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Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation.

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Abstract

This work is a continuation of a recent paper in which we have constructed a multi-parametric family of solutions of the focusing NLS equation given in terms of Wronskians determinants of order $2N$ composed of elementary trigonometric functions. When we perform a special passage to the limit when all the periods tend to infinity, we get a family of quasi-rational solutions. Here we construct Peregrine breathers of orders $N = 4$, 5, and the multi-rogue waves corresponding the 10 or 15 peak formation in frame of the NLS model first explained by Matveev et al. in 2010. In the cases $N = 4$, 5 we get comfortable formulas to study the deformation of higher Peregrine breather of order 4 to the 10 rogue-waves or order 5 to the 15 rogue-waves solution via variation of the free parameters of our construction.

1 Introduction

The nonlinear Schrödinger equation (NLS) was first derived by Zakharov [16] in 1968. The solutions were first given by Zakharov and Schabat. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [12]. The first quasi-rational solutions of NLS equation appeared in 1983 with the work of Peregrine, nowadays called worldwide Peregrine breathers. In 1986 Eleonski,

Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather[3]. Other families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations. In 2010 it has been shown in [6] that rational solutions of NLS equation can be written as a quotient of two wronskians and it has been recovered as particular case, Akhmediev's quasi-rational solutions of NLS equation.

In this paper, we extend a result [10] giving a new representation of the solutions of the NLS equation in terms of a ratio of two wronskians determinants of even order $2N$ composed of elementary functions; the related solutions of NLS are called of order N. When we perform the passage to the limit when some parameter tends to 0, we got families of multi-rogue wave solutions of the focusing NLS equation depending on a certain number of parameters. It allows to recognize the famous Peregrine's breather [14] and also higher order Peregrine's breathers constructed by Akhmediev [1, 4].
We treat in the following the cases of order $N = 4$ and $N = 5$. We get for an arbitrary choice of the parameters the shape of Akhmediev's breathers; we can also get, for particular parameters, the apparition of peaks of similar amplitude for the modulus of the solution v in the $(x; y)$ coordinates.

2 Expression of solutions of NLS equation in terms of Wronskian determinant and quasi-rational limit

2.1 Solutions of NLS equation in terms of Wronskian determinant

We briefly recall results obtained in [10]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

From [10], the solution of the NLS equation can be written in the form

$$v(x,t) = \frac{\det(I + A_3(x,t))}{\det(I + A_1(x,t))} \exp(2it - i\varphi). \tag{2}$$

In (2), the matrix $A_r = (a_{\nu\mu})_{1\leq\nu,\mu\leq 2N}$ is defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda\neq\mu} \left|\frac{\gamma_\lambda+\gamma_\nu}{\gamma_\lambda-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{3}$$

The terms ϵ_ν are defined by :

$$\begin{aligned} \epsilon_\nu = 0, \quad 1\leq\nu\leq N \\ \epsilon_\nu = 1, \quad N+1\leq\nu\leq 2N. \end{aligned} \tag{4}$$

We consider the following functions

$$\begin{aligned} \phi_\nu^r(y) = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), \quad 1\leq\nu\leq N, \\ \phi_\nu^r(y) = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), \quad N+1\leq\nu\leq 2N. \end{aligned} \tag{5}$$

We use the following notations :
$\Theta_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1\leq\nu\leq 2N.$
$W_r(y) = W(\phi_1,\ldots,\phi_{2N})$ is the wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu,\,\mu\in[1,\ldots,2N]}]. \tag{6}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\ldots,2N]}$ defined by

$$\begin{aligned} d_{\nu\mu} = (-1)^{\epsilon_\nu} \textstyle\prod_{\lambda\neq\mu} \left|\frac{\gamma_\lambda+\gamma_\nu}{\gamma_\lambda-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} - ie_\nu), \\ 1\leq\nu\leq 2N, \quad 1\leq\mu\leq 2N, \end{aligned}$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we get the following link between Fredohlm and Wronskian determinants [9]

Theorem 2.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1,\ldots,\phi_{2N})(0), \tag{7}$$

where

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_\nu)}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu-\gamma_\mu)}.$$

It can be deduced the following result :

Theorem 2.2 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \tag{8}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

2.2 Quasi-rational solutions of NLS equation

In the following, we take the limit when the parameters $\lambda_j \to 1$ for $1 \leq j \leq N$ and $\lambda_j \to -1$ for $N+1 \leq j \leq 2N$.
For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.
We consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2\epsilon^2 d_j^2, \quad 1 \leq j \leq N. \tag{9}$$

When ϵ goes to 0, we realize limited expansions at order p, for $1 \leq j \leq N$, of the terms
$\kappa_j = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_j = 4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_j = d_j\epsilon(1-\epsilon^2 d_j^2)^{-1/2}$, $x_{r,j} = (r-1)\ln\frac{1+i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}{1-i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}$,
$\kappa_{N+j} = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_{N+j} = -4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_{N+j} = 1/(d_j\epsilon)(1-\epsilon^2 d_j^2)^{1/2}$, $x_{r,N+j} = (r-1)\ln\frac{1-i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}{1+i\epsilon d_j(1-\epsilon^2 d_j^2)^{-1/2}}$.
The parameters a_j and b_j, for $1 \leq N$ are chosen in the form

$$a_j = \tilde{a}_j\epsilon^{M-1}, \quad b_j = \tilde{b}_j\epsilon^{M-1}, \quad 1 \leq j \leq N, \quad M = 2N. \tag{10}$$

We have the central result given in [10] :

Theorem 2.3 *With the parameters λ_j defined by (9), a_j and b_j chosen as in (10), for $1 \leq j \leq N$, the function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \lim_{\epsilon \to 0} \frac{W_3(0)}{W_1(0)}, \tag{11}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

depending on $3N$ parameters d_j, $\tilde{a}_j$, $\tilde{b}_j$, $1 \leq j \leq N$.

3 Quasi-rational solutions of order N

To get solutions of NLS equation written in the context of fiber optics

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0, \tag{12}$$

from these of (1), we can make the following changes of variables

$$\begin{aligned} t &\to X/2 \\ x &\to T. \end{aligned} \tag{13}$$

In the following, because of the complexity of the expression, we give all the solutions in the case where the parameters d_j are equal to j and all the parameters a_j and b_j are equal to 0.
The solution of NLS equation can be written in the form

$$v_N(x,t) = \frac{n(x,t)}{d(x,t)}\exp(2it - i\varphi) = (1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)})e^{2it}$$

with

$$\begin{aligned} G_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{g}_k(T)X^k \\ H_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{h}_k(T)X^k \\ Q_N(X,T) &= \textstyle\sum_{k=0}^{N(N+1)} \mathbf{q}_k(T)X^k \end{aligned}$$

3.1 Case N=4

In the case of order $N = 4$, we make an expansion at order 7. Taking the limit when $\epsilon \to 0$, the solution of NLS equation (12) takes the form

$$v(x,t) = \frac{n(x,t)}{d(x,t)}\exp(2it - i\varphi).$$

$\alpha_4 = 4, \quad \mathbf{g}_{20} = 0, \quad \mathbf{g}_{19} = 0, \quad \mathbf{g}_{18} = 10, \quad \mathbf{g}_{17} = 0, \quad \mathbf{g}_{16} = 270\,T^2 + 270, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 1800\,T^4$
$-3600\,T^2 + 9000, \quad \mathbf{g}_{13} = 0, \quad \mathbf{g}_{12} = 5880\,T^6 - 54600\,T^4 - 12600\,T^2 + 189000, \quad \mathbf{g}_{11} = 0,$
$\mathbf{g}_{10} = 11340\,T^8 - 176400\,T^6 + 189000\,T^4 - 378000\,T^2 - 1077300, \quad \mathbf{g}_9 = 0,$
$\mathbf{g}_8 = 13860\,T^{10} - 207900\,T^8 + 2356200\,T^6 + 1701000\,T^4 - 56983500\,T^2 - 4819500,$
$\mathbf{g}_7 = 0, \quad \mathbf{g}_6 = 10920\,T^{12} - 18480\,T^{10} + 6967800\,T^8 + 56095200\,T^6 - 342657000\,T^4$
$+198450000\,T^2 - 11907000, \quad \mathbf{g}_5 = 0 \quad \mathbf{g}_4 = 5400\,T^{14} + 163800\,T^{12} + 9034200\,T^{10}$
$+107919000\,T^8 - 615195000\,T^6 + 178605000\,T^4 + 654885000\,T^2 + 178605000, \quad \mathbf{g}_3 = 0,$
$\mathbf{g}_2 = 1530\,T^{16} + 133200\,T^{14} + 5506200\,T^{12} - 116802000\,T^{10} - 1731334500\,T^8$
$+2532222000\,T^6 - 893025000\,T^4 + 4643730000)T^2 + 223256250, \quad \mathbf{g}_1 = 0,$
$\mathbf{g}_0 = 190\,T^{18} + 33150\,T^{16} + 1294200\,T^{14} + 3288600\,T^{12} + 48629700\,T^{10}$
$-2015401500\,T^8 - 1845585000\,T^6 + 14586075000)T^4 + 2098608750T^2 - 44651250,$

$\mathbf{h}_{20} = 0, \quad \mathbf{h}_{19} = 0, \quad \mathbf{h}_{18} = 10\,T, \quad \mathbf{h}_{17} = 0, \quad \mathbf{h}_{16} = 90\,T^3 - 270\,T, \quad \mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 360\,T^5$
$-6000\,T^3 - 5400\,T, \quad \mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 840\,T^7 - 29400\,T^5 + 12600\,T^3 - 138600\,T, \quad \mathbf{h}_{11} = 0,$
$\mathbf{h}_{10} = 1260\,T^9 - 65520\,T^7 + 259560\,T^5 - 529200\,T^3 - 1984500\,T, \quad \mathbf{h}_9 = 0,$
$\mathbf{h}_8 = 1260\,T^{11} - 77700\,T^9 + 718200\,T^7 - 5329800\,T^5 - 6142500\,T^3 + 29767500\,T,$
$\mathbf{h}_7 = 0, \quad \mathbf{h}_6 = 840\,T^{13} - 48720\,T^{11} + 718200\,T^9 + 2973600\,T^7 - 72765000\,T^5$
$+436590000\,T^3 + 146853000\,T, \mathbf{h}_5 = 0, \quad \mathbf{h}_4 = 360\,T^{15} - 12600\,T^{13} + 138600\,T^{11}$
$-5859000\,T^9 - 328293000\,T^7 + 1075599000\,T^5 + 773955000\,T^3 + 535815000T, \quad \mathbf{h}_3 = 0,$
$\mathbf{h}_2 = 90\,T^{17} + 1200\,T^{15} - 189000\,T^{13} - 40143600\,T^{11}$
$-307786500\,T^9 + 2085426000\,T^7 - 4465125000\,T^5 + 4405590000T^3 - 1205583750T, \quad \mathbf{h}_1 = 0,$
$\mathbf{h}_0 = 10\,T^{19} + 930\,T^{17} - 86040\,T^{15} - 7018200\,T^{13} - 48100500\,T^{11} - 542902500\,T^9$
$+6039117000\,T^7 + 12942909000T^5 + 937676250T^3, \quad \mathbf{q}_{20} = 1, \quad \mathbf{q}_{19} = 0,$
$\mathbf{q}_{18} = 10\,T^2 + 10, \quad \mathbf{q}_{17} = 0, \quad \mathbf{q}_{16} = 45\,T^4 - 270\,T^2 + 405, \quad \mathbf{q}_{15} = 0,$
$\mathbf{q}_{14} = 120\,T^6 - 1800\,T^4 + 1800\,T^2 + 16200, \quad \mathbf{q}_{13} = 0, \quad \mathbf{q}_{12} = 210\,T^8 - 4200\,T^6 + 6300\,T^4$
$+113400\,T^2 + 425250, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 252\,T^{10} - 3780\,T^8 + 63000\,T^6$
$+718200\,T^4 + 3005100\,T^2 + 1644300, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 210\,T^{12} + 1260\,T^{10}$
$+255150\,T^8 - 567000\,T^6 + 23388750\,T^4 - 31468500\,T^2 + 17435250, \quad \mathbf{q}_7 = 0,$
$\mathbf{q}_6 = 120\,T^{14} + 5880\,T^{12} + 476280\,T^{10} + 16443000\,T^8 + 162729000\,T^6$
$-154791000\,T^4 + 130977000\,T^2 + 130977000, \quad \mathbf{q}_5 = 0, \quad \mathbf{q}_4 = 45\,T^{16} + 5400\,T^{14}$
$+459900\,T^{12} + 19845000\,T^{10} + 153798750\,T^8 + 702513000\,T^6 - 89302500\,T^4$
$+1250235000T^2 + 111628125, \mathbf{q}_3 = 0, \quad \mathbf{q}_2 = 10\,T^{18}$
$+2250\,T^{16} + 225000\,T^{14} + 4422600\,T^{12} - 99508500\,T^{10} - 224248500\,T^8$
$+9704205000\,T^6 + 15181425000T^4 - 1920003750T^2 + 223256250, \quad \mathbf{q}_1 = 0,$
$\mathbf{q}_0 = T^{20} + 370\,T^{18} + 44325\,T^{16} + 2208600\,T^{14} + 62795250\,T^{12} + 693384300\,T^{10}$
$+6641129250\,T^8 + 4346055000T^6 + 14042818125)T^4 + 2902331250T^2 + 22325625$

The plot of the modulus of v in the (x, t) coordinates gives :

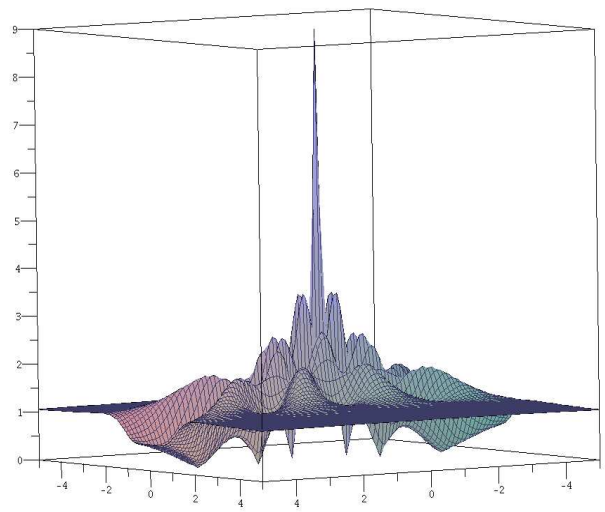

Figure 1: Solution of NLS, N=4a.

If we choose $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 1$, we have the same type of analytical expression for the modulus of v; we don't reproduce here.
We only give shape of the modulus of v in the (x, t) coordinates :

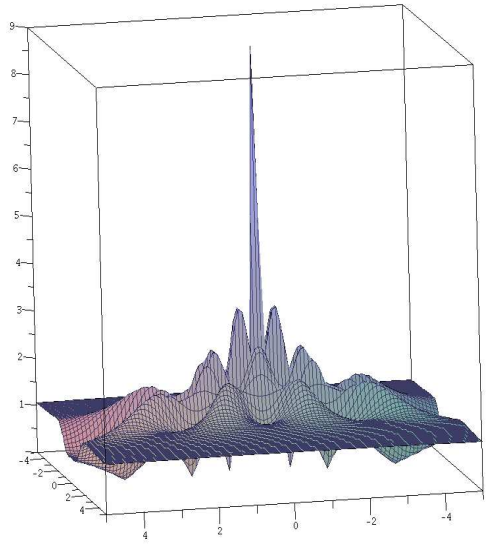

Figure 2: Solution of NLS, N=4b.

If we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, the shape of the modulus of v in the (x, t) coordinates is given by :

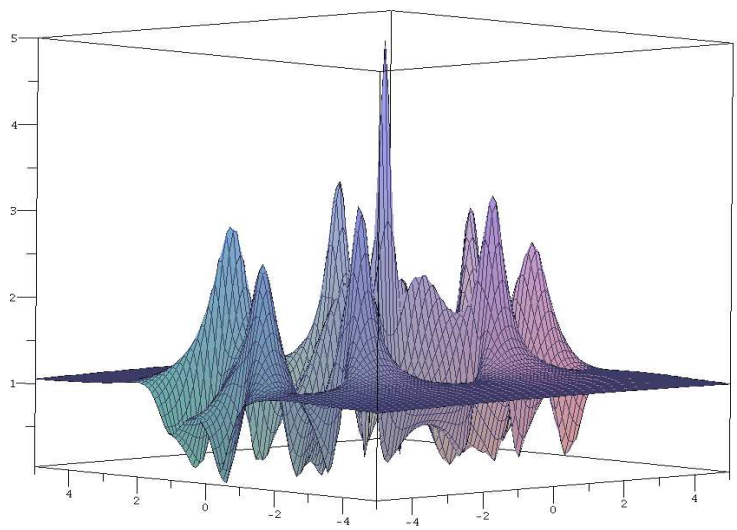

Figure 3: Solution of NLS, N=4c.

In these three plots, we can see the deformation of the initial Akhmediev's breather to multi-rogue waves, by choosing different types of the parameters among the 8 given by our formulation.

3.2 Case N=5

In the case $N = 5$, we realize an expansion at order 9 in ϵ. From (11), we get solutions of NLS equation. Taking the limit when $\epsilon \to 0$, the solution of NLS equation takes the form

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi).$$

$\alpha_5 = 60, \quad \mathbf{g}_{30} = 0, \quad \mathbf{g}_{29} = 0, \quad \mathbf{g}_{28} = 1, \quad \mathbf{g}_{27} = 0, \quad \mathbf{g}_{26} = 42\,T^2 + 42, \quad \mathbf{g}_{25} = 0, \quad \mathbf{g}_{24} = 455\,T^4$
$-1050\,T^2 + 2415, \quad \mathbf{g}_{23} = 0, \quad \mathbf{g}_{22} = 2548\,T^6 - 30660\,T^4 - 13860\,T^2 + 119700, \quad \mathbf{g}_{21} = 0,$
$\mathbf{g}_{20} = 9009\,T^8 - 226380\,T^6 + 171990\,T^4 - 343980\,T^2 + 3221505, \quad \mathbf{g}_{19} = 0, \quad \mathbf{g}_{18} = 22022\,T^{10}$
$-838530\,T^8 + 4142460\,T^6 - 44100\,T^4 - 36713250\,T^2 - 40153050, \quad \mathbf{g}_{17} = 0,$
$\mathbf{g}_{16} = 39039\,T^{12} - 1844766\,T^{10} + 22431465\,T^8 - 9075780\,T^6 - 259473375\,T^4 - 2703484350\,T^2$
$-370010025, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 51480\,T^{14} - 2522520\,T^{12} + 61319160\,T^{10}$
$+39803400\,T^8 - 773955000\,T^6 - 21896973000\,T^4 + 33756345000\,T^2 - 2893401000,$
$\mathbf{g}_{13} = 0, \quad \mathbf{g}_{12} = 51051\,T^{16} - 2023560\,T^{14} + 104367060\,T^{12} + 629483400\,T^{10}$
$+6114046050\,T^8 - 132697164600\,T^6 + 554979316500\,T^4 + 319310019000\,T^2 + 30787036875,$
$\mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 38038\,T^{18} - 589050\,T^{16} + 124369560\,T^{14} + 1700266680\,T^{12}$
$+37748127060\,T^{10} - 446713728300\,T^8 + 2431707075000\,T^6 + 1380509487000\,T^4$
$+4238859255750\,T^2 + 1299806817750, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 21021\,T^{20} + 570570\,T^{18}$
$+112372785\,T^{16} + 1735587000\,T^{14} - 43189665750\,T^{12} - 2318934687300\,T^{10}$
$+10714665764250\,T^8 - 20464596621000\,T^6 + 35015175365625\,T^4 + 40381027706250T^2$
$+5540260123125, \quad \mathbf{g}_7 = 0, \quad \mathbf{g}_6 = 8372\,T^{22} + 769692\,T^{20} + 78618540\,T^{18}$
$+570662820\,T^{16} - 223349124600\,T^{14} - 2950615722600\,T^{12} + 16520555280600\,T^{10}$
$-11401393059000\,T^8 + 147193042090500\,T^6 + 422927620447500T^4 - 99598095922500T^2$
$+17840228332500, \quad \mathbf{g}_5 = 0, \quad \mathbf{g}_4 = 2275\,T^{24} + 415380\,T^{22} + 39897270\,T^{20}$
$-30649500\,T^{18} - 148598863875\,T^{16} - 1555875783000\,T^{14} - 2135859799500\,T^{12}$
$+94593530241000\,T^{10} - 98463038821875\,T^8 + 2611250197762500T^6 - 159203362106250T^4$
$-83293781287500T^2 - 21709549378125, \quad \mathbf{g}_3 = 0, \quad \mathbf{g}_2 = 378\,T^{26} + 114450\,T^{24}$
$+12621420\,T^{22} + 89037900\,T^{20} - 283320450\,T^{18} + 1545272004150\,T^{16}$
$+12633981885000\,T^{14} - 118201467699000\,T^{12} + 1380551057313750\,T^{10}$
$+7814079083238750T^8 + 3521850108367500T^6 + 4776100863187500T^4$
$-1406247137268750T^2 - 13291560843750, \quad \mathbf{g}_1 = 0, \quad \mathbf{g}_0 = 29\,T^{28} + 13230\,T^{26}$
$+1814295\,T^{24} + 74845260\,T^{22} - 764250795\,T^{20} - 204794909550\,T^{18}$
$-3849793565625\,T^{16} - 34193820087000\,T^{14} + 942733356807375\,T^{12}$
$+1889980437035250T^{10} + 13147594251868125T^8 + 3164572952887500T^6$
$-3369410673890625T^4 - 124940671931250T^2 + 3987468253125,$

$\mathbf{h}_{30} = 0, \quad \mathbf{h}_{29} = 0, \quad \mathbf{h}_{28} = T, \quad \mathbf{h}_{27} = 0, \quad \mathbf{h}_{26} = 14\,T^3 - 42\,T, \quad \mathbf{h}_{25} = 0, \quad \mathbf{h}_{24} = 91\,T^5 - 1610\,T^3$
$-1365\,T, \quad \mathbf{h}_{23} = 0, \quad \mathbf{h}_{22} = 364\,T^7 - 14700\,T^5 + 13860\,T^3 - 64260\,T, \quad \mathbf{h}_{21} = 0, \quad \mathbf{h}_{20} = 1001\,T^9$
$-67452\,T^7 + 411894\,T^5 - 97020\,T^3 - 2546775\,T, \quad \mathbf{h}_{19} = 0, \quad \mathbf{h}_{18} = 2002\,T^{11} - 190190\,T^9$
$+2572500\,T^7 - 3342780\,T^5 - 6769350\,T^3 - 39756150\,T, \quad \mathbf{h}_{17} = 0, \quad \mathbf{h}_{16} = 3003\,T^{13}$
$-358974\,T^{11} + 7821765\,T^9 - 39225060\,T^7 - 73327275\,T^5 + 439963650\,T^3 + 2114980875\,T,$
$\mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 3432\,T^{15} - 471240\,T^{13} + 13736520\,T^{11} - 135513000\,T^9 - 793686600\,T^7$
$+2779093800\,T^5 + 51116751000\,T^3 + 24754653000\,T, \quad \mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 3003\,T^{17}$
$-434280\,T^{15} + 14403060\,T^{13} - 248776920\,T^{11} - 793072350\,T^9 - 2707651800\,T^7 + 375945664500\,T^5$
$-81098577000\,T^3 + 297337138875\,T, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 2002\,T^{19} - 275814\,T^{17}$
$+8148168\,T^{15} - 362872440\,T^{13} - 114704100\,T^{11} - 90682521300\,T^9 + 1534457471400\,T^7$
$-1772183107800\,T^5 + 1117897625250\,T^3 + 1493718266250)T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 1001\,T^{21}$

$-113190\,T^{19}+836325\,T^{17}-501931080\,T^{15}-15705928350\,T^{13}-400107348900\,T^{11}$
$+4976480045250\,T^{9}-11450902365000\,T^{7}+30510953731125\,T^{5}-5820156483750T^{3}$
$-21110374254375T,\quad \mathbf{h}_7=0,\quad \mathbf{h}_6=364\,T^{23}-24332\,T^{21}-2084460\,T^{19}$
$-528432660\,T^{17}-31926371400\,T^{15}+150244907400\,T^{13}+11823972489000\,T^{11}$
$-3962494809000\,T^{9}+7158970633500\,T^{7}+132364254802500T^{5}-455536249717500T^{3}$
$-63681344842500T,\quad \mathbf{h}_5=0,\quad \mathbf{h}_4=91\,T^{25}+420\,T^{23}-1450890\,T^{21}-337761900\,T^{19}$
$-18543465675\,T^{17}+274020553800\,T^{15}+5724951088500\,T^{13}+48513868893000\,T^{11}$
$+171111381643125\,T^{9}+1334157649492500T^{7}-1694171881946250T^{5}-515712560737500T^{3}$
$-131586452353125)T,\quad \mathbf{h}_3=0,\quad \mathbf{h}_2=14\,T^{27}+1470\,T^{25}-409500\,T^{23}$
$-111637260\,T^{21}-3311799750\,T^{19}+88973271450\,T^{17}-3045655809000\,T^{15}$
$-34947318861000\,T^{13}+1002802178873250\,T^{11}+1999468016831250T^{9}-6800738923597500T^{7}$
$+4269249343012500T^{5}-1666761729806250T^{3}+204690036993750T,\quad \mathbf{h}_1=0,$
$\mathbf{h}_0=T^{29}+238\,T^{27}-43701\,T^{25}-14070420\,T^{23}-1034990775\,T^{21}-32505382350\,T^{19}$
$+259820563275\,T^{17}+13855420996200\,T^{15}+406907765530875\,T^{13}+497730743291250T^{11}$
$+1983581436965625T^{9}-10570073675332500T^{7}-7864084888813125T^{5}-224184326231250T^{3}$
$+73103584640625)T$

$\mathbf{q}_{30}=1,\quad \mathbf{q}_{29}=0,\quad \mathbf{q}_{28}=15\,T^{2}+15,\quad \mathbf{q}_{27}=0,\quad \mathbf{q}_{26}=105\,T^{4}-630\,T^{2}+945,\quad \mathbf{q}_{25}=0,$
$\mathbf{q}_{24}=455\,T^{6}-7875\,T^{4}+4725\,T^{2}+64575,\quad \mathbf{q}_{23}=0,\quad \mathbf{q}_{22}=1365\,T^{8}-39900\,T^{6}$
$+103950\,T^{4}+548100\,T^{2}+3709125,\quad \mathbf{q}_{21}=0,\quad \mathbf{q}_{20}=3003\,T^{10}-114345\,T^{8}+859950\,T^{6}+4035150\,T^{4}$
$+34827975\,T^{2}+133656075,\quad \mathbf{q}_{19}=0,\quad \mathbf{q}_{18}=5005\,T^{12}-200970\,T^{10}+3649275\,T^{8}+220500\,T^{6}$
$+277333875\,T^{4}+959505750\,T^{2}+1115785125,\quad \mathbf{q}_{17}=0,\quad \mathbf{q}_{16}=6435\,T^{14}-204435\,T^{12}+10174815\,T^{10}$
$+42170625\,T^{8}+2030639625\,T^{6}+7693410375\,T^{4}-27357820875\,T^{2}+24214372875,\quad \mathbf{q}_{15}=0,$
$\mathbf{q}_{14}=6435\,T^{16}-59400\,T^{14}+21035700\,T^{12}+451672200\,T^{10}+2902331250\,T^{8}+79622109000\,T^{6}$
$-319613647500\,T^{4}+191285955000\,T^{2}+463546951875,\quad \mathbf{q}_{13}=0,\quad \mathbf{q}_{12}=5005\,T^{18}$
$+155925\,T^{16}+33585300\,T^{14}+1481098500\,T^{12}+42118035750\,T^{10}+639849435750\,T^{8}$
$-1190848837500\,T^{6}+1787210932500\,T^{4}+4850130403125\,T^{2}+5581517878125,\quad \mathbf{q}_{11}=0$
$\mathbf{q}_{10}=3003\,T^{20}+279510\,T^{18}+40951575\,T^{16}+2550025800\,T^{14}+112585249350\,T^{12}$
$+1486454400900\,T^{10}+2935114197750\,T^{8}+10430710605000\,T^{6}+58973741229375\,T^{4}$
$+49590883833750T^{2}+14657286301875,\quad \mathbf{q}_{9}=0,\quad \mathbf{q}_{8}=1365\,T^{22}+246015\,T^{20}$
$+36850275\,T^{18}+2719544625\,T^{16}+98273999250\,T^{14}-830307854250\,T^{12}$
$-8598553724250\,T^{10}+211739487041250\,T^{8}+162726680615625\,T^{6}$
$+731900852746875T^{4}-370328202365625T^{2}+93610564228125,\quad \mathbf{q}_{7}=0,$
$\mathbf{q}_{6}=455\,T^{24}+134820\,T^{22}+23403870\,T^{20}+1942384500\,T^{18}+36981653625\,T^{16}$
$-1371507795000\,T^{14}+3080287318500\,T^{12}+299367020421000\,T^{10}+3135310421315625\,T^{8}$
$+10570433743012500T^{6}-3151872128081250T^{4}+1114718902762500T^{2}$
$+412481438184375,\quad \mathbf{q}_{5}=0,\quad \mathbf{q}_{4}=105\,T^{26}+46725\,T^{24}+9856350\,T^{22}$
$+950244750\,T^{20}+28094731875\,T^{18}+11015463375\,T^{16}-7594552507500\,T^{14}$
$+191792925292500\,T^{12}+6041183185209375\,T^{10}+13451797993921875T^{8}$
$+29454394197768750T^{6}+13424476452187 50T^{4}+5894807234203125T^{2}$
$+232602314765625,\quad \mathbf{q}_{3}=0,\quad \mathbf{q}_{2}=15\,T^{28}+9450\,T^{26}$
$+2461725\,T^{24}+296748900\,T^{22}+21862740375\,T^{20}+1401063945750\,T^{18}$
$+53045439418125\,T^{16}+801963419355000\,T^{14}+7240875247423125\,T^{12}$
$-18225599206106250T^{10}-19943100941990625T^{8}+193179545303062500T^{6}$

$+118075580755453125T^4 - 5861578332093750T^2 + 299060118984375,$
$\mathbf{q}_1 = 0, \quad \mathbf{q}_0 = T^{30} + 855\,T^{28} + 275625\,T^{26} + 44441775\,T^{24} + 4060783125\,T^{22}$
$+207533751075\,T^{20} + 5923312282125\,T^{18} + 77461769896875\,T^{16}$
$+1691986493491875\,T^{14} + 21127132873153125T^{12} + 60580010182426875T^{10}$
$+225021251512378125T^8 + 50098108080234375T^6 + 67806897644390625T^4$
$+5881515673359375T^2 + 19937341265625$

Here we obtain the Akhmediev's breather of order 5. Again, we can note the presence of $N(N-1)-1$ local maximums; the global maximum of $|v|$ is equal to 11. We represent the modulus of v in the (x,t) coordinates and we get :

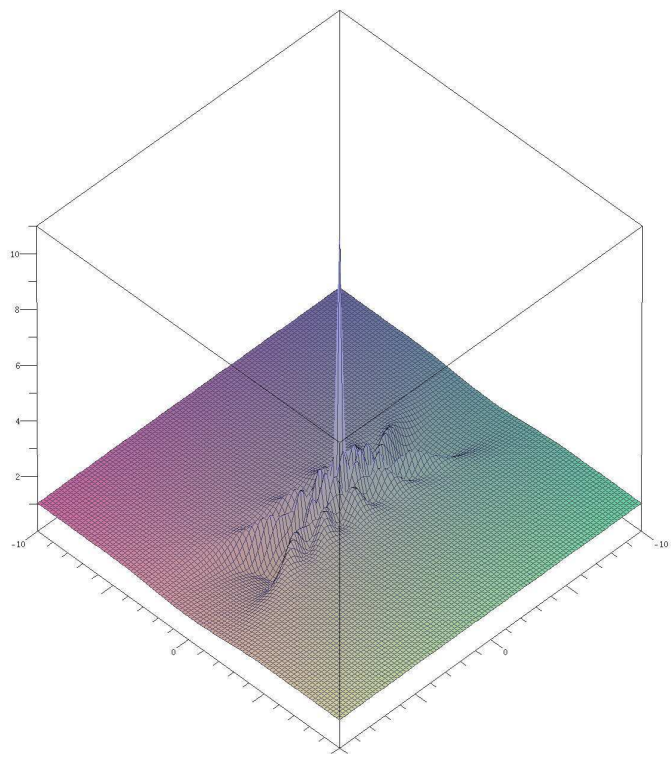

Figure 4: Solution of NLS, N=5a.

In the following cases, we only give the plots for the modulus of v in the (x, t) coordinates.
If we choose $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 1$, we get :

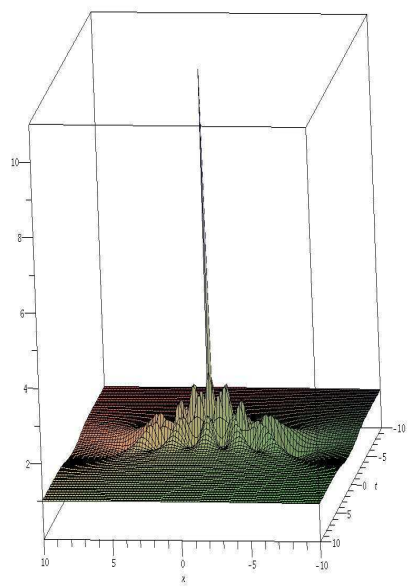

Figure 5: Solution of NLS, N=5b.

If we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = -0$, $a_5 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, we get The shape of the modulus of v in the (x, t) coordinates is given by :

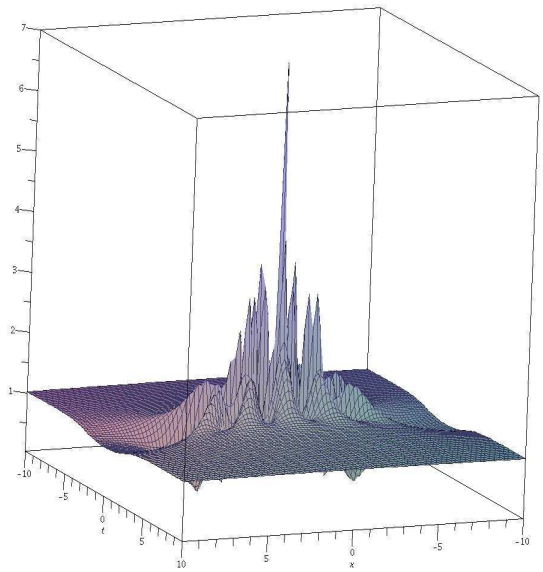

Figure 6: Solution of NLS, N=5c.

3.3 Case N=6

In the case $N = 6$, we realize an expansion at order 9 in ϵ. From (11), we get solutions of NLS equation. Taking the limit when $\epsilon \to 0$, the solution of NLS equation takes the form

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it - i\varphi).$$

$\alpha_6 = 4, \quad \mathbf{g}_{42} = 0, \quad \mathbf{g}_{41} = 0, \quad \mathbf{g}_{40} = 21, \quad \mathbf{g}_{39} = 0, \quad \mathbf{g}_{38} = 1260\,T^2 + 1260, \quad \mathbf{g}_{37} = 0, \quad \mathbf{g}_{36} = 19950\,T^4$
$-49140\,T^2 + 108990, \quad \mathbf{g}_{35} = 0, \quad \mathbf{g}_{34} = 167580\,T^6 - 2249100\,T^4 - 1190700\,T^2 + 9128700, \quad \mathbf{g}_{33} = 0,$
$\mathbf{g}_{32} = 915705\,T^8 - 27439020\,T^6 + 24409350\,T^4 - 48818700\,T^2 + 589098825, \quad \mathbf{g}_{31} = 0, \quad \mathbf{g}_{30} = 3581424\,T^{10}$
$-178128720\,T^8 + 984947040\,T^6 + 60328800\,T^4 - 4186501200\,T^2 + 18503478000, \quad \mathbf{g}_{29} = 0,$
$\mathbf{g}_{28} = 10581480\,T^{12} - 743616720\,T^{10} + 9464477400\,T^8 - 7045768800\,T^6 - 30612897000\,T^4$
$-520740738000\,T^2 - 459479223000, \quad \mathbf{g}_{27} = 0, \quad \mathbf{g}_{26} = 24418800\,T^{14} - 2175329520\,T^{12}$
$+47637843120\,T^{10} - 159272794800\,T^8 - 224542206000\,T^6 - 4616367714000\,T^4 - 32428595430000\,T^2$
$-7322876442000, \quad \mathbf{g}_{25} = 0, \quad \mathbf{g}_{24} = 44971290\,T^{16} - 4661420400\,T^{14} + 152359061400\,T^{12}$
$-1026048416400\,T^{10} - 337581310500\,T^8 - 30371208714000\,T^6 - 186218752545000\,T^4$
$+1100164292010000\,T^2 - 108079377603750, \quad \mathbf{g}_{23} = 0, \quad \mathbf{g}_{22} = 67016040\,T^{18} - 7469967960\,T^{16}$
$+337998679200\,T^{14} - 3317725202400\,T^{12} + 14701701495600\,T^{10} - 134995160034000\,T^8$
$-1193237286444000\,T^6 + 29149053992100000\,T^4 + 20986218417465000\,T^2 + 1152340349265000,$
$\mathbf{g}_{21} = 0, \quad \mathbf{g}_{20} = 81477396\,T^{20} - 8962513560\,T^{18} + 547168642020\,T^{16} - 6239220372000\,T^{14}$
$+108187552103400\,T^{12} - 282142532826000\,T^{10} - 9222667785639000\,T^8 + 257357195058348000\,T^6$
$+18279145208362500\,T^4 + 529488288837585000\,T^2 + 156886140925552500, \quad \mathbf{g}_{19} = 0$
$\mathbf{g}_{18} = 81124680\,T^{22} - 7847559720\,T^{20} + 671406535800\,T^{18} - 6955900320600\,T^{16} + 296997420258000\,T^{14}$
$+830239889094000\,T^{12} - 53402987374506000\,T^{10} + 1441119106966770000\,T^8 - 2734739470632015000\,T^6$
$+4283688113241075000\,T^4 + 11213622020930175000\,T^2 + 4883826305513925000, \quad \mathbf{g}_{17} = 0,$
$\mathbf{g}_{16} = 66134250\,T^{24} - 4572870120\,T^{22} + 650465836020\,T^{20} - 3956090430600\,T^{18}$
$+265628919921750\,T^{16} - 8374317575778000\,T^{14} - 49449016042618500 0\,T^{12} + 7083936614255886000\,T^{10}$
$-27625636046797346250\,T^8 + 58715278628789475000\,T^6 + 100296773020980112500\,T^4$
$+1090634133739515750 00T^2 + 48008615346625106250, \quad \mathbf{g}_{15} = 0, \quad \mathbf{g}_{14} = 43953840\,T^{26}$
$-1085994000\,T^{24} + 525422570400\,T^{22} + 283015101600\,T^{20} - 440867272230000\,T^{18}$
$-70625220688638000\,T^{16} - 1840700971022760000\,T^{14} + 23136462127298520000\,T^{12}$
$-77288502059715870000\,T^{10} + 203718051732368250000\,T^8 + 1193114598558012900000\,T^6$
$+3826674393311385000 00T^4 - 1778554070753402250000T^2 + 450583593605664750000, \quad \mathbf{g}_{13} = 0,$
$\mathbf{g}_{12} = 23604840\,T^{28} + 923030640\,T^{26} + 375409452600\,T^{24} + 2624172616800\,T^{22}$
$-1510226919293400\,T^{20} - 169673051037798000\,T^{18} - 1533701946518973000\,T^{16}$
$+6017129050632084000 0\,T^{14} - 81243452836954485000\,T^{12} + 989384777473422330000\,T^{10}$
$+9218597475208223025000\,T^8 + 11767748128231995900000T^6 - 40367065478563333125000T^4$
$-84257542439914882500 00T^2 + 209728277564314162500 0, \quad \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 10093104\,T^{30}$
$+1259753040\,T^{28} + 242036287920\,T^{26} + 3006820731600\,T^{24} - 1903047548547600\,T^{22}$
$-177522454448262000\,T^{20} + 1673066563892262000\,T^{18} + 8752837151032266600 0\,T^{16}$
$+6620696554494330000\,T^{14} + 58071156589382636700 00\,T^{12} + 232393802954747811300 00\,T^{10}$
$+1376133174966610021500 00T^8 - 2103183399902364895500 00T^6 + 1183354860628415475000 0T^4$
$-8309908689414653475000 0T^2 - 994815686313620325000 0, \quad \mathbf{g}_{9} = 0, \quad \mathbf{g}_{8} = 3357585\,T^{32}$
$+788855760\,T^{30} + 133834793400\,T^{28} + 2706551593200\,T^{26} - 1301950524262500\,T^{24}$
$-74974735386558000\,T^{22} + 3528441325033317000\,T^{20} - 9148978382135130000\,T^{18}$
$-91732847682861779625 0\,T^{16} + 203311861481079697500 00\,T^{14} + 697010070670980239250 00\,T^{12}$
$+887836329346915768050000T^{10} - 517772523069859235062500T^8 + 215285069230716367575000 0T^6$
$-639511342714258678125000T^4 - 797615544339523428750000T^2 - 212630910220158658593 75, \quad \mathbf{g}_{7} = 0,$
$\mathbf{g}_{6} = 837900\,T^{34} + 311725260\,T^{32} + 57825260640\,T^{30} + 2004411679200\,T^{28} - 486305908326000\,T^{26}$
$-2096884640214000\,T^{24} + 2746634505228180000\,T^{22} + 3519359798377222800 0\,T^{20}$
$+5350948351589228850 00\,T^{18} + 1283077121350172507 50000\,T^{16} + 3830217467903955513000 00\,T^{14}$
$+2069806537668865079700000T^{12} - 2878103885627077118550000T^{10} + 1375299493382801118825000 0T^8$
$+8319513974090622187500000T^6 - 1186111617859045820250000 0T^4$

$+1576218096385317649687500T^2 - 26235193683001885312500, \quad \mathbf{g}_5 = 0, \quad \mathbf{g}_4 = 147630\,T^{36}$
$+79512300\,T^{34} + 17590409550\,T^{32} + 1021528519200\,T^{30} - 81006309657000\,T^{28} + 1634306691906000\,T^{26}$
$+606189343425435000\,T^{24} + 11528510197530060000\,T^{22} - 396162299973691597500\,T^{20}$
$+14798306913954275250 00\,T^{18} + 581811329651584456462500\,T^{16} + 12479230555322630715000 00T^{14}$
$+384265929974146076587500 00T^{12} + 7389770241237094037250000T^{10} + 282582672524661537393750 00T^{8}$
$-571168891143567468675000 00T^{6} + 3389031680327926024218750T^{4} + 2208926709003114942187500T^{2}$
$+110589958773237874218750, \quad \mathbf{g}_3 = 0, \quad \mathbf{g}_2 = 16380\,T^{38} + 12074580\,T^{36} + 3288316500\,T^{34}$
$+300491005500\,T^{32} - 226761670800\,T^{30} - 2504351229534000\,T^{28} - 460421913549654000\,T^{26}$
$-26192900747252610000\,T^{24} - 1105363357063008750 00\,T^{22} + 1167000373860202381500 0\,T^{20}$
$+4172050048409443257750 00\,T^{18} + 37391568087958236652500 0T^{16} + 82084510186033493407500 00T^{14}$
$+97521767433772016816250000T^{12} - 248372033459815147632750000T^{10}$
$-439480474711801012586250000T^{8} + 507566738749878299559375 00T^{6} - 116535964348837936546875 00T^{4}$
$+65923105294696343203125 00T^{2} + 6032179569449338593750 0, \quad \mathbf{g}_1 = 0, \quad \mathbf{g}_0 = 861\,T^{40}$
$+835380\,T^{38} + 282059190\,T^{36} + 38110734900\,T^{34} + 757788568425\,T^{32} - 357432351318000\,T^{30}$
$-33079684204539000\,T^{28} - 671694631354782000\,T^{26} + 47369156210904386250\,T^{24}$
$+1080980353533726135000\,T^{22} + 57329971310306073802500\,T^{20} + 1450451085746969465775000T^{18}$
$+57350647803714791396062 50T^{16} + 319705794407152763032500 00T^{14} - 700649607133638933828750 00T^{12}$
$-551776429312196509575000 00T^{10} - 2035703097778171709327343 75)T^{8} - 2096651569905713443218750 0T^{6}$
$+4223818308522276159609375 0)T^{4} + 1146114118195374332812500)T^{2} - 646304953869571992187 5,$

$\mathbf{h}_{42} = 0, \quad \mathbf{h}_{41} = 0, \quad \mathbf{h}_{40} = 21\,T, \quad \mathbf{h}_{39} = 0, \quad \mathbf{h}_{38} = 420\,T^3 - 1260\,T, \quad \mathbf{h}_{37} = 0, \quad \mathbf{h}_{36} = 3990\,T^5$
$-72660\,T^3 - 59850\,T, \quad \mathbf{h}_{35} = 0, \quad \mathbf{h}_{34} = 23940\,T^7 - 1031940\,T^5 + 1190700\,T^3 - 4365900\,T, \quad \mathbf{h}_{33} = 0,$
$\mathbf{h}_{32} = 101745\,T^9 - 7646940\,T^7 + 55526310\,T^5 + 6747300\,T^3 - 308688975\,T, \quad \mathbf{h}_{31} = 0, \quad \mathbf{h}_{30} = 325584\,T^{11}$
$-36414000\,T^9 + 606039840\,T^7 - 487075680\,T^5 - 442940400\,T^3 - 15569593200\,T, \quad \mathbf{h}_{29} = 0,$
$\mathbf{h}_{28} = 813960\,T^{13} - 122608080\,T^{11} + 3460174200\,T^9 - 15134364000\,T^7 - 6941781000\,T^5 + 41650686000\,T^3$
$-133275051000\,T, \quad \mathbf{h}_{27} = 0, \quad \mathbf{h}_{26} = 1627920\,T^{15} - 307676880\,T^{13} + 12556328400\,T^{11}$
$-123859789200\,T^9 - 414363600\,T^7 + 625617594000\,T^5 + 14773276854000\,T^3 + 40215059010000\,T,$
$\mathbf{h}_{25} = 0, \quad \mathbf{h}_{24} = 2645370\,T^{17} - 593676720\,T^{15} + 31525767000\,T^{13} - 527395957200\,T^{11}$
$+752464849500\,T^9 + 2323222398000\,T^7 + 161180546247000\,T^5 + 1150723795410000\,T^3$
$+884942275376250\,T, \quad \mathbf{h}_{23} = 0, \quad \mathbf{h}_{22} = 3527160\,T^{19} - 897198120\,T^{17} + 57072768480\,T^{15}$
$-1415227917600\,T^{13} + 5710878205200\,T^{11} - 31038334110000\,T^9 + 1020560829372000\,T^7$
$+10607242786812000\,T^5 - 10549431670365000\,T^3 + 21705296078655000\,T, \quad \mathbf{h}_{21} = 0,$
$\mathbf{h}_{20} = 3879876\,T^{21} - 1072071000\,T^{19} + 75633978420\,T^{17} - 2635403934240\,T^{15}$
$+18071080057800\,T^{13} - 232668433443600\,T^{11} + 3887539470045000\,T^9 + 73903432233708000\,T^7$
$-332535925127983500\,T^5 - 35419823925255000\,T^3 + 373647041815162500\,T, \quad \mathbf{h}_{19} = 0,$
$\mathbf{h}_{18} = 3527160\,T^{23} - 1014893880\,T^{21} + 72635963400\,T^{19} - 3661723989000\,T^{17} + 22961577870000\,T^{15}$
$-558100069758000\,T^{13} + 8233601126058000\,T^{11} + 401324717365110000\,T^9 - 2629541354603625000\,T^7$
$+1891946762885025000\,T^5 - 2077818312668175000\,T^3 + 469617427731375000T, \quad \mathbf{h}_{17} = 0,$
$\mathbf{h}_{16} = 2645370\,T^{25} - 756854280\,T^{23} + 47845898100\,T^{21} - 4100304839400\,T^{19} - 18503101143450\,T^{17}$
$-1692739675242000\,T^{15} - 39034514859417000\,T^{13} + 1908000279128790000\,T^{11}$
$-13226094269284916250\,T^9 + 41333515336516095000\,T^7 - 30934512876526875 00\,T^5$
$-152813978846731125000T^3 - 136885438789385343750)T, \quad \mathbf{h}_{15} = 0, \quad \mathbf{h}_{14} = 1627920\,T^{27}$
$-437739120\,T^{25} + 17295314400\,T^{23} - 4002542359200\,T^{21} - 118543877298000\,T^{19}$
$-5096839239954000\,T^{17} + 6829841768904000\,T^{15} + 8476647700225320000\,T^{13}$
$-3966200331257241000 0\,T^{11} + 1985918854192465500 00\,T^{9} + 3386764257251067000 00\,T^{7}$
$-1458984265260221700000T^5 - 3931624238736174750000T^3 - 1056176028083310750000T,$

$\mathbf{h}_{13} = 0, \quad \mathbf{h}_{12} = 813960\,T^{29} - 190123920\,T^{27} - 2343879720\,T^{25} - 3521465085600\,T^{23}$
$-214336185247800\,T^{21} - 7685261724294000\,T^{19} + 559933161897879000\,T^{17}$
$+18300777177408264000\,T^{15} - 673194046229293050 00\,T^{13} + 7020284323949612100 00\,T^{11}$
$+96525620962655692500 0\,T^{9} - 501622106907599370000 0T^{7} - 410281094967542048250 00T^{5}$
$+508850425604834775000 0T^{3} - 765950527810686487500 0T, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 325584\,T^{31}$
$-57834000\,T^{29} - 7480453680\,T^{27} - 2671612327440\,T^{25} - 227871922345200\,T^{23} - 4515437156907600\,T^{21}$
$+1063591395097770000\,T^{19} + 15434675218020366000\,T^{17} - 89580321136341882000\,T^{15}$
$+204118336030921389000 0\,T^{13} + 472319063299643175000 0\,T^{11} + 116641003453739572500 00T^{9}$
$-254778861744583996050000T^{7} + 117211546169871233850000T^{5} - 262892994004804432500 00T^{3}$
$-190865517523236517500 00T, \quad \mathbf{h}_{9} = 0, \quad \mathbf{h}_{8} = 101745\,T^{33} - 10024560\,T^{31} - 5022145800\,T^{29}$
$-1608119730000\,T^{27} - 161174098606500\,T^{25} + 335632587066000\,T^{23} + 736078113424269000\,T^{21}$
$-7565148948169890000\,T^{19} - 106455713823272756250\,T^{17} + 581150502429384195 0000\,T^{15}$
$+677936548901582932500 0\,T^{13} + 154597964806237664250000T^{11} - 152695649554003974206 2500T^{9}$
$+132286655446844542875 0000T^{7} - 166506393999097888312 5000)T^{5} + 333570867196457756250000T^{3}$
$+248146906256610376640625T, \quad \mathbf{h}_{7} = 0, \quad \mathbf{h}_{6} = 23940\,T^{35} + 149940\,T^{33} - 1927346400\,T^{31}$
$-708549055200\,T^{29} - 79214033883600\,T^{27} + 823565301174000\,T^{25} + 200617502001084000\,T^{23}$
$-8279951353262820000\,T^{21} - 777838836351965850 00\,T^{19} + 298838300272420639500 0\,T^{17}$
$+903984977552590395000 0\,T^{15} - 415079152248697744500000T^{13} - 626463443444424218325 0000T^{11}$
$-711055582862691008250000T^{9} - 175642478756527825350000 0T^{7} - 332389052423667681750000 0T^{5}$
$+743158139697354864656250 0T^{3} + 510341541351824995312500T, \quad \mathbf{h}_{5} = 0, \quad \mathbf{h}_{4} = 3990\,T^{37}$
$+555660\,T^{35} - 451870650\,T^{33} - 210601490400\,T^{31} - 27164808405000\,T^{29} - 494049363822000\,T^{27}$
$-131254421193000\,T^{25} - 2331351766748340000\,T^{23} - 57607676795659567500\,T^{21}$
$+3182842545894136125000\,T^{19} + 12147149609360590151250 0\,T^{17} - 730868754303114203700000T^{15}$
$-3952170770249132825625000T^{13} - 1881738367410594768075000 0T^{11} - 2460092744691499345312500 0T^{9}$
$-6208459644570939582750000 0T^{7} + 3896203933683738747984375 0T^{5} + 5862704048212428604687500T^{3}$
$+6908281840250313960937 50T, \quad \mathbf{h}_{3} = 0, \quad \mathbf{h}_{2} = 420\,T^{39} + 127260\,T^{37} - 59773140\,T^{35} - 37389170700\,T^{33}$
$-6082652847600\,T^{31} - 456491304234000\,T^{29} - 20958512959818000\,T^{27} + 1456163148715002000\,T^{25}$
$+201140982748216035000\,T^{23} + 4864337674818781005000\,T^{21} + 60289385834313075825000\,T^{19}$
$-641663910397442064825000T^{17} + 3860956945797085030050000T^{15} + 17024232728719929462750000T^{13}$
$-174686108305029234779250000T^{11} - 1331533804064982513187500 00T^{9}$
$+252883712879152705566562500T^{7} - 662913554950789171115625 00T^{5} + 1243490731245056512968750 0T^{3}$
$-784183344028414017187500T, \quad \mathbf{h}_{1} = 0, \quad \mathbf{h}_{0} = 21\,T^{41} + 10500\,T^{39} - 3356010\,T^{37}$
$-2977396380\,T^{35} - 678988637775\,T^{33} - 67009019137200\,T^{31} - 1544931642555000\,T^{29}$
$+190273318160058000\,T^{27} + 15779576666742320250\,T^{25} + 302225493684212595000\,T^{23}$
$+1075022773008800096250 0\,T^{21} + 16316397675737800987500 0T^{19} - 134533446161898484614375 0T^{17}$
$-1040303287910245143015000 0T^{15} - 1159255743354689927388750 00T^{13}$
$-5210329947138656091675000 0T^{11} - 863233355083699959433593 75T^{9} + 2555316345339651335990625 00T^{7}$
$+853527556878926628135937 50T^{5} + 1157603984041944501562500T^{3} - 196045836007103504296875T$

$\mathbf{q}_{42} = 1, \quad \mathbf{q}_{41} = 0, \quad \mathbf{q}_{40} = 21\,T^{2} + 21, \quad \mathbf{q}_{39} = 0, \quad \mathbf{q}_{38} = 210\,T^{4} - 1260\,T^{2} + 1890, \quad \mathbf{q}_{37} = 0,$
$\mathbf{q}_{36} = 1330\,T^{6} - 24570\,T^{4} + 10710\,T^{2} + 193410, \quad \mathbf{q}_{35} = 0, \quad \mathbf{q}_{34} = 5985\,T^{8} - 202860\,T^{6} + 595350\,T^{4}$
$+1984500\,T^{2} + 18356625, \quad \mathbf{q}_{33} = 0, \quad \mathbf{q}_{32} = 20349\,T^{10} - 1012095\,T^{8} + 8136450\,T^{6} + 17662050\,T^{4}$
$+215219025\,T^{2} + 1383295725, \quad \mathbf{q}_{31} = 0, \quad \mathbf{q}_{30} = 54264\,T^{12} - 3444336\,T^{10} + 53290440\,T^{8} - 20109600\,T^{6}$
$+1912264200\,T^{4} + 15369555600\,T^{2} + 65128908600, \quad \mathbf{q}_{29} = 0, \quad \mathbf{q}_{28} = 116280\,T^{14} - 8439480\,T^{12}$
$+217024920\,T^{10} - 694575000\,T^{8} + 14832153000\,T^{6} + 133703703000\,T^{4} + 386251173000\,T^{2}$
$+1034587323000, \quad \mathbf{q}_{27} = 0, \quad \mathbf{q}_{26} = 203490\,T^{16} - 15165360\,T^{14} + 622815480\,T^{12}$

$-2210256720\,T^{10}+56135551500\,T^{8}+1192724190000\,T^{6}+1433519451000\,T^{4}-27693705438000\,T^{2}$
$+38780976953250,\quad \mathbf{q}_{25}=0,\quad \mathbf{q}_{24}=293930\,T^{18}-19631430\,T^{16}+1366583400\,T^{14}$
$+2951613000\,T^{12}+363193267500\,T^{10}+10089021379500\,T^{8}+22829041053000\,T^{6}$
$-475525632015000\,T^{4}+342500315456250\,T^{2}+1376457162806250,\quad \mathbf{q}_{23}=0,\quad \mathbf{q}_{22}=352716\,T^{20}$
$-16521960\,T^{18}+2444960700\,T^{16}+41186426400\,T^{14}+2435977719000\,T^{12}+35777824966800\,T^{10}$
$+398543162031000\,T^{8}-3822883567020000\,T^{6}+7873515567697500\,T^{4}+16086079864575000\,T^{2}$
$+38577274289347500,\quad \mathbf{q}_{21}=0,\quad \mathbf{q}_{20}=352716\,T^{22}-4288284\,T^{20}+3742158420\,T^{18}$
$+149684881500\,T^{16}+9553331403000\,T^{14}+212261247613800\,T^{12}+3849610940821800\,T^{10}$
$-15394646399517000\,T^{8}+71758478683237500\,T^{6}+204046725448912500\,T^{4}+436091149100722500\,T^{2}$
$+674506306475347500,\quad \mathbf{q}_{19}=0,\quad \mathbf{q}_{18}=293930\,T^{24}+11695320\,T^{22}+5003418420\,T^{20}$
$+334607427000\,T^{18}+23101428561750\,T^{16}+807591346254000\,T^{14}+16185920631111000\,T^{12}$
$-8352665505954000\,T^{10}+576663818982693750\,T^{8}+1242059144478075000\,T^{6}+5668373975877112500\,T^{4}$
$+37615010855325750000T^{2}+4325812023235106250,\quad \mathbf{q}_{17}=0,\quad \mathbf{q}_{16}=203490\,T^{26}+22694490\,T^{24}$
$+5786087580\,T^{22}+534631523580\,T^{20}+37855900293750\,T^{18}+1433816319300750\,T^{16}$
$+1641456285777000\,T^{14}-265649781021471000\,T^{12}+5360993203091352750\,T^{10}+69096046130625937 50\,T^{8}$
$+80880110043223837500\,T^{6}+422369703735123375 00T^{4}-113560599693873543750T^{2}$
$+60012457876086581250,\quad \mathbf{q}_{15}=0,\quad \mathbf{q}_{14}=116280\,T^{28}+23904720\,T^{26}+5624073000\,T^{24}$
$+647566164000\,T^{22}+45746991675000\,T^{20}+1145438659134000\,T^{18}-60947538804999000\,T^{16}$
$-613314917245800000\,T^{14}+13256411880737025000\,T^{12}+146890015244661150000\,T^{10}$
$+484908475289220675000\,T^{8}+1198562394651167700000T^{6}-2003316012774759375000T^{4}$
$+7167250886770012500 00T^{2}+708119334693106875000,\quad \mathbf{q}_{13}=0,\quad \mathbf{q}_{12}=54264\,T^{30}$
$+17607240\,T^{28}+4448719800\,T^{26}+609209483400\,T^{24}+44277505939800\,T^{22}+412689587725800\,T^{20}$
$-84804973421553000\,T^{18}-250432798766535000\,T^{16}+2288283894885334 5000\,T^{14}$
$+466654428328441815000\,T^{12}+3892228471345297545000\,T^{10}+20229343597701089775000T^{8}$
$-111438537142351938750 00T^{6}+111541045912878408750 00T^{4}+103811868239979611250 00T^{2}$
$+5208039143851957875000,\quad \mathbf{q}_{11}=0,\quad \mathbf{q}_{10}=20349\,T^{32}+9579024\,T^{30}+2778050520\,T^{28}$
$+446310874800\,T^{26}+36326975409900\,T^{24}+602640375094800\,T^{22}-16736996613922200\,T^{20}$
$+1698321128858046000\,T^{18}+5918814816811540875 0\,T^{16}+9726990665043890700 00\,T^{14}$
$+10020895210571390145000\,T^{12}+5614685803460646297 0000T^{10}+1047534379069700473875 00T^{8}$
$+836351455382287267500 00T^{6}+1969557543513285783750 00T^{4}+5861169026452503225000 00T^{2}$
$+9836753798777262328125,\quad \mathbf{q}_{9}=0,\quad \mathbf{q}_{8}=5985\,T^{34}+3871665\,T^{32}+1327301640\,T^{30}$
$+250676937000\,T^{28}+24928524967500\,T^{26}+1277755297843500\,T^{24}+75968161579863000\,T^{22}$
$+3075359436148923000\,T^{20}+26011936101356493750\,T^{18}+116830305116544009 3750\,T^{16}$
$+2563966393606328287 5000\,T^{14}-6868273378557918442 5000T^{12}-2663234315883694216125 00T^{10}$
$+2701519751233743319687500T^{8}+1281120041892573815625000T^{6}+1364503639750540183125000T^{4}$
$-3235012764336997333593 75T^{2}+4253302124989278539062 5,\quad \mathbf{q}_{7}=0,\quad \mathbf{q}_{6}=1330\,T^{36}$
$+1137780\,T^{34}+466317810\,T^{32}+104251764960\,T^{30}+13208232681000\,T^{28}+1193020745742000\,T^{26}$
$+99210798874341000\,T^{24}+5805544496212692000\,T^{22}+214383530642638549500\,T^{20}$
$+4530497152000032675000\,T^{18}+46797298481837387137500\,T^{16}-2174368519568493171000 00T^{14}$
$+19692526781721872925000T^{12}+533657798449093765755 0000T^{10}+2153047053506552150062 5000T^{8}$
$+3238048459845190356750 0000T^{6}-4144298861975805116718750T^{4}+7309469656059722353125 00T^{2}$
$+1253352866096695907812 50,\quad \mathbf{q}_{5}=0,\quad \mathbf{q}_{4}=210\,T^{38}+230790\,T^{36}+113314950\,T^{34}$
$+30177299250\,T^{32}+4848169221000\,T^{30}+561695828967000\,T^{28}+50626283202243000\,T^{26}$
$+3225603873293865000\,T^{24}+122692742315543137500\,T^{22}+2347942991384156512500\,T^{20}$
$+36794456407482655612500\,T^{18}+266523839860686468637500T^{16}-904411179248760933750 00T^{14}$

$$+5001462949574321566875000T^{12} + 5969745826984951833937500 0T^{10} + 5558835396123469541812500 0)T^{8}$$
$$+627175284723853208732812 50T^{6} + 2437287792703697046093750T^{4} + 2506226987783118058593750)T^{2}$$
$$+5026816307874448828125 0, \quad \mathbf{q}_3 = 0, \quad \mathbf{q}_2 = 21\,T^{40} + 28980\,T^{38} + 17000550\,T^{36} + 5412525300\,T^{34}$$
$$+1069516209825\,T^{32} + 138706771971600\,T^{30} + 10810972830861000\,T^{28} + 389665850910210000\,T^{26}$$
$$-659267012964183750\,T^{24} - 4544658515342165850 00\,T^{22} - 9204165852885701775 00\,T^{20}$$
$$+2216315860557603477750 00T^{18} + 5984763190950898799156250T^{16} + 3244404240239514261525 0000T^{14}$$
$$+1356612363411351198941250 00T^{12} - 1194731497595965722277500 00T^{10}$$
$$-605387147867884252483593 75T^{8} + 2541529600596704902078125 00T^{6}$$
$$+7386978376083043616484375 0T^{4} - 1749332075140308192187500T^{2}$$
$$+4524134677087003945312 5, \quad \mathbf{q}_1 = 0, \quad \mathbf{q}_0 = T^{42} + 1701\,T^{40} + 1186290\,T^{38} + 451782450\,T^{36}$$
$$+105515964225\,T^{34} + 15941427079725\,T^{32} + 1619444848404600\,T^{30} + 114621189126003000\,T^{28}$$
$$+5904248149699121250\,T^{26} + 207513330336568226250\,T^{24} + 5039110473448289647500\,T^{22}$$
$$+110469853983534408247500T^{20} + 1375703927511930449606250)T^{18} + 3784181946524764546781250T^{16}$$
$$+3652804959013126249987500 0T^{14} + 2188776414527986446528750 00T^{12}$$
$$+2547163560565688529573281 25T^{10} + 4398434108492295477916406 25T^{8}$$
$$+4412156359523972157703125 0T^{6} + 2920005781582726260703125 0T^{4}$$
$$+1251677260660737758203125T^{2} + 2154349846231906640625$$

We get the following plot for the modulus of v in function of (x, t). We recognize again $N(N+1)-1$ local maximums and the global maximum is equal to 13 :

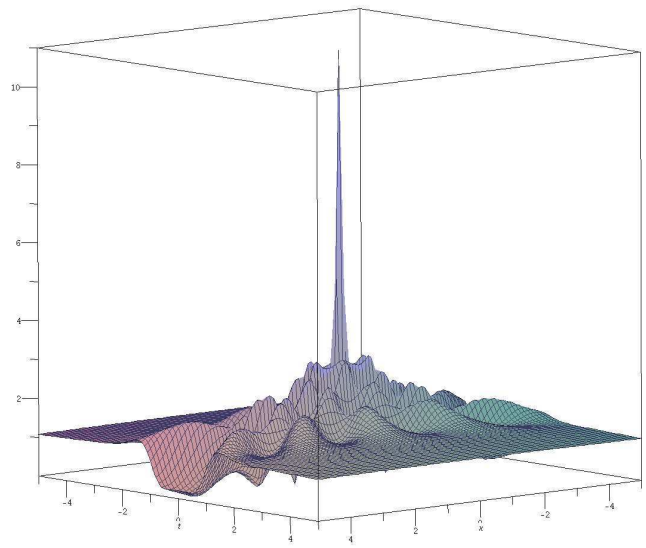

Figure 7: Solution of NLS, N=6a.

In the following, we only represent the modulus of v in the (x, t) coordinates.

If we choose $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $a_6 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 1$, $b_5 = 1$, $b_6 = 1$, we get :

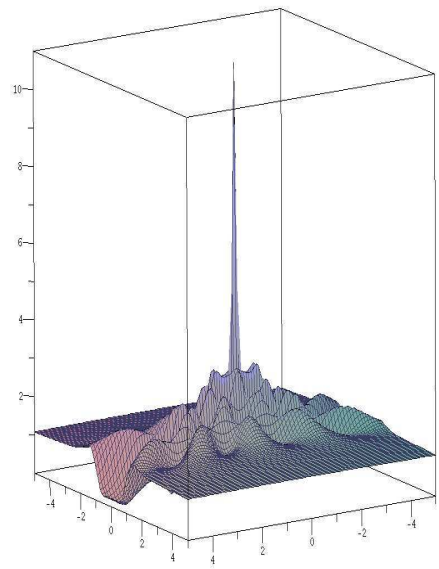

Figure 8: Solution of NLS, N=6b.

If we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, $b_5 = 1000000$, $b_6 = 1000000$, the shape of the modulus of v in the (x, t) coordinates is given by :

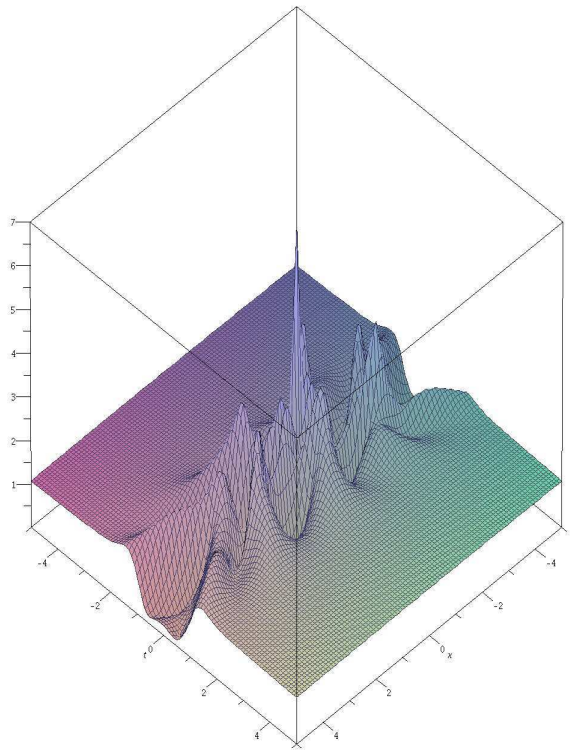

Figure 9: Solution of NLS, N=6c.

4 Conclusion

We have given here with new parameters a family which recover a wide spectrum of solutions of the NLS equation. These solutions also are written as a quotient of wronskians. An other approach has been given in [6].
This method described in the present paper provides a powerful tool to get explicitly solutions of the NLS equation. This new formulation gives an infinite set of non singular solution of NLS equation.
This method with parameters gives as well higher Peregrine's breathers of order N as well solutions with peaks of similar amplitude. It is is reasonable

to conjecture that in general there is $N(N+1)/2$ peaks for the modulus of any solution v in the (x,t) coordinates.
Because of the presence of a lot of redundant parameters $(3N+1)$, the present formulation give more flexibility to pass from Akhmediev's breathers to peaks of similar heights. This present method shows more adapted and efficient than this given in [6] to get all type of solutions.
This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order and the results raise any scepticism about the use of determinants and theta functions.

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