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Pierre Gaillard. Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation. 2012. hal-00589556v3

**HAL Id: hal-00589556**

**<https://hal.science/hal-00589556v3>**

Preprint submitted on 16 Sep 2012

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# Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation.

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April 25, 2011

## Abstract

This work is a continuation of a recent paper in which we have constructed a multi-parametric family of solutions of the focusing NLS equation given in terms of Wronskians determinants of order  $2N$  composed of elementary trigonometric functions. When we perform a special passage to the limit when all the periods tend to infinity, we get a family of quasi-rational solutions. Here we construct Peregrine breathers of orders  $N = 4, 5$ , and the multi-rogue waves corresponding the 10 or 15 peak formation in frame of the NLS model first explained by Matveev et al. in 2010. In the cases  $N = 4, 5$  we get comfortable formulas to study the deformation of higher Peregrine breather of order 4 to the 10 rogue-waves or order 5 to the 15 rogue-waves solution via variation of the free parameters of our construction.

## 1 Introduction

The nonlinear Schrödinger equation (NLS) was first derived by Zakharov [16] in 1968. The solutions were first given by Zakharov and Schabat. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [12]. The first quasi-rational solutions of NLS equation appeared in 1983 with the work of Peregrine, nowadays called worldwide Peregrine breathers. In 1986 Eleonski,

Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather[3]. Other families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations. In 2010 it has been shown in [6] that rational solutions of NLS equation can be written as a quotient of two wronskians and it has been recovered as particular case, Akhmediev's quasi-rational solutions of NLS equation.

In this paper, we extend a result [10] giving a new representation of the solutions of the NLS equation in terms of a ratio of two wronskians determinants of even order  $2N$  composed of elementary functions; the related solutions of NLS are called of order  $N$ . When we perform the passage to the limit when some parameter tends to 0, we got families of multi-rogue wave solutions of the focusing NLS equation depending on a certain number of parameters. It allows to recognize the famous Peregrine's breather [14] and also higher order Peregrine's breathers constructed by Akhmediev [1, 4]. We treat in the following the cases of order  $N = 4$  and  $N = 5$ . We get for an arbitrary choice of the parameters the shape of Akhmediev's breathers; we can also get, for particular parameters, the apparition of peaks of similar amplitude for the modulus of the solution  $v$  in the  $(x; y)$  coordinates.

## 2 Expression of solutions of NLS equation in terms of Wronskian determinant and quasi-rational limit

### 2.1 Solutions of NLS equation in terms of Wronskian determinant

We briefly recall results obtained in [10]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \tag{1}$$

From [10], the solution of the NLS equation can be written in the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \tag{2}$$

In (2), the matrix  $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$  is defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \quad (3)$$

The terms  $\epsilon_\nu$  are defined by :

$$\begin{aligned} \epsilon_\nu &= 0, & 1 \leq \nu \leq N \\ \epsilon_\nu &= 1, & N + 1 \leq \nu \leq 2N. \end{aligned} \quad (4)$$

We consider the following functions

$$\begin{aligned} \phi_\nu^r(y) &= \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & 1 \leq \nu \leq N, \\ \phi_\nu^r(y) &= \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & N + 1 \leq \nu \leq 2N. \end{aligned} \quad (5)$$

We use the following notations :

$$\begin{aligned} \Theta_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, & 1 \leq \nu \leq 2N. \\ W_r(y) &= W(\phi_1, \dots, \phi_{2N}) \text{ is the wronskian} \end{aligned}$$

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]. \quad (6)$$

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined by

$$\begin{aligned} d_{\nu\mu} &= (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} - ie_\nu), \\ &1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N, \end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we get the following link between Fredholm and Wronskian determinants [9]

**Theorem 2.1**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (7)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

It can be deduced the following result :

**Theorem 2.2** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \quad (8)$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

## 2.2 Quasi-rational solutions of NLS equation

In the following, we take the limit when the parameters  $\lambda_j \rightarrow 1$  for  $1 \leq j \leq N$  and  $\lambda_j \rightarrow -1$  for  $N + 1 \leq j \leq 2N$ .

For simplicity, we denote  $d_j$  the term  $\frac{c_j}{\sqrt{2}}$ .

We consider the parameter  $\lambda_j$  written in the form

$$\lambda_j = 1 - 2\epsilon^2 d_j^2, \quad 1 \leq j \leq N. \quad (9)$$

When  $\epsilon$  goes to 0, we realize limited expansions at order  $p$ , for  $1 \leq j \leq N$ , of the terms

$$\begin{aligned} \kappa_j &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_j = 4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_j &= d_j\epsilon(1 - \epsilon^2 d_j^2)^{-1/2}, \quad x_{r,j} = (r - 1) \ln \frac{1 + i\epsilon d_j(1 - \epsilon^2 d_j^2)^{-1/2}}{1 - i\epsilon d_j(1 - \epsilon^2 d_j^2)^{-1/2}}, \\ \kappa_{N+j} &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_{N+j} = -4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_{N+j} &= 1/(d_j\epsilon)(1 - \epsilon^2 d_j^2)^{1/2}, \quad x_{r,N+j} = (r - 1) \ln \frac{1 - i\epsilon d_j(1 - \epsilon^2 d_j^2)^{-1/2}}{1 + i\epsilon d_j(1 - \epsilon^2 d_j^2)^{-1/2}}. \end{aligned}$$

The parameters  $a_j$  and  $b_j$ , for  $1 \leq N$  are chosen in the form

$$a_j = \tilde{a}_j \epsilon^{M-1}, \quad b_j = \tilde{b}_j \epsilon^{M-1}, \quad 1 \leq j \leq N, \quad M = 2N. \quad (10)$$

We have the central result given in [10] :

**Theorem 2.3** *With the parameters  $\lambda_j$  defined by (9),  $a_j$  and  $b_j$  chosen as in (10), for  $1 \leq j \leq N$ , the function  $v$  defined by*

$$v(x, t) = \exp(2it - i\varphi) \lim_{\epsilon \rightarrow 0} \frac{W_3(0)}{W_1(0)}, \quad (11)$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

*depending on  $3N$  parameters  $d_j, \tilde{a}_j, \tilde{b}_j, 1 \leq j \leq N$ .*

### 3 Quasi-rational solutions of order $N$

To get solutions of NLS equation written in the context of fiber optics

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0, \quad (12)$$

from these of (1), we can make the following changes of variables

$$\begin{aligned} t &\rightarrow X/2 \\ x &\rightarrow T. \end{aligned} \quad (13)$$

In the following, because of the complexity of the expression, we give all the solutions in the case where the parameters  $d_j$  are equal to  $j$  and all the parameters  $a_j$  and  $b_j$  are equal to 0.

The solution of NLS equation can be written in the form

$$v_N(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it - i\varphi) = \left(1 - \alpha_N \frac{G_N(2x, 4t) + iH_N(2x, 4t)}{Q_N(2x, 4t)}\right) e^{2it}$$

with

$$G_N(X, T) = \sum_{k=0}^{N(N+1)} \mathbf{g}_k(T) X^k$$

$$H_N(X, T) = \sum_{k=0}^{N(N+1)} \mathbf{h}_k(T) X^k$$

$$Q_N(X, T) = \sum_{k=0}^{N(N+1)} \mathbf{q}_k(T) X^k$$

#### 3.1 Case $N=4$

In the case of order  $N = 4$ , we make an expansion at order 7. Taking the limit when  $\epsilon \rightarrow 0$ , the solution of NLS equation (12) takes the form

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it - i\varphi).$$

$$\begin{aligned}
&\alpha_4 = 4, \quad \mathbf{g}_{20} = 0, \quad \mathbf{g}_{19} = 0, \quad \mathbf{g}_{18} = 10, \quad \mathbf{g}_{17} = 0, \quad \mathbf{g}_{16} = 270T^2 + 270, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 1800T^4 \\
&\quad - 3600T^2 + 9000, \quad \mathbf{g}_{13} = 0, \quad \mathbf{g}_{12} = 5880T^6 - 54600T^4 - 12600T^2 + 189000, \quad \mathbf{g}_{11} = 0, \\
&\mathbf{g}_{10} = 11340T^8 - 176400T^6 + 189000T^4 - 378000T^2 - 1077300, \quad \mathbf{g}_9 = 0, \\
&\mathbf{g}_8 = 13860T^{10} - 207900T^8 + 2356200T^6 + 1701000T^4 - 56983500T^2 - 4819500, \\
&\mathbf{g}_7 = 0, \quad \mathbf{g}_6 = 10920T^{12} - 18480T^{10} + 6967800T^8 + 56095200T^6 - 342657000T^4 \\
&\quad + 198450000T^2 - 11907000, \quad \mathbf{g}_5 = 0 \quad \mathbf{g}_4 = 5400T^{14} + 163800T^{12} + 9034200T^{10} \\
&\quad + 107919000T^8 - 615195000T^6 + 178605000T^4 + 654885000T^2 + 178605000, \quad \mathbf{g}_3 = 0, \\
&\mathbf{g}_2 = 1530T^{16} + 133200T^{14} + 5506200T^{12} - 116802000T^{10} - 1731334500T^8 \\
&\quad + 2532222000T^6 - 893025000T^4 + 4643730000T^2 + 223256250, \quad \mathbf{g}_1 = 0, \\
&\mathbf{g}_0 = 190T^{18} + 33150T^{16} + 1294200T^{14} + 3288600T^{12} + 48629700T^{10} \\
&\quad - 2015401500T^8 - 1845585000T^6 + 14586075000T^4 + 2098608750T^2 - 44651250,
\end{aligned}$$

$$\begin{aligned}
&\mathbf{h}_{20} = 0, \quad \mathbf{h}_{19} = 0, \quad \mathbf{h}_{18} = 10T, \quad \mathbf{h}_{17} = 0, \quad \mathbf{h}_{16} = 90T^3 - 270T, \quad \mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 360T^5 \\
&\quad - 6000T^3 - 5400T, \quad \mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 840T^7 - 29400T^5 + 12600T^3 - 138600T, \quad \mathbf{h}_{11} = 0, \\
&\mathbf{h}_{10} = 1260T^9 - 65520T^7 + 259560T^5 - 529200T^3 - 1984500T, \quad \mathbf{h}_9 = 0, \\
&\mathbf{h}_8 = 1260T^{11} - 77700T^9 + 718200T^7 - 5329800T^5 - 6142500T^3 + 29767500T, \\
&\mathbf{h}_7 = 0, \quad \mathbf{h}_6 = 840T^{13} - 48720T^{11} + 718200T^9 + 2973600T^7 - 72765000T^5 \\
&\quad + 436590000T^3 + 146853000T, \quad \mathbf{h}_5 = 0, \quad \mathbf{h}_4 = 360T^{15} - 12600T^{13} + 138600T^{11} \\
&\quad - 5859000T^9 - 328293000T^7 + 1075599000T^5 + 773955000T^3 + 535815000T, \quad \mathbf{h}_3 = 0, \\
&\mathbf{h}_2 = 90T^{17} + 1200T^{15} - 189000T^{13} - 40143600T^{11} \\
&\quad - 307786500T^9 + 2085426000T^7 - 4465125000T^5 + 4405590000T^3 - 1205583750T, \quad \mathbf{h}_1 = 0, \\
&\mathbf{h}_0 = 10T^{19} + 930T^{17} - 86040T^{15} - 7018200T^{13} - 48100500T^{11} - 542902500T^9 \\
&\quad + 6039117000T^7 + 12942909000T^5 + 937676250T^3, \quad \mathbf{q}_{20} = 1, \quad \mathbf{q}_{19} = 0, \\
&\mathbf{q}_{18} = 10T^2 + 10, \quad \mathbf{q}_{17} = 0, \quad \mathbf{q}_{16} = 45T^4 - 270T^2 + 405, \quad \mathbf{q}_{15} = 0, \\
&\mathbf{q}_{14} = 120T^6 - 1800T^4 + 1800T^2 + 16200, \quad \mathbf{q}_{13} = 0, \quad \mathbf{q}_{12} = 210T^8 - 4200T^6 + 6300T^4 \\
&\quad + 113400T^2 + 425250, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 252T^{10} - 3780T^8 + 63000T^6 \\
&\quad + 718200T^4 + 3005100T^2 + 1644300, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 210T^{12} + 1260T^{10} \\
&\quad + 255150T^8 - 567000T^6 + 23388750T^4 - 31468500T^2 + 17435250, \quad \mathbf{q}_7 = 0, \\
&\mathbf{q}_6 = 120T^{14} + 5880T^{12} + 476280T^{10} + 16443000T^8 + 162729000T^6 \\
&\quad - 154791000T^4 + 130977000T^2 + 130977000, \quad \mathbf{q}_5 = 0, \quad \mathbf{q}_4 = 45T^{16} + 5400T^{14} \\
&\quad + 459900T^{12} + 19845000T^{10} + 153798750T^8 + 702513000T^6 - 89302500T^4 \\
&\quad + 1250235000T^2 + 111628125, \quad \mathbf{q}_3 = 0, \quad \mathbf{q}_2 = 10T^{18} \\
&\quad + 2250T^{16} + 225000T^{14} + 4422600T^{12} - 99508500T^{10} - 224248500T^8 \\
&\quad + 9704205000T^6 + 15181425000T^4 - 1920003750T^2 + 223256250, \quad \mathbf{q}_1 = 0, \\
&\mathbf{q}_0 = T^{20} + 370T^{18} + 44325T^{16} + 2208600T^{14} + 62795250T^{12} + 693384300T^{10} \\
&\quad + 6641129250T^8 + 4346055000T^6 + 14042818125T^4 + 2902331250T^2 + 22325625
\end{aligned}$$

The plot of the modulus of  $v$  in the  $(x, t)$  coordinates gives :

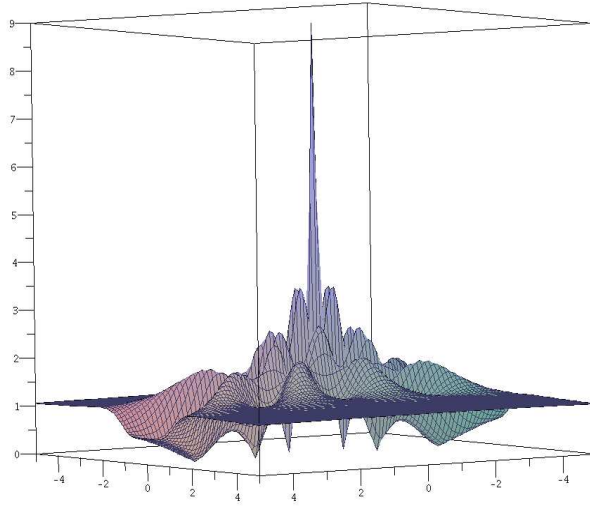


Figure 1: Solution of NLS,  $N=4a$ .

If we choose  $a_1 = 1$ ,  $a_2 = 1$ ,  $a_3 = 1$ ,  $a_4 = 1$ ,  $b_1 = 1$ ,  $b_2 = 1$ ,  $b_3 = 1$ ,  $b_4 = 1$ , we have the same type of analytical expression for the modulus of  $v$ ; we don't reproduce here.

We only give shape of the modulus of  $v$  in the  $(x, t)$  coordinates :



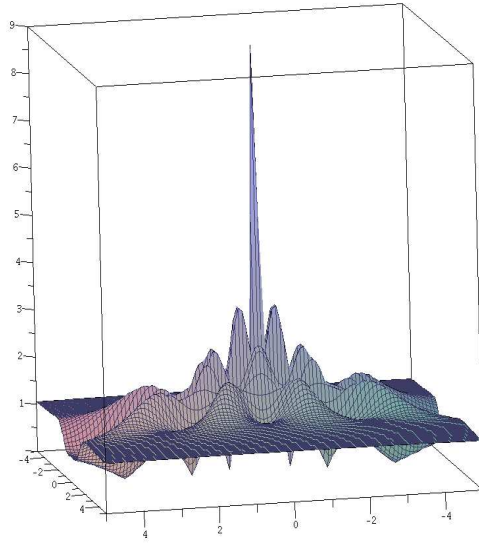


Figure 2: Solution of NLS,  $N=4b$ .

If we choose  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_3 = 0$ ,  $a_4 = 0$ ,  $b_1 = 1000000$ ,  $b_2 = 1000000$ ,  $b_3 = 1000000$ ,  $b_4 = 1000000$ , the shape of the modulus of  $v$  in the  $(x, t)$  coordinates is given by :

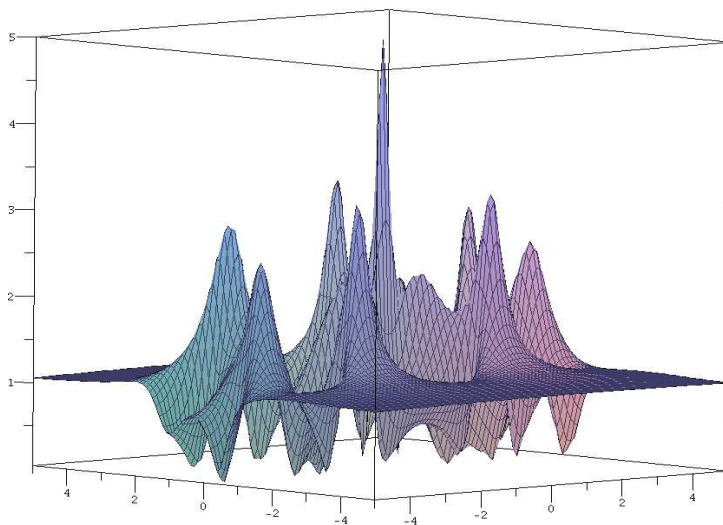


Figure 3: Solution of NLS,  $N=4c$ .

In these three plots, we can see the deformation of the initial Akhmediev's breather to multi-rogue waves, by choosing different types of the parameters among the 8 given by our formulation.

### 3.2 Case $N=5$

In the case  $N = 5$ , we realize an expansion at order 9 in  $\epsilon$ . From (11), we get solutions of NLS equation. Taking the limit when  $\epsilon \rightarrow 0$ , the solution of NLS equation takes the form

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it - i\varphi).$$

$$\begin{aligned}
& \alpha_5 = 60, \quad \mathbf{g}_{30} = 0, \quad \mathbf{g}_{29} = 0, \quad \mathbf{g}_{28} = 1, \quad \mathbf{g}_{27} = 0, \quad \mathbf{g}_{26} = 42T^2 + 42, \quad \mathbf{g}_{25} = 0, \quad \mathbf{g}_{24} = 455T^4 \\
& -1050T^2 + 2415, \quad \mathbf{g}_{23} = 0, \quad \mathbf{g}_{22} = 2548T^6 - 30660T^4 - 13860T^2 + 119700, \quad \mathbf{g}_{21} = 0, \\
& \mathbf{g}_{20} = 9009T^8 - 226380T^6 + 171990T^4 - 343980T^2 + 3221505, \quad \mathbf{g}_{19} = 0, \quad \mathbf{g}_{18} = 22022T^{10} \\
& -838530T^8 + 4142460T^6 - 44100T^4 - 36713250T^2 - 40153050, \quad \mathbf{g}_{17} = 0, \\
& \mathbf{g}_{16} = 39039T^{12} - 1844766T^{10} + 22431465T^8 - 9075780T^6 - 259473375T^4 - 2703484350T^2 \\
& -370010025, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 51480T^{14} - 2522520T^{12} + 61319160T^{10} \\
& +39803400T^8 - 773955000T^6 - 21896973000T^4 + 33756345000T^2 - 2893401000, \\
& \mathbf{g}_{13} = 0, \quad \mathbf{g}_{12} = 51051T^{16} - 2023560T^{14} + 104367060T^{12} + 629483400T^{10} \\
& +6114046050T^8 - 132697164600T^6 + 554979316500T^4 + 319310019000T^2 + 30787036875, \\
& \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 38038T^{18} - 589050T^{16} + 124369560T^{14} + 1700266680T^{12} \\
& +37748127060T^{10} - 446713728300T^8 + 2431707075000T^6 + 1380509487000T^4 \\
& +4238859255750T^2 + 1299806817750, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 21021T^{20} + 570570T^{18} \\
& +112372785T^{16} + 1735587000T^{14} - 43189665750T^{12} - 2318934687300T^{10} \\
& +10714665764250T^8 - 20464596621000T^6 + 35015175365625T^4 + 40381027706250T^2 \\
& +5540260123125, \quad \mathbf{g}_7 = 0, \quad \mathbf{g}_6 = 8372T^{22} + 769692T^{20} + 78618540T^{18} \\
& +570662820T^{16} - 223349124600T^{14} - 2950615722600T^{12} + 16520555280600T^{10} \\
& -11401393059000T^8 + 147193042090500T^6 + 422927620447500T^4 - 99598095922500T^2 \\
& +17840228332500, \quad \mathbf{g}_5 = 0, \quad \mathbf{g}_4 = 2275T^{24} + 415380T^{22} + 39897270T^{20} \\
& -30649500T^{18} - 148598863875T^{16} - 1555875783000T^{14} - 2135859799500T^{12} \\
& +94593530241000T^{10} - 98463038821875T^8 + 2611250197762500T^6 - 159203362106250T^4 \\
& -83293781287500T^2 - 21709549378125, \quad \mathbf{g}_3 = 0, \quad \mathbf{g}_2 = 378T^{26} + 114450T^{24} \\
& +12621420T^{22} + 89037900T^{20} - 283320450T^{18} + 1545272004150T^{16} \\
& +12633981885000T^{14} - 118201467699000T^{12} + 1380551057313750T^{10} \\
& +7814079083238750T^8 + 3521850108367500T^6 + 4776100863187500T^4 \\
& -1406247137268750T^2 - 13291560843750, \quad \mathbf{g}_1 = 0, \quad \mathbf{g}_0 = 29T^{28} + 13230T^{26} \\
& +1814295T^{24} + 74845260T^{22} - 764250795T^{20} - 204794909550T^{18} \\
& -3849793565625T^{16} - 34193820087000T^{14} + 942733356807375T^{12} \\
& +1889980437035250T^{10} + 13147594251868125T^8 + 3164572952887500T^6 \\
& -3369410673890625T^4 - 124940671931250T^2 + 3987468253125,
\end{aligned}$$

$$\begin{aligned}
& \mathbf{h}_{30} = 0, \quad \mathbf{h}_{29} = 0, \quad \mathbf{h}_{28} = T, \quad \mathbf{h}_{27} = 0, \quad \mathbf{h}_{26} = 14T^3 - 42T, \quad \mathbf{h}_{25} = 0, \quad \mathbf{h}_{24} = 91T^5 - 1610T^3 \\
& -1365T, \quad \mathbf{h}_{23} = 0, \quad \mathbf{h}_{22} = 364T^7 - 14700T^5 + 13860T^3 - 64260T, \quad \mathbf{h}_{21} = 0, \quad \mathbf{h}_{20} = 1001T^9 \\
& -67452T^7 + 411894T^5 - 97020T^3 - 2546775T, \quad \mathbf{h}_{19} = 0, \quad \mathbf{h}_{18} = 2002T^{11} - 190190T^9 \\
& +2572500T^7 - 3342780T^5 - 6769350T^3 - 39756150T, \quad \mathbf{h}_{17} = 0, \quad \mathbf{h}_{16} = 3003T^{13} \\
& -358974T^{11} + 7821765T^9 - 39225060T^7 - 73327275T^5 + 439963650T^3 + 2114980875T, \\
& \mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 3432T^{15} - 471240T^{13} + 13736520T^{11} - 135513000T^9 - 793686600T^7 \\
& +2779093800T^5 + 51116751000T^3 + 24754653000T, \quad \mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 3003T^{17} \\
& -434280T^{15} + 14403060T^{13} - 248776920T^{11} - 793072350T^9 - 2707651800T^7 + 375945664500T^5 \\
& -81098577000T^3 + 297337138875T, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 2002T^{19} - 275814T^{17} \\
& +8148168T^{15} - 362872440T^{13} - 114704100T^{11} - 90682521300T^9 + 1534457471400T^7 \\
& -1772183107800T^5 + 1117897625250T^3 + 1493718266250T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 1001T^{21}
\end{aligned}$$

$$\begin{aligned}
& -113190 T^{19} + 836325 T^{17} - 501931080 T^{15} - 15705928350 T^{13} - 400107348900 T^{11} \\
& + 4976480045250 T^9 - 11450902365000 T^7 + 30510953731125 T^5 - 5820156483750 T^3 \\
& - 21110374254375 T, \quad \mathbf{h}_7 = 0, \quad \mathbf{h}_6 = 364 T^{23} - 24332 T^{21} - 2084460 T^{19} \\
& - 528432660 T^{17} - 31926371400 T^{15} + 150244907400 T^{13} + 11823972489000 T^{11} \\
& - 3962494809000 T^9 + 7158970633500 T^7 + 132364254802500 T^5 - 455536249717500 T^3 \\
& - 63681344842500 T, \quad \mathbf{h}_5 = 0, \quad \mathbf{h}_4 = 91 T^{25} + 420 T^{23} - 1450890 T^{21} - 337761900 T^{19} \\
& - 18543465675 T^{17} + 274020553800 T^{15} + 5724951088500 T^{13} + 48513868893000 T^{11} \\
& + 171111381643125 T^9 + 1334157649492500 T^7 - 1694171881946250 T^5 - 515712560737500 T^3 \\
& - 131586452353125 T, \quad \mathbf{h}_3 = 0, \quad \mathbf{h}_2 = 14 T^{27} + 1470 T^{25} - 409500 T^{23} \\
& - 111637260 T^{21} - 3311799750 T^{19} + 88973271450 T^{17} - 3045655809000 T^{15} \\
& - 34947318861000 T^{13} + 1002802178873250 T^{11} + 1999468016831250 T^9 - 6800738923597500 T^7 \\
& + 4269249343012500 T^5 - 1666761729806250 T^3 + 204690036993750 T, \quad \mathbf{h}_1 = 0, \\
& \mathbf{h}_0 = T^{29} + 238 T^{27} - 43701 T^{25} - 14070420 T^{23} - 1034990775 T^{21} - 32505382350 T^{19} \\
& + 259820563275 T^{17} + 13855420996200 T^{15} + 406907765530875 T^{13} + 497730743291250 T^{11} \\
& + 1983581436965625 T^9 - 10570073675332500 T^7 - 7864084888813125 T^5 - 224184326231250 T^3 \\
& + 73103584640625 T
\end{aligned}$$

$$\begin{aligned}
& \mathbf{q}_{30} = 1, \quad \mathbf{q}_{29} = 0, \quad \mathbf{q}_{28} = 15 T^2 + 15, \quad \mathbf{q}_{27} = 0, \quad \mathbf{q}_{26} = 105 T^4 - 630 T^2 + 945, \quad \mathbf{q}_{25} = 0, \\
& \mathbf{q}_{24} = 455 T^6 - 7875 T^4 + 4725 T^2 + 64575, \quad \mathbf{q}_{23} = 0, \quad \mathbf{q}_{22} = 1365 T^8 - 39900 T^6 \\
& + 103950 T^4 + 548100 T^2 + 3709125, \quad \mathbf{q}_{21} = 0, \quad \mathbf{q}_{20} = 3003 T^{10} - 114345 T^8 + 859950 T^6 + 4035150 T^4 \\
& + 34827975 T^2 + 133656075, \quad \mathbf{q}_{19} = 0, \quad \mathbf{q}_{18} = 5005 T^{12} - 200970 T^{10} + 3649275 T^8 + 220500 T^6 \\
& + 277333875 T^4 + 959505750 T^2 + 1115785125, \quad \mathbf{q}_{17} = 0, \quad \mathbf{q}_{16} = 6435 T^{14} - 204435 T^{12} + 10174815 T^{10} \\
& + 42170625 T^8 + 2030639625 T^6 + 7693410375 T^4 - 27357820875 T^2 + 24214372875, \quad \mathbf{q}_{15} = 0, \\
& \mathbf{q}_{14} = 6435 T^{16} - 59400 T^{14} + 21035700 T^{12} + 451672200 T^{10} + 2902331250 T^8 + 79622109000 T^6 \\
& - 319613647500 T^4 + 191285955000 T^2 + 463546951875, \quad \mathbf{q}_{13} = 0, \quad \mathbf{q}_{12} = 5005 T^{18} \\
& + 155925 T^{16} + 33585300 T^{14} + 1481098500 T^{12} + 42118035750 T^{10} + 639849435750 T^8 \\
& - 1190848837500 T^6 + 1787210932500 T^4 + 4850130403125 T^2 + 5581517878125, \quad \mathbf{q}_{11} = 0 \\
& \mathbf{q}_{10} = 3003 T^{20} + 279510 T^{18} + 40951575 T^{16} + 2550025800 T^{14} + 112585249350 T^{12} \\
& + 1486454400900 T^{10} + 2935114197750 T^8 + 10430710605000 T^6 + 58973741229375 T^4 \\
& + 49590883833750 T^2 + 14657286301875, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 1365 T^{22} + 246015 T^{20} \\
& + 36850275 T^{18} + 2719544625 T^{16} + 98273999250 T^{14} - 830307854250 T^{12} \\
& - 8598553724250 T^{10} + 211739487041250 T^8 + 162726680615625 T^6 \\
& + 731900852746875 T^4 - 370328202365625 T^2 + 93610564228125, \quad \mathbf{q}_7 = 0, \\
& \mathbf{q}_6 = 455 T^{24} + 134820 T^{22} + 23403870 T^{20} + 1942384500 T^{18} + 36981653625 T^{16} \\
& - 1371507795000 T^{14} + 3080287318500 T^{12} + 299367020421000 T^{10} + 3135310421315625 T^8 \\
& + 10570433743012500 T^6 - 3151872128081250 T^4 + 1114718902762500 T^2 \\
& + 412481438184375, \quad \mathbf{q}_5 = 0, \quad \mathbf{q}_4 = 105 T^{26} + 46725 T^{24} + 9856350 T^{22} \\
& + 950244750 T^{20} + 28094731875 T^{18} + 11015463375 T^{16} - 7594552507500 T^{14} \\
& + 191792925292500 T^{12} + 6041183185209375 T^{10} + 13451797993921875 T^8 \\
& + 29454394197768750 T^6 + 1342447645218750 T^4 + 5894807234203125 T^2 \\
& + 232602314765625, \quad \mathbf{q}_3 = 0, \quad \mathbf{q}_2 = 15 T^{28} + 9450 T^{26} \\
& + 2461725 T^{24} + 296748900 T^{22} + 21862740375 T^{20} + 1401063945750 T^{18} \\
& + 53045439418125 T^{16} + 801963419355000 T^{14} + 7240875247423125 T^{12} \\
& - 18225599206106250 T^{10} - 19943100941990625 T^8 + 193179545303062500 T^6
\end{aligned}$$

$$\begin{aligned}
& +118075580755453125T^4 - 5861578332093750T^2 + 299060118984375, \\
\mathbf{q}_1 = 0, \quad \mathbf{q}_0 = & T^{30} + 855T^{28} + 275625T^{26} + 44441775T^{24} + 4060783125T^{22} \\
& +207533751075T^{20} + 5923312282125T^{18} + 77461769896875T^{16} \\
& +1691986493491875T^{14} + 21127132873153125T^{12} + 60580010182426875T^{10} \\
& +225021251512378125T^8 + 50098108080234375T^6 + 67806897644390625T^4 \\
& +5881515673359375T^2 + 19937341265625
\end{aligned}$$

Here we obtain the Akhmediev's breather of order 5. Again, we can note the presence of  $N(N - 1) - 1$  local maximums; the global maximum of  $|v|$  is equal to 11. We represent the modulus of  $v$  in the  $(x, t)$  coordinates and we get :

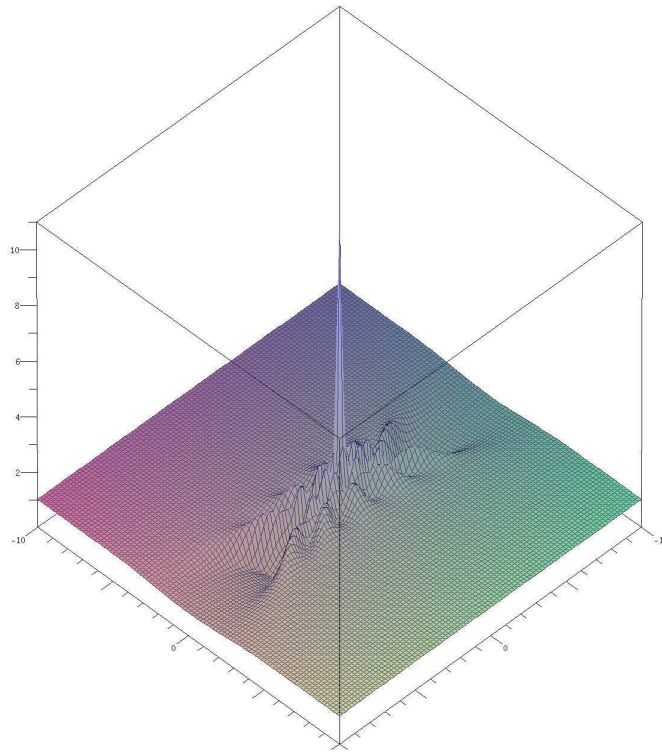


Figure 4: Solution of NLS,  $N=5a$ .

In the following cases, we only give the plots for the modulus of  $v$  in the  $(x, t)$  coordinates.  
If we choose  $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, b_1 = 1, b_2 = 1, b_3 = 1, b_4 = 1$ , we get :

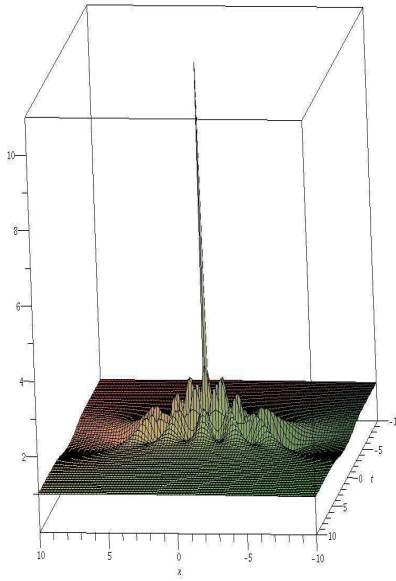


Figure 5: Solution of NLS,  $N=5b$ .

If we choose  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_3 = 0$ ,  $a_4 = -0$ ,  $a_5 = 0$ ,  $b_1 = 1000000$ ,  $b_2 = 1000000$ ,  $b_3 = 1000000$ ,  $b_4 = 1000000$ , we get The shape of the modulus of  $v$  in the  $(x, t)$  coordinates is given by :

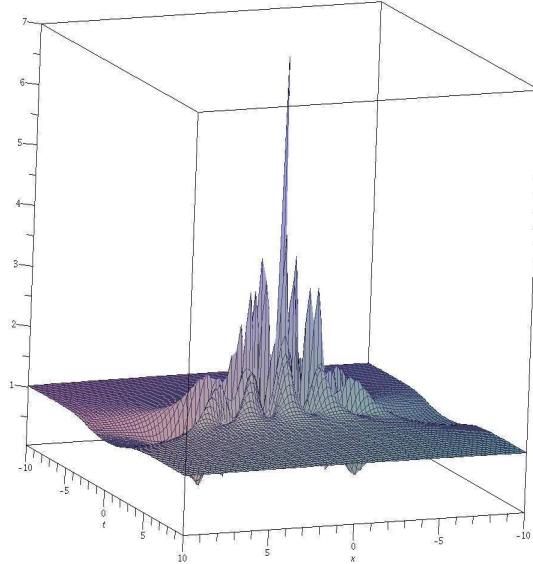


Figure 6: Solution of NLS,  $N=5c$ .

### 3.3 Case $N=6$

In the case  $N = 6$ , we realize an expansion at order 9 in  $\epsilon$ . From (11), we get solutions of NLS equation. Taking the limit when  $\epsilon \rightarrow 0$ , the solution of NLS equation takes the form

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it - i\varphi).$$



$$\begin{aligned}
& \alpha_6 = 4, \quad \mathbf{g}_{42} = 0, \quad \mathbf{g}_{41} = 0, \quad \mathbf{g}_{40} = 21, \quad \mathbf{g}_{39} = 0, \quad \mathbf{g}_{38} = 1260 T^2 + 1260, \quad \mathbf{g}_{37} = 0, \quad \mathbf{g}_{36} = 19950 T^4 \\
& - 49140 T^2 + 108990, \quad \mathbf{g}_{35} = 0, \quad \mathbf{g}_{34} = 167580 T^6 - 2249100 T^4 - 1190700 T^2 + 9128700, \quad \mathbf{g}_{33} = 0, \\
& \mathbf{g}_{32} = 915705 T^8 - 27439020 T^6 + 24409350 T^4 - 48818700 T^2 + 589098825, \quad \mathbf{g}_{31} = 0, \quad \mathbf{g}_{30} = 3581424 T^{10} \\
& - 178128720 T^8 + 984947040 T^6 + 60328800 T^4 - 4186501200 T^2 + 18503478000, \quad \mathbf{g}_{29} = 0, \\
& \mathbf{g}_{28} = 10581480 T^{12} - 743616720 T^{10} + 9464477400 T^8 - 7045768800 T^6 - 30612897000 T^4 \\
& - 520740738000 T^2 - 459479223000, \quad \mathbf{g}_{27} = 0, \quad \mathbf{g}_{26} = 24418800 T^{14} - 2175329520 T^{12} \\
& + 47637843120 T^{10} - 159272794800 T^8 - 224542206000 T^6 - 4616367714000 T^4 - 32428595430000 T^2 \\
& - 7322876442000, \quad \mathbf{g}_{25} = 0, \quad \mathbf{g}_{24} = 44971290 T^{16} - 4661420400 T^{14} + 152359061400 T^{12} \\
& - 1026048416400 T^{10} - 337581310500 T^8 - 30371208714000 T^6 - 186218752545000 T^4 \\
& + 1100164292010000 T^2 - 108079377603750, \quad \mathbf{g}_{23} = 0, \quad \mathbf{g}_{22} = 67016040 T^{18} - 7469967960 T^{16} \\
& + 337998679200 T^{14} - 3317725202400 T^{12} + 14701701495600 T^{10} - 134995160034000 T^8 \\
& - 1193237286444000 T^6 + 29149053992100000 T^4 + 20986218417465000 T^2 + 1152340349265000, \\
& \mathbf{g}_{21} = 0, \quad \mathbf{g}_{20} = 81477396 T^{20} - 8962513560 T^{18} + 547168642020 T^{16} - 6239220372000 T^{14} \\
& + 108187552103400 T^{12} - 282142532826000 T^{10} - 9222667785639000 T^8 + 257357195058348000 T^6 \\
& + 18279145208362500 T^4 + 529488288837585000 T^2 + 156886140925552500, \quad \mathbf{g}_{19} = 0 \\
& \mathbf{g}_{18} = 81124680 T^{22} - 7847559720 T^{20} + 671406535800 T^{18} - 6955900320600 T^{16} + 296997420258000 T^{14} \\
& + 830239889094000 T^{12} - 53402987374506000 T^{10} + 1441119106966770000 T^8 - 2734739470632015000 T^6 \\
& + 4283688113241075000 T^4 + 11213622020930175000 T^2 + 4883826305513925000, \quad \mathbf{g}_{17} = 0, \\
& \mathbf{g}_{16} = 66134250 T^{24} - 4572870120 T^{22} + 650465836020 T^{20} - 3956090430600 T^{18} \\
& + 265628919921750 T^{16} - 8374317575778000 T^{14} - 494490160426185000 T^{12} + 7083936614255886000 T^{10} \\
& - 27625636046797346250 T^8 + 58715278628789475000 T^6 + 100296773020980112500 T^4 \\
& + 109063413373951575000 T^2 + 48008615346625106250, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 43953840 T^{26} \\
& - 1085994000 T^{24} + 525422570400 T^{22} + 283015101600 T^{20} - 440867272230000 T^{18} \\
& - 70625220688638000 T^{16} - 1840700971022760000 T^{14} + 23136462127298520000 T^{12} \\
& - 77288502059715870000 T^{10} + 203718051732368250000 T^8 + 1193114598558012900000 T^6 \\
& + 382667439331138500000 T^4 - 1778554070753402250000 T^2 + 450583593605664750000, \quad \mathbf{g}_{13} = 0, \\
& \mathbf{g}_{12} = 23604840 T^{28} + 923030640 T^{26} + 375409452600 T^{24} + 2624172616800 T^{22} \\
& - 1510226919293400 T^{20} - 169673051037798000 T^{18} - 1533701946518973000 T^{16} \\
& + 60171290506320840000 T^{14} - 81243452836954485000 T^{12} + 989384777473422330000 T^{10} \\
& + 9218597475208223025000 T^8 + 11767748128231995900000 T^6 - 40367065478563333125000 T^4 \\
& - 8425754243991488250000 T^2 + 2097282775643141625000, \quad \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 10093104 T^{30} \\
& + 1259753040 T^{28} + 242036287920 T^{26} + 3006820731600 T^{24} - 1903047548547600 T^{22} \\
& - 177522454448262000 T^{20} + 1673066563892262000 T^{18} + 87528371510322666000 T^{16} \\
& + 6620696554494330000 T^{14} + 5807115658938263670000 T^{12} + 23239380295474781130000 T^{10} \\
& + 137613317496661002150000 T^8 - 210318339990236489550000 T^6 + 11833548606284154750000 T^4 \\
& - 83099086894146534750000 T^2 - 9948156863136203250000, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 3357585 T^{32} \\
& + 788855760 T^{30} + 133834793400 T^{28} + 2706551593200 T^{26} - 1301950524262500 T^{24} \\
& - 74974735386558000 T^{22} + 3528441325033317000 T^{20} - 9148978382135130000 T^{18} \\
& - 917328476828617796250 T^{16} + 20331186148107969750000 T^{14} + 69701007067098023925000 T^{12} \\
& + 887836329346915768050000 T^{10} - 517772523069859235062500 T^8 + 2152850692307163675750000 T^6 \\
& - 639511342714258678125000 T^4 - 797615544339523428750000 T^2 - 21263091022015865859375, \quad \mathbf{g}_7 = 0, \\
& \mathbf{g}_6 = 837900 T^{34} + 311725260 T^{32} + 57825260640 T^{30} + 2004411679200 T^{28} - 486305908326000 T^{26} \\
& - 2096884640214000 T^{24} + 2746634505228180000 T^{22} + 35193597983772228000 T^{20} \\
& + 535094835158922885000 T^{18} + 12830771213501350725000 T^{16} + 38302174679039555130000 T^{14} \\
& + 2069806537668865079700000 T^{12} - 2878103885627077118550000 T^{10} + 13752994933828011188250000 T^8 \\
& + 8319513974090622187500000 T^6 - 11861116178590458202500000 T^4
\end{aligned}$$

$$\begin{aligned}
&+1576218096385317649687500T^2 - 26235193683001885312500, \quad \mathbf{g}_5 = 0, \quad \mathbf{g}_4 = 147630T^{36} \\
&+79512300T^{34} + 17590409550T^{32} + 1021528519200T^{30} - 81006309657000T^{28} + 1634306691906000T^{26} \\
&+606189343425435000T^{24} + 11528510197530060000T^{22} - 396162299973691597500T^{20} \\
&+1479830691395427525000T^{18} + 581811329651584456462500T^{16} + 1247923055532263071500000T^{14} \\
&+3842659299741460765875000T^{12} + 7389770241237094037250000T^{10} + 28258267252466153739375000T^8 \\
&-57116889114356746867500000T^6 + 3389031680327926024218750T^4 + 2208926709003114942187500T^2 \\
&+110589958773237874218750, \quad \mathbf{g}_3 = 0, \quad \mathbf{g}_2 = 16380T^{38} + 12074580T^{36} + 3288316500T^{34} \\
&+300491005500T^{32} - 226761670800T^{30} - 2504351229534000T^{28} - 460421913549654000T^{26} \\
&-26192900747252610000T^{24} - 110536335706300875000T^{22} + 11670003738602023815000T^{20} \\
&+417205004840944325775000T^{18} + 373915680879582366525000T^{16} + 8208451018603349340750000T^{14} \\
&+97521767433772016816250000T^{12} - 248372033459815147632750000T^{10} \\
&-439480474711801012586250000T^8 + 50756673874987829955937500T^6 - 11653596434883793654687500T^4 \\
&+6592310529469634320312500T^2 + 60321795694493385937500, \quad \mathbf{g}_1 = 0, \quad \mathbf{g}_0 = 861T^{40} \\
&+835380T^{38} + 282059190T^{36} + 38110734900T^{34} + 757788568425T^{32} - 357432351318000T^{30} \\
&-33079684204539000T^{28} - 671694631354782000T^{26} + 47369156210904386250T^{24} \\
&+1080980353533726135000T^{22} + 57329971310306073802500T^{20} + 1450451085746969465775000T^{18} \\
&+5735064780371479139606250T^{16} + 31970579440715276303250000T^{14} - 70064960713363893382875000T^{12} \\
&-5517764293121965095750000T^{10} - 203570309777817170932734375T^8 - 20966515699057134432187500T^6 \\
&+42238183085222761596093750T^4 + 1146114118195374332812500T^2 - 6463049538695719921875,
\end{aligned}$$

$$\begin{aligned}
\mathbf{h}_{42} &= 0, \quad \mathbf{h}_{41} = 0, \quad \mathbf{h}_{40} = 21T, \quad \mathbf{h}_{39} = 0, \quad \mathbf{h}_{38} = 420T^3 - 1260T, \quad \mathbf{h}_{37} = 0, \quad \mathbf{h}_{36} = 3990T^5 \\
&-72660T^3 - 59850T, \quad \mathbf{h}_{35} = 0, \quad \mathbf{h}_{34} = 23940T^7 - 1031940T^5 + 1190700T^3 - 4365900T, \quad \mathbf{h}_{33} = 0, \\
\mathbf{h}_{32} &= 101745T^9 - 7646940T^7 + 55526310T^5 + 6747300T^3 - 308688975T, \quad \mathbf{h}_{31} = 0, \quad \mathbf{h}_{30} = 325584T^{11} \\
&-36414000T^9 + 606039840T^7 - 487075680T^5 - 442940400T^3 - 15569593200T, \quad \mathbf{h}_{29} = 0, \\
\mathbf{h}_{28} &= 813960T^{13} - 122608080T^{11} + 3460174200T^9 - 15134364000T^7 - 6941781000T^5 + 41650686000T^3 \\
&-133275051000T, \quad \mathbf{h}_{27} = 0, \quad \mathbf{h}_{26} = 1627920T^{15} - 307676880T^{13} + 12556328400T^{11} \\
&-123859789200T^9 - 414363600T^7 + 625617594000T^5 + 14773276854000T^3 + 40215059010000T, \\
\mathbf{h}_{25} &= 0, \quad \mathbf{h}_{24} = 2645370T^{17} - 593676720T^{15} + 31525767000T^{13} - 527395957200T^{11} \\
&+752464849500T^9 + 2323222398000T^7 + 161180546247000T^5 + 1150723795410000T^3 \\
&+884942275376250T, \quad \mathbf{h}_{23} = 0, \quad \mathbf{h}_{22} = 3527160T^{19} - 897198120T^{17} + 57072768480T^{15} \\
&-1415227917600T^{13} + 5710878205200T^{11} - 31038334110000T^9 + 1020560829372000T^7 \\
&+10607242786812000T^5 - 10549431670365000T^3 + 21705296078655000T, \quad \mathbf{h}_{21} = 0, \\
\mathbf{h}_{20} &= 3879876T^{21} - 1072071000T^{19} + 75633978420T^{17} - 2635403934240T^{15} \\
&+18071080057800T^{13} - 232668433443600T^{11} + 3887539470045000T^9 + 73903432233708000T^7 \\
&-332535925127983500T^5 - 35419823925255000T^3 + 373647041815162500T, \quad \mathbf{h}_{19} = 0, \\
\mathbf{h}_{18} &= 3527160T^{23} - 1014893880T^{21} + 72635963400T^{19} - 3661723989000T^{17} + 22961577870000T^{15} \\
&-558100069758000T^{13} + 8233601126058000T^{11} + 401324717365110000T^9 - 2629541354603625000T^7 \\
&+1891946762885025000T^5 - 2077818312668175000T^3 + 4696174277313750000T, \quad \mathbf{h}_{17} = 0, \\
\mathbf{h}_{16} &= 2645370T^{25} - 756854280T^{23} + 47845898100T^{21} - 4100304839400T^{19} - 18503101143450T^{17} \\
&-1692739675242000T^{15} - 39034514859417000T^{13} + 1908000279128790000T^{11} \\
&-13226094269284916250T^9 + 41333515336516095000T^7 - 3093451287652687500T^5 \\
&-152813978846731125000T^3 - 136885438789385343750T, \quad \mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 1627920T^{27} \\
&-437739120T^{25} + 17295314400T^{23} - 4002542359200T^{21} - 118543877298000T^{19} \\
&-5096839239954000T^{17} + 6829841768904000T^{15} + 8476647700225320000T^{13} \\
&-39662003312572410000T^{11} + 198591885419246550000T^9 + 338676425725106700000T^7 \\
&-1458984265260221700000T^5 - 3931624238736174750000T^3 - 1056176028083310750000T,
\end{aligned}$$

$$\begin{aligned}
& \mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 813960 T^{29} - 190123920 T^{27} - 2343879720 T^{25} - 3521465085600 T^{23} \\
& - 214336185247800 T^{21} - 7685261724294000 T^{19} + 559933161897879000 T^{17} \\
& + 18300777177408264000 T^{15} - 67319404622929305000 T^{13} + 702028432394961210000 T^{11} \\
& + 965256209626556925000 T^9 - 5016221069075993700000 T^7 - 41028109496754204825000 T^5 \\
& + 5088504256048347750000 T^3 - 7659505278106864875000 T, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 325584 T^{31} \\
& - 57834000 T^{29} - 7480453680 T^{27} - 2671612327440 T^{25} - 227871922345200 T^{23} - 4515437156907600 T^{21} \\
& + 1063591395097770000 T^{19} + 15434675218020366000 T^{17} - 89580321136341882000 T^{15} \\
& + 2041183360309213890000 T^{13} + 4723190632996431750000 T^{11} + 11664100345373957250000 T^9 \\
& - 254778861744583996050000 T^7 + 117211546169871233850000 T^5 - 26289299400480443250000 T^3 \\
& - 19086551752323651750000 T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 101745 T^{33} - 10024560 T^{31} - 5022145800 T^{29} \\
& - 1608119730000 T^{27} - 161174098606500 T^{25} + 335632587066000 T^{23} + 736078113424269000 T^{21} \\
& - 7565148948169890000 T^{19} - 106455713823272756250 T^{17} + 5811505024293841950000 T^{15} \\
& + 6779365489015829325000 T^{13} + 154597964806237664250000 T^{11} - 1526956495540039742062500 T^9 \\
& + 1322866554468445428750000 T^7 - 1665063939990978883125000 T^5 + 333570867196457756250000 T^3 \\
& + 248146906256610376640625 T, \quad \mathbf{h}_7 = 0, \quad \mathbf{h}_6 = 23940 T^{35} + 149940 T^{33} - 1927346400 T^{31} \\
& - 708549055200 T^{29} - 79214033883600 T^{27} + 823565301174000 T^{25} + 200617502001084000 T^{23} \\
& - 8279951353262820000 T^{21} - 77783883635196585000 T^{19} + 2988383002724206395000 T^{17} \\
& + 90398497755259039500000 T^{15} - 415079152248697744500000 T^{13} - 6264634434444242183250000 T^{11} \\
& - 711055582862691008250000 T^9 - 1756424787565278253500000 T^7 - 3323890524236676817500000 T^5 \\
& + 7431581396973548646562500 T^3 + 510341541351824995312500 T, \quad \mathbf{h}_5 = 0, \quad \mathbf{h}_4 = 3990 T^{37} \\
& + 555660 T^{35} - 451870650 T^{33} - 210601490400 T^{31} - 27164808405000 T^{29} - 494049363822000 T^{27} \\
& - 131254421193000 T^{25} - 2331351766748340000 T^{23} - 57607676795659567500 T^{21} \\
& + 3182842545894136125000 T^{19} + 121471496093605901512500 T^{17} - 730868754303114203700000 T^{15} \\
& - 3952170770249132825625000 T^{13} - 18817383674105947680750000 T^{11} - 24600927446914993453125000 T^9 \\
& - 62084596445709395827500000 T^7 + 38962039336837387479843750 T^5 + 5862704048212428604687500 T^3 \\
& + 690828184025031396093750 T, \quad \mathbf{h}_3 = 0, \quad \mathbf{h}_2 = 420 T^{39} + 127260 T^{37} - 59773140 T^{35} - 37389170700 T^{33} \\
& - 6082652847600 T^{31} - 456491304234000 T^{29} - 20958512959818000 T^{27} + 1456163148715002000 T^{25} \\
& + 201140982748216035000 T^{23} + 4864337674818781005000 T^{21} + 60289385834313075825000 T^{19} \\
& - 641663910397442064825000 T^{17} + 3860956945797085030050000 T^{15} + 17024232728719929462750000 T^{13} \\
& - 174686108305029234779250000 T^{11} - 133153380406498251318750000 T^9 \\
& + 252883712879152705566562500 T^7 - 66291355495078917111562500 T^5 + 12434907312450565129687500 T^3 \\
& - 784183344028414017187500 T, \quad \mathbf{h}_1 = 0, \quad \mathbf{h}_0 = 21 T^{41} + 10500 T^{39} - 3356010 T^{37} \\
& - 2977396380 T^{35} - 678988637775 T^{33} - 67009019137200 T^{31} - 1544931642555000 T^{29} \\
& + 190273318160058000 T^{27} + 15779576666742320250 T^{25} + 302225493684212595000 T^{23} \\
& + 10750227730088000962500 T^{21} + 163163976757378009875000 T^{19} - 1345334461618984846143750 T^{17} \\
& - 10403032879102451430150000 T^{15} - 115925574335468992738875000 T^{13} \\
& - 52103299471386560916750000 T^{11} - 86323335508369995943359375 T^9 + 255531634533965133599062500 T^7 \\
& + 85352755687892662813593750 T^5 + 1157603984041944501562500 T^3 - 196045836007103504296875 T
\end{aligned}$$

$$\begin{aligned}
& \mathbf{q}_{42} = 1, \quad \mathbf{q}_{41} = 0, \quad \mathbf{q}_{40} = 21 T^2 + 21, \quad \mathbf{q}_{39} = 0, \quad \mathbf{q}_{38} = 210 T^4 - 1260 T^2 + 1890, \quad \mathbf{q}_{37} = 0, \\
& \mathbf{q}_{36} = 1330 T^6 - 24570 T^4 + 10710 T^2 + 193410, \quad \mathbf{q}_{35} = 0, \quad \mathbf{q}_{34} = 5985 T^8 - 202860 T^6 + 595350 T^4 \\
& + 1984500 T^2 + 18356625, \quad \mathbf{q}_{33} = 0, \quad \mathbf{q}_{32} = 20349 T^{10} - 1012095 T^8 + 8136450 T^6 + 17662050 T^4 \\
& + 215219025 T^2 + 1383295725, \quad \mathbf{q}_{31} = 0, \quad \mathbf{q}_{30} = 54264 T^{12} - 3444336 T^{10} + 53290440 T^8 - 20109600 T^6 \\
& + 1912264200 T^4 + 15369555600 T^2 + 65128908600, \quad \mathbf{q}_{29} = 0, \quad \mathbf{q}_{28} = 116280 T^{14} - 8439480 T^{12} \\
& + 217024920 T^{10} - 694575000 T^8 + 14832153000 T^6 + 133703703000 T^4 + 386251173000 T^2 \\
& + 1034587323000, \quad \mathbf{q}_{27} = 0, \quad \mathbf{q}_{26} = 203490 T^{16} - 15165360 T^{14} + 622815480 T^{12}
\end{aligned}$$

$$\begin{aligned}
& -2210256720 T^{10} + 56135551500 T^8 + 1192724190000 T^6 + 1433519451000 T^4 - 27693705438000 T^2 \\
& + 38780976953250, \quad \mathbf{q}_{25} = 0, \quad \mathbf{q}_{24} = 293930 T^{18} - 19631430 T^{16} + 1366583400 T^{14} \\
& + 2951613000 T^{12} + 363193267500 T^{10} + 10089021379500 T^8 + 22829041053000 T^6 \\
& - 475525632015000 T^4 + 342500315456250 T^2 + 1376457162806250, \quad \mathbf{q}_{23} = 0, \quad \mathbf{q}_{22} = 352716 T^{20} \\
& - 16521960 T^{18} + 2444960700 T^{16} + 41186426400 T^{14} + 2435977719000 T^{12} + 35777824966800 T^{10} \\
& + 398543162031000 T^8 - 3822883567020000 T^6 + 7873515567697500 T^4 + 16086079864575000 T^2 \\
& + 38577274289347500, \quad \mathbf{q}_{21} = 0, \quad \mathbf{q}_{20} = 352716 T^{22} - 4288284 T^{20} + 3742158420 T^{18} \\
& + 149684881500 T^{16} + 9553331403000 T^{14} + 212261247613800 T^{12} + 3849610940821800 T^{10} \\
& - 15394646399517000 T^8 + 71758478683237500 T^6 + 204046725448912500 T^4 + 436091149100722500 T^2 \\
& + 674506306475347500, \quad \mathbf{q}_{19} = 0, \quad \mathbf{q}_{18} = 293930 T^{24} + 11695320 T^{22} + 5003418420 T^{20} \\
& + 334607427000 T^{18} + 23101428561750 T^{16} + 807591346254000 T^{14} + 16185920631111000 T^{12} \\
& - 8352665505954000 T^{10} + 576663818982693750 T^8 + 1242059144478075000 T^6 + 5668373975877112500 T^4 \\
& + 3761501085532575000 T^2 + 4325812023235106250, \quad \mathbf{q}_{17} = 0, \quad \mathbf{q}_{16} = 203490 T^{26} + 22694490 T^{24} \\
& + 5786087580 T^{22} + 534631523580 T^{20} + 37855900293750 T^{18} + 1433816319300750 T^{16} \\
& + 1641456285777000 T^{14} - 265649781021471000 T^{12} + 5360993203091352750 T^{10} + 6909604613062593750 T^8 \\
& + 80880110043223837500 T^6 + 42236970373512337500 T^4 - 113560599693873543750 T^2 \\
& + 60012457876086581250, \quad \mathbf{q}_{15} = 0, \quad \mathbf{q}_{14} = 116280 T^{28} + 23904720 T^{26} + 5624073000 T^{24} \\
& + 647566164000 T^{22} + 45746991675000 T^{20} + 1145438659134000 T^{18} - 60947538804999000 T^{16} \\
& - 613314917245800000 T^{14} + 13256411880737025000 T^{12} + 146890015244661150000 T^{10} \\
& + 484908475289220675000 T^8 + 1198562394651167700000 T^6 - 2003316012774759375000 T^4 \\
& + 716725088677001250000 T^2 + 708119334693106875000, \quad \mathbf{q}_{13} = 0, \quad \mathbf{q}_{12} = 54264 T^{30} \\
& + 17607240 T^{28} + 4448719800 T^{26} + 609209483400 T^{24} + 44277505939800 T^{22} + 412689587725800 T^{20} \\
& - 84804973421553000 T^{18} - 250432798766535000 T^{16} + 22882838948853345000 T^{14} \\
& + 466654428328441815000 T^{12} + 3892228471345297545000 T^{10} + 20229343597701089775000 T^8 \\
& - 11143853714235193875000 T^6 + 11154104591287840875000 T^4 + 10381186823997961125000 T^2 \\
& + 5208039143851957875000, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 20349 T^{32} + 9579024 T^{30} + 2778050520 T^{28} \\
& + 446310874800 T^{26} + 36326975409900 T^{24} + 602640375094800 T^{22} - 16736996613922200 T^{20} \\
& + 1698321128858046000 T^{18} + 59188148168115408750 T^{16} + 972699066504389070000 T^{14} \\
& + 10020895210571390145000 T^{12} + 56146858034606462970000 T^{10} + 104753437906970047387500 T^8 \\
& + 83635145538228726750000 T^6 + 196955754351328578375000 T^4 + 58611690264525032250000 T^2 \\
& + 9836753798777262328125, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 5985 T^{34} + 3871665 T^{32} + 1327301640 T^{30} \\
& + 250676937000 T^{28} + 24928524967500 T^{26} + 1277755297843500 T^{24} + 75968161579863000 T^{22} \\
& + 3075359436148923000 T^{20} + 26011936101356493750 T^{18} + 1168303051165440093750 T^{16} \\
& + 25639663936063282875000 T^{14} - 68682733785579184425000 T^{12} - 266323431588369421612500 T^{10} \\
& + 2701519751233743319687500 T^8 + 1281120041892573815625000 T^6 + 1364503639750540183125000 T^4 \\
& - 323501276433699733359375 T^2 + 42533021249892785390625, \quad \mathbf{q}_7 = 0, \quad \mathbf{q}_6 = 1330 T^{36} \\
& + 1137780 T^{34} + 466317810 T^{32} + 104251764960 T^{30} + 13208232681000 T^{28} + 1193020745742000 T^{26} \\
& + 99210798874341000 T^{24} + 5805544496212692000 T^{22} + 214383530642638549500 T^{20} \\
& + 4530497152000032675000 T^{18} + 46797298481837387137500 T^{16} - 217436851956849317100000 T^{14} \\
& + 19692526781721872925000 T^{12} + 5336577984490937657550000 T^{10} + 21530470535065521500625000 T^8 \\
& + 32380484598451903567500000 T^6 - 4144298861975805116718750 T^4 + 730946965605972235312500 T^2 \\
& + 125335286609669590781250, \quad \mathbf{q}_5 = 0, \quad \mathbf{q}_4 = 210 T^{38} + 230790 T^{36} + 113314950 T^{34} \\
& + 30177299250 T^{32} + 4848169221000 T^{30} + 561695828967000 T^{28} + 50626283202243000 T^{26} \\
& + 3225603873293865000 T^{24} + 122692742315543137500 T^{22} + 2347942991384156512500 T^{20} \\
& + 36794456407482655612500 T^{18} + 266523839860686468637500 T^{16} - 90441117924876093375000 T^{14}
\end{aligned}$$

$$\begin{aligned}
& +5001462949574321566875000T^{12} + 59697458269849518339375000T^{10} + 55588353961234695418125000)T^8 \\
& +62717528472385320873281250T^6 + 2437287792703697046093750T^4 + 2506226987783118058593750)T^2 \\
& +50268163078744488281250, \quad \mathbf{q}_3 = 0, \quad \mathbf{q}_2 = 21T^{40} + 28980T^{38} + 17000550T^{36} + 5412525300T^{34} \\
& +1069516209825T^{32} + 138706771971600T^{30} + 10810972830861000T^{28} + 389665850910210000T^{26} \\
& -659267012964183750T^{24} - 454465851534216585000T^{22} - 920416585288570177500T^{20} \\
& +221631586055760347775000T^{18} + 5984763190950898799156250T^{16} + 32444042402395142615250000T^{14} \\
& +135661236341135119894125000T^{12} - 119473149759596572227750000T^{10} \\
& -60538714786788425248359375T^8 + 254152960059670490207812500T^6 \\
& +73869783760830436164843750T^4 - 1749332075140308192187500T^2 \\
& +45241346770870039453125, \quad \mathbf{q}_1 = 0, \quad \mathbf{q}_0 = T^{42} + 1701T^{40} + 1186290T^{38} + 451782450T^{36} \\
& +105515964225T^{34} + 15941427079725T^{32} + 1619444848404600T^{30} + 114621189126003000T^{28} \\
& +5904248149699121250T^{26} + 207513330336568226250T^{24} + 5039110473448289647500T^{22} \\
& +110469853983534408247500T^{20} + 1375703927511930449606250)T^{18} + 3784181946524764546781250T^{16} \\
& +36528049590131262499875000T^{14} + 218877641452798644652875000T^{12} \\
& +254716356056568852957328125T^{10} + 439843410849229547791640625T^8 \\
& +44121563595239721757031250T^6 + 29200057815827262607031250T^4 \\
& +1251677260660737758203125T^2 + 2154349846231906640625
\end{aligned}$$

We get the following plot for the modulus of  $v$  in function of  $(x, t)$ . We recognize again  $N(N + 1) - 1$  local maximums and the global maximum is equal to 13 :

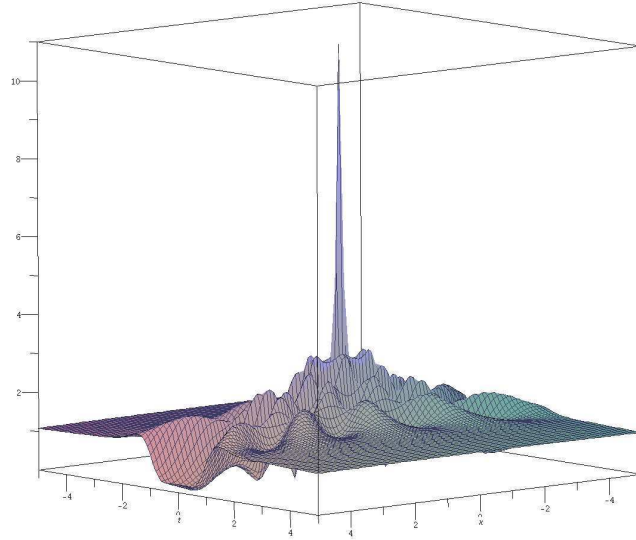


Figure 7: Solution of NLS,  $N=6a$ .

In the following, we only represent the modulus of  $v$  in the  $(x, t)$  coordinates.

If we choose  $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = 1, a_6 = 1, b_1 = 1, b_2 = 1, b_3 = 1, b_4 = 1, b_5 = 1, b_6 = 1$ , we get :

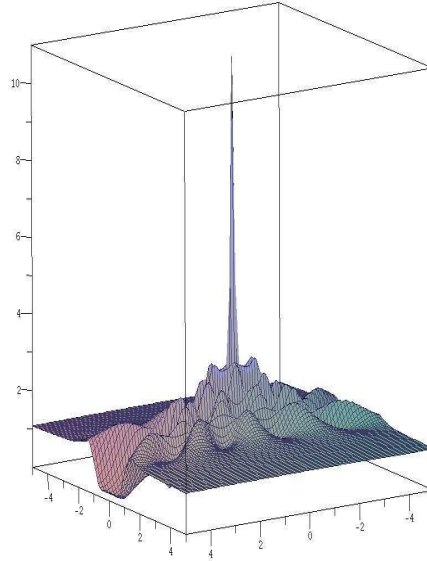


Figure 8: Solution of NLS,  $N=6b$ .

If we choose  $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, b_1 = 1000000, b_2 = 1000000, b_3 = 1000000, b_4 = 1000000, b_5 = 1000000, b_6 = 1000000$ , the shape of the modulus of  $v$  in the  $(x, t)$  coordinates is given by :

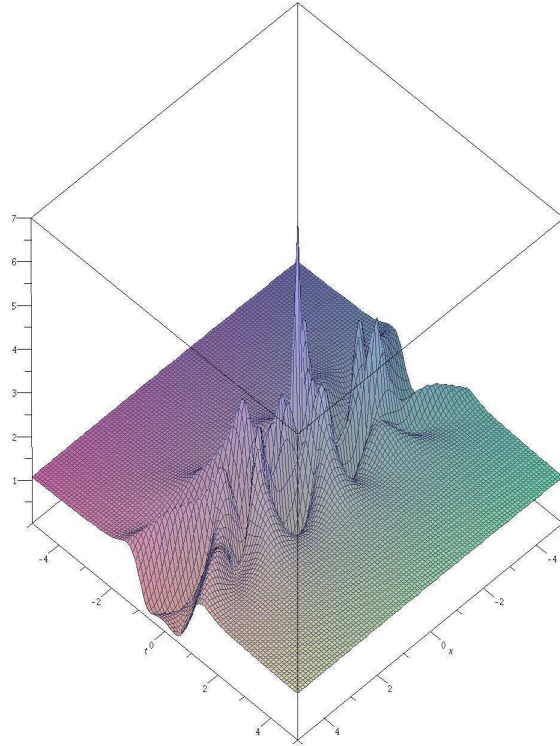


Figure 9: Solution of NLS,  $N=6c$ .

## 4 Conclusion

We have given here with new parameters a family which recover a wide spectrum of solutions of the NLS equation. These solutions also are written as a quotient of wronskians. An other approach has been given in [6].

This method described in the present paper provides a powerful tool to get explicitly solutions of the NLS equation. This new formulation gives an infinite set of non singular solution of NLS equation.

This method with parameters gives as well higher Peregrine's breathers of order  $N$  as well solutions with peaks of similar amplitude. It is reasonable



to conjecture that in general there is  $N(N + 1)/2$  peaks for the modulus of any solution  $v$  in the  $(x, t)$  coordinates.

Because of the presence of a lot of redundant parameters  $(3N + 1)$ , the present formulation give more flexibility to pass from Akhmediev's breathers to peaks of similar heights. This present method shows more adapted and efficient than this given in [6] to get all type of solutions.

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order and the results raise any scepticism about the use of determinants and theta functions.

### **Acknowledgements**

I would like to thank sincerely and deeply V. B. Matveev to introduce me in the rogue wave community and particularly in this subject.

This work has been supported partly by the grant ANR-09-BLAN-0117-01.

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