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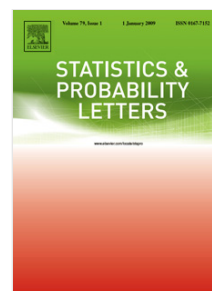
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# Nested classes of $C$ -semi-selfdecomposable distributions

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## Abstract

We get a representation of Lévy measures of  $(C, m)$ -semi-selfdecomposable distributions, which extend  $c$ -semi-selfdecomposable distributions of Maejima and Naito (1998). We prove that for every pair  $(C, m)$  there exists a distribution which is exactly  $(C, m)$ -semi-selfdecomposable. Explicit examples are given.

*Key words:* infinitely divisible, selfdecomposable, semi-selfdecomposable,  $m$ -times selfdecomposable, decomposability semigroup, Lévy measure

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## 1. Introduction and preliminaries

Let  $\mathcal{P}(\mathbb{R}^d)$ ,  $ID(\mathbb{R}^d)$  and  $\mathbb{L}_m(\mathbb{R}^d)$  ( $m = 0, 1, 2, \dots$ ) be the sets of all probability distributions, infinitely divisible distributions and  $(m+1)$ -times selfdecomposable distributions on  $\mathbb{R}^d$ , respectively. A classical result due to Sato (1980) states that an infinitely divisible distribution  $\mu$  belongs to  $\mathbb{L}_m(\mathbb{R}^d)$  ( $m = 0, 1, 2, \dots$ ) if and only if for any  $0 < c < 1$ , there exists  $\mu_c \in \mathbb{L}_{m-1}(\mathbb{R}^d)$  such that

$$\hat{\mu}(z) = \hat{\mu}(cz)\hat{\mu}_c(z), \quad z \in \mathbb{R}^d, \quad (1)$$

where  $\hat{\mu}$  means the characteristic function of  $\mu$ , and with the convention that  $\mathbb{L}_{-1}(\mathbb{R}^d) = ID(\mathbb{R}^d)$ . Let  $C \subset [-1, 1]$ . A distribution  $\mu \in \mathcal{P}(\mathbb{R}^d)$  is called  $(c, m)$ -semi-selfdecomposable and  $(C, m)$ -semi-selfdecomposable if (1) holds with  $\mu_c \in \mathbb{L}_{m-1}(\mathbb{R}^d)$ , for some  $c \in [-1, 1]$ , and for all  $c \in C$ , respectively. Classes of all  $(c, m)$ -semi-selfdecomposable and  $(C, m)$ -semi-selfdecomposable distributions are denoted by  $L(m, \mathbb{R}^d, c)$  and  $L(m, \mathbb{R}^d, C)$ ,

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respectively. Then the class of  $(c, 0)$ -semi-selfdecomposable distributions coincides with the Maejima and Naito (1998) class of  $c$ -semi-selfdecomposable distributions. We call  $\mu \in \mathcal{P}(\mathbb{R}^d)$   $C$ -semi-selfdecomposable if  $\mu$  is  $(C, 0)$ -semi-selfdecomposable. The class  $L(m, \mathbb{R}^d, [0, 1])$  coincides with  $\mathbb{L}_m(\mathbb{R}^d)$ . By definition,  $\mathbb{L}_m(\mathbb{R}^d) \subset L(m, \mathbb{R}^d, C) \subset \mathbb{L}_{m-1}(\mathbb{R}^d)$ . Consequently, we have  $L(m, \mathbb{R}^d, C_1) \subset L(m-1, \mathbb{R}^d, C_2)$ , where  $C_1, C_2 \subset [-1, 1]$ . The class  $L(m, \mathbb{R}^d, C)$  in the case of  $C$  satisfying some additional conditions has been investigated by the author in the preprint Rajba (1997). When  $C \subset [0, 1]$ , this is a particular case of the class studied in the author's previous paper Rajba (2006).

Let  $c \in \mathbb{R}$ ,  $\mathcal{H} \subset \mathcal{P}(\mathbb{R}^d)$  and  $\mu \in \mathcal{H}$ . When (1) is satisfied with  $\mu_c \in \mathcal{H}$ , then we say that  $\mu$  is  $c$ -decomposable under  $\mathcal{H}$ . We denote the set of such laws by  $L_c(\mathcal{H})$ . By  $D(\mu, \mathcal{H})$  we denote the set of all  $c \in [-1, 1]$  such that  $\mu$  is  $c$ -decomposable under  $\mathcal{H}$ . Let  $C \subset [-1, 1]$ . If  $C \subset D(\mu, \mathcal{H})$ , then we say that  $\mu$  is  $C$ -decomposable under  $\mathcal{H}$ . We denote the set of such laws by  $L_C(\mathcal{H})$ . If  $C = D(\mu, \mathcal{H})$ , then we say, that the  $\mu$  is exactly  $C$ -decomposable under  $\mathcal{H}$ . Following Sato (1980) we will say, that a class  $\mathcal{H} \subset \mathcal{P}(\mathbb{R}^d)$  is completely closed if  $\mathcal{H}$  is closed under weak convergence, convolution and type equivalence. If, furthermore,  $\mathcal{H} \subset ID(\mathbb{R}^d)$  and  $\mathcal{H}$  is closed under taking positive powers, then we say, that it is closed in the strong sense. If  $\mathcal{H} \subset \mathcal{P}(\mathbb{R}^d)$  is completely closed and  $\mu \in \mathcal{H}$ , then  $D(\mu, \mathcal{H})$  is a closed multiplicative subsemigroup of  $[-1, 1]$  containing 0 and 1 (see Maejima et al. (1999)). Following Rajba (1997) let  $\mathcal{C}$  be a collection of all closed multiplicative semigroups  $C \subset [-1, 1]$  containing 0 and 1. Our terminology is different from that of Bunge (1997). The concept of the decomposability semigroup associated with probability measures has been introduced by Urbanik (1975).  $D(\mu, \mathcal{H})$  is a subsemigroup of the Urbanik semigroup  $D(\mu)$ . If  $d = 1$ , then  $D(\mu) = D(\mu, \mathcal{P}(\mathbb{R}^1))$  (see also Urbanik (1972), Jurek and Mason (1993)). The Urbanik problem of characterization of those semigroups which are decomposability semigroups for probability measures is still open. Ilinskij (1978) showed, that for every symmetric  $C \in \mathcal{C}$  there is a probability measure  $\mu$  with  $D(\mu) = C$ . We proved a version of the Ilinskij theorem, for decomposability semigroups  $D(\mu, \mathcal{H})$  with an arbitrary (not necessarily symmetric)  $C$  and with  $\mathcal{H} = ID(\mathbb{R})$  (see Niedbalska-Rajba (1981)).

**Proposition 1.1.** *Let  $C \in \mathcal{C}$ . Then there exists a  $\mu \in ID(\mathbb{R})$  such that  $D(\mu, ID(\mathbb{R})) = C$ .*

Notice that  $\mu \in L(m, \mathbb{R}^d, C)$  if and only if  $C \subset D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d))$ . When  $C = D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d))$  we say that  $\mu$  is exactly  $(C, m)$ -semi-selfdecomposable.

By Proposition 1.1, for every  $C \in \mathcal{C}$  there exists  $\mu \in ID(\mathbb{R})$  such that  $\mu$  is exactly  $(C, 0)$ -semi-selfdecomposable in the univariate case. In this note we prove that such a measure exists in general case for  $d \geq 1$  and  $m \geq 0$ .

The following result, which is not used in the sequel but is included for completeness, gives a characterization of the classes  $L(m, \mathbb{R}^d, C)$  in terms of limit distributions of some normed sums. We introduce the following notation. Let  $0 < c < 1$  and  $\mathcal{H} \subset \mathcal{P}(\mathbb{R}^d)$ . Let  $X_1, X_2, \dots$  be independent random variables each with distribution in  $\mathcal{H}$ ,  $a_n \in \mathbb{R}^d$ ,  $b_n > 0$  ( $n = 1, 2, \dots$ ) and  $\lim_{n \rightarrow \infty} b_{n+1}^{-1} b_n = c$ . A distribution  $\mu$  belongs to the class  $\tilde{\mathcal{L}}(\mathcal{H}, c)$  if it is the limit of normed sums  $b_n^{-1} \sum_{j=1}^{k_n} X_j + a_n$ , ( $k_n \in \mathbb{N}, \uparrow \infty$ ), and it belongs to  $\mathcal{L}(\mathcal{H}, c)$  if furthermore  $b_n^{-1} X_j$  are infinitesimal (see Maejima and Naito (1998)). The class  $\mathcal{L}^c(\mathcal{H})$  can be defined as the limits from  $\tilde{\mathcal{L}}(\mathcal{H}, c)$  with  $k_n = n$ .  $\mathcal{L}(\mathcal{H}, c)$ ,  $\tilde{\mathcal{L}}(\mathcal{H}, c)$  and  $\mathcal{L}^c(\mathcal{H})$  are particular cases of the classes studied in Maejima and Naito (1998) and Maejima et al. (1999).

The following result was proved in Maejima and Naito (1998).

**Proposition 1.2.** *Let  $0 < c < 1$  and let  $\mathcal{H} \subset \mathcal{P}(\mathbb{R}^d)$  be completely closed. (i) If  $\mu \in \mathcal{L}(\mathcal{H}, c)$ , then  $\mu \in L_c(\mathcal{H} \cap ID(\mathbb{R}^d))$ . (ii) If  $\mathcal{H}$  is completely closed in the strong sense then the converse of (i) is also true. (iii) If  $\mathcal{H}$  is completely closed in the strong sense, so is  $\mathcal{L}(\mathcal{H}, c)$ . (iv)  $\tilde{\mathcal{L}}(\mathcal{H}, c) = \mathcal{L}^c(\mathcal{H}) = L_c(\mathcal{H})$ . (v)  $\tilde{\mathcal{L}}(\mathcal{H}, c)$  is completely closed. (vi)  $\tilde{\mathcal{L}}(\mathcal{H}, c) = \mathcal{L}(\mathcal{H}, c)$ , whenever  $\mathcal{H}$  is completely closed in the strong sense.*

The following proposition is a direct consequence of the above proposition.

**Proposition 1.3.** *Let  $0 < c < 1$ . (i)  $L(m, \mathbb{R}^d, c)$  is completely closed in the strong sense ( $m \geq 0$ ). (ii)  $L(m, \mathbb{R}^d, c) = L_c(\mathbb{L}_{m-1}(\mathbb{R}^d)) = \mathcal{L}(\mathbb{L}_{m-1}(\mathbb{R}^d), c) = \tilde{\mathcal{L}}(\mathbb{L}_{m-1}(\mathbb{R}^d), c) = \mathcal{L}^c(\mathcal{L}_{m-1}(\mathbb{R}^d))$  ( $m \geq 1$ ). (iii)  $L(0, \mathbb{R}^d, c) = L_c(ID(\mathbb{R}^d)) = \mathcal{L}(\mathcal{P}(\mathbb{R}^d), c) = \mathcal{L}(ID(\mathbb{R}^d), c) = \tilde{\mathcal{L}}(ID(\mathbb{R}^d), c) = \mathcal{L}^c(ID(\mathbb{R}^d))$ .*

In view of Proposition 1.3 we infer that if  $C \subset [0, 1]$  then the class  $L(m, \mathbb{R}^d, C)$  can be described as the class of limit distributions of some normed sums and in terms of the  $C$ -decomposability of measures.  $L(m, \mathbb{R}^d, c)$  can be characterized also in terms of stochastic integral with respect to semi-Lévy process, as a special case of the classes studied in Maejima and Miura (2007) (it coincides with the class  $Q(c, \mathbb{L}_m(\mathbb{R}^d))$  by their terminology).

The class  $\tilde{\mathcal{L}}(\mathcal{P}(\mathbb{R}), c)$  coincides with the Loève class  $L_c$  (Loève (1945)). Bunge (1997) extended the notion of  $c$ -decomposability by introducing the

class of  $C$ -decomposable distributions. The class  $\mathcal{L}(\mathcal{P}(\mathbb{R}^d), c)$  of  $c$ -semi-selfdecomposable distributions was introduced in Maejima and Naito (1998). In Maejima et al. (1999) the notion of operator semi-selfdecomposable distributions was introduced as an extension of  $c$ -semi-selfdecomposable distributions. Finally, we would like to mention recent paper of Maejima and Miura (2007) who consider a characterization of subclasses of semi-selfdecomposable distributions by stochastic integral representations.

$C$ -decomposability of distributions is discussed in various papers, but only with  $C \subset [0, 1]$ . In this note we consider the general case of  $C \subset [-1, 1]$ . Results obtained in this note complement and generalize results of Bunge, Urbanik, Maejima and Naito, Sato, Watanabe and Ilinskij. The methods of our proofs are stimulated by results of Sato (1980). In Sections 2 and 3 we give a characterization of measures in  $L(0, \mathbb{R}^d, C)$  and  $L(m, \mathbb{R}^d, C)$  ( $m > 0$ ), respectively, in terms of properties of their Lévy measures. We give a representation of characteristic functions of distributions from the classes  $L(m, \mathbb{R}^d, C)$  ( $m = 0, 1, 2, \dots$ ). In the case  $C = [0, 1]$  we obtain a representation of  $\mu \in \mathbb{L}_m(\mathbb{R}^d) = L(m, \mathbb{R}^d, [0, 1])$ . We define the  $m$ -spectral measures corresponding to distributions from the classes  $\mathbb{L}_m(\mathbb{R}^d)$ . The family of  $m$ -spectral measures in our representation of  $\mu \in \mathbb{L}_m(\mathbb{R}^d)$  turns very useful in the study of distributions, which are  $C$ -decomposable under  $\mathbb{L}_m(\mathbb{R}^d)$ . It will be shown that these distributions can be described as distributions for which our associated  $m$ -spectral measures are  $C$ -superinvariant. The  $m$ -spectral measures play a role analogous to the role the Lévy measures play in the study of  $C$ -decomposability under infinitely divisible distributions. We give examples of distributions in  $L(m, \mathbb{R}^d, C)$ . It will be shown that for every  $m \geq 0$  and  $C \in \mathcal{C}$  there exists  $\mu \in L(m, \mathbb{R}^d, C)$  such that  $D(\mu, \mathbb{L}_m(\mathbb{R}^d)) = C$ , i.e.  $\mu$  is exactly  $(C, m)$ -semi-selfdecomposable. This generalizes Theorem 1 given in the univariate case, and is a solution of the Urbanik problem of characterization of those semigroups which are decomposability semigroups, for decomposability semigroups  $D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d))$ .

## 2. The class $L(0, \mathbb{R}^d, C)$

If  $\mu \in I(\mathbb{R}^d)$  then its characteristic function  $\hat{\mu}$  has the following Lévy-Khintchine representation  $\hat{\mu}(z) = \exp \{i \langle b, z \rangle - A(z) + \int_{\mathbb{R}^d} g_z(x) \nu(dx)\}$  for  $z \in \mathbb{R}^d$ , where  $b \in \mathbb{R}^d$ ,  $A(z)$  is a nonnegative quadratic form,  $\nu$  is a measure on  $\mathbb{R}^d$  such that  $\nu(\{0\}) = 0$ ,  $g_z(x) = e^{izx} - 1 - izx(1 + |x|^2)^{-1}$  and  $\int_{\mathbb{R}^d} |x|^2(1 + |x|^2)^{-1} \nu(dx) < \infty$ . Moreover  $\gamma$ ,  $A$  and  $\nu$  are uniquely de-

terminated by  $\mu$ . We call the triplet  $(b, A, \nu)$  the Lévy-Khintchine triplet of  $\mu$ . The measure  $\nu$  is called the Lévy measure of  $\mu$ . Let  $S = S^{d-1}$  be the unit sphere in  $\mathbb{R}^d$ , i.e.  $\{x \in \mathbb{R}^d: |x| = 1\}$ , and let  $\mathbb{R}_+, \mathbb{R}_-$  be open intervals  $(0, \infty), (-\infty, 0)$ , respectively. In the sequel of this paper  $\mathbb{R}_+^d$  denotes the set of those  $x \in \mathbb{R}^d$  whose first non-zero coordinate is positive:  $\mathbb{R}_+^d = \bigcup_{i=1}^d \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = \dots = x_{i-1} = 0, x_i > 0\}$ . Then  $\mathbb{R}^d = \mathbb{R}_+^d \cup (-\mathbb{R}_+^d) \cup \{0\}$ . Let  $S_+ = S \cap \mathbb{R}_+^d$ . We denote by  $EB$  the set of all points  $z$  such that  $z = u\xi$ ,  $u \in E$  and  $\xi \in B$ , for  $E \subset \mathbb{R}_- \cup \mathbb{R}_+$  and  $B \subset S_+$ . Given  $C \in \mathcal{C}$  we define  $C$ -invariant and  $C$ -superinvariant measures (see Rajba 1984, 2001). Given  $c \neq 0$  and a  $\sigma$ -finite measure  $\tau$  on  $\mathcal{B}([-\infty, \infty])$ , we say that  $\tau$  is  $c$ -superinvariant if  $\tau \leq T_c^{-1}\tau$ , where  $T_c x = cx$  ( $x \in [-\infty, \infty]$ ),  $T_c \tau(E) = \tau(T_c^{-1}E)$  ( $E \in \mathcal{B}([-\infty, \infty])$ ). We say that  $\tau$  is  $C$ -superinvariant if for any  $c \in C \setminus \{0\}$   $\tau$  is  $c$ -superinvariant. We say that  $\tau$  is  $C$ -invariant on a set  $E_0 \in \mathcal{B}([-\infty, \infty])$  such that  $\tau(E'_0) = 0$ , if  $\tau|_{E_0} = T_c^{-1}\tau|_{E_0}$  for all  $c \in C \setminus \{0\}$ . We say that  $\tau$  is  $C$ -invariant if there exists  $E_0 \in \mathcal{B}([-\infty, \infty])$  such that  $\tau(E'_0) = 0$  and  $\tau$  is  $C$ -invariant on  $E_0$ . In general, we denote by  $\mathcal{B}(T)$  the class of all Borel subsets of  $T$ .

**Remark 2.1.** Notice that using these definitions one can easily see that the class of  $C$ -decomposable under  $ID(\mathbb{R})$  distributions coincides with the class of infinitely divisible distributions for which the Lévy measure is  $C$ -superinvariant (see Rajba (1984, 2001)).

Rajba (2001, p. 283) has given a construction of a measure  $\nu$  on  $\mathbb{R}_- \cup \mathbb{R}_+$  which is exactly  $C$ -superinvariant and it satisfies some additional condition. This result will be useful later on.

**Lemma 2.2.** Let  $C \in \mathcal{C}$ . There exists a non-zero measure  $\nu$  concentrated on  $(-\infty, 0) \cup (0, \infty)$  such that (a)  $\nu$  is  $C$ -superinvariant, (b) there is no  $c \notin C$  for which  $\nu$  is  $c$ -superinvariant, (c)  $\nu(du) \leq \frac{1}{|u|} \chi_{[-1,0) \cup (0,1]}(u) du$ .

Observe that for a measure  $\mu$  in  $ID(\mathbb{R})$  with the Lévy measure  $\nu$  as in the above lemma, we have that  $\mu$  is exactly  $C$ -decomposable under  $ID(\mathbb{R})$ . In the sequel, we will repeatedly make use of the next result on the representation of  $C$ -superinvariant measures which is implicit in Theorem 4.2 of Rajba (2001).

**Proposition 2.3.** Let  $\tau$  be a non-zero  $C$ -superinvariant measure on  $\mathbb{R}_- \cup \mathbb{R}_+$  such that  $0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} h(u) \tau(du) < \infty$ , where  $h(u)$  is a positive real function

on  $\mathbb{R}_- \cup \mathbb{R}_+$ , continuous on  $\mathbb{R}_+$  and such that  $h(u) = h(-u)$  ( $u \neq 0$ ). Then there exists a finite Borel measure  $\gamma$  on  $G_0$  such that

$$\tau = \int_{G_0} \beta \gamma(d\beta),$$

where  $G_0$  is the set of all non-zero  $C$ -invariant measures  $\beta$  concentrated on  $\mathbb{R}_- \cup \mathbb{R}_+$  such that (i)  $\int_{\mathbb{R}_- \cup \mathbb{R}_+} h(u)\beta(du) = 1$ , and (ii) measures  $\beta$  are extreme points, i.e. there are no non-zero  $C$ -invariant measures  $\beta_1, \beta_2$  such that  $\beta = \beta_1 + \beta_2$  and  $\beta_1$  is singular continuous with respect to  $\beta_2$ .

**Remark 2.4.** If  $C = [0, 1]$  then

$$G_0 = \{(A(u))^{-1}x^{-1}\chi_{(0,u)}(x)dx : u > 0\} \cup \{(A(u))^{-1}|x|^{-1}\chi_{(u,0)}(x)dx : u < 0\},$$

where  $A(u) = \int_0^{|u|} h(v)v^{-1}dv$ . Because  $G_0$  and  $\mathbb{R}_- \cup \mathbb{R}_+$  are homeomorphic, we get

$$\tau(dx) = \int_{-\infty}^x \kappa(du) \frac{1}{|x|} \chi_{(-\infty,0)}(x)dx + \int_x^{\infty} \kappa(du) \frac{1}{x} \chi_{(0,\infty)}(x)dx,$$

where  $\kappa(du)$  is a measure on  $\mathbb{R}_- \cup \mathbb{R}_+$  satisfying  $\int_{\mathbb{R}_- \cup \mathbb{R}_+} A(u)\kappa(du) < \infty$ .

From Remark 2.1 and Proposition 2.3 with  $h(u) = u^2(1+u^2)^{-1}$  we obtain a representation of Lévy's measure corresponding to a distribution from the class  $L(0, \mathbb{R}^d, C)$  for  $d = 1$  (see Rajba (2001)). Now we consider the general case of  $d \geq 1$ .

Before we give the theorem on representation, we shall prove a lemma on Lévy measures corresponding to distributions from the classes  $L(0, \mathbb{R}^d, C)$ .

**Lemma 2.5.** Let  $\mu \in ID(\mathbb{R}^d)$ , let  $\nu$  be the Lévy measure of  $\mu$ , and let

$$\nu_B(E) = \nu(EB) \text{ for } E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+), B \in \mathcal{B}(S_+).$$

Then  $\mu \in L(0, \mathbb{R}^d, C)$  if and only if for every  $B \in \mathcal{B}(S_+)$  the measure  $\nu_B$  is  $C$ -superinvariant.

*Proof.* We have, for each  $c \in C \setminus \{0\}$ ,  $\hat{\mu}(z)(\hat{\mu}(cz))^{-1} = \exp\left\{ib_c z - A_c(z) + \int_{\mathbb{R}^d} g_z(x)\nu_c(dx)\right\}$ , where  $b_c \in \mathbb{R}^d$ ,  $A_c(z) = A(z) - A(cz)$ ,  $\nu_c = \nu - T_c\nu$ .



The measure  $\mu$  belongs to the class  $L(0, \mathbb{R}^d, C)$  if and only if for each  $c \in C \setminus \{0\}$ ,  $\hat{\mu}(z)(\hat{\mu}(cz))^{-1}$  is the characteristic function of an infinitely divisible distribution. If an infinitely divisible distribution has the Lévy representation with a signed measure  $\nu$ , then  $\nu$  is nonnegative. Thus  $\mu \in L(0, \mathbb{R}^d, C)$  if and only if  $\nu$  is  $C$ -superinvariant. Hence, if  $\nu$  is  $C$ -superinvariant, then for any fixed  $B \in \mathcal{B}(S_+)$ , we have

$$\nu_B(E) = \nu(EB) \geq \nu(c^{-1}EB) = \nu_B(c^{-1}E) = T_c \nu_B(E),$$

where  $c \in C \setminus \{0\}$  and  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ . Conversely, suppose that  $\nu_B$  is  $C$ -superinvariant holds true. Since for Borel subsets of  $\mathbb{R}^d$  of the form  $EB$ , where  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ ,  $B \in \mathcal{B}(S_+)$ , we have

$$\nu(EB) = \nu_B(E) \geq \nu_B(c^{-1}E) = \nu(c^{-1}EB),$$

this completes the proof.  $\square$

**Theorem 2.6.** *Let  $C \in \mathcal{C}$  and  $\mu \in ID(\mathbb{R}^d)$  with non-zero Lévy measure  $\nu$ . Then  $\mu \in L(0, \mathbb{R}^d, C)$  if and only if  $\nu$  admits representations*

$$\nu(EB) = \int_B \int_E \nu_\xi(du) \lambda(d\xi) = \int_B \int_E \left[ \int_{G_0} \beta \gamma_\xi(d\beta) \right] (du) \lambda(d\xi) \quad (2)$$

for  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ ,  $B \in \mathcal{B}(S_+)$ , where  $\lambda$  is a probability measure on  $S_+$ ,  $G_0$  is the set of extreme  $C$ -invariant measures  $\beta$  satisfying

$$\int_{\mathbb{R}_- \cup \mathbb{R}_+} u^2(1+u^2)^{-1} \beta(du) = 1,$$

and for any fixed  $\xi$ ,  $\nu_\xi(\cdot)$  is a  $C$ -superinvariant measure on  $\mathbb{R}_- \cup \mathbb{R}_+$  satisfying

$$0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} u^2(1+u^2)^{-1} \nu_\xi(du) = b < \infty,$$

$\gamma_\xi(\cdot)$  is a finite Borel measure on  $G_0$  satisfying

$$0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} u^2(1+u^2)^{-1} \left( \int_{G_0} \beta \gamma_\xi(d\beta) \right) (du) = b < \infty,$$

with  $b$  independent of  $\xi$ . For any  $B$  and  $E$ ,  $\nu_\xi(E)$  and  $\gamma_\xi(B)$  are measurable functions of  $\xi$ .

These representations are unique in the sense that if  $\nu \neq 0$  and two pairs  $(\lambda, \gamma_\xi)$  and  $(\tilde{\lambda}, \tilde{\gamma}_\xi)$  (similarly to pairs  $(\lambda, \nu_\xi)$  and  $(\tilde{\lambda}, \tilde{\nu}_\xi)$ ) both satisfy the above conditions, then  $\lambda = \tilde{\lambda}$  and  $\gamma_\xi = \tilde{\gamma}_\xi$  for  $\lambda$ -almost every  $\xi$ .

*Proof.* Assume that  $\mu \in L(0, \mathbb{R}^d, C)$  and  $\nu \neq 0$ . We will use some ideas of the proof of Theorem 3.1 in Sato (1980). Let  $\nu_B(E)$  be as above and let

$$\lambda(B) = b^{-1} \int_{(\mathbb{R}_- \cup \mathbb{R}_+)B} |x|^2 (1 + |x|^2)^{-1} \nu(dx) = b^{-1} \int_{\mathbb{R}_- \cup \mathbb{R}_+} u^2 (1 + u^2)^{-1} \nu_B(du),$$

where  $b$  is a normalizing constant. For each  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ , the measure  $\nu_B(E)$  ( $B \in \mathcal{B}(S_+)$ ) is absolutely continuous with respect to  $\lambda$ .

Hence for each  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$  there exists a non-negative measurable function  $\nu_\xi(E)$  of  $\xi$ , such that  $\nu_B(E) = \int_B \nu_\xi(E) \lambda(d\xi)$  ( $B \in \mathcal{B}(S_+)$ ). By Lemma 2.5, for any  $B \in \mathcal{B}(S_+)$  the measures  $\nu_B$  are  $C$ -superinvariant, so are the measures  $\nu_\xi$  for  $\lambda$ -almost every  $\xi$ . Furthermore, we have  $\int_{\mathbb{R}_- \cup \mathbb{R}_+} u^2 (1 + u^2)^{-1} \nu_\xi(du) = b$  for  $\lambda$ -almost every  $\xi$ . This proves the first representation of (2). Let us put in Proposition (2.3)  $\tau = \nu_\xi$  and  $h(u) = u^2 (1 + u^2)^{-1}$ . The assumption of Proposition 2.3 hold true, hence we obtain the second representation of (2). It is not difficult to prove the uniqueness of the representations (2). Since the converse is obvious, this completes the proof.  $\square$

From Theorem 2.6 and Remark 2.4 with  $h(u) = u^2 (1 + u^2)^{-1}$ , setting  $\kappa_\xi(du_1) = (A_1(u_1))^{-1} \gamma_\xi(du_1)$  ( $u_1 \neq 0$ ) and  $k_\xi(u) = \int_{-\infty}^u \kappa_\xi(du_1) \chi_{(-\infty, 0)}(u) + \int_u^\infty \kappa_\xi(du_1) \chi_{(0, \infty)}(u)$ , where  $A_1(u_1) = \frac{1}{2} \ln(1 + u_1^2)$ , we obtain representations of the Lévy measure of a distribution from the class  $\mathbb{L}_0(\mathbb{R}^d)$ .

**Theorem 2.7.** *Let  $\mu \in ID(\mathbb{R}^d)$  with non-zero Lévy measure  $\nu$ . Then  $\mu \in \mathbb{L}_0(\mathbb{R}^d)$  if and only if  $\nu$  is represented as*

$$\begin{aligned} \nu(EB) &= \int_B \int_E \left( \int_{-\infty}^u \kappa_\xi(du_1) \chi_{(-\infty, 0)}(u) + \int_u^\infty \kappa_\xi(du_1) \chi_{(0, \infty)}(u) \right) \frac{1}{|u|} du \lambda(d\xi) \\ &= \int_B \int_E k_\xi(u) |u|^{-1} du \lambda(d\xi), \end{aligned} \quad (3)$$

for  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ ,  $B \in \mathcal{B}(S_+)$ , where  $\lambda$  is a probability measure on  $S_+$ , and for any fixed  $\xi$ ,  $\kappa_\xi(\cdot)$  is a measure on  $\mathbb{R}_- \cup \mathbb{R}_+$  satisfying

$$0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} \frac{1}{2} \ln(1 + u^2) \kappa_\xi(du_1) = b < \infty,$$

$k_\xi(u)$  is a non-negative function, non-decreasing left-continuous for  $u < 0$ , non-increasing right-continuous for  $u > 0$ , satisfying

$$0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} k_\xi(u) |u| (1 + u^2)^{-1} du = b < \infty,$$

such that  $b$  is independent of  $\xi$ .  $\kappa_\xi$  and  $k_\xi$  are measurable in  $\xi$ . These representations are unique.

The second representation of (3) is equivalent to the Sato (1980, p. 213) representation.

### 3. The class $L(m, \mathbb{R}^d, C)$

Now let us turn to distributions which are  $(C, m)$ -semi-selfdecomposable, for  $m \geq 1$ . First we define a sequence of functions  $A_m(x)$  ( $x \neq 0$ ) as follows:

$$A_0(u_0) = u_0^2 (1 + u_0^2)^{-1},$$

$$A_m(u_m) = \int_0^{|u_m|} A_{m-1}(u_{m-1}) u_{m-1}^{-1} du_{m-1} \quad (m \geq 1).$$

**Theorem 3.1.** *Let  $m \geq 1$ ,  $C \in \mathcal{C}$  and  $\mu \in \mathbb{L}_{m-1}(\mathbb{R}^d)$  with non-zero Lévy measure  $\nu$ . Then*

(i)  $\nu$  is of the form

$$\begin{aligned} \nu(EB) = & \int_B \int_E \left[ \int_{-\infty}^{u_0} \dots \int_{-\infty}^{u_{m-1}} \kappa_\xi^{(m-1)}(du_m) |u_{m-1}|^{-1} du_{m-1} \dots |u_0|^{-1} \right. \\ & \chi_{(-\infty, 0)}(u_0) du_0 + \\ & \left. \int_{u_0}^{\infty} \dots \int_{u_{m-1}}^{\infty} \kappa_\xi^{(m-1)}(du_m) u_{m-1}^{-1} \dots u_0^{-1} \chi_{(0, \infty)}(u_0) du_0 \right] \lambda(d\xi), \end{aligned}$$

for  $E \in \mathcal{B}(\mathbb{R}_- \cup \mathbb{R}_+)$ ,  $B \in \mathcal{B}(S_+)$ , where  $\lambda$  is a probability measure on  $S_+$ ,  $\kappa_\xi^{(m-1)}$  is a measure on  $\mathbb{R}_- \cup \mathbb{R}_+$ , measurable in  $\xi$  and satisfying  $0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} A_m(x) \kappa_\xi^{(m-1)}(dx) = b < \infty$ , for  $\lambda$ -almost every  $\xi$ , such that  $b$  is independent of  $\xi$ .

(ii)  $\mu \in L(m, \mathbb{R}^d, C)$  if and only if the measure  $\kappa_\xi^{(m-1)}$  is  $C$ -superinvariant for  $\lambda$ -almost every  $\xi$ . All these measures are of the form

$$\kappa_\xi^{(m-1)} = \int_{G_0^{(m)}} \beta \gamma_\xi^{(m)}(d\beta),$$

where  $G_0^{(m)}$  is the class of all extreme  $C$ -invariant measures  $\beta$  concentrated on  $\mathbb{R}_- \cup \mathbb{R}_+$  and satisfying  $\int_{\mathbb{R}_- \cup \mathbb{R}_+} A_m(x) \beta(dx) = 1$ ,  $\gamma_\xi^{(m)}$  is

a finite Borel measure on  $G_0^{(m)}$ , measurable in  $\xi$  and satisfying  $0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} A_m(x) (\int_{G_0^{(m)}} \beta \gamma_\xi^{(m)}(d\beta))(dx) = b < \infty$ , such that  $b$  is independent of  $\xi$ . These representations are unique.

*Proof.* (i) We use induction with respect to  $m$ . For  $m = 1$  the representation (i) is given in Theorem 2.7. Assume that the statement (i) of Theorem 3.1 holds true for  $m$ . Let  $\mu \in \mathbb{L}_{m-1}(\mathbb{R}^d)$ , with the pair  $(\lambda, \kappa_\xi^{(m-1)})$  in the representation (i) and satisfying  $0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} A_m(u_m) \kappa_\xi^{(m-1)}(du_m) = b < \infty$ . By definition,  $\mu \in L(m, \mathbb{R}^d, C)$  if and only if for every  $c \in C \setminus \{0\}$  there exists  $\mu_c \in \mathbb{L}_{m-1}(\mathbb{R}^d)$  such that  $\mu = T_c \mu * \mu_c$ . Since the Lévy measure corresponding to  $\mu_c$  can be described in the form given by (i) with  $\kappa_\xi^{(m-1)} - T_c^{-1} \kappa_\xi^{(m-1)}$  in place of  $\kappa_\xi^{(m-1)}$ , we conclude that  $\kappa_\xi^{(m-1)}$  is  $C$ -superinvariant for  $\lambda$  a.e.  $\xi$ . By Proposition 2.3 with  $h(u) = A_m(u)$  we obtain the representation  $\kappa_\xi^{(m-1)} = \int_{G_0^{(m)}} \beta \gamma_\xi^{(m)}(d\beta)$ , where  $G_0^{(m)}, \gamma_\xi^{(m)}$  are as above. Putting  $C = [0, 1]$ , by Remark 2.4 we obtain that the representation given in (i) holds true for  $m + 1$ . This completes the proof of the part (i) of the theorem. (ii) To prove the part (ii) let us take  $\mu \in \mathbb{L}_{m-1}(\mathbb{R}^d)$  with the pair  $(\lambda, \kappa_\xi^{(m-1)})$  in the representation (i). Then, using results from the proof of part (i) of the theorem, we obtain the proof of (ii).  $\square$

**Remark 3.2.** *It is not difficult to prove that*

$$A_{m+1}(v) = \frac{1}{m!} \int_0^{|v|} \frac{u}{1+u^2} \left( \log \frac{|v|}{u} \right)^m du,$$

where  $m = 0, 1, 2, \dots$ . Moreover, we have (see Sato (1980, p. 219)) that

$$A_{m+1}(v) \sim \frac{1}{(m+1)!} (\log v)^{m+1} \quad \text{as } v \rightarrow \infty,$$

and

$$A_{m+1}(v) \sim \frac{1}{m!} \int_0^v u \left( \log \frac{v}{u} \right)^m du = \frac{1}{2^{m+1}} v^2 \quad \text{as } v \rightarrow 0.$$

**Remark 3.3.** *After changing the order of integration in the integral in the representation given in (i) of Theorem 3.1, we can write (see Sato (1980, p. 220)).*

$$\nu(EB) = \int_B \int_E \int_{\mathbb{R}_- \cup \mathbb{R}_+} \frac{m+1}{|u|} \left[ \log \left( \frac{v}{u} \right)_+ \right]_+^m \Gamma_\xi^{(m)}(dv) du \lambda(d\xi),$$

where  $x_+ = x\chi_{(0,\infty)}(x)$  ( $x \in \mathbb{R}$ ),  $\Gamma_\xi^{(m)}(\cdot)$  is a measure on  $\mathbb{R}_- \cup \mathbb{R}_+$  satisfying

$$0 < \int_{\mathbb{R}_- \cup \mathbb{R}_+} \int_{\mathbb{R}_- \cup \mathbb{R}_+} (m+1) \frac{|u|}{1+u^2} \left[ \log \left( \frac{v}{u} \right)_+ \right]_+^m du \Gamma_\xi^{(m)}(dv) = b < \infty,$$

with  $b$  independent of  $\xi$ .  $\Gamma_\xi^{(m)}$  is measurable in  $\xi$ .

**Remark 3.4.** We will call  $\kappa_\xi^{(m)}$  ( $\xi \in S_+$ ) the  $m$ -spectral measures of a distribution  $\mu \in \mathbb{L}_m(\mathbb{R}^d)$ . Note that the family  $\{\kappa_\xi^{(m)}\}_\xi$  can be helpful to study the  $C$ -decomposability under the class  $\mathbb{L}_m(\mathbb{R}^d)$  in the same way as the family  $\{\nu_\xi\}_\xi$  is to study the  $C$ -decomposability under the class  $ID(\mathbb{R}^d)$  (see Theorem 2.6). By Theorem 3.1 we can see that a measure  $\mu \in \mathbb{L}(\mathbb{R}^d)$  with the pair  $(\lambda, \kappa_\xi^{(m)})$ , is  $C$ -decomposable under  $\mathbb{L}_m(\mathbb{R}^d)$  if and only if the  $\kappa_\xi^{(m)}$  is  $C$ -superinvariant for  $\lambda$  a.e.  $\xi$ .

**Examples.** Let  $\mu \in L(m, \mathbb{R}^d, C)$ , ( $m = 0, 1, 2, \dots$ ) with the pair  $(\lambda, \kappa_\xi^{(m-1)})$ .

a) Let  $C = [q, 1]$ , where  $-1 \leq q < 0$ . Then  $G_0^{(m)} = \{A_{(x,y)}^{(m)} \frac{1}{|u_m|} \chi_{[y,0]}(u_m) du_m + A_{(x,y)}^{(m)} \frac{1}{u_m} \chi_{(0,x]} du_m : (x, y) \in D_q\}$  and  $\kappa_\xi^{(m-1)}(du_m) = \int_{-\infty}^{u_m} \int_{y/q}^{y/q} A_{(x,y)}^{(m)} \gamma_\xi^{(m)y}(dx) \gamma_{\xi,2}^{(m)}(dy) \frac{1}{|u_m|} \chi_{(-\infty,0)}(u_m) du_m + \int_{u_m}^{\infty} \int_{x/q}^{x/q} A_{(x,y)}^{(m)} \gamma_\xi^{(m)x}(dy) \gamma_{\xi,1}^{(m)}(dx) \frac{1}{|u_m|} \chi_{(0,\infty)}(u_m) du_m$ , where  $A_{(x,y)}^{(m)} = (A_{m+1}(x) + A_{m+1}(y))^{-1}$ ,  $D_q = \{(x, y) : x > 0, x/q \leq y \leq xq\}$ ,  $\gamma_\xi^{(m)}(d(x, y)) = \gamma_\xi^{(m)y}(dx) \gamma_{\xi,2}^{(m)}(dy) = \gamma_\xi^{(m)x}(dy) \gamma_{\xi,1}^{(m)}(dx)$  and  $\gamma_\xi^{(m)}(d(x, y))$  is a finite measure concentrated on the set  $D_q$ . Let  $x_0 > 0$ . Putting  $\gamma_\xi^{(m)}(d(x, y)) = \delta_{x_0}(x) \delta_{x_0 q}(y)$ , where  $\delta_x$  ( $x \in \mathbb{R}^d$ ) denotes the probability measure concentrated at a point  $x$ , we have  $D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d)) = [q, 1]$ .

b) Let  $C = \{c^k\}_{k=0}^m \cup \{0\}$ , where  $0 < |c| < 1$ . Then  $G_0^{(m)} = \{[\sum_{k=0}^{\infty} A_m(xc^k)]^{-1} \sum_{k=0}^{\infty} \delta_{xc^k} : x \in \mathbb{R}_- \cup \mathbb{R}_+\}$  and  $\kappa_\xi(du_m) = \int_{\mathbb{R}_- \cup \mathbb{R}_+} [\sum_{k=0}^{\infty} A_m(xc^k)]^{-1} \sum_{k=0}^{\infty} \delta_{xc^k}(u_m) \gamma_\xi(dx)$ , where  $\gamma_\xi$  is a finite measure on  $\mathbb{R}_- \cup \mathbb{R}_+$ .

c) Let  $C = \{c_0^k\}_{k=3}^{\infty} \cup \{0, 1\}$ , where  $0 < |c_0| < 1$ . Put  $C_1 = C$ ,  $C_2 = \{c_0^k\}_{k=2}^{\infty} \cup \{0, 1\}$ ,  $C_3 = \{c_0^k\}_{k=0}^{\infty} \cup \{0\}$ . Then  $G_0^{(m)} = \bigcup_{i=1}^3 \{[\sum_{c \in C_i \setminus \{0\}} A_m(xc)]^{-1} \sum_{c \in C_i \setminus \{0\}} \delta_{xc}(u_m) : x \in \mathbb{R}_- \cup \mathbb{R}_+\}$  and  $\kappa_\xi^{(m)}(du_m) = \int_{\mathbb{R}_- \cup \mathbb{R}_+} \int_{\{1,2,3\}} [\sum_{c \in C_i \setminus \{0\}} A_m(xc)]^{-1} \sum_{c \in C_i \setminus \{0\}} \delta_{xc}(u_m) \gamma_\xi(d(x, i))$ , where  $\gamma_\xi$  is a finite measure on  $(\mathbb{R}_- \cup \mathbb{R}_+) \times \{1, 2, 3\}$ .

The following theorem gives a characterization of those semigroups which are decomposability semigroups  $D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d))$ :

**Theorem 3.5.** *Let  $C \in \mathcal{C}$  and  $m \geq 0$ . Then there exists a distribution  $\mu \in L(m, \mathbb{R}^d, C)$  such that*

$$D(\mu, \mathbb{L}_{m-1}(\mathbb{R}^d)) = C.$$

*Proof.* Let  $\mu \in \mathbb{L}_{m-1}(\mathbb{R}^d)$  with the pair  $(\lambda, \kappa_\xi^{(m-1)})$  in the representation (3.1). Let  $\lambda$  be a non-zero measure. Lemma 2.2 enables us to construct a family of measures  $\{\kappa_\xi^{(m-1)}\}_\xi$  (measurable in  $\xi$ ) such that for any  $\xi$ ,  $\kappa_\xi^{(m-1)} \geq T_c \kappa_\xi^{(m-1)}$  for each  $c \in C \setminus \{0\}$ , and  $\kappa_\xi^{(m-1)} \not\geq T_c \kappa_\xi^{(m-1)}$  for each  $c \in (-1, 1) \setminus C$ . By Remark 3.4 this proves the theorem.  $\square$

**Remark 3.6.** *By definition, every measure  $\mu \in L(m, \mathbb{R}^d, C)$  is  $C$ -decomposable under  $\mathbb{L}_{m-1}(\mathbb{R}^d)$ . Theorem 3.1 gives a representation of measures which are  $C$ -decomposable under  $\mathbb{L}_{m-1}(\mathbb{R}^d)$ . Moreover, from Theorem 3.5 we can see that for every  $C \in \mathcal{C}$  there exists  $\mu \in L(m, \mathbb{R}^d, C)$  such that  $\mu$  is exactly  $C$ -decomposable under  $\mathbb{L}_{m-1}(\mathbb{R}^d)$ ,*

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