



HAL
open science

Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation. 2011. hal-00589556v1

HAL Id: hal-00589556

<https://hal.science/hal-00589556v1>

Preprint submitted on 29 Apr 2011 (v1), last revised 16 Sep 2012 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation.

⁺Pierre Gaillard, ⁺ Université de Bourgogne, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

April 25, 2011

Abstract

This work is a continuation of a recent paper in which we have constructed a multi-parametric family of solutions of the focusing NLS equation given in terms of wronskians determinants of order $2N$ composed of elementary trigonometric functions. When we perform a special passage to the limit when all the periods tend to infinity, we get a family of quasi-rational solutions. Here we construct Peregrine breathers of orders $N = 4, 5, 6$ and the multi-rogue waves corresponding in the frame of the NLS model first explained by Matveev et al. in 2010. In the cases $N = 4, 5, 6$ we get convenient formulas to study the deformation of higher Peregrine breather of order 4, 5 and 6 to the multi-rogue waves via variation of the free parameters of our construction.

1 Introduction

The nonlinear Schrödinger equation (NLS) was first derived by Zakharov [17] in 1968. The solutions were first given by Zakharov and Schabat. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [12]. The first quasi-rational solutions of NLS equation appeared in 1983 with the work of Peregrine, nowadays called worldwide Peregrine breathers. In 1986 Eleonski,

Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather[3]. Other families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations. In 2010 it has been shown in [6] that rational solutions of NLS equation can be written as a quotient of two wronskians and it has been recovered as particular case, Akhmediev's quasi-rational solutions of NLS equation.

In this paper, we extend a result [10] giving a new representation of the solutions of the NLS equation in terms of a ratio of two wronskians determinants of even order $2N$ composed of elementary functions; the related solutions of NLS are called of order N . When we perform the passage to the limit when some parameter tends to 0, we got families of multi-rogue wave solutions of the focusing NLS equation depending on a certain number of parameters. It allows to recognize the famous Peregrine's breather [14] and also higher order Peregrine's breathers constructed by Akhmediev [1, 4]. We treat in the following the cases of order $N = 4$, $N = 5$ and $N = 6$. We get for an arbitrary choice of the parameters the shape of Akhmediev's breathers; we can also get, for particular parameters, the apparition of peaks of similar amplitude for the modulus of the solution v in the $(x; y)$ coordinates.

2 Expression of solutions of NLS equation in terms of Wronskian determinant and quasi-rational limit

2.1 Solutions of NLS equation in terms of Wronskian determinant

We briefly recall results obtained in [10]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \tag{1}$$

From [10], the solution of the NLS equation can be written in the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \tag{2}$$

In (2), the matrix $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$ is defined by

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} + e_\nu). \quad (3)$$

We consider the following functions

$$\begin{aligned} \phi_\nu^r(y) &= \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & 1 \leq \nu \leq N, \\ \phi_\nu^r(y) &= \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & N + 1 \leq \nu \leq 2N. \end{aligned} \quad (4)$$

We use the following notations :

$$\Theta_\nu = \kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \leq \nu \leq 2N.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]. \quad (5)$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$ defined by

$$\begin{aligned} d_{\nu\mu} &= \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} - ie_\nu), \\ &1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N, \end{aligned}$$

with

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we get the following link between Fredholm and Wronskian determinants obtained in [9] :

Theorem 2.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (6)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

It can be deduced the following result :

Theorem 2.2 *The function v defined by*

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \quad (7)$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

2.2 Quasi-rational solutions of NLS equation

In the following, we take the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N + 1 \leq j \leq 2N$.

For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.

We consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2\epsilon^2 d_j^2, \quad 1 \leq j \leq N. \quad (8)$$

When ϵ goes to 0, we realize limited expansions at order p , for $1 \leq j \leq N$, of the terms

$$\begin{aligned} \kappa_j &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_j = 4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_j &= d_j\epsilon(1 - \epsilon^2 d_j^2)^{-1/2}, \quad x_{r,j} = (r - 1) \ln(1 + 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{-1/2}), \\ \kappa_{N+j} &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_{N+j} = -4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_{N+j} &= 1/(d_j\epsilon)(1 - \epsilon^2 d_j^2)^{1/2}, \quad x_{r,N+j} = (r - 1) \ln(1 - 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{-1/2}), \end{aligned}$$

The parameters a_j and b_j , for $1 \leq N$ are chosen in the form

$$a_j = \tilde{a}_j\epsilon^{M-1}, \quad b_j = \tilde{b}_j\epsilon^{M-1}, \quad 1 \leq j \leq N, \quad M = 2N. \quad (9)$$

We have the central result given in [10] :

Theorem 2.3 *With the parameters λ_j defined by (8), a_j and b_j chosen as in (9), for $1 \leq j \leq N$, the function v defined by*

$$v(x, t) = \exp(2it - i\varphi) \lim_{\epsilon \rightarrow 0} \frac{W_3(0)}{W_1(0)}, \quad (10)$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

depending on $3N$ parameters $d_j, \tilde{a}_j, \tilde{b}_j, 1 \leq j \leq N$.

3 Quasi-rational solutions of order N

In this section we choose $x_{0i} = 0, t_{0i} = 0$ for $1 \leq i \leq 2N$.

To get solutions of NLS equation written in the context of fiber optics

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0, \quad (11)$$

from these of (1), we can make the following changes of variables

$$\begin{aligned} t &\rightarrow X/2 \\ x &\rightarrow T. \end{aligned} \tag{12}$$

In the following, because of the complexity of the expression, we give if it is possible, all the solutions in the case where the parameters d_j are equal to j . (1).

3.1 Case N=4

In the case of order $N = 4$, we make an expansion at order 7. Taking the limit when $\epsilon \rightarrow 0$, the solution of NLS equation (11) takes the form

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it - i\varphi).$$

In this case, the analytical expression of v takes about 13 pages of usual format. We can't reproduce in the text.

Meanwhile, we can give the analytical expression for particular choices of the parameters.

If we choose $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 0$, we get :

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it - i\varphi),$$

with

$$\begin{aligned} N(x, t) = & -(395779571712000i)t^7 + 48039505920000t^6 + (10716300000i)t - 11341179360000t^4 \\ & + (1883933717299200i)t^{13} - (53014155264000i)t^5 + (19203609600000t^4 - 1311675120000t^2 - 6958006272000t^6 - \\ & 2679075000 - 1181263645900800t^{12} + 2748779069440t^{18} - 330497733427200t^{14} - 66486093742080t^{16} \\ & + (202937204736000i)t^{13} - (4511324160000i)t^3 + (2693991392870400i)t^{11} - (24739011624960i)t^{17} \\ & + 1542243483648000t^{10} + 1756650405888000t^8 - (20615843020800i)t^{15} + (73156608000000i)t^5 + (77157360000i)t \\ & + (1290950148096000i)t^9 - (546681913344000i)t^7)x^2 + (2061584302080t^4 + 11430720000 \\ & + 19920273408000t^4 - (14431090114560i)t^{13} - (7153090560000i)t^3 + (19074908160000i)t^5 - (12472182374400i)t^7 \\ & - 47933344972800t^8 + 36923296972800t^{10} - (48197586124800i)t^9 - (150377472000i)t - 16161649459200t^6 \\ & - 678730752000t^2 + (52312701665280i)t^{11} - 40587440947200t^12)x^6 + (-44236800 + 754974720t^4 \end{aligned}$$

$$\begin{aligned}
&+(283115520i)t-(1509949440i)t^3-1415577600t^2)x^{16}+963558457344000t^8+((1415577600i)t \\
&\quad -324403200+8053063680t^6-37748736000t^4-(24159191040i)t^5+4246732800t^2 \\
&+(25165824000i)t^3)x^{14}+523098390528000t^{10}+(-(13212057600i)t^3+235615027200t^4-1354752000 \\
&+56371445760t^8-(225485783040i)t^7+10734796800t^2-465064427520t^6+(493250150400i)t^5+(9083289600i)t)x^{12} \\
&+((5588700364800i)t^5+9398592000+1086898176000t^4-(5411658792960i)t^{11}-10477161676800t^6 \\
&+18232639488000t^8-(12049396531200i)t^7+(20857434931200i)t^9-14543833006080t^{10} \\
&\quad +901943132160t^{12}+(402554880000i)t^3+804722688000t^2-(121927680000i)t)x^8 \\
&\quad -26800595927040t^{19}-(240045120000i)t^3-(10995116277760i)t^9+(32514048000i)t \\
&+3223742054400t^6-(1352914698240i)t^9+(4396972769280i)t^7+(138726604800i)t^3+6096384000 \\
&- (1088673546240i)t^5+74008166400t^2-3297729576960t^8-9909043200t^4+270582939648t^0)x^{10}-87873660000t^2 \\
&\quad -(63909113364480i)t^{17}+(-7864320-(41943040i)t+41943040t^2)x^{18}+1048576x^{20} \\
&+1099511627776t^{20}+(806992478208000i)t^{11}-379138238054400t^{16}+((54116587929600i)t^{13}-52425444556800t^{12} \\
&+207309570048000t^6-(3170119680000i)t^3-(37205154201600i)t^1-273331047628800t^{10}-(24739011624960i)t^{15} \\
&- (70490456064000i)t^5-(137168640000i)t+(98297708544000i)t^9-9644670000+(344240160768000i)t^7 \\
&+3092376453120t^{16}-291375415296000t^8-69578470195200t^{14}-3292047360000t^4-350542080000t^2)x^4 \\
&+832835262873600t^{12}-796770120499200t^{14}+(569274531840000i)t^9+(369538986147840i)t^{15}+200930625
\end{aligned}$$

$$\begin{aligned}
D(x, t) = &1048576x^{20}+(2621440+41943040t^2)x^{18}+(-283115520t^2+26542080+754974720t^4)x^{16}+(265420800 \\
&+471859200t^2-7549747200t^4+8053063680t^6)x^{14}+(7431782400t^2-70464307200t^6+1741824000+6606028800t^4 \\
&+56371445760t^8)x^{12}+(188271820800t^4-253671505920t^8+270582939648t^{10}+1683763200+49235558400t^2 \\
&+264241152000t^6)x^{10}+(4463424000+338228674560t^{10}+4280706662400t^8+901943132160t^{12}-128894976000t^2 \\
&+1532805120000t^4-594542592000t^6)x^8+(134120448000t^2+42658430976000t^6-2536095744000t^4 \\
&+6313601925120t^{12}+8382528000+31962609745920t^{10}+68966940672000t^8+2061584302080t^{14})x^6 \\
&+(23192823398400t^{14}+332943851520000t^{10}+46039891968000t^6+123453466214400t^{12}-365783040000t^4 \\
&+161269678080000t^8+3092376453120t^{16}+320060160000t^2+1786050000)x^4+(-58785398784000t^8 \\
&-417368899584000t^{10}+296795661926400t^{12}+15545779200000t^4+158993694720000t^6-122880240000t^2 \\
&+893025000+2748779069440t^{18}+38654705664000t^{16}+241591910400000t^{14})x^2+22325625 \\
&+17801441280000t^6+1053529473024000t^{12}+3594961440000t^4+727066135756800t^{10}+592866548121600t^{14} \\
&+1099511627776t^{20}+25426206392320t^{18}+190374425395200t^{16}+46437300000t^2+435233046528000t^8
\end{aligned}$$

We get the Akhmediev breather of order 4. The plot of the modulus of v in the (x, t) coordinates gives $N(N + 1)$ local maximums; we can evaluate the global maximum equal to 9 :

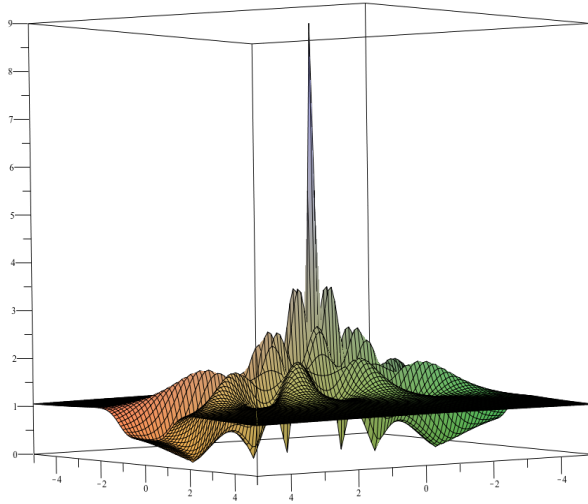


Figure 1: Solution of NLS, N=4 a.

If we choose $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 1$, we have the same type of analytical expression for the modulus of v ; we don't reproduce here.

We only give shape of the modulus of v in the (x, t) coordinates :

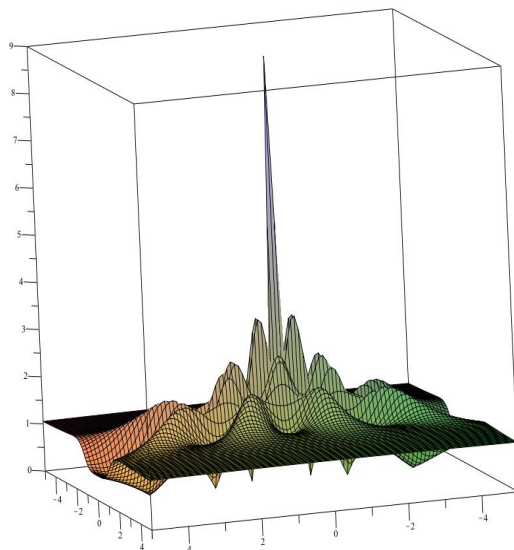


Figure 2: Solution of NLS, N=4 b.

If we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, for the same reason, we only give the shape of the modulus of v in the (x, t) coordinates is given by :

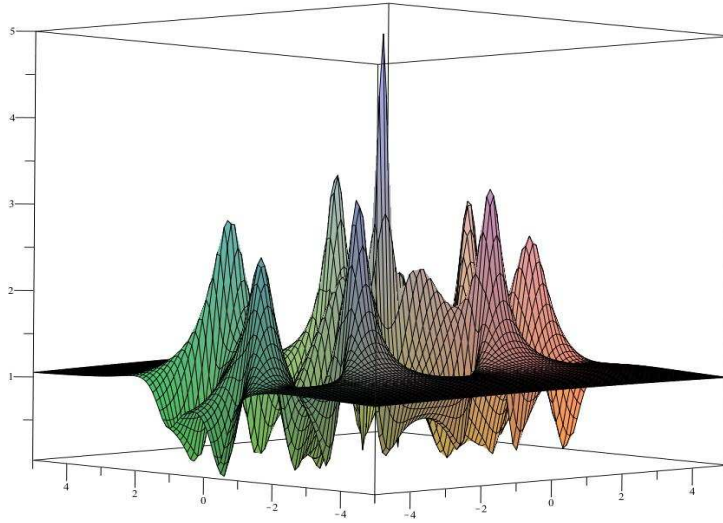


Figure 3: Solution of NLS, $N=4$ c.

In these three plots, we can see the deformation of the initial Akhmediev breather to multi-rogue waves, by choosing different types of the parameters among the 8 given by our formulation.

3.2 Case $N=5$

In the case $N = 5$, we realize an expansion at order 9 in ϵ . From (10), we get solutions of NLS equation. Taking the limit when $\epsilon \rightarrow 0$, the solution of NLS equation (11) takes the form

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it - i\varphi).$$

The analytical expressions being too long, we give only them for particular choices of parameters.

If we choose $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 0, b_5 = 0$, we get :

$$\begin{aligned}
N(x, t) = & 18889465931478580854784t^{30} + 17592186044416x^{30} - 605412580955104110182400000t^8 - 9380226632990588928000000t^6 \\
& + 1132342702045986816000000t^4 + 3506950894841856000000t^2 + (55395572453349783502848t^{20} - 13694342076892712219443200t^{10} \\
& - (6014504159045222400000i)t + 2974978989460384697548800t^{12} + (23929556785294749990912000i)t^9 - 1024642960060317696000000t^4 \\
& + 5497603512646245757747200t^{16} - 9309545291393500446720000t^6 + (24513482783505340012953600i)t^{13} - 1062519926119464960000 \\
& + (484295008141751353344000i)t^{11} - 2309036601781454918123520t^{18} + (4769878378074493377576960i)t^{17} \\
& + 32697208502457678692352000t^8 - (72020006110872207360000i)t^3 - 54959655518868602880000t^2 \\
& - (553955724533497835028480i)t^{19} - (25307307481980884484096000i)t^7 - (8807060728452527916318720i)t^{15} \\
& + (1826752437725914202112000i)t^5 - 22122347001668801711308800t^4)x^{10} + 21705195782799224104550400000t^{16} \\
& + (- (434766952714721034240i)t^5 + (10500782721859584000i)t + 54097238723974397952t^{10} - 737334960492631818240t^8 \\
& + 1016318179971956736000t^6 + (6400477087609651200i)t^3 + (1139163007745230110720i)t^7 - 27638423787405312000t^4 \\
& + 15246591016933785600t^2 - (270486193619871989760i)t^9 - 1024508184389222400)x^{20} + (4474736705877024832487424000i)t^{21} \\
& + (- (1055531162664960i)t - 197912092999680 + 1055531162664960t^2)x^{29} + (7916247049317226905600000i)t^5 \\
& + (7419049882145060290560t^{16} - (1594806375882424320000i)t - 7877376947540459520000t^2 - 908454125999168579174400t^{10} \\
& + (572152777784866897920000i)t^9 + (509346309216497408409600i)t^{13} - (927956795300510485708800i)t^{11} \\
& - (52690745717466071040000i)t^3 + 68321228985854853120000t^4 - (59352399057160482324480i)t^{15} \\
& + 138603785986070544384000t^6 + (209440234922399760384000i)t^7 + 171033922413527040000 - 226851717550204728115200t^{14} \\
& + 9044622232503975936000t^8 - (45834689216702840832000i)t^5 + 776361578603554956902400t^2)x^{14} \\
& + (973325387811016309682995200i)t^{23} + (14104458243735552000000i)t^3 + (262466757462293993226240000t^8 \\
& + 7468309112126617681920000t^6 - 7657800350944355942400000t^4 + 118960605919759564800000t^2 + 12348349674722966686924800t^{16} \\
& - (1611507562279266429173760i)t^{23} - (8527502183042226585600000i)t^5 + (1834231213761704755200000i)t^3 \\
& - (7379410738537656483840000i)t^7 + (6732692652022512148807680i)t^{21} + (36048281392391823713894400i)t^{19} \\
& - 760715037774609163223040000t^{10} + (65352136762463103221760000i)t^9 - (3120142857158167741071360000i)t^{11} \\
& + (2156753915104980910473216000i)t^{15} + (16025935163896627200000i)t + (571163791359936001631846400i)t^{17} \\
& - (634352727169645009698816000i)t^{13} - 689891979313152000000 + 134292296856605535764480t^{24} + 3168655831830796374638592000t^{12} \\
& + 3385986254051716141940736000t^{14} - 26260842509408148022886400t^{20} - 199940216583912302208614400t^{18} \\
& - 6779178447088260218880000t^{22})x^6 - 907420350015602939658240000t^{10} + ((90050002314854400i)t \\
& - (1536853372840181760i)t^5 + (1699405171890585600i)t^3 + 297857699964518400t^2 - 22079567875276800 \\
& + 512284457613393920t^6 - 2475220576449331200t^4)x^{24} + (402436987932672000000 + 332595057410306998272000000t^8 \\
& - 136602098065224813772800000t^6 + 731127543005970432000000t^4 + 45686179963404288000000t^2 + 10050849128814676725989376000t^{16} \\
& + 100487493230237224796160000t^{10} + (27286548100357187174400000i)t^5 - (6042864278644433312808960000i)t^{13} \\
& + (519133261524369408000000i)t^3 - (343809808336711109836800000i)t^7 + (1460298592300752703586304000i)t^{19} \\
& + (100365736909141692540518400i)t^{21} - (464857950657480700723200i)t^{23} - (3200493777375232870318080000i)t^{11} \\
& - (705521076605372753510400000i)t^9 + (5010740706245254985023488000i)t^{17} - (1611507562279266429173760i)t^{25} \\
& - (4627795739945763568877568000i)t^{15} + (8278703751757824000000i)t - 6624225796869099985305600t^{24}
\end{aligned}$$

$$\begin{aligned}
& +1407130850301005581516800000t^{12} + 6034682374224875869962240000t^{14} - 416086906642894976502988800t^{20} \\
& + 539237634440278069739520000t^{18} - 69481736812335318486220800t^{22} + 123962120175328186859520t^2x^4 \\
& \quad + (-59109745109237760i)t^3 - 1731730813747200 - 55415386039910400t^2 + (11083077207982080i)t \\
& \quad + 29554872554618880t^4x^{26} + (574110117449990594887680000t^{10} + 123924431432410629931008000t^{12} \\
& \quad + (21250398522389299200000i)t - (328301860508856578211840000i)t^9 + 754032281291089300684800t^{18} \\
& \quad + (2652495489921289090498560000i)t^{13} - 187447798759056998400000t^2 + 100719222642454151823360t^{22} \\
& \quad - (3615814040026204864512000i)t^{17} + 23890368656849491722240000t^6 - 1469962939270378291200000t^4 \\
& \quad + (422325775175211804524544000i)t^{11} + (93740181582407270400000i)t^3 + (7829951106386940552806400i)t^{19} \\
& \quad - (1107911449066995670056960i)t^{21} - 6599147896409054773248000t^{14} + (135629797491220237044940800i)t^{15} \\
& \quad - 4681991171778313432596480t^{20} + (47214001120669492838400000i)t^7 - (7862612906697063137280000i)t^5 \\
& \quad - 1001620947743539200000 - 72468727197550625842790400t^{16} - 118510986917599437127680000t^8x^8 \\
& + (8783473670339729855348736000i)t^{19} - (4214712085961158353223680i)t^{27} - 297066020686567237169971200t^{24} \\
& + (-54262746010280263680t^6 + 1516996192842547200t^2 - (24589653965442908160i)t^7 + 6147413491360727040t^8 \\
& + 34191293186624716800t^4 - 238654152769536000 - (3657415478634086400i)t^3 + (62065232364699648000i)t^5 \\
& \quad + (1059819258013286400i)t^2x^{22} + (-545026421387427840000i)t + (4837451489737703424000i)t^5 \\
& \quad + (97000510358141785866240i)t^{11} + 6394158434802991104000t^4 - (132097867272673453670400i)t^9 \\
& \quad - 91740947404476225945600t^8 + 11325789671913632563200t^6 + (41403273187402815897600i)t^7 \\
& \quad + 136077135313427339673600t^{10} - (1298333729375385508480i)t^{13} - 45878619763985979801600t^{12} \\
& \quad + 49837699246325760000 + 1854762470536265072640t^{14} - (1814044308651638784000i)t^3 \\
& \quad + 2317626664647671808000t^2x^{16} - (28745499138048000000i)t - (14624828633331484330033152000i)t^{15} \\
& \quad - (283341988972178712821760i)t^{29} + (48368469765910866910248960i)t^{25} + (7186374784512000000 \\
& \quad - 2099327812978762108108800000t^8 - 4867127054896516300800000t^6 - 2826801026509307904000000t^4 \\
& \quad + 82327109531369472000000t^2 - 11166360257961750171746304000t^{16} - 6944698416599625262694400000t^{10} \\
& \quad - 81262980499310844995174400t^{24} + 15759259819131533430620160000t^{12} - (3219495903461376000000i)t \\
& - (1653889840453699376250880000i)t^{11} + (12859138468604237379010560000i)t^{15} - (6508011309204729810124800i)t^{25} \\
& \quad + (419454323422396416000000i)t^3 + (438134268982282577510400000i)t^7 + (9221996027343603907952640000i)t^{13} \\
& \quad - (991696961402625494876160i)t^{27} - (2061035938005167112192000000i)t^9 - (6010499882222309854937088000i)t^{17} \\
& + (3579604828806168434442240000i)t^{19} + (1930634966540786604598886400i)t^{21} + (113309125472760920801280000i)t^{23} \\
& \quad - (17190268412538912768000000i)t^5 + 772728085947074299822080000t^{14} + 1190425059366988941361152000t^{20} \\
& \quad + 6386388776384723297501184000t^{18} - 530962203922070260993228800t^{22} + 70835497243044678205440t^{28} \\
& \quad - 3904806785522837886074880t^2x^2 - (511165044043202691072000000i)t^9 - 9740812251630034401361920000t^{12} \\
& - (205222774063702546841600000i)t^{11} - (26843989178012838962135040000i)t^{13} - (4387979219806808916885504000i)t^{17} \\
& \quad - 3593187392256000000 + (170242952819488535347200000i)t^7 + 16464595962633925070684160000t^{14} \\
& + 4564646798971790373263769600t^{20} + 20503770921695624558542848000t^{18} - 12391959898086405584486400t^{22} \\
& \quad - 1044823584334909003530240t^{28} + (1861193095820391481344000t^6 - (4788949827578757120000i)t \\
& \quad + (837112579672932089856000i)t^9 - (3891936938939693767065600i)t^{13} - 541364273413179310080000t^4 \\
& \quad + 2792211953131347089817600t^{14} + (4201468665788923851571200i)t^{11} - 15363603467662786560000t^2
\end{aligned}$$

$$\begin{aligned}
& -(207733396700061688135680i)t^{17} + (20898944158028267520000i)t^3 + 23081488522229076459520t^{18} \\
& + 306035902636288376832000t^{10} + (1877590316327480642764800i)t^{15} - 5382843124711562831462400t^{12} \\
& + 1200696044596248969216000t^8 + (178625678284116983808000i)t^7 + 250604339960217600000 \\
& -(1550087360736468664320000i)t^5 - 837924614573806520893440t^{16} + (360648258159829319680t^{12} \\
& - 4368616543070296473600t^6 + 15188692328906961715200t^8 + 15139622816317440000 + (40980327375568896000i)t \\
& - 6855784676749832355840t^{10} - (2163889548958975918080i)t^{11} + (12848094196943919513600i)t^9 \\
& + 217311652821860352000t^2 + 307841128105771008000t^4 + (111644685562281984000i)t^3 \\
& + (882102114983293747200i)t^5 - (10861415663822438400000i)t^7)x^{18} - 38234566441577787634483200t^{26} \\
& D(x, t) = 18889465931478580854784t^{30} + 17592186044416x^{30} + 241614729037663646515200000t^8 \\
& + 3362027121813749760000000t^6 + 284402742017458176000000t^4 + 1541804044677120000000t^2 \\
& + 5450887469422271161958400000t^{16} + (7419049882145060290560t^{16} + 51058351312522444800000t^8 \\
& + 127134421975850758963200t^{10} + 87545434673548099584000t^6 + 94736370681477542707200t^{12} \\
& + 821566920909127680000t^2 + 124432437403975680000 - 21963682613884354560000t^4 - 4280221085852919398400t^{14})x^{14} \\
& + 1040756650099481688145920000t^{10} + (-554153860399104000t^4 + 512284457613393920t^6 + 20780769764966400t^2 \\
& + 17750240840908800)x^{24} + 326653399296000000 + 204951512401595827067289600t^{24} + (65970697666560 \\
& + 1055531162664960t^2)x^{28} + 580738206390010804961280000t^{12} + (45399667230773556019200000t^6 \\
& + 26982825014023047736197120t^{20} + 329157519925544742813696000t^{10} + 18701879810929459200000t^2 \\
& + 2486990036017521748869120t^{22} - 846074231955180748800000t^4 + 134292296856605535764480t^{24} \\
& + 215456891557737450700800000t^8 + 139963553766563635003392000t^{18} + 54188987577307282538496000t^{12} \\
& - 386045124656108163563520000t^{14} + 166550561485094752616448000t^{16} + 432518136525619200000)x^6 \\
& + (2967493922437201920000t^8 + 2114748541141254144000t^4 + 25999984897818624000 + 1854762470536265072640t^{14} \\
& - 3682773559285949399040t^{12} + 11455823260640933314560t^{10} - 470003783791804416000t^2 \\
& + 8930847518040784896000t^6)x^{16} + 7441435294537031645921280000t^{14} + 3738595696032685094653132800t^{20} \\
& + (60513601555582156800t^6 + 2296193884146892800 + 17746777379281305600t^4 + 9573440871097958400t^2 \\
& - 128741024847919841280t^8 + 54097238723974397952t^{10})x^{20} + (602642323184025600t^2 + 6147413491360727040t^8 \\
& + 1828707739317043200t^4 - 11230851570755174400t^6 + 254889129148416000)x^{22} + (-37816839207478018179072000t^{10} \\
& + 100719222642454151823360t^{22} + 1134543935823413835202560t^{20} - 24852304967919206400000t^2 \\
& + 110646686600627100844032000t^{14} + 48990960639069741121536000t^{16} + 2795623085722986086400000t^6 \\
& - 58427720984423376617472000t^{12} + 785872556615584972800000t^4 + 10621368624544019487129600t^{18} \\
& + 58202507015295011389440000t^8 + 392631163984281600000)x^8 + (843514205937109743697920000t^{12} \\
& + 123962120175328186859520t^{26} + 31626414950727234355200000t^6 + 1660587789415640098406400000t^{10} \\
& + 3447696467376315197030400t^{24} + 231100129825171439616000000t^8 + 24724613561647104000000t^2 \\
& + 273889401728689288249344000t^{20} + 45454391487726784767590400t^{22} + 60975301201920000000 \\
& + 506109696013295985623040000t^{18} - 534419122544104156692480000t^{14} + 12402309187740836560896000t^{16} \\
& + 90090136450105344000000t^4)x^4 + (2950871966563758715699200t^{16} + 3227192189276550856704000t^8 \\
& + 26149982367173726070374400t^{10} + 322253089752659798261760t^{18} + 55395572453349783502848t^{20} \\
& + 11484295242665150958796800t^{14} + 245908458260398080000 + 13311951515355709440000t^2
\end{aligned}$$

$$\begin{aligned}
& +253290289902932459520000t^4 + 716792974760245985280000t^6 + 31689930438754648562073600t^{12})x^{10} \\
& +6669056746644308068663296000t^{18} + (1039038488248320 - 11083077207982080t^2 + 29554872554618880t^4)x^{26} \\
& +1170441047589273489899520000t^{22} + (19599203957760000000 - 1252453640643678516019200000t^{10} \\
& +2789147703944884204339200t^{26} - 6146310361153536000000t^2 + 7961426529817146918174720000t^{12} \\
& +14930963824823248135127040000t^{16} + 342128188278426772596326400t^{22} + 1575376470498366367727616000t^{20} \\
& +70835497243044678205440t^{28} + 51856239333300240384000000t^6 + 6309831064001936822894592000t^{18} \\
& +14108289674109167264071680000t^{14} - 85654966326676527513600000t^8 + 45410811054852645951897600t^{24} \\
& +1980979522659680256000000t^4)x^2 + 1009405835713386664427520t^{28} + (605018978260505277235200t^{14} \\
& +5207787865689292800000t^2 - 327338035938695577600000t^6 + 23081488522229076459520t^{18} \\
& +44942321401455653683200t^{16} + 374569324196659200000 + 1667568663174750142464000t^{12} \\
& +30704050024564654080000t^4 + 2814087578532150509568000t^8 + 2963793282957456703488000t^{10})x^{12} \\
& +(1027179595635779174400t^8 - 905088417112648581120t^{10} + 65936733065183232000t^2 \\
& +304931820338675712000t^4 + 3879077022793728000t^6 + 360648258159829319680t^{12} \\
& +4792260621238272000)x^{18} + 20337535341264780656640000t^{26}.
\end{aligned}$$

We obtain the Akhmediev breather of order 5. Again, we can note the presence of $N(N + 1) - 1$ local maximums; the global maximum of $|v|$ is equal to 11 :

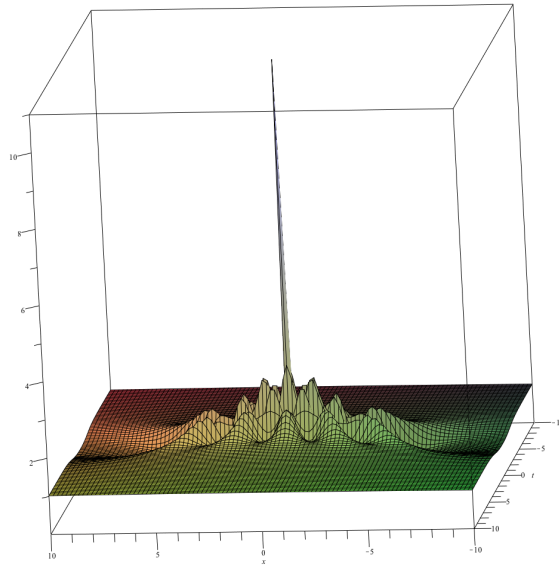


Figure 4: Solution of NLS, $N=5$ a.

In the following cases, we only give the plots for the modulus of v in the (x, t) coordinates.

If we choose $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = 1, b_1 = 1, b_2 = 1, b_3 = 1, b_4 = 1, b_5 = 1$, we get :

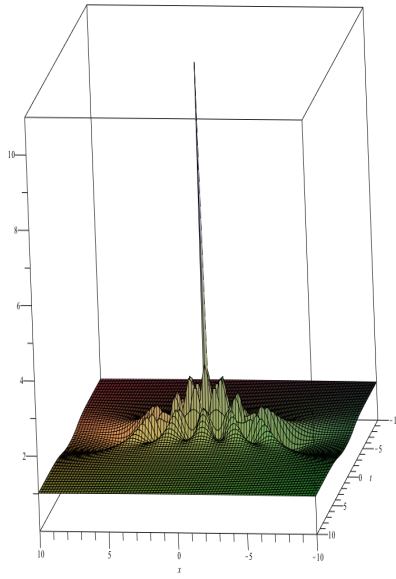


Figure 5: Solution of NLS, N=5 b.

In the case where $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, $b_5 = 1000000$, we get The shape of the modulus of v in the (x, t) coordinates is given by :

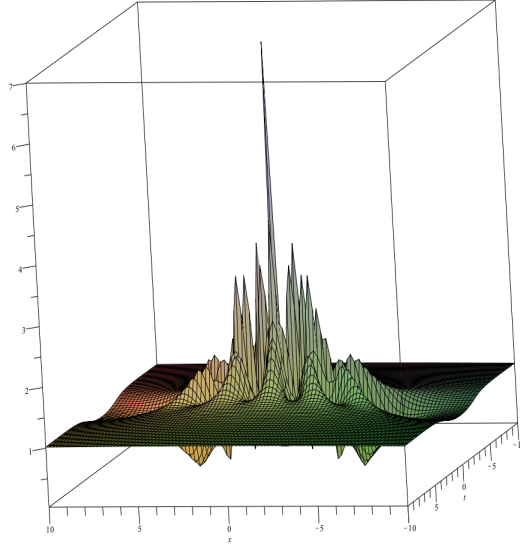


Figure 6: Solution of NLS, N=5 c.

Our formulation with 10 parameters again make possible to get a deformation of Akhmediev breather to obtain multi-rogue waves.

3.3 Case N=6

In the case $N = 6$, we realize an expansion at order 9 in ϵ . From (10), we get solutions of NLS equation. Taking the limit when $\epsilon \rightarrow 0$, the solution of NLS equation (11) takes the form

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it - i\varphi).$$

We give analytical expression of v in the following case : we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0$, $b_5 = 0$, $b_6 = 0$, we get

$$N(x, t) = -(-80071106228346824067907584000000t^8 x^8 - 341862050367943999488000t^4 x^{26})$$

$$\begin{aligned}
& +5833280126811999691154204393472000t^{22}x^6 - 15590430582092097270885384192000t^{16}x^{14} \\
& +1578813021748933517125877760000t^{10}x^{16} - 384381742239657641813999616t^{32}x^{10} \\
& -117220069376598451153006913126400000t^{24}x^2 - 142471752915773030400000x^{22} - 709429917325492677726720000000t^4x^6 \\
& +380692222792533049677447168t^{20}x^{20} - 6003371153223348445863129120768000t^{28}x^2 - 57336486357486331997297049600t^{16}x^{18} \\
& +397800127501274234734999555276800000t^{18}x^2 - 167164394438381123855066333184000t^{18}x^{12} \\
& +101282529609939830597222400t^{10}x^{22} - 12577694624499351158784000t^8x^{24} \\
& +209348911860207722305859616768000000t^{20}x^2 + 311955071120179200x^{34} + 9540199756962201600x^{30} \\
& +226904280603480704146672189440000t^{12}x^{10} - 69182893465796404348284504637440000t^{20}x^4 \\
& -16322002359249329455104000t^6x^{22} - 551741126442850713600t^2x^{30} \\
& -10511849301398468930982421463040000t^{18}x^6 - 33947568448408047679402357555200000t^{18}x^4 \\
& -82190313059811212101063680000000t^8 + 576522030352621731840000000t^2x^{16} + 17415814527696544727040t^6x^{30} \\
& +26768153105602468581998592000t^{14}x^{18} + 3878882819130028045683916800t^{12}x^{20} - 3275138962547785765220253696000t^{18}x^{14} \\
& -3606691997144060226850258944000000t^{24}x^4 + 60235149526832166510919680000000t^8x^4 \\
& +886860669870899270105274776000000t^{12}x^6 - 29064418237400190004779968102400t^{20}x^{12} \\
& +172356479738448445440000000t^2x^{18} - 1101408938529969675896063262720000t^{18}x^8 - 5473567973976711218646220800t^{10}x^{20} \\
& +35776684678416328647000000000t^4 - 14737667925867336450000000x^6 + 865865406873600x^{38} \\
& -1938319329515352397991638990848000t^{24}x^6 - 147985231562978344972873157836800t^{22}x^{10} \\
& +2783454017978870795480334336000000t^{12}x^4 - 32661858646516799700000000x^8 + 17074228651321446767511207936000000t^{12}x^2 \\
& -574740295332151689216000t^6x^{26} - 9917174038210516353024000t^6x^{24} - 50820486401353805136000000x^{10} \\
& -3665884543785271110625394688000000t^{10}x^2 - 49242956390085427200000x^{20} - 3515458568357285376747101591961600t^{30} \\
& +533244673939321531687500000t^2 + 28216962161043182714880000t^2x^{20} + 3107035033487762130926378680320000t^{20}x^8 \\
& -643855095469597985910711071539200t^{30}x^2 - 771081978815447040000x^{28} + 866978725483223130293340733440000t^{22}x^2 \\
& -780645876774648274975414498099200t^{20}x^{10} - 1021166865913610214329517342720000t^{28}x^4 \\
& -904537357828423629139265716224000t^{26}x^6 - 467333661870917302036847394816000t^{24}x^8 \\
& -257979458356789691547648000t^8x^{22} - 6406275691988765429760000000t^2x^{10} - 11743071850186317748800000000t^2x^8 \\
& -40196491256559835664946806390784000t^{26}x^2 + 5707699630013745524160000000t^6x^6 \\
& +1620346841146711477800000000t^2x^4 + 1799588748353206110300000000t^2x^2 \\
& +9138653608566228875054206156800t^{34}x^2 + 17927619210928982016000000x^{14} + 4468120594522299585213235200000t^8x^{12} \\
& -10498041106592184076861440t^{12}x^{26} + 602218372516180635156480000t^4x^{16} + 239840100586017594644889600000t^6x^{14} \\
& -181007670580914341412864000000t^4x^{12} - 17602857560677810176000t^4x^{28} - 10605193913696256000000t^2x^{28} \\
& -56580697170227032951357440000t^{12}x^{18} + 3987077385111625728000000x^{18} - 7834776665591977122642001920000t^{10}x^{14} \\
& -5076881258565254152825759334400000t^{24} + 351417647058818328713471459328000t^{18}x^{10} \\
& +39525148590423603074511667200000t^{10}x^{10} - 8378857034730260398080t^{14}x^{28} + 13029752662919612928000000x^{12} \\
& +419122741387123923473557094400t^{22}x^{14} - 14777436277309440t^4x^{38} - 110527331700896945305190400t^{12}x^{22} \\
& +42352188056877527550708940800t^{18}x^{18} - 56421242364564286293934080000t^{10}x^{18} \\
& +36354420568666567491566174208t^{30}x^{10} + 418503408638170614145744896000000t^{14}x^2 \\
& -137279193657020586362142720t^{28}x^{14} + 678222481557682755993600t^8x^{28} + 354295859065251043265740800000t^8x^{14} \\
& -14057436215111227785648223027200t^{30}x^4 + 162844574638002327262624481280t^{38} \\
& +82274724986110217960494517452800000t^{16} + 5251091426009956221793704345600t^{28}x^8
\end{aligned}$$

$$\begin{aligned}
& -(136508037042965776957440000i)t^9x^{22} + (1123085157075517440i)t^5x^{36} \\
& + (780775413924304584934686720i)t^{25}x^{16} + (37280690843833713900681953280000i)t^{13}x^{14} \\
& + (5086556838405655082355916800i)t^{13}x^{20} + (38461974951039427193276989440i)t^{37}x^2 \\
& - 30346203387962327040000x^{24} + 2032754815873112384471040t^{14}x^{26} + 45139566440644005963025612800t^{32}x^8 \\
& + 41817877468425911867937914880t^{34}x^6 + 20007277373390447443968t^{10}x^{30} - 4398046511104x^{42} \\
& + 6273466752227312137500000x^4 + 4133396115076165926912000000t^6x^{16} - 256254494826438427875999744t^{30}x^{12} \\
& + 13850398478334325017381175296000t^{34} - 227895775089131520t^2x^{36} - 16258866372646639822582264627200t^{26}x^8 \\
& - 5179782650849745327868398796800000t^{20}x^6 + (71571079609727920285679616i)t^{21}x^{20} \\
& - (65805770922393600i)tx^{36} - (16974444206427155258419445760000000i)t^{13}x^4 \\
& + (469215993944894582292480i)t^{15}x^{26} - (43236949334398003203035627520i)t^{25}x^{12} \\
& + (79760559602785867370437017600i)t^{23}x^{14} + (1016409728027076102720000000i)tx^8 \\
& - (68278335792570765322223616000i)t^{29}x^{10} + (1086118711882205459774674305024000i)t^{17}x^{10} \\
& + (3843817422396576418139996160i)t^{29}x^{12} - (45175176486008433672192000000i)t^7x^{18} \\
& - (16215119484413274827129125601280000i)t^{21}x^4 + (369435906932736i)tx^{40} \\
& + (2832286521765898413366312960i)t^{33}x^6 - (365310252524231479413696783974400i)t^{27}x^6 \\
& - (6891582102039545315328000000i)t^3x^{10} - (72616321866698588160i)t^5x^{34} \\
& + (2030995376952577013506375680i)t^{39}x^2 + (5450002512461762172939141120i)t^{17}x^{20} \\
& - (1129082691647499620434323701760i)t^{35}x^2 - (2142181495067608350720i)t^5x^{30} \\
& + (58651999243111822786560i)t^{13}x^{28} - (448820561475591313729526326886400i)t^{31}x^2 \\
& + (6268757128294013337600000i)tx^{20} + (25852438384792656811130880000i)t^{15}x^{18} \\
& - (1086117568431372977955840000000i)t^5x^2 + (7235421030393555610616463360i)t^{35}x^6 \\
& + (458218744086811115520i)t^9x^{32} - (1745946085939980673351680000000i)t^5x^8 \\
& - 1882705701688384657758879744000t^{12}x^{16} + 1442551066296549589490073600t^{22}x^{18} \\
& + 4554276425157839688000000000t^4x^4 - 12337578956837438544240000000t^4x^2 \\
& + 14822610651470630486016000000t^4x^{14} - 3176951330081070357438617616384000t^{26}x^4 \\
& - 253874422119072126688296960t^{38}x^4 + (24013635781478107564800000000i)t^3x^4 \\
& + (55162564826717880975989145600i)t^{21}x^{16} - (1482275812375285041090841804800i)t^{29}x^8 \\
& - (62158588719209584432725255782400i)t^{31}x^4 + (55993158217452313538753200128000000i)t^{19}x^4 \\
& - (572411985417732096000i)tx^{28} + (363651043103876590652620800000i)t^7x^{14} \\
& + (182231043480357760401408000i)t^5x^{22} - (2275409010059868092973554073600i)t^{31}x^6 \\
& + (121759029608014621025280000000i)t^3x^6 + (260258471308101528311562240i)t^{23}x^{18} \\
& + (1969718255603417088000000i)tx^{18} + (22194017923492874142351360000000i)t^7x^8 \\
& + (6150107875834522269023993856i)t^{31}x^{10} + (5865199924311182278656i)t^{11}x^{30} \\
& + 304159304532499302649143125606400000t^{18} + 11043537362179637510940917760t^{38}x^2 \\
& - 29684566402368649311509544960000t^{14}x^{14} + 239476186779159813851594096640000t^{14}x^{12} \\
& + 11860323462109574495837827891200t^{32}x^4 - 338878217849090531655680t^{18}x^{24} \\
& - 7003909504890127521335476224000t^{24}x^{10} - 2434164287594765531672582553600t^{22}x^{12}
\end{aligned}$$

$$\begin{aligned}
& -5319016096575029456694474178560000t^{16}x^8 + 1244038685257223188297600204800000t^{16}x^6 \\
& -47295743577338098483200t^6x^{28} + 1836096457455697920t^4x^{36} - 1626615445675634551947264t^{20}x^{22} \\
& -1692754945579509234206370693120000t^{22}x^8 - 54400278399993364405215559680000t^{12}x^{12} \\
& + 4553926564682913546240000t^2x^{22} + 148959751303875713185087488000t^{20}x^{16} \\
& + 30056590690344115436375978803200000t^{14}x^6 - 2852919586917590947866869760t^{18}x^{20} \\
& -103519019531854247320734400512000t^{16}x^{12} - 10451183887931957865349120000t^{16}x^{16} \\
& -21554988228400687174913894645760000t^{22}x^4 - 290232638614071372247547904000000t^{10} \\
& + 921550020610531832465215979520t^{32}x^6 - 101549768847628850675318784t^{40}x^2 \\
& -504281339250167885290536960000t^8x^{16} + 3826649604707605351652917248000000t^{14}x^8 \\
& -369435906932736t^2x^{40} + 16667909082316800x^{36} + 9182793192693972428579143680t^{16}x^{20} \\
& -83765271854643888757248000000t^6x^2 - 238152498374515252372340893286400t^{32} \\
& -444958817402333593625333760000000t^8x^2 - 374361719025172480t^6x^{36} + 1089353185242843709440000t^2x^{24} \\
& + (465874449007264546835084083200000i)t^{13}x^8 + (4823614020262370407077642240i)t^{37}x^4 \\
& - (7131005536638246684865855488000000i)t^{13}x^6 + (663307352538500994247837089792000000i)t^{15}x^2 \\
& + (21860900352587055513600000000i)t^3x^8 + (2031462368091062245785600000i)t^5x^{18} \\
& - (266246547151911911424000i)t^7x^{28} - (185821084269058298524880338944000i)t^{25}x^8 \\
& + (7418734806093004800i)t^3x^{32} + (2862213347615440896000i)t^3x^{28} \\
& - (552177703400611963207680i)t^{11}x^{28} - (42778916079528612640849920i)t^{15}x^{24} \\
& - (5542613928474067253329920i)t^{13}x^{26} + (26954043769812418560i)t^7x^{34} \\
& - (13961497204251709617450516480i)t^{23}x^{16} + (5236753980771532800i)t^3x^{34} \\
& + (3049903960641814784901120i)t^{17}x^{24} + (561076209770717318721212252160000i)t^{13}x^{10} \\
& + (176852015110408037400000000i)t^4x^4 - (1236013710365367044407296000i)t^{19}x^{20} \\
& - (56114676712486862314820075520i)t^{27}x^{12} + (13237501624287485755392000i)t^9x^{24} \\
& - (9322396457300067560386650439680000i)t^{21}x^6 - (12320637217113300526845143285760i)t^{25}x^{10} \\
& - (871337957585499407646720000000i)t^5x^6 + (342297333433095735591911628472320000i)t^{21}x^2 \\
& - (4680371916583523208155627520i)t^{21}x^{18} + (3411782612506572724843315200000i)t^9x^{14} \\
& + (29554872554618880i)t^3x^{38} - (5051252967285585457994268672000000i)t^{11}x^4 \\
& - (97910776635488142675148800000i)t^5x^{14} - (2133898812152520447982200422400i)t^{33}x^4 \\
& - (6726600004333369028140597248000000i)t^{11}x^6 - (267483574424095948800i)t^3x^{30} \\
& + (41984482557941552951077109760i)t^{35}x^4 + (16266154456756345519472640i)t^{19}x^{22} \\
& - (47339646149515730623408373760i)t^{31}x^8 - (148448764770164128559923200i)t^{11}x^{24} \\
& - (393978418837264322831453257728000i)t^{15}x^{10} - (295458101549236503129253478400i)t^{19}x^{16} \\
& + (4274394850752626341969920000i)t^9x^{20} - 260093950641466110684168192000000t^{14}x^{10} \\
& + 11209026792613700640398376960t^{26}x^{14} - 17793896935440197600500802125824000t^{28} \\
& - 28006548001014786328125 + (20935891052923967084534169600i)t^{19}x^{18} + (12733345087949378692800000000i)t^3x^2 \\
& - (551960058360795226548228587520i)t^{27}x^{10} - (2135496646505306400837599232000i)t^{19}x^{14} \\
& - (52281692339772407826929732812800i)t^{29}x^6 - (1278248237987266560i)t^3x^{36} \\
& - (258599751610208611498721280i)t^{17}x^{22} + (42646262462432422133760i)t^7x^{30}
\end{aligned}$$

$$\begin{aligned}
& +(1921908711198288209069998080i)t^{27}x^{14} - (2152422258407783792640i)t^7x^{32} \\
& - (11722986284931477432824758272000000i)t^{11}x^2 + (4916208569464755956219904000000i)t^5x^{10} \\
+ & (7687634844793152836279992320i)t^{33}x^8 - (1200089453926809600i)tx^{34} - (5541538603991040i)tx^{38} \\
& + (265155523025203543340588256460800000i)t^{19}x^2 - (40998519207767310336000i)t^9x^{30} \\
& + (973952452359812756275200i)t^9x^{28} - (6040880073988867584348590702592000i)t^{27}x^2 \\
& - (476463860665945350840778752000i)t^{15}x^{16} + (14137168975845394061721600i)t^{11}x^{26} \\
& - (83330559414840018454393651200i)t^{17}x^{16} - (8715857820114190388428800i)t^9x^{26} \\
- & (65679609891970644542822862028800i)t^{23}x^{10} - (15444549824224090909762702540800i)t^{21}x^{12} \\
& - (501102268458389167267098132480000i)t^{29}x^4 - (1822390385308493414400i)t^7x^{26} \\
& - (176379232651551233950597565644800000i)t^{17}x^2 - (121754530052938137600i)t^3x^{30} \\
& - (7632568926974509056000i)t^5x^{28} - (21212944836172185600i)tx^{32} \\
& - (11868804176214003144500183040i)t^{15}x^{20} - (1428230923169793746927616000i)t^5x^{20} \\
- & (1593404980586920754439782400i)t^{13}x^{22} - (4059972824910431180685941145600i)t^{23}x^{12} \\
& - (1153654289755375740948106444800i)t^{21}x^{14} + (110315298314213807452323840000i)t^9x^{18} \\
- & (44141341633992947685829430476800i)t^{33}x^2 - (29664573099192904243142983680000i)t^{27}x^8 \\
& + (976829175879916584960i)t^5x^{32} - (32299446070763337039929671680i)t^{25}x^{14} \\
& + (145661776222191697920000i)tx^{22} - (39272801034337134745485312000i)t^{13}x^{18} \\
- & (686702447560384727557865472000i)t^{13}x^{16} + (4806073882099297446065602560000i)t^{15}x^{14} \\
+ & (321950676862368497336869453824000i)t^{15}x^{12} + (226464613738450274921425498275840000i)t^{23}x^2 \\
& - (81342884389055836953403352678400i)t^{21}x^{10} - (143534785864018526208000000i)tx^{16} \\
& - (276870208705871413248000000i)tx^{14} + (253802963770735067136000i)t^3x^{26} \\
& + (4942321068014944911360000i)t^3x^{24} - (11327366003901081845760000i)t^3x^{22} \\
- & (9507936586815535841280000i)t^3x^{20} - (558485756052497241499238400000000i)t^9x^2 \\
& + (2554410040510642126848000i)t^7x^{24} + (280529624686808077959168000i)t^7x^{22} \\
& + (5078605192094849421017088000i)t^7x^{20} + (177526096599051062602629120000i)t^7x^{16} \\
- & (1346531590074221867512627200000i)t^7x^{12} - (830392007014836338688000000i)t^5x^{16} \\
- & (688337455098696593257267200000i)t^5x^{12} + (2553416209978975025879040000000i)t^5x^4 \\
& + (4145743852606982365957324800000i)t^9x^{12} + (12524232380160862819713024000000i)t^9x^{10} \\
- & (4098892631724525344186695680000000i)t^9x^8 - (47718132406773061546672128000000i)t^9x^6 \\
- & (412735073577221378801664000000000i)t^9x^4 + (18279630702336234246484525056000000i)t^{13}x^2 \\
& - (4093146367802632849430937600i)t^{11}x^{20} - (65100417802656175439216640000000i)t^7x^4 \\
& + (66291948028992606848040960000000i)t^7x^2 - (908890277446267721597583360000i)t^9x^{16} \\
& + (2097868492701958252045271040000i)t^{11}x^{16} - (10902208455765898661241815040000i)t^{11}x^{14} \\
& + (48243026527976085634217410560000i)t^{11}x^{12} + (81143797205972851403784192000000i)t^{11}x^{10} \\
- & (1872788723070122441857145241600000i)t^{17}x^8 + (663993202870949744757178368000000i)t^{11}x^8 \\
- & (2105194565269620861922795585536000i)t^{29}x^2 - (569600135887715790718104502272000i)t^{27}x^4 \\
& + (237376536542879789562992787456000i)t^{25}x^6 - (9457877798008398928725147648000i)t^{25}x^4
\end{aligned}$$

$$\begin{aligned}
& +(26231903255774124633802269523968000i)t^{25}x^2 + (24184876704701330485753675776000i)t^{23}x^8 \\
& +(3614003629024326552068246470656000i)t^{23}x^6 - (10499474947997371531042372976640000i)t^{23}x^4 \\
& +(6389817349089724587377374003200000i)t^{15}x^8 + (24848549853847487146295820288000000i)t^{15}x^6 \\
& -(50224918358402156413513315123200000i)t^{15}x^4 + (3315001117333115557858900967424000i)t^{21}x^8 \\
& -(34611341837775193966142029824000i)t^{19}x^{12} + (1197497452659195739512483348480000i)t^{19}x^{10} \\
& -(2129400123998763522583261347840000i)t^{19}x^8 - (5738530825456039112050999296000i)t^{17}x^{14} \\
& +(157607173704729467077603098624000i)t^{17}x^{12} + (13143047438973691263056955310080000i)t^{17}x^6 \\
& +(133559322398266650037448815411200000i)t^{17}x^4 - (74018468157868179633922375680000i)t^{13}x^{12} \\
& -(17097919982892090127495987200000i)t^7x^{10} - (7366979512184196839768064000000i)t^7x^6 \\
& -(501973337906011496448000000i)tx^{12} - (312714063910070710272000000i)tx^{10} \\
& +(522589738344268795200000000i)tx^6 - (50187734017818497100000000i)tx^2 \\
& -(139440026561558033203200000i)t^3x^{18} - (2563793130931038978048000000i)t^3x^{16} \\
& -(16490427271028092698624000000i)t^3x^{14} + (5335683438790152290304000000i)t^3x^{12} \\
& +(43180590813665034240000i)tx^{26} + (171968454786061172736000i)t^5x^{26} \\
& +(11076242798116488609792000i)t^5x^{24} - 9012722919683961913344000t^{14}x^{24} \\
& -78591102059488214620569600t^{10}x^{24} + 1315871590749296394240t^8x^{32} - 524237317699453990853760000000t^6 \\
& +24522724634493022093505986560000000t^{14} + 9374582034547518740130850406400t^{30}x^6 \\
& +8652194022058481664000000x^{16} - 2889102380234435076096000000t^2x^{12} - 4568266985082716160000x^{26} \\
& -582312415127830734946959360t^{16}x^{22} + 27707693019955200t^2x^{38} + 1024797651488330815547375616000000t^{10}x^8 \\
& -109543867036748218368000t^2x^{26} - 13956755653155003314303139840t^{26}x^{12} \\
& +1028899809518611239536689152000t^{24}x^{12} - (3136733376113656068750000i) \\
& t + (406199075390515402701275136i)t^{41} - (14060356871129413671142796820480i)t^{35} \\
& +(12693721105953606334414848000i)t^{39} - (253572946242805728537854607360i)t^{37} \\
& +(179400689678910125855408652288000000i)t^{19} - (924506802373649140305119674368000000i)t^{17} \\
& +(71064895596404886819735909433344000i)t^{25} + (85068913796130252196170505912320000i)t^{23} \\
& +(189120006247548024160067675750400000i)t^{21} + (349604887297608346884480000000i)t^5 \\
& +(296346619914737792400000000i)t^3 - (1236098226654234031630975185715200i)t^{31} \\
& +(13710637516226960614134317580288000i)t^{27} - (200402074079873642284062238310400i)t^{33} \\
& -(874148309544138151997472768000000i)t^{11} - (44680685994957750725922678374400000i)t^{15} \\
& -(31118534408803516039720599552000000i)t^{13} - 837413738790263716939038720000000t^{12} \\
& +21827596489442869265741905920000000t^{14}x^4 - 21688205942341794025963520t^{24}x^{18} \\
& +82026364415156176583393280t^{18}x^{22} + 4461206888113542973787996160t^{24}x^{16} \\
& -12581738988371940359233307934720000t^{22} - 13079267440752624654090240000t^6x^{18} \\
& +784183344028414017187500x^2 - 1156846770416611491840000t^4x^{24} - 6738510942453104640t^8x^{34} \\
& -1130716815140272736758136832000000t^{10}x^6 - 459306963730169856000t^4x^{30} \\
& +22670936310385070781990174720t^{28}x^{12} + 21794914383199203466590289920000t^{12}x^{14} \\
& -898325080077249596620800t^{10}x^{28} - 618167015515602501526093824000t^{14}x^{16}
\end{aligned}$$

$$\begin{aligned}
& -45969512204299105569079296000t^{24}x^{14} + 87933827260455321600t^4x^{32} \\
& -241754408013083730331985510400t^{18}x^{16} + 9772923577259908191757742899200t^{32}x^2 \\
& +61444580041052651520t^6x^{34} - 1854683701523251200t^2x^{34} - 77122158784324793482916501913600000t^{16}x^2 \\
& +2406535838623662321671098859520t^{26}x^{10} - 28208920857359155200t^2x^{32} - 3922239420529166499840000000t^4x^{10} \\
& -27759545995008961606645186560t^{22}x^{16} - 50563806593382432117463449600000t^{10}x^4 \\
& +89125941971739511955128320000t^8x^{18} + 976683049559901695744409600000t^6x^{12} \\
& -58651999243111822786560t^{16}x^{26} - 1477195572245748938440704000t^8x^{20} \\
& +591198160146252383519433031680t^{36}x^2 + 13564393772444526890188800t^{10}x^{26} \\
& +27180430318123159563565793280t^{36}x^4 + 166829522934218514821504040960t^{28}x^{10} \\
& +3194465160736712672975530229760t^{36} + 966830160095454660592140288000t^{34}x^4 \\
& -91643748817362223104t^{10}x^{32} - 23937352718546582211371257036800t^{28}x^6 \\
& +1279503167224870826982607159296000t^{16}x^{10} + 235292930319472442081280000000t^6x^6 \\
& +228627036523314177638400t^{12}x^{28} - 60059647224946506533437440t^{26}x^{16} \\
& +7686358541099158352560128000000t^6x^8 - 53888758652974116372480t^8x^{30} \\
& -977533320718530379776t^{12}x^{30} - 19342813113834066795298816t^{42} - 3879347444249210377784524800000t^6x^{10} \\
& -257068171959178028359680000000t^4x^8 + 4179430850376499200x^{32} + 14376685446051712456458240t^{16}x^{24} \\
& +140431627964630793463529472000000t^8x^6 + (401867327451515734248652800i)t^{11}x^{22} \\
& +(36211760706281352600748032000i)t^{11}x^{18} - (65963755169576024776114176000i)t^{17}x^{18} \\
& +(141979432513769079570432000i)t^{13}x^{24} - 2052834294940463333376000000t^2x^{14} \\
& -2073987321006894612480t^6x^{32} - 803711151345709001549964902400t^{20}x^{14} \\
& -19083105725960179439585280000000t^6x^4 + 1475838932049899227290009600t^{14}x^{22} \\
& -9713761369774005468266496000t^{14}x^{20} - 452213814399597225663528960t^{34}x^8 + 69269232549888x^{40} \\
& +280507056646219587020390400000t^{10}x^{12} + 1492444278814740435060129792000000t^{12}x^8 \\
& +769495633791371182080000000t^4x^{18} + 4113162625508371681247232000t^6x^{20} \\
& +(1028131595438122357563064320i)t^{15}x^{22} + (237549883224301240320000i)t^{24} \\
& -(5473554208670333781702806077440000i)t^{19}x^6 + 539566201623545739097826918400t^{30}x^8 \\
& -6506461782702538207789056t^{22}x^{20} + 141611701718037306903131219558400000t^{16}x^4 \\
& -(1781184913849238098131214663680000i)t^{29} - (90516577854024576866304000000000i)t^9 \\
& +(16746521200817938995548160000000i)t^7 + 130676991329371088131551815270400000t^{20} \\
& +29910409415857677027350937600000t^8x^{10} - 3048843404359167167692800t^8x^{26} \\
& -35146293073997620838400000t^4x^{20} - 38690544532792538522217640624128000t^{26} \\
& +2107157703588298651512864768t^{40} + 116740110638455833231360000t^4x^{22} \\
& -401967835021864200589803520t^{36}x^6 - 10489757533115490019521331200t^{20}x^{18} \\
& -42184962622881792000t^4x^{34} + 170710247836455760664985600t^{12}x^{24}); \\
D(x, t) = & (45323980394714778162954240000000t^8x^8 + 24627676640899497984000t^4x^{26} \\
& +6536462007456578404383675383808000t^{22}x^6 - 4288801766427224085507342336000t^{16}x^{14} \\
& +368404647701690338586394624000t^{10}x^{16} + 384381742239657641813999616t^{32}x^{10} \\
& -742268668480789473023659868160000t^{24}x^2 + 161804815860907376640000x^{22}
\end{aligned}$$

$$\begin{aligned}
& -6790019255461159103232000000t^4x^6 - 4944054841461468177629184t^{20}x^{20} \\
& + 3116050765605001976242539331584000t^{28}x^2 + 26009896265605858335916032000t^{16}x^{18} \\
& + 60921626487686420870861527449600000t^{18}x^2 - 23870477918779431749595168768000t^{18}x^{12} \\
& + 157352538270124324631347200t^{10}x^{22} + 11092996319640910036992000t^8x^{24} \\
& - 4048034951690653322202537000960000t^{20}x^2 + 315364416159744000x^{34} + 69931633115293286400x^{30} \\
& + 172157668824188616584126791680000t^{12}x^{10} + 41305449926112714726330020659200000t^{20}x^4 \\
& - 65676639587066896711680000t^6x^{22} + 264046954624214630400t^2x^{30} \\
& + 19925337192920352081977750323200000t^{18}x^6 + 40455932657724327818987097292800000t^{18}x^4 \\
& + 28825577773415107644072960000000t^8 - 119076919384603145011200000t^2x^{16} \\
& - 88442956119696998400t^6x^{30} + 56829188844642993916870656000t^{14}x^{18} \\
& + 3734139358041821340814540800t^{12}x^{20} + 1289649279612910746118127616000t^{18}x^{14} \\
& + 14526828401811079008028238807040000t^{24}x^4 + 58288613843255631982755840000000t^8x^4 \\
& + 21144689625774893695185715200000t^{12}x^6 + 1858588673501596499770657996800t^{20}x^{12} \\
& + 15776879049053621452800000t^2x^{18} + 457606819270524444235687526400000t^{18}x^8 \\
& + 4232691991847276125146316800t^{10}x^{20} + 747521480085177922740000000t^4 \\
& + 8021458343018853810000000x^6 + 519519244124160x^{38} + 1787222867366445406442219372544000t^{24}x^6 \\
& + 10856203874861428872202892083200t^{22}x^{10} + 1342569987536088015694725120000000t^{12}x^4 \\
& + 10888453439972553060000000x^8 + 9104071459689094366598529024000000t^{12}x^2 \\
& + 32785352890867775360000t^6x^{26} + 1568799755546822443008000t^6x^{24} + 1007283588994791662400000x^{10} \\
& - 501106709929274941281140736000000t^{10}x^2 + 707271124818693980160000x^{20} \\
& + 1867092791250438641571509934489600t^{30} + 20026836170571804131250000t^2 \\
& + 7316395404151027138560000t^2x^{20} + 865636725666914269306697023488000t^{20}x^8 \\
& + 639672080962623609632348071526400t^{30}x^2 + 277719919821324288000x^{28} \\
& - 31980191244095515157394295357440000t^{22}x^2 - 18844182928440318306122779852800t^{20}x^{10} \\
& + 647591200264023821608437153792000t^{28}x^4 + 343864818302012316956699394048000t^{26}x^6 \\
& + 92071972531818619699269206016000t^{24}x^8 + 109550710205924732043264000t^8x^{22} \\
& + 960293933293978128384000000t^2x^{10} - 1325061228272434107840000000t^2x^8 \\
& + 7019595923832916022361996656640000t^{26}x^2 + 748489692780515568960000000t^2x^6 \\
& + 64159410887247822300000000t^2x^4 - 111957252808979724300000000t^2x^2 \\
& + 6389982016100992830288376627200t^{34}x^2 + 116018271796118630400000000x^{14} \\
& + 5430273073229572585449062400000t^8x^{12} + 701227890912144151019520t^{12}x^{26} \\
& + 708618775142017164902400000t^4x^{16} + 80434160738159540620492800000t^6x^{14} \\
& + 11695926375914239033344000000t^4x^{12} + 9188048507825553408000t^4x^{28} \\
& + 1658936156076638208000t^2x^{28} + 71186431760663987349356544000t^{12}x^{18} \\
& + 1133985667018943692800000x^{18} + 2523551246339044311407001600000t^{10}x^{14} \\
& + 58409809823636206689745549393920000t^{24} + 119508725048133376351659884544000t^{18}x^{10} \\
& + 60287229757947398671596257280000t^{10}x^{10} + 8378857034730260398080t^{14}x^{28} \\
& + 21332128333217619456000000x^{12} + 186648239032841056095830016000t^{22}x^{14} + \\
& 14777436277309440t^4x^{38} + 171416692930800481468416000t^{12}x^{22} + 6027751534210401769095168000t^{18}x^{18}
\end{aligned}$$

$$\begin{aligned}
& -2295963211679982289944576000t^{10}x^{18} + 11308915469050980093369778176t^{30}x^{10} \\
& + 34836525267081102400438665216000000t^{14}x^2 + 137279193657020586362142720t^{28}x^{14} \\
& - 12219092621800243200000t^8x^{28} + 520666510730106735113011200000t^8x^{14} \\
& + 89432936845822803438402011136000t^{30}x^4 + 89633538159414902728886845440t^{38} \\
& + 16252937702437484582525730816000000t^{16} + 4624173302020412626742280192000t^{28}x^8 \\
& + 23093119135147622400000x^{24} - 273194838579757700874240t^{14}x^{26} + 18283421028899504936284323840t^{32}x^8 \\
& + 21492056547517699724956139520t^{34}x^6 - 3877977581534696177664t^{10}x^{30} + 4398046511104x^{42} \\
& + 804290609259911812500000x^4 + 21711089220782970529382400000t^6x^{16} \\
& + 256254494826438427875999744t^{30}x^{12} + 31142815803988284981674351001600t^{34} \\
& + 11775769533480960t^2x^{36} + 28740632513159451112667873280000t^2x^8 \\
& + 15085899823696223162390563258368000t^{20}x^6 - 292085023149255531215978496000t^{12}x^{16} \\
& + 53935143725034198301409280t^{22}x^{18} + 998313079891434310080000000t^4x^4 \\
& + 7564265857109036663280000000t^4x^2 - 840251636564522434560000000t^4x^{14} \\
& + 3648008162636396412554547560448000t^{26}x^4 + 253874422119072126688296960t^{38}x^4 \\
& + 94537654042279934894166717235200000t^{18} + 8758667563107988370746245120t^{38}x^2 \\
& - 2697387532000929170797363200000t^{14}x^{14} + 25159947500789794169652510720000t^{14}x^{12} \\
& + 8906766657607955147189649408000t^{32}x^4 + 338878217849090531655680t^{18}x^{24} \\
& + 10470537786849459916081043865600t^{24}x^{10} + 3190530548022051252724354252800t^{22}x^{12} \\
& + 1284562789522580451354476544000000t^{16}x^8 + 12863543457321089775721827532800000t^{16}x^6 \\
& + 16308124688452681728000t^6x^{28} - 43224001111301120t^4x^{36} + 1626615445675634551947264t^{20}x^{22} \\
& + 342130184183092637503793922048000t^{22}x^8 + 32068248131267736673022115840000t^{12}x^{12} \\
& + 1079518545924902092800000t^2x^{22} + 38524261286006536729924730880t^{20}x^{16} \\
& - 3735536672399445680577734246400000t^{14}x^6 + 269650932253973825055621120t^{18}x^{20} \\
& - 4405660387524677485524418560000t^{16}x^{12} + 403583415082537113240993792000t^{16}x^{16} \\
& + 34534936785834022537771823923200000t^{22}x^4 + 267089457768372741558583296000000t^{10} \\
& + 550530899077293867881925181440t^{32}x^6 + 101549768847628850675318784t^{40}x^2 \\
& + 29676525841394574557184000000t^8x^{16} + 1761944289373158958952349696000000t^{14}x^8 \\
& + 369435906932736t^2x^{40} + 13291033995509760x^{36} + 674120776546396850356224000t^{16}x^{20} \\
& + 416404209761764131156480000000t^6x^2 + 294067425509390107565690034585600t^{32} \\
& - 15869860849067864948305920000000t^8x^2 + 374361719025172480t^6x^{36} \\
& + 91939228359642316800000t^2x^{24} + 267373483487109126151030702080000t^{14}x^{10} \\
& + 1763857007974744770824110080t^{26}x^{14} + 8259327114186084142196937719808000t^{28} \\
& + 6154544491010705483366400t^{14}x^{24} + 6389363531939347169280000t^{10}x^{24} \\
& - 284879416553971384320t^8x^{32} + 180721924486101900316800000000t^6 \\
& + 9805423648517498549009645568000000t^{14} + 7692422509798564361512880701440t^{30}x^6 \\
& + 3932976439367210188800000x^{16} + 680341459697530380288000000t^2x^{12} \\
& + 2602547308142788608000x^{26} + 44044496389822028238028800t^{16}x^{22} - 5541538603991040t^2x^{38} \\
& - 71490651801908749987007692800000t^{10}x^8 - 29735889790316838912000t^2x^{26} \\
& + 82064395606244341723221196800t^{26}x^{12} + 702370714222287966351812198400t^{24}x^{12}
\end{aligned}$$

$$\begin{aligned}
& +367215746822415666584852889600000t^{12} - 388441643701022205897867264000000t^{14}x^4 \\
& +21688205942341794025963520t^{24}x^{18} - 4762130745563535365898240t^{18}x^{22} \\
& +418639448913360682053795840t^{24}x^{16} + 88648968947267503649373667983360000t^{22} \\
& +2154349846231906640625 + 133365085130776778508800000t^6x^{18} + 180965387083480157812500x^2 \\
& -2042367037914155581440000t^4x^{24} + 6738510942453104640t^8x^{34} + 358131686186596444493001523200000t^{10}x^6 \\
& +525639180819942604800t^4x^{30} + 5196740001990108249524797440t^{28}x^{12} \\
& +3643894751364567984085401600000t^{12}x^{14} + 61087084302631981547520t^{10}x^{28} \\
& +28876804363165060911071232000t^{14}x^{16} + 25936458820714974771412992000t^{24}x^{14} \\
& +19419629345361100800t^4x^{32} + 170487818456707150066483200000t^{18}x^{16} \\
& +78916367221302480288978095308800t^{32}x^2 - 14275003443880919040t^6x^{34} \\
& +545495206330368000t^2x^{34} + 102817448717754853936727064576000000t^{16}x^2 \\
& +2058245621187256415713178419200t^{26}x^{10} + 14789738781632102400t^2x^{32} \\
& +51630769268674678849536000000t^4x^{10} + 6670904798520590070794158080t^{22}x^{16} \\
& +1001557152044251656675655680000000t^{10}x^4 + 9907008973268534584934400000t^8x^{18} \\
& -186962840836126122442752000000t^6x^{12} + 58651999243111822786560t^{16}x^{26} \\
& -1057912045110554643136512000t^8x^{20} + 321131310041398187750798131200t^{36}x^2 \\
& -155532989696642729902080t^{10}x^{26} + 17437999369303766701902397440t^{36}x^4 \\
& +204983947865102952783182561280t^{28}x^{10} + 2133482299428731345274352435200t^{36} \\
& +535114721888049703907701555200t^{34}x^4 + 91643748817362223104t^{10}x^{32} \\
& +60912221983103393139076890624000t^{28}x^6 + 260312228549486582307760373760000t^{16}x^{10} \\
& +8488349754576575808798720000000t^6x^6 - 38008038983200753582080t^{12}x^{28} \\
& +60059647224946506533437440t^{26}x^{16} + 1343351729047547481292800000000t^6x^8 \\
& +3749981339475152732160t^8x^{30} + 977533320718530379776t^{12}x^{30} + 19342813113834066795298816t^{42} \\
& +350791225471574901522432000000t^6x^{10} + 89424110534691401441280000000t^4x^8 \\
& +5941209899571609600x^{32} - 1414593613323999620628480t^{16}x^{24} + 90305338687107457092157440000000t^8x^6 \\
& +187885181646143815680000000t^2x^{14} + 143137942141088563200t^6x^{32} \\
& +824104538643404730610483200000t^{20}x^{14} + 4110255945966244388751360000000t^6x^4 \\
& +46371793646940589758873600t^{14}x^{22} + 2689023734168603609530368000t^{14}x^{20} \\
& +452213814399597225663528960t^{34}x^8 + 23089744183296x^{40} + 16716993992988126079164088320000t^{10}x^{12} \\
& -294990095408936873523727564800000t^{12}x^8 + 380398138248276423475200000t^4x^{18} \\
& +308200319155218205900800000t^6x^{20} + 391750298587119499897303203840t^{30}x^8 \\
& +6506461782702538207789056t^{22}x^{20} + 18315378812895833262524674867200000t^{16}x^4 \\
& +121462888973612955113913644482560000t^{20} + 7029884218031297562201292800000t^8x^{10} \\
& +2468867664234739138560000t^8x^{26} + 54773375791185631641600000t^4x^{20} \\
& +26590369766887902946198699376640000t^{26} + 2056382819164484226175205376t^{40} \\
& +8454122966951909130240000t^4x^{22} + 401967835021864200589803520t^{36}x^6 \\
& +1442137173241003254459924480t^{20}x^{18} + 2618376990385766400t^4x^{34} + 830805200433869488128000t^{12}x^{24}
\end{aligned}$$

We get the following plot for the modulus of v in function of (x, t) . We recognize again $N(N + 1) - 1$ local maximums and the global maximum is equal to 13 :

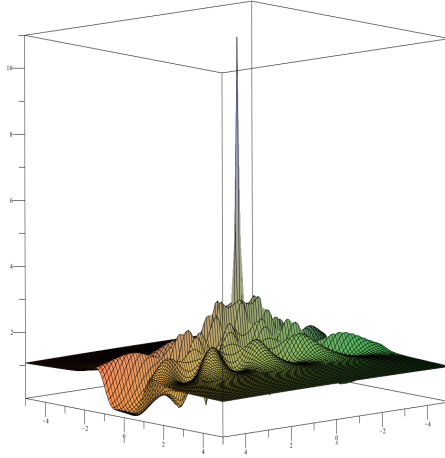


Figure 7: Solution of NLS, N=6 a.

In the following, we only represent the modulus of v in function of v in the (x, t) coordinates.

If we choose $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = 1, a_6 = 1, b_1 = 1, b_2 = 1, b_3 = 1, b_4 = 1, b_5 = 1, b_6 = 1$, we get :

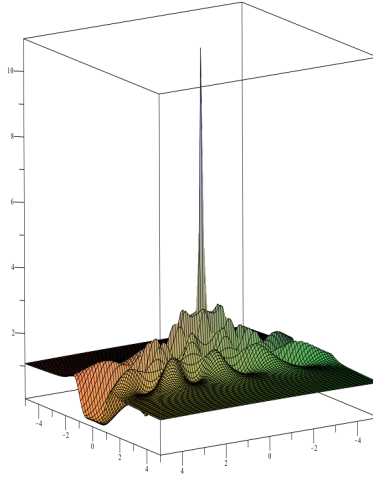


Figure 8: Solution of NLS, N=6 b.

If we choose $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $b_1 = 1000000$, $b_2 = 1000000$, $b_3 = 1000000$, $b_4 = 1000000$, $b_5 = 1000000$, $b_6 = 1000000$, we get The shape of the modulus of v in the (x, t) coordinates is given by :

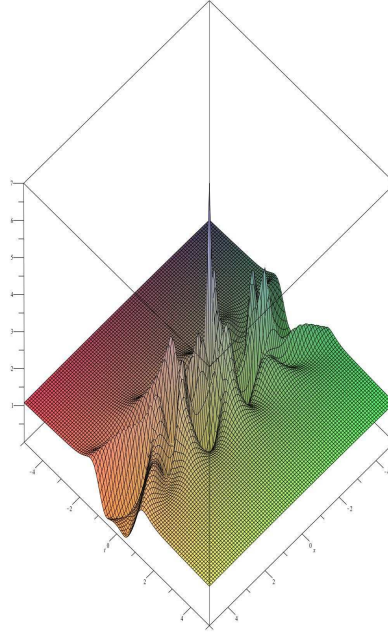


Figure 9: Solution of NLS, $N=6$ c.

We see again in this deformation how to get multi-rogue waves from Akhmediev breathers.

4 Conclusion

We have given here an extension of a previous result exposed in [9] which gives with new parameters a family which recover a wide spectrum of solutions of the NLS equation.

This method described in the present paper provides a powerful tool to get explicitly solutions of the NLS equation. It gives an infinite set of non singular solution of NLS equation.

We can get as well higher Peregrine's breathers of order N as well solutions with peaks of similar amplitude.

For those who were sceptic about the use of theta functions and determinants,

it is the proof of this efficiency of this method.

References

- [1] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, *Physical Review E*, V. **80**, N. 026601, (2009).
- [2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, *Th. Math. Phys.*, V. **72**, N. 2, 183-196, (1987).
- [3] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, *Sov. Phys. J.E.T.P.*, V. **62**, 894-899, (1985).
- [4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, *J. Phys. A : Math. Theor.*, V. **43**, 122002, 1-9, (2010).
- [5] E.D. Belokolos, A.i. Bobenko, A.R. its, V.Z. Enolskij and V.B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).
- [6] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Special Topics*, V. **185**, 247-258, (2010).
- [7] P. Dubard, V.B. Matveev, Multi-rogue waves solutions of the focusing NLS equation and the KP-i equation, *Nat. Hazards Earth Syst. Sci.*, V. **11**, 667-672, (2011).
- [8] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions to the NLS equation, *Soviet Doklady 1986 sect. Math. Phys.*, V. **287**, 606-610, (1986).
- [9] P. Gaillard, Quasi-rational solutions of the NLS equation and rogue waves, halshs-00536287, 2011

- [10] P. Gaillard, Families of quasi-rational solutions of the NLS equation as an extension of higher order Peregrine solutions, hal-00573955, 2011
- [11] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, Teore. i Mat. Fiz., V. **74.**, N. 1, 29-45, (1988).
- [12] A.R. Its, V.P. Kotlyarov, Explicit expressions for the solutions of nonlinear Schrödinger equation, Dokl. Akad. Nauk. SSSR, S. A, V. **965.**, N. 11, (1976).
- [13] V.B. Matveev, M.A. Salle, Darboux transformations and solitons, Series in Nonlinear Dynamics, Springer Verlag, Berlin, (1991).
- [14] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B, V. **25**, 16-43, (1983).
- [15] R.R.. Rosales, Exact solutions of some nonlinear evolution equations, Studies In Appl. Math., V. **59**, 117-151, (1978).
- [16] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Of Appl. Mech. And Tech. Physics, V. **9**, N. 2, 190-194, (1969)
- [17] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, J. Appl. Tech. Phys, V. **9**, 86-94, (1968)
- [18] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, Sov. Phys. JETP, V. **34**, 62-69, (1972)