue measure on $\mathbb{R}$ such that the shot noise process is a stationary one. In order to derive more precise sample paths properties and especially crossings rates, mainly two properties have been extensively exhibited and used. The first one is the Markov property, which is valid, choosing a non continuous positive causal kernel function that is 0 for negative time. This is the case in particular of the exponential kernel $g(t) = e^{-t}\mathbf{1}_{t\geq 0}$ for which explicit distributions and crossings rates can be obtained [21]. A simple formula for the expected numbers of level crossings is valid for more general kernels of this type but resulting shot noise processes are non differentiable [4, 16]. The infinitely divisible property is the second main tool. Actually, this allows to establish convergence to a Gaussian process as the intensity increases [22, 15]. Sample paths properties of Gaussian processes have been extensively studied and fine results are known concerning the level crossings of smooth Gaussian processes (see [2, 9] for instance).

The goal of the paper is to study the crossings of a shot noise process in the general case when the kernel function g is smooth. In this setting we lose Markov's property but the shot noise process inherits smoothness properties. Integral formulas for the number of level crossings of smooth processes was generalized to the non Gaussian case by [18] but it uses assumptions that rely on properties of some densities, which may not be valid for shot noise processes. We derive integral formulas for the mean number of crossings function and pay a special interest in the continuity of this function with respect to the level. Exploiting further on normal convergence, we exhibit a Gaussian regime for the mean number of crossings function when the intensity goes to infinity. A particular example, which

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is studied in detail, concerns the shot noise process where $\beta = 1$ almost surely and g is a Gaussian kernel of width σ:

$$g(t) = g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}.$$

Such a model has many applications because it is solution of the heat equation (we consider σ as a variable), and it thus models a diffusion from random sources (the points of the Poisson point process).

The paper is organized as follows. In Section 2, we consider crossings for general smooth processes. We give an explicit formula for the Fourier transform of the mean number of crossings function of a process X in terms of the characteristic function of $(X(t), X'(t))$. One of the difficulties is then to obtain results for the mean number of crossings of a given level α and not only for almost every α. Thus we focus on the continuity property of the mean number of crossings function. Section 3 is devoted to crossings for a smooth shot noise process X defined by (1). In order to get the continuity of the mean number of crossings function, we study the question of the existence and the boundedness of a probability density for $X(t)$. In Section 4, we show how, and in which sense, the mean number of crossings function converges to the one of a Gaussian process when the intensity λ goes to infinity. We give rates of this convergence. Finally, in Section 5, we study in detail the case of a Gaussian kernel of width σ. We are mainly interested in the mean number of local extrema of this process, as a function of σ. Thanks to the heat equation, and also to scaling properties between σ and λ, we prove that the mean number of local extrema is a decreasing function of σ, and give its asymptotics as σ is small or large.

2. Crossings of smooth processes

The goal of this section is to investigate crossings of general smooth processes in order to get results for smooth shot noise processes. This is a very different situation from the one studied in [21, 4, 16] where shot noise processes are non differentiable. However, crossings of smooth processes have been extensively studied especially in the Gaussian processes realm (see [2] for instance) which are second order processes. Therefore in the whole section we will consider second order processes which are both almost surely and mean square continuously differentiable (see Section 2.2 of [1] for instance). This implies in particular that the derivatives are also second order processes. Moreover, most of known results on crossings are based on assumptions on density probabilities, which are not well-adapted for shot noise processes. In this section, we revisit these results with a more adapted point of view based on characteristic functions.

When X is an almost surely continuously differentiable process on $\mathbb{R}$, we can consider its multiplicity function on an interval $[a, b]$ defined by

$$\forall \alpha \in \mathbb{R},\ \ N_X(\alpha, [a, b]) = \#\{t \in [a, b];\ X(t) = \alpha\}. \tag{2}$$

This defines a positive random process taking integer values. Let us briefly recall some points of "vocabulary". For a given level $\alpha \in \mathbb{R}$, a point $t \in [a, b]$ such that $X(t) = \alpha$ is called "crossing" of the level α. Then $N_X(\alpha, [a, b])$ counts the number of crossings of the level α in the interval $[a, b]$. Now we have to distinguish three different types of crossings (see for instance [9]): the up-crossings that are points for which $X(t) = \alpha$ and $X'(t) > 0$, the down-crossings that are points for which $X(t) = \alpha$ and $X'(t) < 0$ and the tangencies that are points for which $X(t) = \alpha$ and $X'(t) = 0$.

Let us also recall that according to Rolle's theorem, whatever the level α is,

$$N_X(\alpha, [a, b]) \leq N_{X'}(0, [a, b]) + 1 \text{ a.s.}$$

Note that when there are no tangencies of X' for the level 0, then $N_{X'}(0, [a, b])$ is the number of local extrema for X, which corresponds to the sum of the number of local minima (up zero-crossings of X') and of the number of local maxima (down zero-crossings of X').

Dealing with random processes, one may be more interested in the mean number of crossings. We will denote by $C_X(\alpha, [a, b])$ the mean number of crossings of the level α by the process X in $[a, b]$:

$$C_X(\alpha, [a, b]) = \mathbb{E}\left(N_X(\alpha, [a, b])\right) = \mathbb{E}(\#\{t \in [a, b] \text{ such that } X(t) = \alpha\}). \tag{3}$$

Let us emphasize that this function is no more with integer values and can be continuous with respect to α. When moreover X is a stationary process, by the additivity of means, we get $C_X(\alpha,[a,b]) = (b-a)C_X(\alpha,[0,1])$ for $\alpha \in \mathbb{R}$. In this case $C_X(\alpha,[0,1])$ corresponds to the mean number of crossings of the level α per unit length. Let us also recall that when X is a strictly stationary ergodic process, the ergodic theorem states that $(2T)^{-1}N_X(\alpha,[-T,T]) \underset{T\to+\infty}{\longrightarrow} C_X(\alpha,[0,1])$ a.s. (see [9] for instance).

2.1. **A Fourier approach for the mean number of crossings function.** One way to obtain results on crossings for almost every level α is to use the well-known *co-area formula* which is in fact valid in the more general framework of bounded variations functions (see for instance [12]). When X is an almost surely continuously differentiable process on $[a,b]$, for any bounded and continuous function h on $\mathbb{R}$, we have:

$$\int_{\mathbb{R}} h(\alpha)N_X(\alpha,[a,b])\,d\alpha = \int_a^b h(X(t))|X'(t)|\,dt \quad a.s. \tag{4}$$

In particular when $h=1$ this shows that $\alpha \mapsto N_X(\alpha,[a,b])$ is integrable on $\mathbb{R}$ and $\int_{\mathbb{R}} N_X(\alpha,[a,b])\,d\alpha = \int_a^b |X'(t)|\,dt$ is the total variation of X on $[a,b]$. Moreover, taking the expected values we get by Fubini's theorem that

$$\int_{\mathbb{R}} C_X(\alpha,[a,b])\,d\alpha = \int_a^b \mathbb{E}(|X'(t)|)dt.$$

Therefore, when the total variation of X on $[a,b]$ has finite expectation, the function $\alpha \mapsto C_X(\alpha,[a,b])$ is integrable on $\mathbb{R}$. This is the case when X is also mean square continuously differentiable since then the function $t \mapsto \mathbb{E}(|X'(t)|)$ is continuous on $[a,b]$. Let us emphasize that this implies in particular that $C_X(\alpha,[a,b]) < +\infty$ for almost every level $\alpha \in \mathbb{R}$ but one cannot conclude for a fixed given level. However, it allows to use Fubini's theorem such that, taking expectation in (4), for any bounded continuous function h:

$$\int_{\mathbb{R}} h(\alpha)C_X(\alpha,[a,b])\,d\alpha = \int_a^b \mathbb{E}(h(X(t))|X'(t)|)dt. \tag{5}$$

In the following theorem we obtain a closed formula for the Fourier transform of the mean number of crossings function, which only involves characteristic functions of the process. This can be helpful, when considering shot noise processes, whose characteristic functions are well-known.

Theorem 1. *Let $a,b \in \mathbb{R}$ with $a<b$. Let X be an almost surely and mean square continuously differentiable process on $[a,b]$. Then $\alpha \mapsto C_X(\alpha,[a,b]) \in L^1(\mathbb{R})$ and its Fourier transform $u \mapsto \widehat{C_X}(u,[a,b])$ is given by*

$$\widehat{C_X}(u,[a,b]) = \int_a^b \mathbb{E}\left(e^{iuX(t)}|X'(t)|\right)dt. \tag{6}$$

Moreover, if ψ_t denotes the joint characteristic function of $(X(t),X'(t))$, then $\widehat{C_X}(u,[a,b])$ can be computed by

$$\begin{aligned}\widehat{C_X}(u,[a,b]) &= -\frac{1}{\pi}\int_a^b\int_0^{+\infty}\frac{1}{v}\left(\frac{\partial\psi_t}{\partial v}(u,v) - \frac{\partial\psi_t}{\partial v}(u,-v)\right)dv\,dt \\ &= -\frac{1}{\pi}\int_a^b\int_0^{+\infty}\frac{1}{v^2}\left(\psi_t(u,v)+\psi_t(u,-v)-2\psi_t(u,0)\right)dv\,dt.\end{aligned}$$

Proof. Choosing in Equation (5) h of the form $h(x)=\exp(iux)$ for any u real, shows that $\widehat{C_X}(u,[a,b]) = \int_a^b \mathbb{E}\left(e^{iuX(t)}|X'(t)|\right)dt$. Let us now identify the right-hand term. Let $\mu_t(dx,dy)$ denote the law of $(X(t),X'(t))$. Then the joint characteristic function $\psi_t(u,v)$ of $(X(t),X'(t))$ is

$$\psi_t(u,v) = \mathbb{E}\left(\exp(iuX(t)+ivX'(t))\right) = \int_{\mathbb{R}^2} e^{iux+ivy}\mu_t(dx,dy).$$

Since the random vector $(X(t), X'(t))$ has moments of order two, then ψ_t is twice continuously differentiable on $\mathbb{R}^2$. Now, let us consider the integral

$$\begin{aligned} I_A &= \int_0^A \frac{1}{v}\left(\frac{\partial\psi_t}{\partial v}(u,v) - \frac{\partial\psi_t}{\partial v}(u,-v)\right) dv = \int_{v=0}^A \int_{x,y\in\mathbb{R}^2} \frac{iye^{iux+ivy} - iye^{iux-ivy}}{v}\mu_t(dx,dy)\, dv \\ &= -2\int_{v=0}^A \int_{\mathbb{R}^2} ye^{iux}\frac{\sin(vy)}{v}\mu_t(dx,dy)\, dv = -2\int_{\mathbb{R}^2} ye^{iux}\int_{v=0}^{Ay}\frac{\sin(v)}{v}\, dv\mu_t(dx,dy) \end{aligned}$$

The order of integration has been reversed thanks to Fubini's Theorem ($|ye^{iux}\frac{\sin(vy)}{v}| \le y^2$ which is integrable on $[0,A]\times\mathbb{R}^2$ with respect to $dv \times \mu_t(dx,dy)$, since $X'(t)$ is a second order random variable). As A goes to $+\infty$, then $\int_{v=0}^{Ay}\frac{\sin(v)}{v}\, dv$ goes to $\frac{\pi}{2}\mathrm{sign}(y)$, and moreover for all A, x and y, we have $|ye^{iux}\int_{v=0}^{Ay}\frac{\sin(v)}{v}\, dv| \le 3|y|$, thus by Lebesgue's dominated convergence theorem, the limit of $-\frac{1}{\pi}I_A$ exists as A goes to infinity and its value is:

$$\lim_{A\to+\infty} -\frac{1}{\pi}\int_0^A \frac{1}{v}\left(\frac{\partial\psi_t}{\partial v}(u,v) - \frac{\partial\psi_t}{\partial v}(u,-v)\right) dv = \int_{\mathbb{R}^2} |y|e^{iux}\mu_t(dx,dy) = \mathbb{E}\left(e^{iuX(t)}|X'(t)|\right).$$

The second expression in the proposition is simply obtained by integration by parts in the above formula. □

The last expression considerably simplifies when X is a stationary Gaussian process almost surely and mean square continuously differentiable on $\mathbb{R}$. By independence of $X(t)$ and $X'(t)$ we get $\psi_t(u,v) = \phi_X(u)\phi_{X'}(v)$ where ϕ_X, respectively $\phi_{X'}$, denotes the characteristic function of $X(t)$, resp. $X'(t)$ (independent of t by stationarity). Then, the Fourier transform of the mean number of crossings function is given by

$$\widehat{C_X}(u,[a,b]) = -\frac{b-a}{\pi}\phi_X(u)\int_{\mathbb{R}} \frac{1}{v}\frac{\partial\phi_{X'}}{\partial v}(v)\, dv.$$

By the inverse Fourier transform we get a weak Rice's formula

$$C_X(\alpha,[a,b]) = \frac{b-a}{\pi}\left(\frac{m_2}{m_0}\right)^{1/2} e^{-(\alpha-\mathbb{E}(X(0)))^2/2m_0}, \text{ for a.e. } \alpha\in\mathbb{R}, \tag{7}$$

where $m_0 = \mathrm{Var}(X(t))$ and $m_2 = \mathrm{Var}(X'(t))$. Let us quote that in fact Rice's formula holds for all level $\alpha \in \mathbb{R}$ and as soon as X is a.s. continuous (see Exercise 3.2 of [2]) in the sense that $C_X(\alpha,[a,b]) = +\infty$ if $m_2 = +\infty$.

However, in general, the knowledge of $\widehat{C_X}(u,[a,b])$ only allows to get almost everywhere results on $C_X(\alpha,[a,b])$ itself, which can still be used in practice as explained in [25].

2.2. **Mean number of crossings for a given level.** One way to derive results on $C_X(\alpha,[a,b])$ for a given level α is to use Kac's counting formula (see Lemma 3.1 [2]), which we recall now. When X is almost surely continuously differentiable on $[a,b]$ such that for $\alpha\in\mathbb{R}$

$$\mathbb{P}(\exists t\in[a,b] \text{ s.t. } X(t)=\alpha \text{ and } X'(t)=0) = 0 \quad\text{and}\quad \mathbb{P}(X(a)=\alpha) = \mathbb{P}(X(b)=\alpha) = 0, \tag{8}$$

then,

$$N_X(\alpha,[a,b]) = \lim_{\delta\to 0}\frac{1}{2\delta}\int_a^b \mathbf{1}_{|X(t)-\alpha|<\delta}|X'(t)|dt \quad \text{a.s.} \tag{9}$$

The first part of assumption (8) means that the number of tangencies for the level α is 0 almost surely. The following proposition gives a simple criterion to check this.

Proposition 1. *Let $a,b\in\mathbb{R}$ with $a\le b$. Let X be a real valued random process almost surely $\mathcal{C}^2$ on $[a,b]$. Let us assume that there exists $\phi\in L^1(\mathbb{R})$ and $c>0$ such that*

$$\forall t\in[a,b],\ \left|\mathbb{E}\left(e^{iuX(t)}\right)\right| \le c\phi(u).$$

Then,

$$\forall \alpha \in \mathbb{R},\ \ \mathbb{P}\left(\exists t \in [a,b], X(t) = \alpha \text{ and } X'(t) = 0\right) = 0.$$

Proof. Let $M > 0$ and let denote A_M the event corresponding to

$$\max_{t\in[a,b]} \left|X''(t)\right| \leq 2M$$

such that $\mathbb{P}\left(\exists t \in [a,b], X(t) = \alpha, X'(t) = 0\right) = \lim_{M\to+\infty} \mathbb{P}\left(\exists t \in [a,b], X(t) = \alpha, X'(t) = 0, A_M\right)$. Let us assume that there exists $t \in [a,b]$ such that $X(t) = \alpha$ and $X'(t) = 0$. Then for any $n \in \mathbb{N}$ there exists $k_n \in [2^n a, 2^n b] \cap \mathbb{Z}$ such that $|t - 2^{-n}k_n| \leq 2^{-n}$ and, by the first-order Taylor's formula,

$$|X(2^{-n}k_n) - \alpha| \leq 2^{-2n}M. \tag{10}$$

Therefore, let us denote

$$B_n = \bigcup_{k_n\in[2^n a,2^n b]\cap\mathbb{Z}} \left\{|X(2^{-n}k_n) - \alpha| \leq 2^{-2n}M\right\}.$$

Since $(B_n \cap A_M)_{n\in\mathbb{N}}$ is a decreasing sequence we get

$$\mathbb{P}\left(\exists t \in [a,b]; X(t) = \alpha, X'(t) = 0, A_M\right) \leq \lim_{n\to+\infty} \mathbb{P}(B_n \cap A_M).$$

But, according to assumption, for any $n \in \mathbb{N}$ the random variable $X(2^{-n}k_n)$ admits a uniformly bounded density function. Therefore, there exists $c' > 0$ such that

$$\mathbb{P}\left(|X(2^{-n}k_n) - \alpha| \leq 2^{-2n}M\right) \leq c'2^{-2n}M.$$

Hence $\mathbb{P}(B_n \cap A_M) \leq (b - a + 1)c'2^{-n}M$, which yields the result. □

Now taking expectation in (9) gives an upper bound on $C_X(\alpha, [a,b])$, according to Fatou's Lemma:

$$C_X(\alpha, [a,b]) \leq \liminf_{\delta\to 0} \frac{1}{2\delta} \int_a^b \mathbb{E}\left(\mathbf{1}_{|X(t)-\alpha|<\delta}|X'(t)|\right) dt.$$

This upper bound is not very tractable without assumptions on the existence of a bounded joint density for the law of $(X(t), X'(t))$. As far as shot noise processes are concerned, one can exploit the infinite divisibility property by considering the mean number of crossings function of the sum of independent processes. The next proposition gives an upper bound in this setting. Another application of this proposition will be seen in Section 5 where we will decompose a shot noise process into the sum of two independent processes (for which crossings are easy to compute) by partitioning the set of points of the Poisson process.

Proposition 2 (Crossings of a sum of independent processes)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Let $n \geq 2$ and X_j be independent real-valued processes almost surely and mean square two times continuously differentiable on $[a,b]$ for $1 \leq j \leq n$. Assume that there exist constants c_j and probability measures $d\mu_j$ on $\mathbb{R}$ such that if $dP_{X_j(t)}$ denotes the law of $X_j(t)$, then*

$$\forall t \in [a,b],\ \ dP_{X_j(t)} \leq c_j d\mu_j,\ \ \text{for } 1 \leq j \leq n.$$

Let X be the process obtained by $X = \sum_{j=1}^{n} X_j$ and assume that X satisfies (8) for $\alpha \in \mathbb{R}$. Then

$$C_X(\alpha, [a,b]) \leq \sum_{j=1}^{n} \left(\prod_{i\neq j} c_i\right) (C_{X_j'}(0, [a,b]) + 1). \tag{11}$$

Moreover, in the case where all the X_j are stationary on $\mathbb{R}$:

$$C_X(\alpha, [a,b]) \leq \sum_{j=1}^{n} C_{X_j'}(0, [a,b]).$$

Proof. We first need an elementary result. Let f be a C^1 function on $[a,b]$, then for all $\delta > 0$, and for all $x \in \mathbb{R}$, we have:

$$\frac{1}{2\delta}\int_a^b \mathbf{1}_{|f(t)-x|\leq\delta}|f'(t)|\,dt \leq N_{f'}(0,[a,b])+1. \tag{12}$$

This result (that can be found as an exercise at the end of Chapter 3 of [2]) can be proved this way: let $a_1 < \ldots < a_n$ denote the points at which $f'(t) = 0$ in $[a,b]$. On each interval $[a,a_1]$, $[a_1,a_2]$, ..., $[a_n,b]$, f is monotonic and thus $\int_{a_i}^{a_{i+1}} \mathbf{1}_{|f(t)-x|\leq\delta}|f'(t)|\,dt \leq 2\delta$. Summing up these integrals, we have the announced result.

For the process X, since it satisfies the conditions of Kac's formula (8), by (9) and Fatou's Lemma,

$$C_X(\alpha,[a,b]) \leq \liminf_{\delta\to 0}\frac{1}{2\delta}\int_a^b \mathbb{E}(\mathbf{1}_{|X(t)-\alpha|\leq\delta}|X'(t)|)\,dt.$$

Now, for each $\delta > 0$, we have $\mathbb{E}(\mathbf{1}_{|X(t)-\alpha|\leq\delta}|X'(t)|) \leq \sum_{j=1}^{n}\mathbb{E}(\mathbf{1}_{|X_1(t)+\ldots+X_n(t)-\alpha|\leq\delta}|X_j'(t)|)$. Then, thanks to the independence of $X_1,\ldots,X_n$ and to the bound on the laws of $X_j(t)$, we get:

$$\begin{aligned}
&\int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+\ldots+X_n(t)-\alpha|\leq\delta}|X_1'(t)|)\,dt\\
=\ &\int_a^b\int_{\mathbb{R}^{n-1}} \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\ldots+x_n-\alpha|\leq\delta}|X_1'(t)|\,|\,X_2(t)=x_2,\ldots,X_n(t)=x_n)\,dP_{X_2(t)}(x_2)\ldots dP_{X_n(t)}(x_n)\,dt\\
\leq\ &\left(\prod_{j=2}^{n} c_j\right)\int_{\mathbb{R}^{n-1}}\int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\ldots+x_n-\alpha|\leq\delta}|X_1'(t)|)\,dt\,d\mu_2(x_2)\ldots d\mu_n(x_n).
\end{aligned}$$

Now, (12) holds almost surely for X_1, taking expectation we get

$$\frac{1}{2\delta}\int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\ldots+x_n-\alpha|\leq\delta}|X_1'(t)|)\,dt \leq C_{X_1'}(0,[a,b])+1.$$

Using the fact the $d\mu_j$ are probability measures we get

$$\frac{1}{2\delta}\int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+\ldots+X_n(t)-\alpha|\leq\delta}|X_1'(t)|)\,dt \leq \left(\prod_{j=2}^{n} c_j\right)(C_{X_1'}(0,[a,b])+1).$$

We obtain similar bounds for the other terms. Since this holds for all $\delta > 0$, we have the bound (11) on the expected number of crossings of the level α by the process X.

When the X_j are stationary, things become simpler: we can take $c_j = 1$ for all $1 \leq j \leq n$, and also by stationarity we have that for all $p \geq 1$ integer: $C_X(\alpha,[a,b+p(b-a)]) = (p+1)C_X(\alpha,[a,b])$. Now using (11) for all p, then dividing by $(p+1)$, we have that for all p: $C_X(\alpha,[a,b]) \leq \sum_{j=1}^{n} C_{X_j'}(0,[a,b]) + \frac{n}{p+1}$. Finally, letting p go to infinity, we have the result. □

As previously seen, taking the expectation in Kac's formula only allows us to get an upper bound for C_X. However, under stronger assumptions (see Theorem 2 of [18]), one can justify the interversion of the limit and the expectation. In particular one has to assume that $(X(t),X'(t))$ admits a density p_t continuous in a neighborhood of $\{\alpha\}\times\mathbb{R}$. The Rice's formula states that

$$C_X(\alpha,[a,b]) = \int_a^b\int_{\mathbb{R}} |z|p_t(\alpha,z)\,dz\,dt < +\infty, \tag{13}$$

such that, under appropriate assumptions, one can prove that the mean number of crossings function $\alpha \mapsto C_X(\alpha,[a,b])$ is continuous on $\mathbb{R}$.

3. Crossings of smooth shot noise processes

From now, we focus on a shot noise process X given by the formal sum (1), which can also be written as the stochastic integral

$$X(t) = \int_{\mathbb{R}\times\mathbb{R}} zg(t-s)N(ds,dz), \tag{14}$$

where N is a Poisson random measure of intensity $\lambda\nu(ds)F(dz)$, with F the law of the impulse β (see [17] chapter 10 for instance). We focus in this paper on stationary shot noise processes for which $\nu(ds) = ds$ is the Lebesgue measure. Such processes are obtained as the almost sure limit of truncated shot noise processes defined for $\nu_T(ds) = \mathbf{1}_{[-T,T]}(s)ds$, as T tends to infinity. Therefore, from now on and in all the paper, the measure $\nu(ds)$ is the Lebesgue measure ds or the measure $\nu_T(ds)$. Then, assuming that the random impulse β is an integrable random variable of $L^1(\Omega)$ and that the kernel function g is an integrable function of $L^1(\mathbb{R})$, is enough to ensure the almost sure convergence of the infinite sum (see also Campbell's Theorem and [15]). When moreover $\beta \in L^2(\Omega)$ and $g \in L^2(\mathbb{R})$ the process X defines a second order process.

3.1. **Regularity and Fourier transform of the mean number of crossings function.** Under further regularity assumptions on the kernel function we obtain the following sample paths regularity for the shot noise process itself.

Proposition 3. *Let $\beta \in L^2(\Omega)$. Let $g \in \mathcal{C}^2(\mathbb{R})$ such that $g, g', g'' \in L^1(\mathbb{R})$. Then X is almost surely and mean square continuously differentiable on $\mathbb{R}$ with*

$$X'(t) = \sum_i \beta_i g'(t-\tau_i),\ \forall t \in \mathbb{R}.$$

Proof. Let $A > 0$ and remark that for any $s \in \mathbb{R}$ and $|t| \le A$, since $g \in \mathcal{C}^1(\mathbb{R})$,

$$|g(t-s)| = \left|\int_0^t g'(u-s)du + g(-s)\right| \le \int_{-A}^{A} |g'(u-s)|du + |g(-s)|,$$

such that by Fubini's theorem, since $g, g' \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \sup_{t\in[-A,A]} |g(t-s)|ds \le 2A\int_{\mathbb{R}} |g'(s)|ds + \int_{\mathbb{R}} |g(s)|ds < +\infty.$$

Therefore, since $\beta \in L^1(\Omega)$, the series $\sum_i \beta_i \sup_{t\in[-A,A]} |g(t-\tau_i)|$ converges almost surely which means that $\sum_i \beta_i g(\cdot - \tau_i)$ converges uniformly on $[-A, A]$ almost surely. This implies that the sample paths of X are almost surely continuous on $\mathbb{R}$. Similarly, since $g' \in \mathcal{C}^1(\mathbb{R})$ and $g', g'' \in L^1(\mathbb{R})$, almost surely the series $\sum_i \beta_i g'(\cdot - \tau_i)$ converges uniformly on $[-A, A]$ and therefore X is continuously differentiable on $[-A, A]$ with $X'(t) = \sum_i \beta_i g'(t-\tau_i)$ for all $t \in [-A, A]$. Note that the same holds true on $[-A+n, A+n]$ for any $n \in \mathbb{Z}$, which concludes for the almost sure continous differentiability on $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[-A+n, A+n]$.
Now, let us be concerned with the mean square continuous differentiability. First, $g, g' \in L^1(\mathbb{R})$ implies that $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ such that X is a second order process since $\beta \in L^2(\Omega)$. Its covariance function is given by $S(t,t') = \mathrm{Cov}\,(X(t), X(t')) = \lambda\mathbb{E}(\beta^2)\int_{\mathbb{R}} g(t-s)g(t'-s)\nu(ds)$. Similarly we also have that $g' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and X' is a second order process. According to Theorem 2.2.2 of [1] it is sufficient to remark that assumptions on g ensure that $\frac{\partial^2 S}{\partial t\partial t'}$ exists and is finite at any point $(t,t) \in \mathbb{R}^2$ with $\frac{\partial^2 S}{\partial t\partial t'}(t,t) = \lambda\mathbb{E}(\beta^2)\int_{\mathbb{R}} g'(t-s)g'(t-s)\nu(ds)$. Therefore, for all $t \in \mathbb{R}$, the limit $\lim_{h\to 0}\frac{X(t+h)-X(t)}{h}$ exists in $L^2(\Omega)$ and is equal to $X'(t)$ by unicity. Moreover, the covariance function of X' is given by $(t,t') \mapsto \lambda\mathbb{E}(\beta^2)\int_{\mathbb{R}} g'(t-s)g'(t'-s)\nu(ds)$. ◻

Iterating this result one can obtain higher order smoothness properties. In particular it is straightforward to obtain the following result for Gaussian kernels.

Example (Gaussian kernel): Let $\beta \in L^2(\Omega)$, $g(t) = g_1(t) = \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)$ and X given by (1). Then, the process X is almost surely and mean square smooth on $\mathbb{R}$. Moreover, for any $n \in \mathbb{N}$,

$$\forall t \in \mathbb{R},\ \ X^{(n)}(t) = \sum_i \beta_i g_1^{(n)}(t-\tau_i) = \sum_i \beta_i (-1)^n H_n(t-\tau_i) g_1(t-\tau_i)\,,$$

where H_n is the Hermite polynomial of order n.

From now on, in order to work with almost sure and mean square continuously differentiable process, we make the following assumption :

$$(\mathbf{A}) \qquad g \in \mathcal{C}^2(\mathbb{R}) \text{ with } g, g', g'' \in L^1(\mathbb{R}).$$

Therefore, choosing $\beta \in L^2(\Omega)$, the shot noise process X satisfies the assumptions of Theorem 1 such that the Fourier transform of its mean number of crossings function can be written with respect to ψ_t, the joint characteristic function of $(X(t), X'(t))$, given by (see Lemma 10.2 of [17] for instance)

$$(15) \qquad \forall u, v \in \mathbb{R},\ \psi_t(u,v) = \mathbb{E}(e^{iuX(t)+vX'(t)}) = \exp\left(\int_{\mathbb{R}\times\mathbb{R}} [e^{iz(ug(t-s)+vg'(t-s))} - 1]\,\lambda\nu(ds)F(dz)\right)$$

In order to get stronger results on the mean number of crossings function we first have to investigate the existence of a density when considering a shot noise process X, or more precisely a shot noise vector-valued process (X, X'). Then we consider a $\mathbb{R}^d$-valued shot noise process given on $\mathbb{R}$ by

$$(16) \qquad Y(t) = \sum_i \beta_i h(t-\tau_i),$$

where $h : \mathbb{R} \mapsto \mathbb{R}^d$ is a given (deterministic) measurable vectorial function in $L^1(\mathbb{R})$. In this setting one can recover X given by (1) with $d = 1$ and $h = g$, or recover (X, X') -if it exists- with $d = 2$ and $h = (g, g')$. It will be particularly helpfull to see Y as the almost sure limit of a truncated shot noise process Y_T defined for $\nu_T(ds) = \mathbf{1}_{[-T,T]}(s)ds$, as $T > 0$ tends to infinity. Therefore, from now on and in all the paper, we use the following notations.

Notations. For any $T > 0$, we denote by Y_T, respectively X_T when $d = 1$, the shot noise process given by (16), respectively (1), obtained for $\nu_T(ds) = \mathbf{1}_{[-T,T]}(s)ds$. We simply denote by Y, respectively X when $d = 1$, the shot noise process obtained for ν the Lebesgue measure.

3.2. **Existence of a density and continuity of the mean number of crossings function.** Let us remark that for $d \geq 1$ and $T > 0$, the shot noise process Y_T satisfies

$$(17) \qquad Y_T(\cdot) = \sum_{|\tau_i|\leq T} \beta_i h(\cdot - \tau_i) \stackrel{fdd}{=} \sum_{i=1}^{\gamma_T} \beta_i h(\cdot - U_T^{(i)}),$$

where

$$(18) \qquad \gamma_T = \#\{i; \tau_i \in [-T,T]\}$$

is a Poisson random variable of parameter $\lambda\nu_T(\mathbb{R}) = 2\lambda T$ and $\{U_T^{(i)}\}$ are i.i.d. with uniform law on $[-T, T]$ independent from γ_T and $\{\beta_i\}$. Here and in the sequel the convention is that $\sum_{i=1}^{0} = 0$.
Moreover, for any $M > T$, one can write Y_M as the sum of two independent processes Y_T and $Y_M - Y_T$ such that the existence of a density for the random vector $Y_T(t)$ implies the existence of a density for the random vector $Y_M(t)$ and therefore for $Y(t)$. Note also that by stationarity $Y(s)$ will also admit a density for any $s \in \mathbb{R}$. Such a remark can be used for instance to establish an integral equation to compute or approximate the density in some examples [21, 20, 14]. However the shot noise process may not have a density. For example, when h has compact support, there exists $A > 0$

such that $h(s) = 0$ for $|s| > A$. Then, for any $T \geq A$, we get $X_T(0) = X_A(0) = X(0)$ such that $\mathbb{P}(X_T(0) = 0) = \mathbb{P}(X(0) = 0) \geq \mathbb{P}(\gamma_A = 0) > 0$, which proves that $X_T(0)$ and $X(0)$ don't have a density. Such a behavior is extremely linked to the number of points of the Poisson process $\{\tau_i\}$ that are thrown in the interval of study. Therefore, by conditioning we obtain the following criterion.

Proposition 4. *If there exists $m \geq 1$ such that for all $T > 0$ large enough, conditionally on $\{\gamma_T = m\}$, the random variable $Y_T(0)$ admits a density, then, conditionally on $\{\gamma_T \geq m\}$, the random variable $Y_T(0)$ admits a density. Moreover, $Y(0)$ admits a density.*

Proof. Let $T > 0$ be large enough. First, let us remark that conditionally on $\{\gamma_T = m\}$, $Y_T(0) \stackrel{d}{=} \sum_{i=1}^{m} \beta_i h(U_T^{(i)})$. Next, notice that if a random vector V in $\mathbb{R}^d$ admits a density f_V then, for U_T with uniform law on $[-T, T]$ and β with law F, independent of V, the random vector $W = V + \beta h(U_T)$ admits $w \in \mathbb{R}^d \mapsto \frac{1}{2T} \int_{\mathbb{R}} \int_{-T}^{T} f_V(w - zh(t)) dt F(dz)$ for density. Therefore, by induction the assumption implies that $\sum_{i=1}^{n} \beta_i h(U_T^{(i)})$ has a density, for any $n \geq m$. This proves that, conditionally on $\{\gamma_T \geq m\}$, the random variable $Y_T(0)$ admits a density.
To prove that $Y(0)$ admits a density, we follow the same lines as in [3], proof of Proposition A.2. Let $A \subset \mathbb{R}^d$ be a Borel set with Lebesgue measure 0, since $Y_T(0)$ and $Y(0) - Y_T(0)$ are independent

$$\mathbb{P}(Y(0) \in A) = \mathbb{P}(Y_T(0) + (Y(0) - Y_T(0)) \in A) = \int_{\mathbb{R}^d} \mathbb{P}(Y_T(0) \in A - y) \mu_T(dy).$$

with μ_T the law of $Y(0) - Y_T(0)$. But for any $y \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{P}(Y_T(0) \in A - y) &= \mathbb{P}\left(\sum_{i=1}^{\gamma_T} \beta_i h(U_T^{(i)}) \in A - y \right) \\ &= \sum_{n=0}^{+\infty} \mathbb{P}\left(\sum_{i=1}^{\gamma_T} \beta_i h(U_T^{(i)}) \in A - y \,|\, \gamma_T = n \right) \mathbb{P}(\gamma_T = n) \\ &= \sum_{n=0}^{m-1} \mathbb{P}\left(\sum_{i=1}^{n} \beta_i h(U_T^{(i)}) \in A - y \right) \mathbb{P}(\gamma_T = n), \end{aligned}$$

since $A - y$ has Lebesgue measure 0 and $\sum_{i=1}^{n} \beta_i (h(U_T^{(i)}))$ has a density for any $n \geq m$. Hence, for any $T > 0$ large enough,

$$\mathbb{P}(Y(0) \in A) \leq \mathbb{P}(\gamma_T \leq m - 1).$$

Letting $T \to +\infty$ we conclude that $\mathbb{P}(Y(0) \in A) = 0$ such that $Y(0)$ admits a density. □

Let us emphasize that $Y_T(0)$ does not admit a density since $\mathbb{P}(Y_T(0) = 0) \geq \mathbb{P}(\gamma_T = 0) > 0$. Let us also mention that Breton in [6] gives a similar assumption for real-valued shot noise series in his Proposition 2.1. In particular his Corollary 2.1. can be adapted in our vector-valued setting.

Corollary 1. *Let $h : \mathbb{R} \mapsto \mathbb{R}^d$ be an integrable function and $\beta = 1$ a.s. Let us define $h_d : \mathbb{R}^d \mapsto \mathbb{R}^d$ by $h_d(x) = h(x_1) + \ldots + h(x_d)$, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. If the h_d image measure of the d-dimensional Lebesgue measure is absolutely continuous with respect to the d-dimensional Lebesgue measure then the random vector $Y(0)$, given by (16), admits a density.*

Proof. Let $A \subset \mathbb{R}^d$ a Borel set with Lebesgue measure 0 then the assumptions ensure that $\int_{\mathbb{R}^d} \mathbf{1}_{h_d(x) \in A} dx = 0$. Therefore, for any $T > 0$, using the notations of Proposition 4,

$$\mathbb{P}\left(\sum_{i=1}^{d} h(U_T^{(i)}) \in A \right) = \frac{1}{(2T)^d} \int_{[-T,T]^d} \mathbf{1}_{h_d(x) \in A} dx = 0.$$

Hence $\sum_{i=1}^{d} h(U_T^{(i)})$ admits a density and Proposition 4 gives the conclusion. $\square$

Example (Gaussian kernel): let $g(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$, $\beta = 1$ a.s. and X given by (1). Let us consider $h = (g, g')$ and $h_2 : (x_1, x_2) \in \mathbb{R}^2 \mapsto h(x_1) + h(x_2)$. The Jacobian of h_2 is

$$J(h_2)(x_1, x_2) = \frac{1}{2\pi}(1 + x_1x_2)(x_1 - x_2) \exp(-(x_1^2 + x_2^2)/2).$$

Hence, the h_2 image measure of the 2-dimensional Lebesgue measure is absolutely continuous with respect to the 2-dimensional Lebesgue measure. Then, for any $t \in \mathbb{R}$, the law of the random vector $(X(t), X'(t))$ is absolutely continuous with respect to the Lebesgue measure. Note that in particular this implies the existence of a density for $X(t)$. However this density is not bounded (and therefore not continuous) in a neighborhood of 0 as proved in the following proposition.

Proposition 5. *Let assume for sake of simplicity that $\beta = 1$ a.s. and let g denote the kernel function of the shot noise process. Then,*

(1) *If g is such that there exist $\alpha > 1$ and $A > 0$ such that $\forall |s| > A$, $|g(s)| \leq e^{-|s|^\alpha}$, then $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon < \varepsilon_0$:*

$$\mathbb{P}(|X(t)| \leq \varepsilon) \geq \frac{1}{2} e^{-2\lambda T_\varepsilon} \text{ where } T_\varepsilon \text{ is defined by } T_\varepsilon = (-\log \varepsilon)^{1/\alpha}.$$

(2) *If g is such that there exists $A > 0$ such that $\forall |s| > A$, $|g(s)| \leq e^{-|s|}$ and if $\lambda < 1/4$ then $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon < \varepsilon_0$:*

$$\mathbb{P}(|X(t)| \leq \varepsilon) \geq \left(1 - \frac{\lambda}{(1 - 2\lambda)^2}\right) e^{-2\lambda T_\varepsilon} \text{ where } T_\varepsilon \text{ is defined by } T_\varepsilon = -\log \varepsilon.$$

This implies in both cases that $\mathbb{P}(|X(t)| \leq \varepsilon)/\varepsilon$ goes to $+\infty$ as ε goes to 0, and thus the density of $X(t)$ -if it exists- is not bounded in a neighborhood of 0.

Proof. We start with the first case. Let $\varepsilon > 0$ and let $T_\varepsilon = (-\log \varepsilon)^{1/\alpha}$. Assume that ε is small enough to have $T_\varepsilon > A$. We have by definition $X(t) \stackrel{d}{=} X(0) \stackrel{d}{=} \sum_i g(\tau_i)$. If we denote $X_{T_\varepsilon}(0) = \sum_{|\tau_i| \leq T_\varepsilon} g(\tau_i)$ and $R_{T_\varepsilon}(0) = \sum_{|\tau_i| > T_\varepsilon} g(\tau_i)$, then $X_{T_\varepsilon}(0)$ and $R_{T_\varepsilon}(0)$ are independent and $X(0) = X_{T_\varepsilon}(0) + R_{T_\varepsilon}(0)$. We also have: $\mathbb{P}(|X(0)| \leq \varepsilon) \geq \mathbb{P}(|X_{T_\varepsilon}(0)| = 0 \text{ and } |R_{T_\varepsilon}(0)| \leq \varepsilon) = \mathbb{P}(|X_{T_\varepsilon}(0)| = 0) \times \mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon)$. Now, on the one hand, we have: $\mathbb{P}(|X_{T_\varepsilon}(0)| = 0) \geq \mathbb{P}(\text{ there are no } \tau_i \text{ in } [-T_\varepsilon, T_\varepsilon]) = e^{-2\lambda T_\varepsilon}$. On the other hand, the first moments of the random variable $R_{T_\varepsilon}(0)$ are given by: $\mathbb{E}(R_{T_\varepsilon}(0)) = \lambda \int_{|s|>T_\varepsilon}^{+\infty} g(s)\, ds$ and $\mathrm{Var}(R_{T_\varepsilon}(0)) = \lambda \int_{|s|>T_\varepsilon}^{+\infty} g^2(s)\, ds$. Now, we use the following inequality on the tail of $\int e^{-s^\alpha}$:

$$\forall T > 0,\ e^{-T^\alpha} = \int_T^{+\infty} \alpha s^{\alpha-1} e^{-s^\alpha}\, ds \geq \alpha T^{\alpha-1} \int_T^{+\infty} e^{-s^\alpha}\, ds.$$

Thus, we obtain bounds for the tail of $\int g$ and of $\int g^2$:

$$\int_T^{+\infty} e^{-s^\alpha}\, ds \leq \frac{e^{-T^\alpha}}{\alpha T^{\alpha-1}} \text{ and } \int_T^{+\infty} \left(e^{-s^\alpha}\right)^2 ds \leq \frac{e^{-2T^\alpha}}{2\alpha T^{\alpha-1}}.$$

Back to the moments of $R_{T_\varepsilon}(0)$, since $T_\varepsilon = (-\log \varepsilon)^{1/\alpha}$ we have:

$$|\mathbb{E}(R_{T_\varepsilon}(0))| \leq \frac{2\lambda\varepsilon}{\alpha T_\varepsilon^{\alpha-1}} \text{ and } \mathrm{Var}(R_{T_\varepsilon}(0)) \leq \frac{\lambda\varepsilon^2}{\alpha T_\varepsilon^{\alpha-1}}.$$

We can take ε small enough in such a way that we can assume that $|\mathbb{E}(R_{T_\varepsilon}(0))| < \varepsilon$. Then, using Chebyshev's inequality, we have

$$\begin{aligned}\mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon) &= \mathbb{P}(-\varepsilon - \mathbb{E}(R_{T_\varepsilon}(0)) \leq R_{T_\varepsilon}(0) - \mathbb{E}(R_{T_\varepsilon}(0)) \leq \varepsilon - \mathbb{E}(R_{T_\varepsilon}(0))) \\ &\geq 1 - \mathbb{P}(|R_{T_\varepsilon}(0) - \mathbb{E}(R_{T_\varepsilon}(0))| \geq \varepsilon - |\mathbb{E}(R_{T_\varepsilon}(0))|) \\ &\geq 1 - \frac{\mathrm{Var}(R_{T_\varepsilon}(0))}{(\varepsilon - |\mathbb{E}(R_{T_\varepsilon}(0))|)^2} \geq 1 - \frac{\lambda}{\alpha T_\varepsilon^{\alpha-1}(1 - 2\lambda/\alpha T_\varepsilon^{\alpha-1})^2},\end{aligned}$$

which is larger than $1/2$ for T_ε large enough (i.e. for ε small enough).

For the second case, we can make exactly the same computations by setting $\alpha = 1$, and get $\mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon) \geq 1 - \lambda/(1-2\lambda)^2$, which is > 0 when $\lambda < 1/4$. □

Such a feature is particularly bothersome when considering crossings of these processes since most of known results are based on the existence of a bounded density for each marginal of the process. However this is again linked to the number of points of the Poisson process $\{\tau_i\}$ that are thrown in the interval of study. By conditioning, the characteristic functions are proved to be integrable such that conditional laws have continuous bounded densities. The main tool is Proposition 10, postponed in Appendix, established using the classical stationary phase estimate for oscillatory integrals (see [26] for example).

Proposition 6. *Let assume for sake of simplicity that $\beta = 1$ a.s., let $T > 0$, $a < b$, and assume that $g \in L^1(\mathbb{R})$ is a function of class $\mathcal{C}^2$ on $[-T+a, T+b]$ such that*

$$(19)\quad m = \min_{s\in[-T+a,T+b]} \sqrt{g'(s)^2 + g''(s)^2} > 0 \text{ and } n_0 = \#\{s \in [-T+a, T+b] \text{ s. t. } g''(s) = 0\} < +\infty.$$

Then, conditionally on $\{\gamma_T \geq k_0\}$ with $k_0 \geq 3$, for all $t \in [a,b]$ and $M \geq T$, the law of $X_M(t)$ admits a continuous bounded density. Therefore, for any $t \in \mathbb{R}$, the law of $X(t)$, conditionally on $\{\gamma_T \geq k_0\}$, admits a continuous bounded density.

Proof. Actually, we will prove that conditionally on $\{\gamma_T \geq k_0\}$, the law of the truncated process $X_T(t) = \sum_{|\tau_i|\leq T} g(t - \tau_i)$ admits a continuous bounded density for $t \in [a,b]$. The result will follow, using the fact that for $M \geq T$, $X_M(t) = X_T(t) + (X_M(t) - X_T(t))$, with $X_M(t) - X_T(t)$ independent of $X_T(t)$. So let us denote ψ_{t,k_0}^T the characteristic function of $X_T(t)$ conditionally on $\{\gamma_T \geq k_0\}$. Then, for all $u \in \mathbb{R}$, we get

$$\begin{aligned}\psi_{t,k_0}^T(u) &= \frac{1}{\mathbb{P}(\gamma_T \geq k_0)} \sum_{k\geq k_0} \mathbb{E}\left(e^{iuX_T(t)} | \gamma_T = k\right) \mathbb{P}(\gamma_T = k) \\ &= \frac{1}{\mathbb{P}(\gamma_T \geq k_0)} \sum_{k\geq k_0} \left(\frac{1}{2T}\int_{-T}^{T} e^{iug(t-s)} ds\right)^k e^{-2\lambda T} \frac{(2\lambda T)^k}{k!}\end{aligned}$$

Therefore,

$$(20)\quad \left|\psi_{t,k_0}^T(u)\right| \leq (2T)^{-k_0} \left|\int_{-T+t}^{T+t} e^{iug(s)} ds\right|^{k_0}.$$

Hence, using Proposition 10 on $[-T+t, T+t] \subset [-T+a, T+b]$, one can find C a positive constant that depends on T, k_0, λ, m and n_0 such that for any $|u| > 1/m$

$$\left|\psi_{t,k_0}^T(u)\right| \leq C|u|^{-k_0/2}.$$

Then, since $k_0 \geq 3$, ψ_{t,k_0}^T is integrable on $\mathbb{R}$ and thanks to Fourier inverse Theorem it is the characteristic function of a bounded continuous density. □

Using similar ideas we obtain the following result concerning the continuity of the mean number of crossings function.

Theorem 2. *Assume for sake of simplicity that $\beta = 1$ a.s. and that g is a function of class $\mathcal{C}^4$ on $\mathbb{R}$ satisfying (**A**). Let $T > 0$, $a \le b$, and assume that for all $s \in [-T+a, T+b]$, the matrice $\Phi(s) = \begin{pmatrix} g'(s) & g''(s) \\ g''(s) & g^{(3)}(s) \end{pmatrix}$ and its component-wise derivative $\Phi'(s) = \begin{pmatrix} g''(s) & g^{(3)}(s) \\ g^{(3)}(s) & g^{(4)}(s) \end{pmatrix}$ are invertible. Then, conditionally on $\{\gamma_T \ge k_0\}$ with $k_0 \ge 8$, for all $M \ge T$, the mean number of crossings function $\alpha \mapsto \mathbb{E}\left(N_{X_M}(\alpha,[a,b])|\gamma_T \ge k_0\right)$ is continuous on $\mathbb{R}$. Moreover*

$$\mathbb{E}\left(N_{X_M}(\alpha,[a,b])|\gamma_T \ge k_0\right) \underset{M\to+\infty}{\longrightarrow} \mathbb{E}\left(N_{X}(\alpha,[a,b])|\gamma_T \ge k_0\right),$$

uniformly on $\alpha \in \mathbb{R}$.

Proof. The result follows from Rice's formula. To establish it we use Theorem 2 of [18] and thus we have to check assumptions *i)* to *iii)* related to joint densities. Let $t \in [a,b]$ and $M \ge T$. We write $X_M(t) = X_T(t) + (X_M(t) - X_T(t))$ with $X_M - X_T$ independent of X_T. We adopt the convention that $X_\infty = X$. Let us write for $M \in [T, +\infty]$ and ε small enough

$$\psi_{t,\varepsilon,k_0}^M = \psi_{t,\varepsilon,k_0}^T \psi_{t,\varepsilon}^{T,M}$$

with $\psi_{t,\varepsilon,k_0}^M$ the characteristic function of $(X_M(t), (X_M(t+\varepsilon) - X_M(t))/\varepsilon)$, conditionally on $\{\gamma_T \ge k_0\}$. Note that, $X_M - X_T$ is independent of γ_T such that $\psi_{t,\varepsilon}^{T,M}$ is just the characteristic function of $(X_M(t) - X_T(t), ((X_M - X_T)(t+\varepsilon) - (X_M - X_T)(t))/\varepsilon)$. First we prove that there exists $C > 0$ such that, for all $0 \le j \le 3$, for all $M \ge T$ and $\varepsilon > 0$ small enough,

$$\left|\frac{\partial^j}{\partial v^j}\psi_{t,\varepsilon,k_0}^M(u,v)\right| \le C(1+\sqrt{u^2+v^2})^{-(k_0-3)/2}. \tag{21}$$

Let us remark that, since $g', g'' \in L^1(\mathbb{R})$ by (**A**), one has $g, g' \in L^\infty(\mathbb{R})$. It implies in particular that $g, g' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^3(\mathbb{R})$ such that the above partial derivatives exist. Moreover, by Leibniz formula, for $0 \le j \le 3$, one has

$$\frac{\partial^j}{\partial v^j}\psi_{t,\varepsilon,k_0}^M(u,v) = \sum_{l=0}^{j} \begin{pmatrix} j \\ l \end{pmatrix} \frac{\partial^l}{\partial v^l}\psi_{t,\varepsilon,k_0}^T(u,v)\frac{\partial^{j-l}}{\partial v^{j-l}}\psi_{t,\varepsilon}^{T,M}(u,v). \tag{22}$$

On the one hand

$$\left|\frac{\partial^{j-l}}{\partial v^{j-l}}\psi_{t,\varepsilon}^{T,M}(u,v)\right| \le \mathbb{E}\left(\left|\frac{(X_M - X_T)(t+\varepsilon) - (X_M - X_T)(t)}{\varepsilon}\right|^{j-l}\right),$$

with

$$\left|\frac{(X_M - X_T)(t+\varepsilon) - (X_M - X_T)(t)}{\varepsilon}\right| \le \sum_{T<|\tau_i|\le M} |g_\varepsilon(t-\tau_i)|,$$

where $g_\varepsilon(s) = \frac{1}{\varepsilon}\int_0^\varepsilon g'(s+x)dx$ is such that $g_\varepsilon \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|g_\varepsilon\|_\infty \le \|g'\|_\infty$ and $\|g_\varepsilon\|_1 \le \|g'\|_1$. Then using the moment formula established in [5], one can find $c > 0$ such that for all $0 \le j \le 3$, with $(j-1)_+ = \max(0, j-1)$,

$$\left|\frac{\partial^j}{\partial v^j}\psi_{t,\varepsilon}^{T,M}(u,v)\right| \le \mathbb{E}\left(\left(\sum_{T<|\tau_i|\le M}|g_\varepsilon(t-\tau_i)|\right)^j\right) \le c\max(1,\|g'\|_\infty)^{(j-1)_+}\max(1,\lambda\|g'\|_1)^j. \tag{23}$$

On the other hand,

$$\begin{aligned}\mathbb{P}(\gamma_T \ge k_0)\psi_{t,\varepsilon,k_0}^T(u,v) &= \sum_{k\ge k_0}\mathbb{E}\left(e^{iuX_T(t)+iv(X_T(t+\varepsilon)-X_T(t))/\varepsilon}|\gamma_T = k\right)\mathbb{P}(\gamma_T = k) \\ &= \sum_{k\ge k_0}\chi_{t,\varepsilon}^T(u,v)^k\mathbb{P}(\gamma_T = k)\end{aligned}$$

where

$$\chi_{t,\varepsilon}^T(u,v) = (2T)^{-1}\int_{-T+t}^{T+t} e^{iug(s)+ivg_\varepsilon(s)}ds,$$

is the characteristic function of $(g(t-U_T), g_\varepsilon(t-U_T))$, with U_T a uniform random variable on $[-T,T]$. It follows that $\left|\chi^T_{t,\varepsilon}(u,v)\right| \le 1$, so that one can find $c>0$ such that for all $0 \le j \le 3$,

$$\left|\frac{\partial^j}{\partial v^j}\psi^T_{t,\varepsilon,k_0}(u,v)\right| \le c\max(1,\|g'\|_\infty)^{(j-1)_+}\max(1,\lambda\|g'\|_1)^j\frac{\mathbb{P}(\gamma_T \ge k_0-j)}{\mathbb{P}(\gamma_T\ge k_0)}\left|\chi^T_{t,\varepsilon}(u,v)\right|^{k_0-j}.$$

This, together with (23) and (22), implies that one can find $c>0$ such that for all $0\le j\le 3$,

$$\left|\frac{\partial^j}{\partial v^j}\psi_{t,\varepsilon,k_0}(u,v)\right| \le c\max(1,\|g'\|_\infty)^{(j-1)_+}\max(1,\lambda\|g'\|_1)^j\frac{\mathbb{P}(\gamma_T \ge k_0-j)}{\mathbb{P}(\gamma_T\ge k_0)}\left|\chi^T_{t,\varepsilon}(u,v)\right|^{k_0-j}. \tag{24}$$

Moreover, let $\Phi_\varepsilon(s) = \begin{pmatrix} g'(s) & g'_\varepsilon(s) \\ g''(s) & g''_\varepsilon(s)\end{pmatrix}$ and $\Phi'_\varepsilon(s) = \begin{pmatrix} g''(s) & g''_\varepsilon(s) \\ g^{(3)}(s) & g^{(3)}_\varepsilon(s)\end{pmatrix}$. Then $\det\Phi_\varepsilon(s)$ converges to $\det\Phi(s)$ as $\varepsilon\to 0$, uniformly in $s\in[-T-a,T+b]$. The assumption on Φ ensures that one can find ε_0 such that for $\varepsilon\le\varepsilon_0$, the matrix $\Phi_\varepsilon(s)$ is invertible for all $s\in[-T-a,T+b]$. The same holds true for $\Phi'_\varepsilon(s)$. Denote $m = \min\limits_{s\in[-T-a,T+b],\varepsilon\le\varepsilon_0} \| \Phi_\varepsilon(s)^{-1}\|^{-1} > 0$, where $\|\cdot\|$ is the matricial norm induced by the Euclidean one. According to Proposition 10 with $n_0=0$,

$$\forall (u,v)\in\mathbb{R}^2 \text{ s. t. } \sqrt{u^2+v^2} > \frac{1}{m},\ \left|\chi^T_{t,\varepsilon}(u,v)\right| = (2T)^{-1}\left|\int_{-T+t}^{T+t} e^{iug(s)+ivg_\varepsilon(s)}ds\right| \le \frac{24\sqrt{2}}{\sqrt{m\sqrt{u^2+v^2}}}.$$

Therefore, one can find a constant $c_{k_0}>0$ such that, for all $0\le j\le 3$, $\left|\frac{\partial^j}{\partial v^j}\psi^T_{t,\varepsilon,k_0}(u,v)\right|$ is less than

$$c_{k_0}(2T)^{-k_0+3}\max(1,\|g'\|_\infty)^{(j-1)_+}\max(1,\lambda\|g'\|_1)^j\frac{\mathbb{P}(\gamma_T\ge k_0-j)}{\mathbb{P}(\gamma_T\ge k_0)}\left(1+\sqrt{u^2+v^2}\right)^{-(k_0-3)/2}.$$

Letting ε tends to 0 we obtain the same bounds as (21) for ψ^M_{t,k_0} the characteristic function of $(X_M(t), X'_M(t))$ conditionally on $\{\gamma_T\ge k_0\}$. Since $k_0\ge 8$, (21) for $j=0$ ensures that $\psi^M_{t,\varepsilon,k_0}\in L^1(\mathbb{R}^2)$, respectively $\psi^M_{t,k_0}\in L^1(\mathbb{R}^2)$, such that, conditionally on $\{\gamma_T\ge k_0\}$, $(X_M(t),(X_M(t+\varepsilon)-X_M(t))/\varepsilon)$, respectively $(X_M(t),X'_M(t))$, admits $p^M_{t,\varepsilon,k_0}(x,z) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ixu-izv}\psi^M_{t,\varepsilon,k_0}(u,v)dudv$, respectively $p^M_{t,k_0} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}e^{-ixu-izv}\psi^M_{t,k_0}(u,v)dudv$, as density. Moreover,
i) $p^M_{t,\varepsilon,k_0}(x,z)$ is continuous in (t,x) for each z,ε, according to Lebesgue's dominated convergence theorem using the fact that X_M is almost surely continuous on $\mathbb{R}$.
ii) Since X_M is almost surely continuously differentiable on $\mathbb{R}$ we clearly have for any $(u,v)\in\mathbb{R}^2$, $\psi^M_{t,\varepsilon,k_0}(u,v)\to\psi^M_{t,k_0}(u,v)$ as $\varepsilon\to 0$. Then by Lebesgue's dominated convergence theorem, using (21) for $j=0$ we check that $p^M_{t,\varepsilon,k_0}(x,z)\to p^M_{t,k_0}(x,z)$ as $\varepsilon\to0$, uniformly in (t,x) for each $z\in\mathbb{R}$.
iii) For any $z\ne 0$, integrating by parts we get

$$p^M_{t,\varepsilon,k_0}(x,z) = \frac{i}{4\pi^2z^3}\int_{\mathbb{R}^2} e^{-ixu-izv}\frac{\partial^3}{\partial v^3}\psi^M_{t,\varepsilon,k_0}(u,v)dudv,$$

such that by (21) for $j=3$, we check that $p^M_{t,\varepsilon,k_0}(x,z)\le Ch(z)$ for all t,ε,x with $h(z)=(1+|z|^3)^{-1}$ satisfying $\int_{\mathbb{R}}|z|h(z)dz<+\infty$ and C a positive constant.
Therefore, Theorem 2 of [18] implies that

$$\mathbb{E}\left(N_{X_M}(\alpha,[a,b])|\gamma_T\ge k_0\right) = \int_a^b\int_{\mathbb{R}}|z|p^M_{t,k_0}(\alpha,z)\,dz\,dt,$$

which concludes the proof, using $p^M_{t,k_0}(\alpha,z) = \frac{i}{4\pi^2z^3}\int_{\mathbb{R}^2}e^{-i\alpha u-izv}\frac{\partial^3}{\partial v^3}\psi^M_{t,k_0}(u,v)dudv$, and (21) for $j=3$. □

Note that, despite we have closed forms, these crossings formulas are not very tractable for general shot noise processes. However, as the intensity λ of the shot noise process X tends to infinity, due to its infinitely divisible property and since it is of second order, we obtain, after renormalization, a Gaussian process at the limit. It is then natural to hope the same kind of asymptotics for the mean number of crossings function. This behavior is studied in detail in the next section.

4. High intensity and Gaussian field

4.1. **General feature.** It is well-known that, as the intensity λ of the Poisson process goes to infinity, the shot noise process converges to a normal process. Precise bounds on the distance between the law of $X(t)$ and the normal distribution are given by Papoulis in [22]. Moreover, Heinrich and Schmidt in [15] give conditions of normal convergence for a wide class of shot noise processes (not restricted to 1d, nor to Poisson processes). In this section we obtain a stronger result for smooth stationary shot noise processes by considering convergence in law in the space of continuous functions. In all this section we continue to assume that X is a stationary shot noise process obtained for ν the Lebesgue measure on $\mathbb{R}$, and we will denote X_λ the strictly stationary shot noise process given by (1) with intensity $\lambda > 0$. Let us define the normalized shot noise process

$$Z_\lambda(t) = \frac{1}{\sqrt{\lambda}} \left(X_\lambda(t) - \mathbb{E}(X_\lambda(t)) \right), t \in \mathbb{R}. \tag{25}$$

Then, we obtain the following result.

Proposition 7. *Let $\beta \in L^2(\Omega)$ and g satisfying ($\mathbf{A}$). Then,*

$$Y_\lambda = \begin{pmatrix} Z_\lambda \\ Z'_\lambda \end{pmatrix} \underset{\lambda \to +\infty}{\overset{\text{fdd}}{\longrightarrow}} \sqrt{\mathbb{E}(\beta^2)} \begin{pmatrix} B \\ B' \end{pmatrix},$$

where B is a stationary centered Gaussian process almost surely and mean square continuously differentiable, with covariance function

$$Cov\left(B(t), B(t')\right) = \int_{\mathbb{R}} g(t-s)g(t'-s)ds.$$

When, moreover $g'' \in L^p(\mathbb{R})$ for $p > 1$, the convergence holds in distribution on the space of continuous functions on compact sets endowed with the topology of the uniform convergence.

Proof. We begin with the proof of the finite dimensional distributions convergence. Let k be an integer with $k \geq 1$ and let $t_1, \ldots, t_k \in \mathbb{R}$ and $w_1 = (u_1, v_1), \ldots, w_k = (u_k, v_k) \in \mathbb{R}^2$.
Let us write

$$\sum_{j=1}^{k} Y_\lambda(t_j) \cdot w_j = \frac{1}{\sqrt{\lambda}} \left(\sum_i \beta_i \widetilde{g}(\tau_i) - \mathbb{E}\left(\sum_i \beta_i \widetilde{g}(\tau_i) \right) \right),$$

for $\widetilde{g}(s) = \sum_{j=1}^{k} \left(u_j g(t_j - s) + v_j g'(t_j - s) \right)$. Therefore

$$\log \mathbb{E}\left(e^{i\sum_{j=1}^{k} Y_\lambda(t_j)\cdot w_j} \right) = \lambda \int_{\mathbb{R}\times\mathbb{R}} \left(e^{iz\left(\frac{\widetilde{g}(s)}{\sqrt{\lambda}}\right)} - 1 - iz\frac{\widetilde{g}(s)}{\sqrt{\lambda}} \right) dsF(dz).$$

Note that as $\lambda \to +\infty$,

$$\lambda \left(e^{iz\left(\frac{\widetilde{g}(s)}{\sqrt{\lambda}}\right)} - 1 - iz\frac{\widetilde{g}(s)}{\sqrt{\lambda}} \right) \to -\frac{1}{2} z^2 \widetilde{g}(s)^2,$$

with for all $\lambda > 0$

$$\left| \lambda \exp\left(iz\left(\frac{\widetilde{g}(s)}{\sqrt{\lambda}} \right) - 1 - iz\frac{\widetilde{g}(s)}{\sqrt{\lambda}} \right) \right| \leq \frac{1}{2} z^2 \widetilde{g}(s)^2.$$

By the dominated convergence theorem, since $\widetilde{g} \in L^2(\mathbb{R})$ and $\beta \in L^2(\Omega)$, we get that, as $\lambda \to +\infty$,

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^{k} Y_\lambda(t_j) \cdot w_j \right) \right) \to \exp\left(-\frac{1}{2}\mathbb{E}(\beta^2) \int_{\mathbb{R}} \widetilde{g}(s)^2 ds \right).$$

Let us identify the limiting process. Let us recall that X_λ is a second order process with covariance function given by $\text{Cov}(X_\lambda(t), X_\lambda(t')) = \lambda \mathbb{E}(\beta^2) S(t - t')$ with $S(t) = \int_{\mathbb{R}} g(t-s)g(-s)ds$. Hence one can define B to be a stationary Gaussian centered process with $(t, t') \mapsto S(t - t')$ as covariance function. The assumptions on g ensure that the function S is twice differentiable. Therefore B is mean square differentiable with B' a stationary Gaussian centered process with $(t,t') \mapsto -S''(t-t') = \int_{\mathbb{R}} g'(t-t'-s)g'(-s)ds$ as covariance function. Moreover

$$\mathbb{E}\left((B'(t)-B'(t'))^2\right) = 2\left(S''(0)-S''(t-t')\right) \leq 2\|g'\|_\infty \|g''\|_1 |t-t'|,$$

such that by Theorem 3.4.1 of [1] the process B' is almost surely continuous on $\mathbb{R}$. Therefore as in [11] p. 536, one can check that almost surely $B(t) = B(0) + \int_0^t B'(s)ds$, such that B is almost surely continuously differentiable. We conclude for the fdd convergence by noticing that

$$\int_{\mathbb{R}} \widetilde{g}(s)^2 ds = \operatorname{Var}\left(\sum_{j=1}^k u_j B(t_j) + v_j B'(t_j)\right).$$

Let us prove the convergence in distribution on the space of continuous functions on compact sets endowed with the topology of the uniform convergence. It is enough to prove the tightness of the sequence $(Y_\lambda)_\lambda$ according to Lemma 14.2 and Theorem 14.3 of [17]. Let $t, s \in \mathbb{R}$ and remark that for any $q \geq 1$, on the one hand

$$\mathbb{E}\left((Z_\lambda(t)-Z_\lambda(t'))^2\right) = \mathbb{E}(\beta^2)\int_{\mathbb{R}} \left(g(t-s)-g(t'-s)\right)^2 ds \leq \mathbb{E}(\beta^2)\|g'\|_q \|g'\|_1 |t-t'|^{2-1/q}.$$

On the other hand,

$$\mathbb{E}\left((Z'_\lambda(t)-Z'_\lambda(t'))^2\right) = \mathbb{E}(\beta^2)\int_{\mathbb{R}} \left(g'(t-s)-g'(t'-s)\right)^2 ds \leq \mathbb{E}(\beta^2)\|g''\|_q \|g''\|_1 |t-t'|^{2-1/q}.$$

Note that assuming that $g'' \in L^p(\mathbb{R})$ allows us to choose $q = p > 1$ in the second upper bound such that $2 - 1/q > 1$. Moreover assumption (**A**) implies that $g' \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^p(\mathbb{R})$ such that one can also choose $q = p$ in the first upper bound. Then, $(Y_\lambda)_\lambda$ satisfies a Kolmogorov-Chentsov criterion which implies its tightness according to Corollary 14.9 of [17]. □

In particular, when $a < b$, the functional $(f,g) \mapsto \int_a^b h(f(t))|g(t)|dt$ is clearly continuous and bounded on $\mathcal{C}([a,b],\mathbb{R}) \times \mathcal{C}([a,b],\mathbb{R})$ for any continuous bounded function h on $\mathbb{R}$. Then, Proposition 7 implies that

$$\int_a^b \mathbb{E}\left(h\left(Z_\lambda(t)\right)\left|Z'_\lambda(t)\right|\right) dt \underset{\lambda\to+\infty}{\longrightarrow} \int_a^b \mathbb{E}\left(h(B(t))|B'(t)|\right) dt.$$

By the co-area formula (4), this means the weak convergence of the mean number of crossings function, i.e.

$$C_{Z_\lambda}(\cdot,[a,b]) \underset{\lambda\to+\infty}{\rightharpoonup} C_B(\cdot,[a,b]).$$

This implies also the pointwise convergence of Fourier transforms. Such a result can be compared to the classical central limit theorem. Numerous of improved results can be obtained under stronger assumptions than the classical ones. This is the case for instance for the rate of convergence derived by Berry-Esseen Theorem or the convergence of the densities. We refer to [13] chapter 15 and 16. Adapting the technical proofs allows us to get similar results for crossings in the next section.

4.2. **High intensity: rate of convergence for the mean number of crossings function.** Let us remark that only $\mathbb{E}(\beta^2)$ appears in the limit field. For sake of simplicity we may assume that $\beta = 1$ a.s. Note that, according to Rice's formula [9], as recalled in Equation (7), since the limit Gaussian field is stationary, $C_B(\alpha,[a,b]) = (b-a)C_B(\alpha,[0,1])$ with

$$C_B(\alpha,[0,1]) = \frac{1}{\pi}\left(\frac{m_2}{m_0}\right)^{1/2} e^{-\alpha^2/2m_0},\ \forall \alpha \in \mathbb{R},$$

where $m_0 = \operatorname{Var}(B(t)) = \int_{\mathbb{R}} g(s)^2 ds$ and $m_2 = \operatorname{Var}(B'(t)) = \int_{\mathbb{R}} g'(s)^2 ds$. Moreover its Fourier transform is given by $\widehat{C_B}(u,[0,1]) = \sqrt{\frac{2m_2}{\pi}} e^{-m_0 u^2/2}$. We obtain the following rate of convergence, for which the proof is postponed to the Appendix.

Proposition 8. *Let $\beta = 1$ a.s. and let g satisfy (**A**). There exist three constants a_1, a_2 and a_3 (depending only on g and its derivative) such that*

$$\forall \lambda > 0, \forall u \in \mathbb{R} \text{ such that } |u| < a_1\sqrt{\lambda} \text{ then } \left|\widehat{C_{Z_\lambda}}(u,[0,1]) - \sqrt{\frac{2m_2}{\pi}} e^{-m_0u^2/2}\right| \leq \frac{a_2 + a_3|u|}{\sqrt{\lambda}},$$

where $m_0 = \int_{\mathbb{R}} g(s)^2 ds$ and $m_2 = \int_{\mathbb{R}} g'(s)^2 ds$.

Let us emphasize that this implies the uniform convergence of the Fourier transform of the mean number of crossings functions on any fixed interval. Moreover, taking $u = 0$, the previous upper bound may be a bit refined such that the following corollary is in force.

Corollary 2. *Let $\beta = 1$ a.s. and let g satisfy (**A**). The mean total variation of the process satisfies:*

$$\forall \lambda > 0, \quad \left|\frac{\mathbb{E}(|X_\lambda'(t)|)}{\sqrt{\lambda}} - \sqrt{\frac{2m_2}{\pi}}\right| \leq \frac{14m_3}{3\pi m_2\sqrt{\lambda}},$$

where $m_2 = \int_{\mathbb{R}} g'(s)^2 ds$ and $m_3 = \int_{\mathbb{R}} |g'(s)|^3\, ds$.

Under additional assumptions we obtain the following uniform convergence for the mean number of crossings function. The proof is inspired by Theorem 2 p.516 of [13] concerning the central limit theorem for densities.

Theorem 3. *Let $\beta = 1$ a.s. Let us assume moreover that g is a function of class $\mathcal{C}^4$ on $\mathbb{R}$ satisfying (**A**) such that for all $s \in [-1,2]$, $\Phi(s) = \begin{pmatrix} g'(s) & g''(s) \\ g''(s) & g^{(3)}(s) \end{pmatrix}$ and $\Phi'(s) = \begin{pmatrix} g''(s) & g^{(3)}(s) \\ g^{(3)}(s) & g^{(4)}(s) \end{pmatrix}$ are invertible.*
Let $\gamma_\lambda = \#\{i; \tau_{\lambda,i} \in [-1,1]\}$ with $\{\tau_{\lambda,i}\}_i$ the points of a Poisson point process with intensity $\lambda > 0$. Then

$$C_{Z_\lambda}(\alpha,[0,1]|\gamma_\lambda \geq \lambda) \underset{\lambda\to+\infty}{\longrightarrow} C_B(\alpha,[0,1]) = \frac{1}{\pi}\left(\frac{m_2}{m_0}\right)^{1/2} e^{-\alpha^2/2m_0}, \text{ uniformly in } \alpha \in \mathbb{R},$$

where $m_0 = \int_{\mathbb{R}} g(s)^2 ds$ and $m_2 = \int_{\mathbb{R}} g'(s)^2 ds$.

Proof. Let $\lambda \geq 8$. Then, according to Theorem 1, $\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda)$ and $\widehat{C_B}(u,[0,1])$ are integrable such that $C_{Z_\lambda}(\alpha,[0,1]|\gamma_\lambda \geq \lambda)$ and $C_B(\alpha,[0,1])$ are bounded continuous functions with, for any $\alpha \in \mathbb{R}$,

$$|C_{Z_\lambda}(\alpha,[0,1]\,|\,\gamma_\lambda \geq \lambda) - C_B(\alpha,[0,1])| \leq \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda) - \widehat{C_B}(u,[0,1])\right| du.$$

Let $u \in \mathbb{R}$, then

$$\widehat{C_{Z_\lambda}}(u,[0,1]) - \widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda) \;=\; \frac{1}{\mathbb{P}(\gamma_\lambda \geq \lambda)}\mathbb{E}\left(e^{iuZ_\lambda(0)}|Z_\lambda'(0)|\mathbf{1}_{\gamma_\lambda<\lambda}\right) - \frac{\mathbb{P}(\gamma_\lambda < \lambda)}{\mathbb{P}(\gamma_\lambda \geq \lambda)}\widehat{C_{Z_\lambda}}(u,[0,1])$$

Note that $\left|\widehat{C_{Z_\lambda}}(u,[0,1])\right| \leq \mathbb{E}\left(|Z_\lambda'(0)|\right)$, which is bounded according to Corollary 2, while by Cauchy-Schwarz inequality,

$$\left|\mathbb{E}\left(e^{iuZ_\lambda(0)}|Z_\lambda'(0)|\mathbf{1}_{\gamma_\lambda<\lambda}\right)\right| \leq \mathbb{E}\left(Z_\lambda'(0)^2\right)^{1/2}\mathbb{P}(\gamma_\lambda < \lambda)^{1/2},$$

with $\mathbb{E}\left(Z_\lambda'(0)^2\right) = \mathrm{Var}(Z_\lambda'(0)) \leq \max(1,\|g'\|_\infty)\|g'\|_1$. Therefore, one can find $c_1 > 0$ such that

$$\left|\widehat{C_{Z_\lambda}}(u,[0,1]) - \widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda)\right| \leq c_1\frac{\mathbb{P}(\gamma_\lambda < \lambda)^{1/2}}{\mathbb{P}(\gamma_\lambda \geq \lambda)}.$$

According to Markov's inequality,

$$\mathbb{P}(\gamma_\lambda < \lambda) = \mathbb{P}\left(e^{-\ln(2)\gamma_\lambda} > e^{-\ln(2)\lambda}\right) \leq \mathbb{E}\left(e^{-\ln(2)(\gamma_\lambda-\lambda)}\right) = \exp\left(-(1-\ln(2))\lambda\right).$$

Choosing λ large enough such that in particular $\frac{\mathbb{P}(\gamma_\lambda<\lambda)^{1/2}}{\mathbb{P}(\gamma_\lambda\geq\lambda)} \leq \frac{1}{\sqrt{\lambda}}$, according to Proposition 8, one can find c_2 such that for all $|u| < \lambda^{1/8}$,

$$|\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda) - \widehat{C_B}(u,[0,1])| \leq c_2\lambda^{-3/8}.$$

Thus we may conclude that

$$\int_{|u|<\lambda^{1/8}} \left|\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda) - \widehat{C_B}(u,[0,1])\right| du \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

Now, let us be concerned with the remaining integral for $|u| \geq \lambda^{1/8}$. According to Theorem 1,

$$\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda \geq \lambda) = \frac{e^{-iu\sqrt{\lambda}\int_{\mathbb{R}} g}}{\sqrt{\lambda}} \widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}},[0,1]\,|\,\gamma_\lambda \geq \lambda\right),$$

with $\widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}},[0,1]\,|\,\gamma_\lambda \geq \lambda\right)) = \int_0^1 \mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}X_\lambda(t)}|X'_\lambda(t)|\,|\,\gamma_\lambda\geq\lambda\right)dt$ and

$$\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}X_\lambda(t)}|X'_\lambda(t)|\,|\,\gamma_\lambda\geq\lambda\right) = -\frac{1}{\pi}\int_0^{+\infty}\frac{1}{v}\left(\frac{\partial\psi_{t,\lambda}}{\partial v}\left(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}}\right) - \frac{\partial\psi_{t,\lambda}}{\partial v}\left(\frac{u}{\sqrt{\lambda}},-\frac{v}{\sqrt{\lambda}}\right)\right)dv,$$

where $\psi_{t,\lambda}$ is the characteristic function of $(X_\lambda(t), X'_\lambda(t))$ conditionally on $\{\gamma_\lambda \geq \lambda\}$. Integrating by parts we obtain

$$\begin{aligned}&\int_0^1\frac{1}{v}\left(\frac{\partial\psi_{t,\lambda}}{\partial v}\left(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}}\right) - \frac{\partial\psi_{t,\lambda}}{\partial v}\left(\frac{u}{\sqrt{\lambda}},-\frac{v}{\sqrt{\lambda}}\right)\right)dv\\ &= -\frac{1}{\sqrt{\lambda}}\int_0^1 \ln(v)\left(\frac{\partial^2\psi_{t,\lambda}}{\partial v^2}\left(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}}\right) - \frac{\partial^2\psi_{t,\lambda}}{\partial v^2}\left(\frac{u}{\sqrt{\lambda}},-\frac{v}{\sqrt{\lambda}}\right)\right)dv.\end{aligned}$$

Then, according to (24), one can find a positive constant $c_3 > 0$ such that

$$\begin{aligned}\left|\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}X_\lambda(t)}|X'_\lambda(t)|\,|\,\gamma_\lambda\geq\lambda\right)\right| &\leq c_3\lambda^2\frac{\mathbb{P}(\gamma_\lambda\geq\lambda-2)}{\mathbb{P}(\gamma_\lambda\geq\lambda)}\\ &\times\int_{\mathbb{R}}\left|\chi_t\left(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}}\right)\right|^{\lambda-2}\left(\frac{1}{\sqrt{\lambda}}|\ln(|v|)|\mathbf{1}_{0\leq|v|\leq1} + |v|^{-1}\mathbf{1}_{|v|\geq1}\right)dv,\end{aligned}$$

where $\chi_t(u,v) = \frac{1}{2}\int_{-1+t}^{1+t} e^{iug(s)+ivg'(s)}ds$, is the characteristic function of $(g(t-U), g'(t-U))$, with U a uniform random variable on $[-1,1]$. Then,

$$\begin{aligned}&\int_{|u|\geq\lambda^{1/8}}\left|\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda\geq\lambda) - \widehat{C_B}(u,[0,1])\right|du\\ \leq\ &\int_{|u|\geq\lambda^{1/8}}\left|\widehat{C_{Z_\lambda}}(u,[0,1]\,|\,\gamma_\lambda\geq\lambda)\right|du + \int_{|u|\geq\lambda^{1/8}}\left|\widehat{C_B}(u,[0,1])\right|du\\ =\ &I_1(\lambda) + I_2(\lambda).\end{aligned}$$

Now, for $\theta \in [0,2\pi]$, let us consider the random variable $V_{t,\theta} = \cos(\theta)g(t-U) + \sin(\theta)g'(t-U)$ such that for any $r > 0$, $\chi_t(r\cos(\theta), r\sin(\theta)) = \mathbb{E}(e^{irV_{t,\theta}}) := \varphi_{t,\theta}(r)$. By a change of variables in polar coordinates, since $\lambda > 1$, we get

$$I_1(\lambda) \leq c_4(\lambda)\int_{\lambda^{1/8}}^{+\infty}\int_0^{2\pi}\left|\varphi_{t,\theta}\left(\frac{r}{\sqrt{\lambda}}\right)\right|^{\lambda-2} r\left(|\ln(r|\sin(\theta)|)| + 1\right)d\theta dr,$$

with $c_4(\lambda) = c_3\lambda^{3/2}\frac{\mathbb{P}(\gamma_\lambda\geq\lambda-2)}{\mathbb{P}(\gamma_\lambda\geq\lambda)}$. Since $\det\Phi(s) \neq 0$ for any $s \in [-1+t, 1+t]$, we have the following property (see [13] p.516): there exists $\delta > 0$ such that

$$|\varphi_{t,\theta}(r)| \leq e^{-\frac{\kappa(t)}{4}r^2},\ \forall r \in (0,\delta], \forall\theta\in[0,2\pi], \text{ and } \eta = \sup_{r>\delta,\theta\in[0,2\pi]}|\varphi_{t,\theta}(r)| < 1,$$

with $\kappa(t) = \min_{\theta\in[0,2\pi]} \mathrm{Var}(V_{t,\theta}) > 0$. Note also that according to Proposition 10, $|\varphi_{t,\theta}(r)| \leq 24\sqrt{\frac{2}{m}} r^{-1/2}$ for any $r > m$ with $m = \min_{s\in[-1,2]} \|\Phi(s)^{-1}\|^{-1}$, which may be assumed to be larger than δ. Then, for λ large enough such that $\lambda^{1/8} \in (e, \delta\sqrt{\lambda})$,

$$I_1(\lambda) \quad \leq \quad c_5(\lambda) \left(\int_{\lambda^{1/8}}^{\delta\sqrt{\lambda}} e^{-\frac{\kappa(t)}{8}\lambda^{1/4}} r \ln(r) dr + \int_{\delta\sqrt{\lambda}}^{m\sqrt{\lambda}} \eta^{\lambda-2} r \ln(r) dr + \left(24\sqrt{\frac{2}{m}}\right)^5 \int_{m\sqrt{\lambda}}^{+\infty} \eta^{\lambda-7} r^{-3/2} \ln(r) dr \right)$$

with $c_5(\lambda) = c_4(\lambda) \left(\int_0^{2\pi} (2 + |\ln(|\sin(\theta)|)|)\, d\theta \right)$. This enables us to conclude that $I_1(\lambda) \underset{\lambda\to+\infty}{\longrightarrow} 0$. This concludes the proof since clearly $I_2(\lambda) \underset{\lambda\to+\infty}{\longrightarrow} 0$. $\square$

Notice that to obtain the convergence in Theorem 3 without the conditioning on $\{\gamma_\lambda \geq \lambda\}$ (which is an event of probability going to 1 exponentially fast as λ goes to infinity), one simply needs to have an upper-bound polynomial in λ on the second moment of the number of crossings $N_{Z_\lambda}(\alpha, [0, 1])$.

5. The Gaussian kernel

In this section we will be interested in a real application of shot noise processes in Physics. Indeed, each time a physical model is given by sources that produce each a potential in such a way that the global potential at a point is the sum of all the individual potentials, then this can be modeled as a shot noise process. In particular, we will be interested here in the temperature produced by sources of heat. Assuming that the sources are randomly placed as a Poisson point process of intensity λ on the real line $\mathbb{R}$, then the temperature after a time σ^2 on the line is given by the following shot noise process $X_{\lambda,\sigma}$:

$$t \in \mathbb{R} \mapsto X_{\lambda,\sigma}(t) = \sum_i \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\tau_i)^2/2\sigma^2},$$

where the $\{\tau_i\}$ are the points of a Poisson process of intensity $\lambda > 0$ on $\mathbb{R}$. In the following, we will denote by g_σ the Gaussian kernel of width σ defined for all $t \in \mathbb{R}$ by

$$g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}.$$

We will be interested in the crossings of $X_{\lambda,\sigma}$ because they provide informations on the way the temperature is distributed on the line. The number of local extrema of $X_{\lambda,\sigma}$ is also interesting for practical applications since it measures the way the temperature fluctuates on the line. In a first part, we will be interested in the crossings of $X_{\lambda,\sigma}$ when λ and σ are fixed, and then, in a second part, we will study how the number of crossings evolves when these two parameters change. From the point of view of applications, this amounts to describe the fluctuations of the temperature on the line when the time (recall that σ^2 represents the time) increases, or when the number of sources changes.

5.1. **Crossings and local extrema of $X_{\lambda,\sigma}$.**

We assume in this subsection that $\lambda > 0$ and $\sigma > 0$ are fixed. Since the Gaussian kernel g_σ, and its derivatives are smooth functions which belong to all L^p spaces, many results of the previous sections about crossings can be applied here. In particular, we have:

- The function $\alpha \mapsto C_{X_{\lambda,\sigma}}(\alpha, [a, b])$ belongs to $L^1(\mathbb{R})$ (by Theorem 1).
- For any $T > 0$, the function $\alpha \mapsto C_{X_{\lambda,\sigma}}(\alpha, [a, b]|\gamma_T \geq 8)$ is continuous (by Theorem 2), with $\gamma_T = \#\{\tau_i \in [-T, T]\}$.

This second point comes from the fact that the Gaussian kernel satisfies the hypothesis of Theorem 2. Indeed, the derivatives of g_σ are given by $g_\sigma^{(k)}(s) = \frac{1}{\sigma\sqrt{2\pi}} e^{-s^2/2\sigma^2} \cdot \frac{(-1)^k}{\sigma^k} H_k(\frac{s}{\sigma})$, where the H_k's are the Hermite polynomials ($H_1(x) = x$; $H_2(x) = x^2 - 1$; $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$)

Thus, using the notations of Theorem 2 we get $\det\Phi(s) = \frac{-1}{\sigma^4}(\frac{s^2}{\sigma^2}+1)\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-s^2/2\sigma^2}\right)^2 < 0$ and $\det\Phi'(s) = \frac{-1}{\sigma^6}(\frac{s^4}{\sigma^4}+3)\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-s^2/2\sigma^2}\right)^2 < 0$. These two matrices are thus invertible for all $s \in \mathbb{R}$.

The first point implies that for almost every $\alpha \in \mathbb{R}$, the expected number of crossings of the level α by $X_{\lambda,\sigma}$ is finite. We will now prove in the following proposition that in fact, for every $\alpha \in \mathbb{R}$, $C_{X_{\lambda,\sigma}}(\alpha,[a,b]) < +\infty$, by considering the zero-crossings of the derivative $X'_{\lambda,\sigma}$ and using Rolle's Theorem.

In the sequel, we will denote by $\rho(\lambda,\sigma)$ the mean number of local extrema of $X_{\lambda,\sigma}$ in the interval $[0,1]$. It is the mean number of local extrema per unit length.

Proposition 9. *We have*

$$\mathbb{P}(\exists t \in [0,1] \text{ such that } X'_{\lambda,\sigma}(t) = 0 \text{ and } X''_{\lambda,\sigma}(t) = 0) = 0,$$

which implies that the local extrema of $X_{\lambda,\sigma}$ are exactly the points where the derivative vanishes, in other words $\rho(\lambda,\sigma) = \mathbb{E}(N_{X'_{\lambda,\sigma}}(0,[0,1]))$. Moreover, we have the following bounds:

$$\forall \alpha \in \mathbb{R},\ \ C_{X_{\lambda,\sigma}}(\alpha,[0,1]) \leq \rho(\lambda,\sigma) \leq (3\lambda(2+2\sigma)+1)e^{\lambda}.$$

Proof. For the first part of the proposition, we use Proposition 10 (in the Appendix) with the kernel function $h = g'_\sigma$ on the interval $[-T+1,T]$ for $T>0$. For this function we can compute $h'(s) = \frac{1}{\sigma^3\sqrt{2\pi}}(-1+\frac{s^2}{\sigma^2})e^{-s^2/2\sigma^2}$ and $h''(s) = \frac{1}{\sigma^4\sqrt{2\pi}}(3\frac{s}{\sigma}-\frac{s^3}{\sigma^3})e^{-s^2/2\sigma^2}$, and thus $n_0 = 3$ and $m(\sigma,T) = \min_{s\in[-T,T+1]}\sqrt{h'(s)^2+h''(s)^2} > 0$ (we don't need to have an exact value for it but notice that it is of the order of $e^{-T^2/2\sigma^2}$ when T is large). Finally, as in (20), we get that there is a constant $c(T,\sigma)$ which depends continuously on σ and T such that

$$|\mathbb{E}(e^{iuX'_{\lambda,\sigma}(t)}|\gamma_T \geq 3)| \quad \leq \quad \frac{c(T,\sigma)^3}{(1+\sqrt{|u|})^3},$$

with $\gamma_T = \#\{\tau_i \in [-T,T]\}$. We can now use Proposition 1 and we get that for all $T>1$:

$$\mathbb{P}(\exists t \in [0,1] \text{ such that } X'_{\lambda,\sigma}(t) = 0 \text{ and } X''_{\lambda,\sigma}(t) = 0|\gamma_T \geq 3) = 0.$$

Since the events $\{\gamma_T \geq 3\}$ are an increasing sequence of events such that $\mathbb{P}(\gamma_T \geq 3)$ goes to 1 as T goes to infinity, we obtain that $\mathbb{P}(\exists t \in [0,1]$ such that $X'_{\lambda,\sigma}(t) = 0$ and $X''_{\lambda,\sigma}(t) = 0) = 0$.

For the second part of the proposition, the left-hand inequality is simply a consequence of Proposition 2 for the process $X'_{\lambda,\sigma}$ and $n=1$.

To obtain the right-hand inequality (the bound on $\rho(\lambda,\sigma)$), we will apply Proposition 2 to the process $X'_{\lambda,\sigma}$ for the crossings of the level 0 on the interval $[0,1]$. We already know by the first part of the proposition and by Corollary 1 that the condition (8) for Kac's formula is satisfied by $X'_{\lambda,\sigma}$. Then we write for all $t \in [0,1]$:

$$X'_{\lambda,\sigma}(t) = \sum_{\tau_i\in\mathbb{R}} g'_\sigma(t-\tau_i) = \frac{1}{\sigma\sqrt{2\pi}}\sum_{\tau_i\in[-\sigma,1+\sigma]}\frac{-(t-\tau_i)}{\sigma^2}e^{-(t-\tau_i)^2/2\sigma^2} + \frac{1}{\sigma\sqrt{2\pi}}\sum_{\tau_i\in\mathbb{R}\setminus[-\sigma,1+\sigma]}\frac{-(t-\tau_i)}{\sigma^2}e^{-(t-\tau_i)^2/2\sigma^2}$$

Let $Y_1(t)$ (resp. $Y_2(t)$) denote the first (resp. second) term. We then have

$$Y'_2(t) = \frac{1}{\sigma\sqrt{2\pi}}\sum_{\tau_i\in\mathbb{R}\setminus[-\sigma,1+\sigma]}\left(\frac{(t-\tau_i)^2}{\sigma^4}-\frac{1}{\sigma^2}\right)e^{-(t-\tau_i)^2/2\sigma^2}.$$

Since $(t-\tau_i)^2 > \sigma^2$ for all $t \in [0,1]$ and all $\tau_i \in \mathbb{R}\setminus[-\sigma,1+\sigma]$, we get $Y'_2(t) > 0$ on $[0,1]$ and thus $N_{Y'_2}(0,[0,1]) = 0$ a.s. Note that when the event $\#\{\tau_i \in [-\sigma,1+\sigma]\} = 0$ holds, then $X'_{\lambda,\sigma} = Y_2$ such that $N_{X'_{\lambda,\sigma}}(0,[0,1]) \leq 1$. On the other hand, let us work conditionally on $\#\{\tau_i \in [-\sigma,1+\sigma]\} \geq 1$. The probability of this event is $1-e^{-\lambda(1+2\sigma)}$. To study the zero-crossings of Y'_1, we first need an elementary lemma.

Lemma 1. *Let $n \geq 1$ be an integer. Let $P_1,\dots,$ P_n be n real non-zero polynomials and let $a_1,\dots,$ a_n be n real numbers, then*

$$\#\{t \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} P_i(t)e^{a_i t} = 0\} \leq \sum_{i=1}^{n} \deg(P_i) + n - 1.$$

This elementary result can be proved by induction on n. For $n = 1$, it is obviously true. Assume the result holds for $n \geq 1$, then we prove it for $n+1$ in the following way. For $t \in \mathbb{R}$, $\sum_{i=1}^{n+1} P_i(t)e^{a_i t} = 0 \iff f(t) := P_{n+1}(t)+\sum_{i=1}^{n} P_i(t)e^{(a_i-a_{n+1})t} = 0$. Let k denote the degree of P_{n+1}. Thanks to Rolle's Theorem, we have that $N_f(0,\mathbb{R}) \leq N_{f'}(0,\mathbb{R}) + 1 \leq N_{f''}(0,\mathbb{R}) + 2 \leq \dots \leq N_{f^{(k+1)}}(0,\mathbb{R}) + k + 1$. But $f^{(k+1)}$ can be written as $f^{(k+1)}(t) = \sum_{i=1}^{n} Q_i(t)e^{(a_i-a_{n+1})t}$, where the Q_i are polynomials of degree $\deg(Q_i) \leq \deg(P_i)$. Thus by induction $N_{f^{(k+1)}}(0,\mathbb{R}) \leq \sum_{i=1}^{n} \deg(P_i) + n - 1$, and then $N_f(0,\mathbb{R}) \leq \sum_{i=1}^{n} \deg(P_i) + n - 1 + k + 1 \leq \sum_{i=1}^{n+1} \deg(P_i) + n$. This proves the result for $n+1$.

Thanks to this lemma, we get that $N_{Y_1'}(0,[0,1]) \leq 3\#\{\tau_i \in [-\sigma,1+\sigma]\}) - 1$ such that

$$\mathbb{E}(N_{Y_1'}(0,[0,1])|\#\{\tau_i \in [-\sigma,1+\sigma]\} \geq 1) \leq 3\lambda(1+2\sigma)/(1-e^{-\lambda(1+2\sigma)}) - 1.$$

To use Proposition 2, we need to obtain uniform bounds on the laws of $Y_1(t)$ and of $Y_2(t)$ when $t \in [0,1]$. As in the notations of the proposition, we will denote these constants by c_1 and c_2. Let us start with Y_1. Let U be a random variable following the uniform distribution on $[-1-\sigma,1+\sigma]$. For $t \in [0,1]$, we can write U as $U = \eta_t U_t + (1-\eta_t)V_t$, where U_t is uniform on $[-1-\sigma+t,\sigma+t]$, V_t is uniform on $[-1-\sigma,-1-\sigma+t] \cup [\sigma+t,\sigma+1]$ and η_t is an independent Bernoulli random variable with parameter $\frac{1+2\sigma}{2+2\sigma}$. We then have $g_\sigma'(U) = \eta_t g_\sigma'(U_t) + (1-\eta_t)g_\sigma'(V_t)$. Thus the law of $g_\sigma'(U)$ is the mixture of the law of $g_\sigma'(U_t)$ and of the one of $g_\sigma'(V_t)$, with respective weights $\frac{1+2\sigma}{2+2\sigma}$ and $1-\frac{1+2\sigma}{2+2\sigma}$. Consequently

$$\forall t \in [0,1], \forall x \in \mathbb{R},\ \ dP_{g_\sigma'(U_t)}(x) \leq \frac{2+2\sigma}{1+2\sigma} dP_{g_\sigma'(U)}(x).$$

The law of $Y_1(t)$ conditionally on $\#\{\tau_i \in [-\sigma,1+\sigma]\} \geq 1$ can be written as

$$dP_{Y_1(t)}(x) = \frac{1}{1-e^{-\lambda(1+2\sigma)}} \sum_{k=1}^{+\infty} e^{-\lambda(1+2\sigma)} \frac{(\lambda(1+2\sigma))^k}{k!} (dP_{g_\sigma'(U_t)} * \ldots * dP_{g_\sigma'(U_t)})(x).$$

Thus, if we write $f_0 = dP_{g_\sigma'(U)}$, we get

$$dP_{Y_1(t)}(x) \leq \frac{1}{1-e^{-\lambda(1+2\sigma)}} \sum_{k=1}^{+\infty} e^{-\lambda(1+2\sigma)} \frac{(\lambda(1+2\sigma))^k}{k!} \left(\frac{2+2\sigma}{1+2\sigma}\right)^k (f_0 * \ldots * f_0)(x) = e^{\lambda} \frac{1-e^{-\lambda(2+2\sigma)}}{1-e^{-\lambda(1+2\sigma)}} \tilde{f}_0(x),$$

where $\tilde{f}_0(x)dx$ is a probability measure on $\mathbb{R}$. This shows that we can take $c_1 = e^{\lambda}\frac{1-e^{-\lambda(2+2\sigma)}}{1-e^{-\lambda(1+2\sigma)}}$.

For $Y_2(t)$, we first notice that $Y_2(t)$ can be decomposed as the sum of two independent random variables in the following way:

$$Y_2(t) = \sum_{\tau_i \in (-\infty,-1-\sigma+t] \cup [1+\sigma+t,+\infty)} g_\sigma'(t-\tau_i) + \sum_{\tau_i \in (-\sigma-1+t,-\sigma) \cup (1+\sigma,1+\sigma+t)} g_\sigma'(t-\tau_i).$$

The first random variable in the sum above has a law that does not depend on t. For the second random variable, using the same trick as above (*i.e.* decompose here a uniform random variable on the interval $(-1-\sigma,-\sigma) \cup (\sigma,1+\sigma)$ as a mixture with weights $1/2$ and $1/2$ of two uniform random variables: one on $(-1-\sigma,-1-\sigma+t) \cup (t+\sigma,1+\sigma)$, and the other one on the rest), we obtain that $c_2 = e^{\lambda}$.

And finally the bound on the expectation of the number of local extrema is

$$\begin{aligned}
\rho(\lambda,\sigma) &\leq \left(c_1 \frac{3\lambda(1+2\sigma)}{1-e^{-\lambda(1+2\sigma)}} + c_2\right)(1-e^{-\lambda(1+2\sigma)}) + e^{-\lambda(1+2\sigma)} \\
&\leq e^{\lambda}\frac{2+2\sigma}{1+2\sigma}(3\lambda(1+2\sigma)) + e^{\lambda} = (3\lambda(2+2\sigma)+1)e^{\lambda}.
\end{aligned}$$

□

5.2. **Scaling properties.** An interesting property of the shot noise process with Gaussian kernel is that we have two scale parameters: the intensity λ of the Poisson point process and the width σ of the Gaussian kernel. These two parameters are linked in the sense that changing one of them amounts to change the other one in an appropriate way. These scaling properties are described more precisely in the following lemma.

Lemma 2. *We have the following scaling properties for the process $X_{\lambda,\sigma}$:*

(1) *Changing σ and λ in a proportional way: for all $c > 0$,*

$$\{X_{\frac{\lambda}{c},c\sigma}(t); t \in \mathbb{R}\} \stackrel{fdd}{=} \{\frac{1}{c} X_{\lambda,\sigma}(\frac{t}{c}); t \in \mathbb{R}\}.$$

(2) *Increasing the width of the Gaussian kernel: for all σ_1 and σ_2, we have*

$$\{X_{\lambda,\sqrt{\sigma_1^2+\sigma_2^2}}(t); t \in \mathbb{R}\} \stackrel{a.s.}{=} \{(X_{\lambda,\sigma_1} * g_{\sigma_2})(t); t \in \mathbb{R}\}.$$

(3) *Increasing the intensity of the Poisson process: for all $c > 0$, we have*

$$\{X_{\lambda\sqrt{1+c^2},\sigma}(t); t \in \mathbb{R}\} \stackrel{fdd}{=} \{\sqrt{1+c^2} \cdot (X_{\lambda,\sigma} * g_{c\sigma})(t\sqrt{1+c^2}); t \in \mathbb{R})\}.$$

(4) *The mean number $\rho(\lambda, \sigma)$ of local extrema of $X_{\lambda,\sigma}$ per unit length satisfies:*

$$\forall c > 0, \quad c\rho(\lambda, c\sigma) = \rho(c\lambda, \sigma).$$

Proof. For the first property, let $\{\tau_i\}$ be a Poisson point process of intensity λ/c on the line. Then

$$X_{\frac{\lambda}{c},c\sigma}(t) = \sum_i \frac{1}{c\sigma\sqrt{2\pi}} e^{-(t-\tau_i)^2/2c^2\sigma^2} = \frac{1}{c} \sum_i g_\sigma\left(\frac{t}{c} - \frac{\tau_i}{c}\right).$$

Since the points $\{\tau_i/c\}$ are now the points of a Poisson process on intensity λ on the line, we obtain the first scaling property. The second property comes simply from the fact that if g_{σ_1} and g_{σ_2} are two Gaussian kernels of respective width σ_1 and σ_2, then their convolution is the Gaussian kernel of width $\sqrt{\sigma_1^2 + \sigma_2^2}$. The third property is just a consequence of combining the first and second properties. For the fourth property, we first compute

$$X'_{\lambda,c\sigma}(t) = \frac{1}{c\sigma\sqrt{2\pi}} \sum_{\tau_i} \frac{-(t-\tau_i)}{c^2\sigma^2} e^{-(t-\tau_i)^2/2c^2\sigma^2} = \frac{1}{c^2\sigma\sqrt{2\pi}} \sum_{\tau_i} \frac{-(t/c-\tau_i/c)}{\sigma^2} e^{-(t/c-\tau_i/c)^2/2\sigma^2},$$

where the $\{\tau_i\}$ are the points of a Poisson point process of intensity λ on $\mathbb{R}$. Then, since the $\{\tau_i/c\}$ are now the points of a Poisson point process of intensity $c\lambda$ on $\mathbb{R}$, we have that the expected number of points $t \in [0, c]$ such that $X'_{\lambda,c\sigma}(t) = 0$ (which, by definition, equals $c\rho(\lambda, c\sigma)$), also equals the expected number of points $t \in [0, 1]$ such that $X'_{c\lambda,\sigma}(t) = 0$ (which is $\rho(c\lambda, \sigma)$). □

To study how $\rho(\lambda, \sigma)$ varies when λ and σ vary, we first can use the result on high intensity and convergence to the crossings of a Gaussian process obtained in Theorem 3. Indeed, if the second moment of $N_{X'_{\lambda,\sigma}}(0)$ is bounded by a polynomial in λ, then we will get

$$\rho(\lambda, \sigma) \underset{\lambda\to+\infty}{\longrightarrow} \frac{1}{\sigma\pi}\sqrt{\frac{3}{2}}.$$

And thanks to the scaling properties, this also will imply that $\rho(\lambda, \sigma)$ is equivalent to $\frac{1}{\sigma\pi}\sqrt{\frac{3}{2}}$ as σ goes to $+\infty$. These two facts have been empirically checked and are illustrated on Figure 1. Now, notice that we can also observe on the left-hand figure another regime when λ is small. Indeed, $\rho(\lambda, \sigma)$ seems to be almost linear for small values of λ. Notice also on the right-hand figure that $\rho(\lambda, \sigma)$ seems to be a decreasing function of σ (this then indicates that, as time goes by, the temperature on the line fluctuates less and less). The study of these two facts is the aim of the next section.

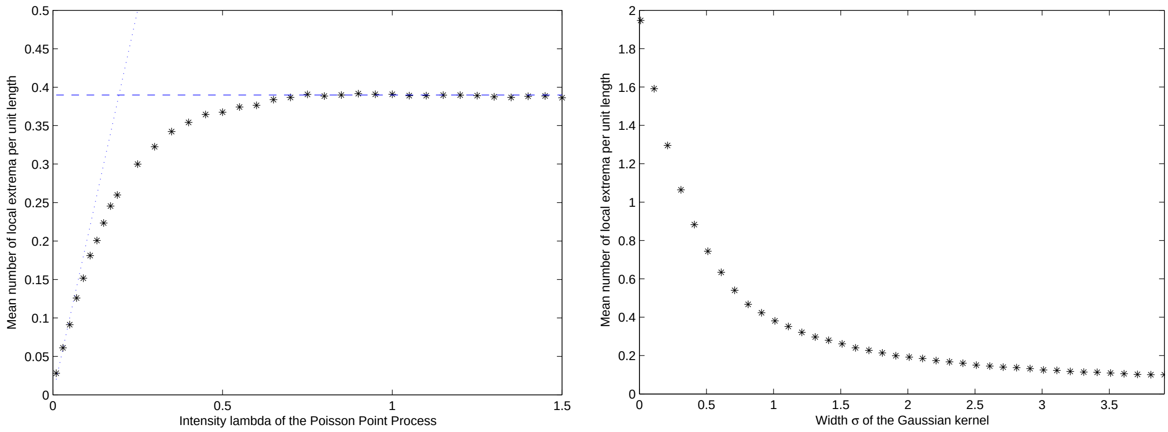

FIGURE 1. On the left: empirical mean number of local extrema of $X_{\lambda,\sigma}$ per unit length as a function of λ (here $\sigma = 1$ and we have taken the mean value from 50 samples on the interval $[-100, 100]$). The horizontal dashed line is the constant $\frac{1}{\pi}\sqrt{\frac{3}{2}}$ and the dotted line is the map $\lambda \mapsto 2\lambda$. On the right: empirical mean number of local extrema of $X_{\lambda,\sigma}$ per unit length as a function of σ (here $\lambda = 1$ and we have taken the mean value from 10 samples on the interval $[-100, 100]$).

5.3. **Heat equation and local extrema.** In this subsection we assume first that $\lambda > 0$ is fixed. As we already mentioned it in the introduction of Section 5, one of the main features of the shot noise process $X_{\lambda,\sigma}$ is that it can be seen in a dynamic way, which means that we can study how it evolves as the width σ of the Gaussian kernel changes and consider it as a random field indexed by the variable (σ, t). Then, the main tool is the heat equation which is satisfied by the Gaussian kernel:

$$\forall \sigma > 0,\ \forall t \in \mathbb{R},\ \ \frac{\partial g_\sigma}{\partial \sigma}(t) = \sigma\, g_\sigma''(t) \text{ and also consequently } \frac{\partial g_\sigma'}{\partial \sigma}(t) = \sigma\, g_\sigma^{(3)}(t). \tag{26}$$

Since the Gaussian kernel g_σ is a very smooth function, both in $\sigma > 0$ and $t \in \mathbb{R}$, by the same type of proof as the ones in Proposition 3, we have that $(\sigma, t) \mapsto X_{\lambda,\sigma}(t)$ is almost surely and mean square smooth on $(0, +\infty) \times \mathbb{R}$ with

$$\frac{\partial X_{\lambda,\sigma}}{\partial \sigma}(t) = \sum_i \frac{\partial g_\sigma}{\partial \sigma}(t - \tau_i) = \sigma\, X_{\lambda,\sigma}''(t) \text{ and also } \frac{\partial X_{\lambda,\sigma}'}{\partial \sigma}(t) = \sigma\, X_{\lambda,\sigma}^{(3)}(t). \tag{27}$$

We will see in the following that this equation will be of great interest to study the crossings of $X_{\lambda,\sigma}$.

The convolution of a 1d function with a Gaussian kernel of increasing width σ (which amounts to apply the heat equation) is a very common smoothing technique in signal processing. One of its main property is generally formulated by the wide-spread idea that "Gaussian convolution in 1d cannot create new extrema" (and it is in some sense the only kernel that has this property - see [27]). This has been studied (together with its extension in higher dimension) for applications in image processing by Lindeberg in [19], and also by other authors (for instance to study mixtures of Gaussian distributions as in [8] and [7]). However, in most cases, the correct mathematical framework for the validity of this property is not exactly stated. Thus we start here with a lemma giving the conditions under which one can obtain properties for the zero-crossings of a function solution of the heat equation. The result, which proof is postponed in the Appendix, is stated under a general form for a function h in the two variables σ and t. But we have to keep in mind that we will want to apply this to $h(\sigma, t) = X_{\lambda,\sigma}'(t)$ to follow the local extrema of the shot noise process when σ evolves.

Lemma 3. *Let $\sigma_0 > 0$ and $(\sigma, t) \mapsto h(\sigma, t)$ be a $\mathcal{C}^2$ function defined on $(0, \sigma_0] \times [a, b]$, which satisfies the heat equation:*

$$\forall (\sigma, t) \in (0, \sigma_0] \times \mathbb{R},\ \ \frac{\partial h}{\partial \sigma}(\sigma, t) = \sigma\, \frac{\partial^2 h}{\partial t^2}(\sigma, t).$$

We assume that

(a) *There are no* $t \in [a,b]$ *such that* $h(\sigma_0, t) = 0$ *and* $\frac{\partial h}{\partial t}(\sigma_0, t) = 0$,
(b) *There are no* $(\sigma, t) \in (0, \sigma_0] \times [a,b]$ *such that* $h(\sigma, t) = 0$ *and* $\nabla h(\sigma, t) = 0$.

Then we have the following properties for the zero-crossings of h*:*

i) *Global curves: If* $t_0 \in (a,b)$ *is such that* $h(\sigma_0, t_0) = 0$*, there exists* $\sigma_0^- < \sigma_0$ *and a maximal continuous path* $\sigma \mapsto \Gamma_{t_0}(\sigma)$ *defined on* $(\sigma_0^-, \sigma_0]$ *such that* $\Gamma_{t_0}(\sigma_0) = t_0$ *and for all* $\sigma \in (\sigma_0^-, \sigma_0]$ *we have* $h(\sigma, \Gamma_{t_0}(\sigma)) = 0$*. Moreover, if* $\Gamma_{t_0}(\sigma)$ *stays within some compact set of* $\mathbb{R}$ *for all* σ*, then* $\sigma_0^- = 0$*.*
ii) *Non-intersecting curves: If* $\widetilde{t_0} \neq t_0$ *is another point in* (a,b) *such that* $h(\sigma_0, \widetilde{t_0}) = 0$*, then for all* $\sigma \in (0, \sigma_0]$ *we have* $\Gamma_{t_0}(\sigma) \neq \Gamma_{\widetilde{t_0}}(\sigma)$*.*
iii) *Local description of the curves: If* $(\sigma_1, t_1) \in (0, \sigma_0] \times \mathbb{R}$ *is such that* $h(\sigma_1, t_1) = 0$ *then there exist: else a* $\mathcal{C}^1$ *function* η *defined on a neighborhood of* σ_1 *and such that* $h(\sigma, \eta(\sigma)) = 0$ *in this neighborhood of* σ_1*; or a* $\mathcal{C}^1$ *function* ξ *defined on a neighborhood of* t_1 *and such that* $h(\xi(t), t) = 0$ *in this neighborhood of* t_1*, and moreover if* $\xi'(t_1) = 0$ *then* $\xi''(t_1) < 0$ *(it is a local maximum).*

The properties stated in Lemma 3 are illustrated on Figure 2, where the different types of curves formed by the set of points $\{(t, \sigma) \in \mathbb{R}^2; h(\sigma, t) = 0\}$ are shown for some h satisfying the heat equation.

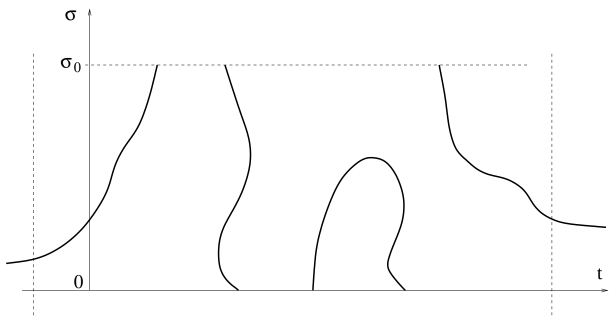

FIGURE 2. Curves of $h(\sigma, t) = 0$ for some h satisfying the heat equation, in the (t, σ) domain: here t is along the horizontal axis and σ is along the vertical one. According to Lemma 3, the zeros-crossings of h are a set of non-intersecting curves, that are locally else functions of σ or functions of t with no local minima.

Let us consider again the shot noise process $X_{\lambda,\sigma}$. We now give the main result for the number of local extrema of $X_{\lambda,\sigma}$ as a function of σ. The intensity λ is assumed to be fixed.

Theorem 4. *Let* $\sigma_0 > 0$ *and* $a \leq b$*. Then,*

$$\mathbb{P}(\exists (\sigma, t) \in (0, \sigma_0) \times [a,b] \text{ such that } X'_{\lambda,\sigma}(t) = 0 \text{ and } \nabla X'_{\lambda,\sigma}(t) = 0) = 0.$$

Moreover, if we assume that for all $0 < \sigma_1 < \sigma_0$

$$\mathbb{E}(\#\{\sigma \in [\sigma_1, \sigma_0] \text{ such that } X'_{\lambda,\sigma}(0) = 0\}) < +\infty,$$

then the function $\sigma \mapsto \rho(\lambda, \sigma)$*, which gives the mean number of local extrema of* $X_{\lambda,\sigma}$ *per unit length, is decreasing and it has the limit* 2λ *as* σ *goes to* 0*.*

Proof. Let us denote $Y(\sigma, t) := X'_{\lambda,\sigma}(t)$ for all $(\sigma, t) \in (0, +\infty) \times \mathbb{R}$. We first check that the assumptions (a) and (b) of Lemma 3 are satisfied almost surely for Y. Assumption (a) is already given by Proposition 9. For assumption (b), we first notice that since $Y(\sigma, t)$ satisfies the heat equation, we have

$$\{Y(\sigma, t) = 0 \text{ and } \nabla Y(\sigma, t) = 0\} = \left\{Y(\sigma, t) = 0 \text{ and } Y'(\sigma, t) = 0 \text{ and } Y''(\sigma, t) = 0\right\}.$$

Then a slight modification of the proof of Proposition 1, using the second-order Taylor formula in (10), allows us to conclude that $\mathbb{P}(\exists (\sigma, t) \in (0, \sigma_0) \times [a,b]$ such that $Y(\sigma, t) = 0$ and $\nabla Y(\sigma, t) = 0) = 0$, using the same integrability bound for the characteristic function of $Y(\sigma, t)$ as the one obtained in the proof of Proposition 9 (and considering first $(\sigma,t)\in(\sigma_1,\sigma_0)$ for $\sigma_1>0$, and conditioning by $\{\gamma_T\geq 3\}$). This also proves the first part of the theorem.

Let $0<\sigma_1<\sigma_0$ be fixed. By assumption, we have $\mathbb{E}(\#\{\sigma\in[\sigma_1,\sigma_0]$ such that $X'_{\lambda,\sigma}(0)=0\})<+\infty$. Notice that by stationarity this expected value is independent of the value of t (taken as 0 above). Let $T>0$ and let us consider the zeros of $Y(\sigma,t)=X'_{\lambda,\sigma}(t)$ for $(\sigma,t)\in[\sigma_1,\sigma_0]\times[0,T]$. Let $t_0\in[0,T]$ be such that $Y(\sigma_0,t_0)=0$. By Lemma 3, there is a continuous path $\sigma\mapsto\Gamma_{t_0}(\sigma)$ that will: else "cross the left or right boundary of the domain", *i.e.* be such that there exists $\sigma\in$ such that $\Gamma_{t_0}(\sigma)=0$ or T, or will be defined until σ_1 and such that $\Gamma_{t_0}(\sigma_1)\in[0,T]$. We thus have:

$$\rho(\sigma_0,[0,T])\leq 2\mathbb{E}(\#\{\sigma\in[\sigma_1,\sigma_0] \text{ such that } X'_{\lambda,\sigma}(0)=0)+\rho(\sigma_1,[0,T]).$$

Dividing both sides by T and letting T go to infinity then shows that $\rho(\sigma_0)\leq\rho(\sigma_1)$. Thus the function $\sigma\mapsto\rho(\lambda,\sigma)$ is decreasing.

To find the limit of $\rho(\lambda,\sigma)$ as σ goes to 0 (that exists thanks to the bound of Proposition 9), instead of looking at the local extrema of $X_{\lambda,\sigma}$ in $[0,1]$, we will only look at the local maxima (which are the down-crossings of 0 by the derivative) in $[0,1]$. Let $D_{X'_{\lambda,\sigma}}(0,[0,1])$ be the random variable that counts these local maxima, and let $\rho^-(\lambda,\sigma)=\mathbb{E}(D_{X'_{\lambda,\sigma}}(0,[0,1]))$. By stationarity of $X_{\lambda,\sigma}(t)$ and because between any two local maxima, there is a local minima, we have that $\rho^-(\lambda,\sigma)=\frac{1}{2}\rho(\lambda,\sigma)$. Now, we introduce "barriers" in the following way: let E_{σ_0} be the event "there are no points of the Poisson point process in the intervals $[-2\sigma_0,2\sigma_0]$ and $[1-2\sigma_0,1+2\sigma_0]$". If we assume that E_{σ_0} holds, then $X''_{\lambda,\sigma}(t)>0$ for all t in $[-\sigma_0,\sigma_0]\cup[1-\sigma_0,1+\sigma_0]$ and all $\sigma\leq\sigma_0$, and therefore there are no local maxima of $X_{\lambda,\sigma}$ in these intervals. Then by Lemma 3, we can follow all the local maxima of $X_{\lambda,\sigma}$ in $[0,1]$ from $\sigma=\sigma_0$ down to $\sigma=0$. Thus $\sigma\mapsto D_{X'_{\lambda,\sigma}}(0,[0,1])\mathbf{1}_{E_{\sigma_0}}$ is a decreasing function of σ for $\sigma\leq\sigma_0$. Moreover, we can also check that the set of local maxima of $X_{\lambda,\sigma}(t)$ in $[0,1]$ converges, as σ goes to 0, to the set of points of the Poisson process in $[0,1]$. This implies in particular that $D_{X'_{\lambda,\sigma}}(0,[0,1])$ goes to $\#\{\tau_i\in[0,1]\}$ as σ goes to 0. Thus by monotone convergence, it implies that $\rho^-(\lambda,\sigma|E_{\sigma_0})$ goes to $\mathbb{E}(\#\{\tau_i\in[0,1]\}|E_{\sigma_0})$. Since the sequence of events E_{σ_0} is an increasing sequence of events as σ_0 decreases to 0, we finally get:

$$\lim_{\sigma\to 0}\rho^-(\lambda,\sigma)=\lim_{\sigma_0\to 0}\mathbb{E}(\#\{\tau_i\in[0,1]\}|E_{\sigma_0})=\mathbb{E}(\#\{\tau_i\in[0,1]\})=\lambda.$$

□

Thus, under the assumption that $\mathbb{E}(\#\{\sigma\in[\sigma_1,\sigma_0]$ such that $X'_{\lambda,\sigma}(0)=0\})<+\infty$ for all $0<\sigma_1<\sigma_0$, Theorem 4 asserts that the function $\sigma\mapsto\rho(\lambda,\sigma)$ is a decreasing function with limit 2λ when $\sigma\to 0$. This fact was empirically observed on Figure 1, and is also illustrated on Figure 3 where we "follow" the local extrema as σ evolves. Now, these properties can be translated, using the scaling relations of Lemma 2, into the following properties on $\lambda\mapsto\rho(\lambda,\sigma)$:

$$\forall c\geq 1,\ \ \rho(c\lambda,\sigma)\leq c\rho(\lambda,\sigma);\ \ \rho(\lambda,\sigma)\leq 2\lambda \ \text{ and }\ \frac{\rho(\lambda,\sigma)}{2\lambda}\underset{\lambda\to 0}{\longrightarrow}1.$$

This shows the second asymptotic linear regime observed for small values of the intensity λ.

6. Appendix

6.1. Stationary phase estimate for oscillatory integrals.

Proposition 10 (Stationary phase estimate for oscillatory integrals)**.** *Let $a<b$ and let φ be a function of class $\mathcal{C}^2$ defined on $[a,b]$. Assume that φ' and φ'' cannot simultaneously vanish on $[a,b]$ and denote $m=\min_{s\in[a,b]}\sqrt{\varphi'(s)^2+\varphi''(s)^2}>0$. Let us also assume that $n_0=\#\{s\in[a,b]$ s. t. $\varphi''(s)=0\}<+\infty$. Then*

$$\forall u\in\mathbb{R} \text{ s.t. } |u|>\frac{1}{m},\ \ \left|\int_a^b e^{iu\varphi(s)}\,ds\right|\leq\frac{8\sqrt{2}(2n_0+1)}{\sqrt{m|u|}}.$$

Now, let φ_1 and φ_2 be two functions of class $\mathcal{C}^3$ defined on $[a,b]$. Assume that the derivatives of these functions are linearly independent, in the sense that for all $s\in[a,b]$, the matrix $\Phi(s)=$

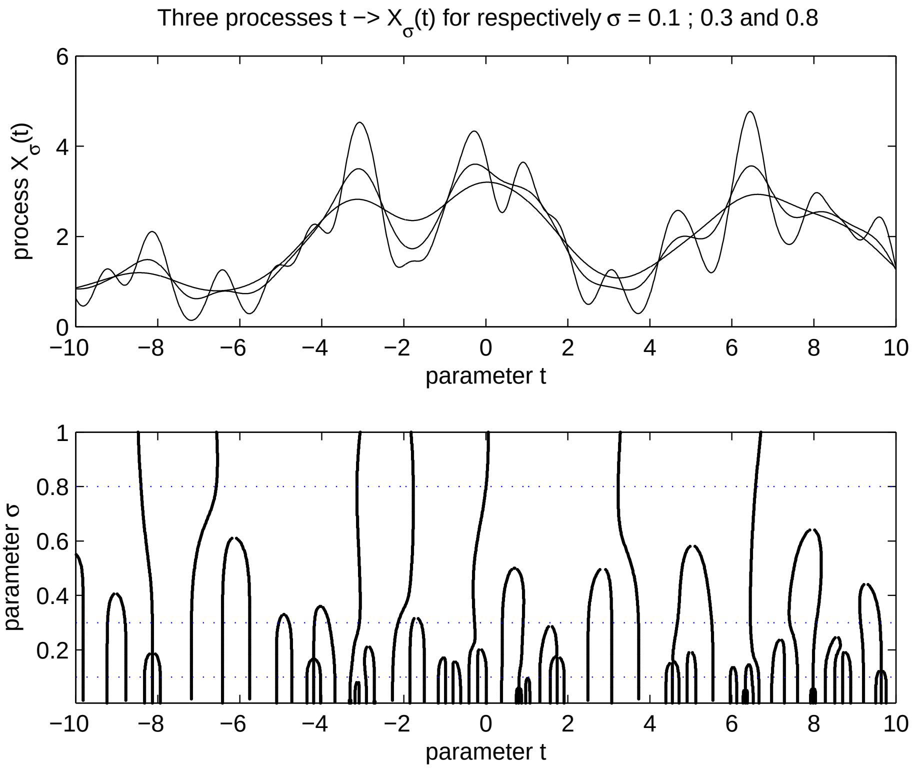

FIGURE 3. Top: three processes $t \mapsto X_{\lambda,\sigma}(t)$ obtained from the same Poisson point process of intensity $\lambda = 2$ and for a Gaussian kernel of respective width $\sigma = 0.1; 0.3$ and 0.8. Bottom: evolution of the local extrema of $t \mapsto X_{\lambda,\sigma}(t)$ as σ goes from 0 to 1. The three values $\sigma = 0.1; 0.3$ and 0.8 are plotted as dotted line. They indicate the local extrema of the three processes above.

$\begin{pmatrix} \varphi_1'(s) & \varphi_2'(s) \\ \varphi_1''(s) & \varphi_2''(s) \end{pmatrix}$ *is invertible. Denote* $m = \min_{s\in[a,b]} \| \Phi(s)^{-1} \|^{-1} > 0$, *where* $\| \cdot \|$ *is the matricial norm induced by the Euclidean one. Assume moreover that there exists* $n_0 < +\infty$ *such that* $\#\{s \in [a,b] \text{ s.t. } \det(\Phi'(s)) = 0\} \le n_0$, *where* $\Phi'(s) = \begin{pmatrix} \varphi_1''(s) & \varphi_2''(s) \\ \varphi_1^{(3)}(s) & \varphi_2^{(3)}(s) \end{pmatrix}$. *Then*

$$\forall (u,v) \in \mathbb{R}^2 \text{ s.t. } \sqrt{u^2+v^2} > \frac{1}{m}, \quad \left| \int_a^b e^{iu\varphi_1(s)+iv\varphi_2(s)}\, ds \right| \le \frac{8\sqrt{2}(2n_0+3)}{\sqrt{m\sqrt{u^2+v^2}}}.$$

Proof. For the first part of the proposition, by assumption, $[a,b]$ is the union of the three compact sets

$$\left\{s\in[a,b]; |\varphi''| \ge m/2\right\}, \left\{s\in[a,b]; |\varphi'| \ge m/2 \text{ and } \varphi'' \ge 0\right\} \text{ and } \left\{s\in[a,b]; |\varphi'| \ge m/2 \text{ and } \varphi'' \le 0\right\}.$$

Therefore there exists $1 \le n \le 2n_0 + 1$ and a subdivision $(a_i)_{0\le i\le n}$ of $[a,b]$ such that $[a_{i-1}, a_i]$ is included in one of the previous subsets for any $1 \le i \le n$. If $[a_{i-1}, a_i] \subset \{s \in [a,b]; |\varphi''(s)| \ge m/2\}$, according to Proposition 2 p.332 of [26]

$$\left| \int_{a_{i-1}}^{a_i} e^{iu\varphi(s)}\, ds \right| = \left| \int_{a_{i-1}}^{a_i} e^{iu(m/2)(2\varphi(s)/m)}\, ds \right| \le 8\frac{\sqrt{2}}{\sqrt{m|u|}};$$

otherwise,

$$\left| \int_{a_{i-1}}^{a_i} e^{iu\varphi(s)}\, ds \right| \le \frac{6}{m|u|}$$

The result follows from summing up these n integrals.

For the second part of the proposition, we use polar coordinates, and write $(u,v)=(r\cos\theta, r\sin\theta)$. For $\theta\in[0,2\pi)$, let φ_θ be the function defined on $[a,b]$ by $\varphi_\theta(s)=\varphi_1(s)\cos\theta+\varphi_2(s)\sin\theta$. Then $\begin{pmatrix}\varphi'_\theta(s)\\ \varphi''_\theta(s)\end{pmatrix}=\Phi(s)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$, and thus $1=\|\ \Phi(s)^{-1}\begin{pmatrix}\varphi'_\theta(s)\\ \varphi''_\theta(s)\end{pmatrix}\|$. This implies that for all $s\in[a,b]$, $\sqrt{\varphi'_\theta(s)^2+\varphi''_\theta(s)^2}\geq 1/\ \|\ \Phi(s)^{-1}\ \|\geq m$. Moreover, thanks to Rolle's Theorem, the number of points $s\in[a,b]$ such that $\varphi''_\theta(s)=0$ is bounded by one plus the number of $s\in[a,b]$ such that $\varphi''_1(s)\varphi'''_2(s)-\varphi'''_1(s)\varphi''_2(s)=0$, that is by $1+n_0$. Thus, we can apply the result of the first part of the proposition to each function φ_θ and the obtained bound will depend only on m, n_0 and $r=\sqrt{u^2+v^2}$. □

6.2. **Proof of Proposition 8.** For $k\geq 0$ and $l\geq 0$ integers, let us denote $m_{kl}=\int|g(s)|^k|g'(s)|^l\,ds$. We will also simply denote $m_0=m_{20}=\int g(s)^2\,ds$ and $m_2=m_{02}=\int g'(s)^2\,ds$.

Let $\psi_\lambda(u,v)$ denote the joint characteristic function of $(Z_\lambda(t),Z'_\lambda(t))$, then

$$\psi_\lambda(u,v)=\mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}X_\lambda+i\frac{v}{\sqrt{\lambda}}X'_\lambda})e^{-iu\sqrt{\lambda}\int g}=\exp\left(\lambda\int_{\mathbb{R}}(e^{i\frac{u}{\sqrt{\lambda}}g(s)+i\frac{v}{\sqrt{\lambda}}g'(s)}-1-i\frac{u}{\sqrt{\lambda}}g(s))\,ds\right).$$

We now use the fact $\int g'=0$, and we thus have $\psi_\lambda(u,v)=\exp(H_\lambda(u,v))$ where

$$H_\lambda(u,v)=\lambda\int_{\mathbb{R}}\left(e^{i\frac{u}{\sqrt{\lambda}}g(s)+i\frac{v}{\sqrt{\lambda}}g'(s)}-1-i\frac{u}{\sqrt{\lambda}}g(s)-i\frac{v}{\sqrt{\lambda}}g'(s)\right)ds.$$

We need to notice that

$$\forall(u,v)\in\mathbb{R}^2,\ \ |\psi_\lambda(u,v)|=|\exp(H_\lambda(u,v))|=|\mathbb{E}(e^{iuZ_\lambda+ivZ'_\lambda})|\leq 1.$$

In the following, we will also need these simple bounds:

$$\forall x\in\mathbb{R},\ \ |e^{ix}-1-ix+\frac{x^2}{2}|\leq\frac{|x|^3}{3!}\quad\text{and}\quad\forall z\in\mathbb{C},\ \ |e^z-1|\leq|z|e^{|z|}. \tag{28}$$

We first estimate $H_\lambda(u,0)$. We have

$$H_\lambda(u,0)=\lambda\int(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1-i\frac{u}{\sqrt{\lambda}}g(s))\,ds=-\frac{1}{2}u^2m_0+K_\lambda(u),$$

where $K_\lambda(u)=\lambda\int(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1-i\frac{u}{\sqrt{\lambda}}g(s)+\frac{1}{2}\frac{u^2}{\lambda}g^2(s))\,ds$. Then, thanks to the simple bounds (28), we get

$$|K_\lambda(u)|\leq\frac{|u|^3m_{30}}{6\sqrt{\lambda}}\ \text{and consequently}\ |e^{H_\lambda(u,0)}-e^{-\frac{1}{2}u^2m_0}|\leq\frac{|u|^3m_{30}}{6\sqrt{\lambda}}e^{-\frac{1}{2}u^2m_0}e^{\frac{|u|^3m_{30}}{6\sqrt{\lambda}}}.$$

We then estimate $H_\lambda(u,v)-H_\lambda(u,0)$:

$$\begin{aligned}H_\lambda(u,v)-H_\lambda(u,0)&=\lambda\int(e^{i\frac{u}{\sqrt{\lambda}}g(s)+i\frac{v}{\sqrt{\lambda}}g'(s)}-e^{i\frac{u}{\sqrt{\lambda}}g(s)})\,ds\\&=\lambda\int e^{i\frac{u}{\sqrt{\lambda}}g(s)}(e^{i\frac{v}{\sqrt{\lambda}}g'(s)}-1-i\frac{v}{\sqrt{\lambda}}g'(s))\,ds\\&=-\frac{v^2}{2}\int g'(s)^2e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds+F_\lambda(u,v),\end{aligned}$$

where $F_\lambda(u,v)=\lambda\int e^{i\frac{u}{\sqrt{\lambda}}g(s)}(e^{i\frac{v}{\sqrt{\lambda}}g'(s)}-1-i\frac{v}{\sqrt{\lambda}}g'(s)+\frac{v^2}{2\lambda}g'(s)^2)\,ds$. And again, thanks to the simple bounds (28), we get: $|F_\lambda(u,v)|\leq\frac{|v|^3m_{03}}{6\sqrt{\lambda}}$. This implies that

$$\begin{aligned}\left|e^{H_\lambda(u,v)-H_\lambda(u,0)}-e^{-\frac{v^2}{2}\int g'(s)^2e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}\right|&\leq\left|e^{-\frac{v^2}{2}\int g'(s)^2e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}\right|\cdot\left|e^{F_\lambda(u,v)}-1\right|\\&\leq\frac{|v|^3m_{03}}{6\sqrt{\lambda}}e^{-\frac{v^2}{2}\int g'(s)^2cos(\frac{u}{\sqrt{\lambda}}g(s))\,ds+\frac{|v|^3m_{03}}{6\sqrt{\lambda}}}.\end{aligned}$$

Let us now compute $\widehat{C_{Z_\lambda}}(u,[0,1])$. By Proposition 1, we know that

$$-\pi\widehat{C_{Z_\lambda}}(u,[0,1]) = \int_0^{+\infty} \frac{1}{v^2}\left(\psi_\lambda(u,v)+\psi_\lambda(u,-v)-2\psi_\lambda(u,0)\right) dv.$$

Let $V>0$ be a real number. We split the integral above in two parts, and write it as the sum of the integral between 0 and V, and of the integral between V and $+\infty$. Since for all (u,v), we have $|\psi_\lambda(u,v)|\le 1$, we get

$$\left|\int_V^{+\infty} \frac{1}{v^2}\left(\psi_\lambda(u,v)+\psi_\lambda(u,-v)-2\psi_\lambda(u,0)\right) dv\right| \le 4\int_V^{+\infty}\frac{1}{v^2}\,dv = \frac{4}{V}.$$

On the other hand, let $I_V(u)$ denote the integral between 0 and V. We have

$$I_V(u) = \int_0^V \frac{1}{v^2}e^{H_\lambda(u,0)}\left(e^{H_\lambda(u,v)-H_\lambda(u,0)}+e^{H_\lambda(u,-v)-H_\lambda(u,0)}-2\right) dv.$$

We then decompose this into:

$$\begin{aligned} I_V(u) &= \int_0^V \frac{1}{v^2}e^{H_\lambda(u,0)}\left(e^{H_\lambda(u,v)-H_\lambda(u,0)}+e^{H_\lambda(u,-v)-H_\lambda(u,0)}-2e^{-\frac{v^2}{2}\int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}\right) dv \\ &+\int_0^V \frac{1}{v^2}e^{H_\lambda(u,0)}\left(2e^{-\frac{v^2}{2}\int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}-2e^{-\frac{v^2}{2}m_2}\right) dv \\ &+\int_0^V \frac{1}{v^2}\left(e^{H_\lambda(u,0)}-e^{-\frac12 u^2 m_0}+e^{-\frac12 u^2 m_0}\right)\left(2e^{-\frac{v^2}{2}m_2}-2\right) dv. \end{aligned}$$

Using the bounds we computed above, we get that

$$\begin{aligned} \left|I_V(u)-2e^{-\frac12 u^2 m_0}\int_0^V \frac{e^{-\frac{v^2}{2}m_2}-1}{v^2}\,dv\right| &\le 2\int_0^V \frac{vm_{03}}{6\sqrt{\lambda}}e^{-\frac{v^2}{2}\int g'(s)^2\cos(\frac{u}{\sqrt{\lambda}}g(s))\,ds+\frac{|v|^3 m_{03}}{6\sqrt{\lambda}}}\,dv \\ &+2\int_0^V \frac{1}{v^2}\left|e^{-\frac{v^2}{2}\int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}-2e^{-\frac{v^2}{2}m_2}\right| dv \\ &+2\left|e^{H_\lambda(u,0)}-e^{-\frac12 u^2 m_0}\right|\int_0^V \frac{1-e^{-\frac{v^2}{2}m_2}}{v^2}\,dv. \end{aligned}$$

Let $J_V^{(n)}(u)$, for $n=1,2,3$ respectively denote the three terms above. To give an upper bound for $J_V^{(1)}(u)$, we will need the following basic inequality: $\forall x\in\mathbb{R}$, $\cos(x)\ge 1-\frac{x^2}{2}$. This gives us the bound:

$$J_V^{(1)}(u)\le 2\int_0^V \frac{vm_{03}}{6\sqrt{\lambda}}e^{-\frac{v^2 m_2}{2}+\frac{v^2}{2}\frac{u^2 m_{22}}{2\lambda}+\frac{|v|^3 m_{03}}{6\sqrt{\lambda}}}\,dv.$$

For the second term, we use

$$\begin{aligned} \left|e^{-\frac{v^2}{2}\int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)}\,ds}-2e^{-\frac{v^2}{2}m_2}\right| &\le e^{-\frac{v^2}{2}m_2}\left|e^{-\frac{v^2}{2}\int g'(s)^2(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1)\,ds}-1\right| \\ &\le e^{-\frac{v^2}{2}m_2}\left|\frac{v^2}{2}\int g'(s)^2(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1)\,ds\right| e^{\left|\frac{v^2}{2}\int g'(s)^2(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1)\,ds\right|}. \end{aligned}$$

But $\left|\int g'(s)^2(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1)\,ds\right|\le\int g'(s)^2\frac{|u|}{\sqrt{\lambda}}g(s)\,ds=\frac{|u|}{\sqrt{\lambda}}m_{12}$ and thus

$$J_V^{(2)}(u)\le\frac{|u|}{\sqrt{\lambda}}m_{12}\int_0^V e^{-\frac{v^2}{2}m_2+\frac{v^2}{2}\frac{|u|}{\sqrt{\lambda}}m_{12}}\,dv.$$

For the third term, we use an integration by parts to obtain that

$$\int_0^V \frac{1-e^{-\frac{v^2}{2}m_2}}{v^2}\,dv = \frac{e^{-\frac{V^2}{2}m_2}-1}{V}+\int_0^V m_2 e^{-\frac{v^2}{2}m_2}\,dv \le \frac12\sqrt{2\pi m_2},$$

which gives

$$J_V^{(3)}(u) \leq \sqrt{2\pi m_2}\frac{|u|^3 m_{30}}{6\sqrt{\lambda}} e^{-\frac{1}{2}u^2 m_0 + \frac{|u|^3 m_{30}}{6\sqrt{\lambda}}}.$$

Moreover we also have

$$\left| 2\int_0^V \frac{1-e^{-\frac{v^2}{2}m_2}}{v^2}\,dv - \sqrt{2\pi m_2} \right| \leq \frac{1-e^{-\frac{V^2}{2}m_2}}{V} + \int_V^{+\infty} m_2 e^{-\frac{v^2}{2}m_2}\,dv \leq \frac{2}{V}.$$

The partial conclusion of all these estimates is that

$$\left| \pi\widehat{C_{Z_\lambda}}(u,[0,1]) - \sqrt{2\pi m_2}e^{-m_0u^2/2} \right| \leq \frac{4}{V} + \frac{2e^{-m_0u^2/2}}{V} + J_V^{(1)}(u) + J_V^{(2)}(u) + J_V^{(3)}(u).$$

We now have to choose V in an appropriate way. The choice of V will be given by the bound on $J_V^{(1)}(u)$. Assume in the following that u satisfies the condition $(U1)$ given by: $\frac{u^2 m_{22}}{2\lambda} \leq \frac{m_2}{4}$, and let us set

$$V = \frac{3\sqrt{\lambda}m_2}{4m_{03}}.$$

Then for all $v \in [0,V]$, $-\frac{v^2 m_2}{2} + \frac{v^2}{2}\frac{u^2 m_{22}}{2\lambda} + \frac{|v|^3 m_{03}}{6\sqrt{\lambda}} \leq -\frac{v^2 m_2}{4}$, and thus

$$J_V^{(1)}(u) \leq \frac{m_{03}}{3\sqrt{\lambda}} \int_0^V v e^{-\frac{v^2 m_2}{4}}\,dv \leq \frac{2m_{03}}{3m_2\sqrt{\lambda}}.$$

For the term $J_V^{(2)}(u)$, we notice that if u satisfies the condition $(U2)$ given by: $\frac{|u|}{\sqrt{\lambda}}m_{12} \leq \frac{m_2}{2}$, then for all $V>0$, we can bound $J_V^{(2)}(u)$ by:

$$J_V^{(2)}(u) \leq \frac{|u|}{\sqrt{\lambda}}m_{12}\int_0^V e^{-\frac{v^2}{4}m_2}\,dv. \leq \frac{|u|}{\sqrt{\lambda}}m_{12}\sqrt{\frac{\pi}{m_2}}.$$

Finally, for the third term, we have that if u satisfies the condition $(U3)$ given by: $\frac{|u|m_{30}}{3\sqrt{\lambda}} \leq \frac{1}{2}m_0$, then $J_V^{(3)}(u)$ can be bounded, independently of V, by

$$J_V^{(3)}(u) \leq \sqrt{2\pi m_2}\frac{|u|^3 m_{30}}{6\sqrt{\lambda}}e^{-\frac{1}{4}u^2 m_0} \leq \sqrt{2\pi m_2}\frac{2|u|m_{30}}{3m_0\sqrt{\lambda}}e^{-1},$$

because of the fact that for all $x \geq 0$, then $xe^{-x} \leq e^{-1}$.
The final conclusion of all these computations is that if we set $a_1 = \min(\sqrt{\frac{m_2}{2m_{22}}}, \frac{m_2}{2m_{12}}, \frac{3m_0}{2m_{30}})$, then for all u and $\lambda > 0$ we have

$$|u| \leq a_1\sqrt{\lambda} \Longrightarrow |\pi\widehat{C_{Z_\lambda}}(u,[0,1]) - \sqrt{2\pi m_2}e^{-m_0u^2/2}| \leq \frac{a_2}{\sqrt{\lambda}} + \frac{a_3|u|}{\sqrt{\lambda}},$$

where $a_2 = \frac{24m_{30}+2m_{03}}{3m_2}$ and $a_3 = m_{12}\sqrt{\frac{\pi}{m_2}} + \frac{2\sqrt{2\pi m_2}m_{30}e^{-1}}{3m_0}$.

6.3. **Proof of Lemma 3.** The proof of this lemma relies upon the implicit function theorem. Let us start with the proof of i): let (σ_0,t_0) be a point such that $h(\sigma_0,t_0)=0$. By Assumption (a), we have that $\frac{\partial h}{\partial t}(\sigma_0,t_0) \neq 0$. Then, thanks to the implicit function theorem, there exist two open intervals $I=(\sigma_0^-,\sigma_0^+)$ and $J=(t_0^-,t_0^+)$ containing respectively σ_0 and t_0, and a C^1 function $\eta : I \to J$ such that $\eta(\sigma_0)=t_0$ and $\forall(\sigma,t)\in I\times J$, $h(\sigma,t)=0 \Leftrightarrow t=\eta(\sigma)$. Let us now denote $\eta=\Gamma_{t_0}$. We need to prove that we can take $\sigma_0^-=0$ when Γ_{t_0} remains bounded. Assume we cannot: the maximal interval on which Γ_{t_0} is defined is (σ_0^-,σ_0^+) with $\sigma_0^->0$. By assumption, there is an $M_0>0$ such that for all $\sigma\in(\sigma_0^-,\sigma_0^+)$, then $|\Gamma_{t_0}(\sigma)| \leq M_0$. We can thus find a sub-sequence (σ_k) converging to σ_0^- as k goes to infinity and a point $t_1\in[-M_0,M_0]$ such that $\Gamma_{t_0}(\sigma_k)$ goes to t_1 as k goes to infinity. By continuity of h, we have $h(\sigma_0^-,t_1)=0$. Now, we also have $\frac{\partial h}{\partial t}(\sigma_0^-,t_1)=0$. Indeed, if it were $\neq 0$, we could again apply the implicit function theorem in the same way at the point (σ_0^-,t_1), and get a contradiction with the maximality of $I=(\sigma_0^-,\sigma_0^+)$. Then, by Assumption (b), we have $\frac{\partial h}{\partial \sigma}(\sigma_0^-,t_1)\neq 0$. We can again apply the implicit function theorem, and we thus obtain that there exist two open intervals $I_1 = (\sigma_1^-, \sigma_1^+)$ and $J_1 = (t_1^-, t_1^+)$ containing respectively σ_0^- and t_1, and a C^1 function $\xi : J_1 \to I_1$ such that $\xi(t_1) = \sigma_0^-$ and $\forall(\sigma, t) \in I_1 \times J_1$, $h(\sigma, t) = 0 \Leftrightarrow \sigma = \xi(t)$. Moreover we can compute the derivatives of ξ at t_1. We start from the implicit definition of ξ: $h(\xi(t), t) = 0$. By differentiation, we get $\xi'(t)\frac{\partial h}{\partial \sigma}(\xi(t), t) + \frac{\partial h}{\partial t}(\xi(t), t) = 0$. Taking the value at $t = t_1$, we get $\xi'(t_1) = 0$. We can again differentiate, and find $\xi''(t)\frac{\partial h}{\partial \sigma}(\xi(t), t) + \xi'(t)^2\frac{\partial^2 h}{\partial \sigma^2}(\xi(t), t) + 2\xi'(t)\frac{\partial^2 h}{\partial \sigma \partial t}(\xi(t), t) + \frac{\partial^2 h}{\partial t^2}(\xi(t), t) = 0$. Taking again the value at $t = t_1$, we get

$$\xi''(t_1) = -\frac{1}{\xi(t_1)} = -\frac{1}{\sigma_0^-} < 0.$$

Thus it shows that ξ has a strict local maximum at t_1: there exist a neighborhood U_1 of $\sigma_0^- = \xi(t_1)$ and a neighborhood V_1 of t_1 such that for all points in $U_1 \times V_1$, then $h(\sigma, t) = 0$ implies $\sigma = \xi(t) \leq \xi(t_1) = \sigma_0^-$, which is in contradiction with the definition of Γ_{t_0} on (σ_0^-, σ_0^+). This ends the proof of i), and also of iii).

For ii): assume that t_0 and $\widetilde{t_0}$ are two points such that $h(\sigma_0, t_0) = h(\sigma_0, \widetilde{t_0}) = 0$ and such that there exists $\sigma_1 < \sigma_0$ such that $\Gamma_{t_0}(\sigma_1) = \Gamma_{\widetilde{t_0}}(\sigma_1) = t_1$. Then, if $\frac{\partial h}{\partial t}(\sigma_1, t_1) \neq 0$, the implicit function theorem implies that $\Gamma_{t_0}(\sigma) = \Gamma_{\widetilde{t_0}}(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_0]$ and in particular $t_0 = \widetilde{t_0}$. But now, if $\frac{\partial h}{\partial t}(\sigma_1, t_1) = 0$, then as above this implies that $\frac{\partial h}{\partial \sigma}(\sigma_1, t_1) \neq 0$ and using again the implicit function theorem, this would be in contradiction with the fact $\Gamma_{t_0}(\sigma)$ is defined for $\sigma \in [\sigma_1, \sigma_0]$.

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