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Control of the bilinear Schrödinger equation for fully coupling potentials

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Abstract: We present a general result of approximate controllability for the bilinear Schrödinger equation (with wave function varying in the unit sphere of an infinite dimensional Hilbert space), under the hypothesis that the Schrödinger operator has discrete spectrum and that the control potential couples all eigenstates. The control method is based on a tracking procedure for the Galerkin approximations, lifted in $SU(n)$. The method allows to estimate the L^1 norm of the control laws achieving controllability.

Keywords: Quantum Control, Control of the Schrödinger equation, Bilinear control systems on Lie groups

1. INTRODUCTION

In this paper we are concerned with the controllability problem for the Schrödinger equation

$$i\frac{d\psi}{dt} = (H_0 + u(t)H_1)\psi. \tag{1}$$

Here ψ belongs to the Hilbert sphere of a complex Hilbert space $\mathcal{H}$ and H_0, H_1 are self-adjoint operators on $\mathcal{H}$. The control u is scalar-valued and represents the action of an external field. The most studied case is the one in which $H_0 = -\Delta + V(x)$, $H_1 = W(x)$, where x belongs to a domain $D \subset \mathbf{R}^n$ with suitable boundary conditions and V, W are functions (identified with the corresponding multiplicative operators) characterizing respectively the autonomous dynamics and the coupling of the system with the control u. However, equation (1) can be used to describe more general controlled dynamics. For instance, a quantum particle on a Riemannian manifold subject to an external field (in this case Δ is the Laplace–Beltrami operator) or a two-level ion trapped in a harmonic potential (the so-called Eberly and Law model Law and Eberly [1996], Bloch et al. [2010]). In the last case, as in many others relevant physical situations, the operator H_0 cannot be written as the sum of a Laplacian plus a potential.

The controllability problem consists in establishing whether, for every pair of states ψ_0 and ψ_1, there exist a control $u(\cdot)$ and a time T such that the solution of (1) with initial condition $\psi(0) = \psi_0$ satisfies $\psi(T) = \psi_1$. Unfortunately the answer to this problem is negative when $\mathcal{H}$ is infinite dimensional. Indeed, Ball, Marsden, and Slemrod proved in Ball et al. [1982] a result which implies (see Turinici [2000]) that equation (1) is not controllable in (the Hilbert sphere of) $\mathcal{H}$. Moreover, they proved that in the case in which H_0 is the sum of the Laplacian and a potential in a domain D of $\mathbf{R}^n$, equation (1) is neither controllable in the Hilbert sphere $\mathbf{S}$ of $L^2(D, \mathbf{C})$ nor in the natural functional space where the problem is formulated, namely the intersection of $\mathbf{S}$ with the Sobolev spaces $H^2(D, \mathbf{C})$ and $H^1_0(D, \mathbf{C})$. Hence one has to look for weaker controllability properties as, for instance, approximate controllability or controllability between the eigenstates of H_0 (which are the most relevant physical states).

However, in certain cases one can describe quite precisely the set of states that can be connected by admissible paths. Indeed in Beauchard [2005], Beauchard and Coron [2006] the authors prove that, in the case in which H_0 is the Laplacian on the interval $[-1, 1]$, with Dirichlet boundary conditions, and H_1 is the operator of multiplication by x, the system is exactly controllable near the eigenstates in $H^7(D, \mathbf{C}) \cap \mathbf{S}$ (with suitable boundary conditions). This result was then refined in Beauchard and Laurent [2010], where the authors proved that the exact controllability holds in $H^3(D, \mathbf{C}) \cap \mathbf{S}$, for a large class of control potentials (see also Nersesyan and Nersisyan [2010]).

In dimension larger than one (for H_0 equal to the sum of the Laplacian and a potential) or for more general situations, the exact description of the reachable set appears to be more difficult and at the moment only approximate controllability results are available.

In Chambrion et al. [2009] an approximate controllability result for (1) was proved via finite dimensional geometric control techniques applied to the Galerkin approximations. The main hypothesis is that the spectrum of H_0 is discrete and without rational resonances, which means that the gaps between the eigenvalues of H_0 are $\mathbf{Q}$-linearly independent. Another crucial hypothesis appearing naturally is that the operator H_1 couples all eigenvectors of H_0.

The main advantages of that result with respect to those previously known are that: **i)** it does not need H_0 to be

of the form $-\Delta + V$; **ii)** it can be applied to the case in which H_1 is an unbounded operator; **iii)** the control is a bounded function with arbitrarily small bound; **iv)** it allows to prove controllability for density matrices and it can be generalized to prove approximate controllability results for a system of Schrödinger equations controlled by the same control (see Chambrion [2009]).

Results similar to those presented in Chambrion et al. [2009] have been obtained, with different techniques, in Nersesyan [2010] (see also Ito and Kunisch [2007], Beauchard and Nersesyan [2010], Nersesyan [2009], Nersesyan and Nersisyan [2010]). They require less restrictive hypotheses on the spectrum of H_0 (which is still assumed to be discrete) but they do not admit H_1 unbounded and do not work for the density matrices. However, it should be noticed that Nersesyan [2010] proves approximate controllability with respect to some Sobolev norm H^s, while the results given in Chambrion et al. [2009] permit to get approximate controllability in the weaker norm L^2. As it happens for the results in Chambrion et al. [2009], the sufficient conditions for controllability obtained in Nersesyan [2010] are generic.

In this paper we prove the approximate controllability of (1) under less restrictive hypotheses on the spectrum of H_0 than those in Chambrion et al. [2009]. More precisely, assume that H_0 has discrete spectrum $(\lambda_k)_{k\in\mathbf{N}}$, that the operator H_1 couples all eigenvectors of H_0, and that $|\lambda_j - \lambda_k| \neq |\lambda_l - \lambda_m|$ for every $(j,k),(l,m) \in \mathbf{N}^2 \setminus \{(p,p) : p \in \mathbf{N}\}$ such that $\{j,k\} \neq \{l,m\}$, then the system is approximately controllable.

The idea of the proof is the following. We recover approximate controllability for the system defined on an infinite-dimensional Hilbert space through fine controllability properties of the finite-dimensional Galerkin approximations, which allow us to pass to the limit as $N \to \infty$. More precisely, we prove that, for $n, N \in \mathbf{N}$ with $N \gg n \gg 1$, for given initial and final conditions ψ_0, ψ_1 in the Hilbert sphere of $\mathcal{H}$ which are linear combinations of the first n eigenvectors of H_0, it is possible to steer ψ_0 to ψ_1 in the Galerkin approximation of order N in such a way that the projection on the last $N-n$ components has arbitrarily small norm along the trajectory. This kind of controllability for the Galerkin approximation of order N is proved in two steps: firstly, thanks to a time-dependent change of variables we transform the system in a driftless one, nonlinear in the control, and we prove the result up to phases. The change of variables was already introduced in Agrachev and Chambrion [2006], Chambrion et al. [2009]; the technical novelty of this paper is the convexification analysis for the transformed system, which allows to conclude the controllability with less restrictive non-resonance hypotheses. Secondly, the control of phases is obtained via a classical method, using as pivot an eigenstate of H_0 and exploiting the time-reversibility properties of the Schrödinger equation.

2. FRAMEWORK AND MAIN RESULTS

2.1 Settings and notations

As in Chambrion et al. [2009], we use an abstract framework instead of a presentation in terms of partial differential equations. The advantage is that this presentation is very versatile and applies without modification for Schrödinger equation on a (possibly not bounded) domain of $\mathbf{R}^n$ or on a manifold as $\mathbb{S}^1$. Hereafter $\mathbf{N}$ denotes the set of strictly positive integers.

Definition 1. Let $\mathcal{H}$ be an Hilbert space with scalar product $\langle\cdot,\cdot\rangle$, (A,B) be a pair of (possibly unbounded) linear operators of $\mathcal{H}$, with domains $D(A)$ and $D(B)$, and $\delta > 0$. Let us introduce the formal controlled equation

$$\frac{d\psi}{dt}(t) = (A + u(t)B)\psi(t), \quad u(t) \in [0,\delta]. \tag{2}$$

We say that (A,B,Φ,δ) satisfies $(\mathfrak{A})$ if the following assumptions are verified:

- $(\mathfrak{A}1)$ $\Phi = (\phi_k)_{k\in\mathbf{N}}$ is an Hilbert basis of $\mathcal{H}$ made of eigenvectors of A associated with the family of eigenvalues $(i\lambda_k)_{k\in\mathbf{N}}$;
- $(\mathfrak{A}2)$ $\phi_k \in D(B)$ for every $k \in \mathbf{N}$;
- $(\mathfrak{A}3)$ $A + uB : \mathrm{span}\{\phi_k \mid k \in \mathbf{N}\} \to \mathcal{H}$ is essentially skew-adjoint for every $u \in [0,\delta]$;
- $(\mathfrak{A}4)$ $\langle \phi_j, B\phi_k\rangle \neq 0$ for every $j \neq k$, $j,k \in \mathbf{N}$;
- $(\mathfrak{A}5)$ $|\lambda_j - \lambda_k| \neq |\lambda_l - \lambda_m|$ for every $(j,k),(l,m) \in \mathbf{N}^2 \setminus \{(p,p) : p \in \mathbf{N}\}$ such that $\{j,k\} \neq \{l,m\}$.

A crucial consequence of assumption $(\mathfrak{A}3)$ is that, for every constant u in $[0,\delta]$, $A + uB$ generates a group of unitary transformations $e^{t(A+uB)} : \mathcal{H} \to \mathcal{H}$. The unit sphere of $\mathcal{H}$ is invariant for all these transformations.

Definition 2. Let (A,B,Φ,δ) satisfy $(\mathfrak{A})$. For any piecewise constant control function $u : [0,T] \to [0,\delta]$, the *solution* of (2) with initial condition $\psi_0 \in \mathcal{H}$ is

$$\begin{aligned}\psi(t) =& e^{(t-\sum_{l=1}^{j-1} t_l)(A+u_jB)} \circ e^{t_{j-1}(A+u_{j-1}B)} \circ \\ & \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0),\end{aligned} \tag{3}$$

where $\sum_{l=1}^{j-1} t_l \leq t < \sum_{l=1}^{j} t_l$ and $u(\tau) = u_j$ if $\sum_{l=1}^{j-1} t_l \leq \tau < \sum_{l=1}^{j} t_l$.

Notice that a solution $\psi(\cdot)$ of (2) as in (3) satisfies, for every $n \in \mathbf{N}$ and almost every $t \in [0,T]$ the differential equation

$$\frac{d}{dt}\langle \phi_n, \psi(t)\rangle = -\langle (A + u(t)B)\phi_n, \psi(t)\rangle. \tag{4}$$

As already recalled, exact controllability is hopeless in general.

Definition 3. Let (A,B,Φ,δ) satisfy $(\mathfrak{A})$. We say that (2) is *approximatly controllable* if for every ψ_0, ψ_1 in the unit sphere of $\mathcal{H}$ and every $\varepsilon > 0$ there exist $k \in \mathbf{N}$, $t_1,\ldots,t_k > 0$, and $u_1,\ldots,u_k \in U$ such that

$$\|\psi_1 - e^{t_k(A+u_kB)} \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0)\| < \varepsilon.$$

2.2 Main results

Theorem 2.1. Let (A,B,Φ,δ) satisfy $(\mathfrak{A})$. Then (2) is approximately controllable.

Theorem 2.2. Let (A,B,Φ,δ) satisfy $(\mathfrak{A})$. For every $\varepsilon > 0$ and for every $j,k \in \mathbf{N}$, $j \neq k$, there exist a piecewise constant control $u : [0,T_u] \to [0,\delta]$ and $\theta \in \mathbb{S}^1$ such that the trajectory $\psi(t)$ of (2) with initial condition ϕ_j and corresponding to the control u satisfies $\|\psi(T_u) - e^{i\theta}\phi_k\| < \varepsilon$ and

$$\|u\|_{L^1} \leq \frac{5\pi}{4|\langle \phi_j, B\phi_k\rangle|}.$$

3. CONVEXIFICATION PROCEDURE

3.1 Time-reparametrization

We denote by PC the set of piecewise constant functions $u : [0,\infty) \to [0,\infty)$ such that there exist $u_1, \dots, u_p > 0$ and $0 = t_1 < \dots < t_{p+1} = T_u$ for which

$$u : t \mapsto \sum_{j=1}^{p} u_j \chi_{[t_j, t_{j+1})}(t).$$

Let us identify $u = \sum_{j=1}^{p} u_j \chi_{[t_j,t_{j+1})}$ with the finite sequence $(u_j, \tau_j)_{1\le j\le p}$ where $\tau_j = t_{j+1} - t_j$ for every $1 \le j \le p$.

We define the map

$$\begin{aligned} \mathcal{P} : PC &\to PC \\ (u_j, \tau_j)_{1\le j\le p} &\mapsto \left(\frac{1}{u_j}, u_j\tau_j\right), \end{aligned}$$

which satisfies the following easily verifiable properties.

Proposition 4. For every $u \in PC$, $\mathcal{P} \circ \mathcal{P}(u) = u$ and $\|\mathcal{P}(u)\|_{L^1} = \sum_{i=1}^{p} \tau_j$.

Assume that (A, B, Φ) satisfies $(\mathfrak{A})$. In analogy with Definition 2, we define, for every $u = \sum_{j=1}^{p} u_j \chi_{[t_j,t_{j+1})} \in PC$ such that $u(t) \in [0, \delta]$ for every $t \ge 0$, the solution of

$$\frac{d\psi}{dt}(t) = (u(t)A + B)\psi(t), \tag{5}$$

with initial condition $\psi_0 \in \mathcal{H}$ as

$$\psi(t) = e^{(t-t_l)(u_l A+B)} \circ \dots \circ e^{t_1(u_1 A+B)}(\psi_0)\,,$$

where $t_l \le t \le t_{l+1}$.

System (5) is the time reparametrization of system (2) induced by the transformation $\mathcal{P}$, as stated in the following proposition.

Proposition 5. Let $u = (u_j, \tau_j)_{1\le j\le p}$ belong to PC and ψ_0 be a point of $\mathcal{H}$. Let ψ be the solution of (2) with control u and initial condition ψ_0, and $\widetilde{\psi}$ be the solution of (5) with control $\mathcal{P}(u)$ and initial condition ψ_0. Then $\psi(T_u) = \widetilde{\psi}(\|u\|_{L^1})$.

Proof. It is enough to remark that, if $u \ne 0$, for every $t \in [0,\infty)$, $e^{t(A+uB)} = e^{tu\left(\frac{1}{u}A+B\right)}$.

Remark 6. As a consequence of Proposition 5 it is equivalent to prove controllability for (2) with control $u \in [0, \delta]$ and to prove controllability for system (5) with control $u \in [1/\delta, \infty)$.

3.2 Convexification

For every positive integer N let the matrices $A^{(N)} = \mathrm{diag}(i\lambda_1, \dots, i\lambda_N)$ and

$$B^{(N)} = (\langle B\phi_j, \phi_k\rangle)_{j,k=1}^{N} =: (b_{jk})_{j,k=1}^{N}\,,$$

be the Galerkyn approximations at order N of A and B, respectively. Let $t \mapsto \psi(t)$ be a solution of

$$\dot{\psi} = (uA^{(N)} + B^{(N)})\psi\,,$$

corresponding to a control function u and consider $v(t) = \int_0^t u(\tau)d\tau$. Denote by $\mathrm{d}(B)$ the diagonal of $B^{(N)}$ and let $\hat{B}^{(N)} = B^{(N)} - \mathrm{d}(B)$. Then $y : t \mapsto e^{-v(t)A^{(N)} - t\mathrm{d}(B)}\psi(t)$, is a solution of

$$\dot{y}(t) = e^{-v(t)A^{(N)} - t\mathrm{d}(B)}\hat{B}^{(N)} e^{v(t)A^{(N)} + t\mathrm{d}(B)} y(t). \tag{Θ_N}$$

Let us set $\vartheta_N(t, v) = e^{-vA^{(N)} - t\mathrm{d}(B)}\hat{B}^{(N)} e^{vA^{(N)} + t\mathrm{d}(B)}$.

Lemma 7. Let K be a positive integer and $\gamma_1, \dots, \gamma_K \in \mathbf{R} \setminus \{0\}$ be such that $|\gamma_1| \ne |\gamma_j|$ for $j = 2, \dots, K$. Let

$$\varphi(t) = (e^{it\gamma_1}, \dots, e^{it\gamma_K}).$$

Then, for every $t_0 \in \mathbf{R}$, we have

$$\overline{\mathrm{conv}\,\varphi([t_0, \infty))} \supseteq \nu\mathbb{S}^1 \times \{(0, \dots, 0)\}\,,$$

where $\nu = \prod_{k=2}^{\infty} \cos\left(\frac{\pi}{2k}\right) > 0$. Moreover, for every $R > 0$ and $\xi \in \mathbb{S}^1$ there exists a sequence $(t_k)_{k\in\mathbf{N}}$ such that $t_{k+1} - t_k > R$ and

$$\lim_{h\to\infty} \frac{1}{h} \sum_{k=1}^{h} \varphi(t_k) = (\nu\xi, 0, \dots, 0)\,.$$

Proof. Since

$$\varphi(t - t_0) = (e^{-it_0\gamma_1} e^{it\gamma_1}, \dots, e^{-it_0\gamma_K} e^{it\gamma_K}), \tag{6}$$

it is enough to prove the lemma for $t_0 = 0$. We can suppose that $|\gamma_1| = 1$ and, up to a reordering of the indexes, that there exist n and $\tilde{n}$ such that $1 \le n \le \tilde{n} \le K$, $|\gamma_i| \ne |\gamma_j|$ for every $i, j \in \{1, \dots, n\}$, $\gamma_2, \dots, \gamma_{\tilde{n}} \in \mathbf{Z}$, $\gamma_{\tilde{n}+1}, \dots, \gamma_K \in \mathbf{R} \setminus \mathbf{Z}$, and $\{|\gamma_{n+1}|, \dots, |\gamma_{\tilde{n}}|\} \subset \{|\gamma_2|, \dots, |\gamma_n|\}$.

Consider the 2^{n-1} real numbers defined as follows: let

$$\bar{t}_1 = 0,$$

and for $k \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, 2^{k-1}\}$,

$$\bar{t}_{2^{k-1}+j} = \bar{t}_j + \frac{\pi}{|\gamma_{k+1}|}\,.$$

Up to a reordering of the $\bar{t}_j$, we can suppose that $0 = \bar{t}_1 < \bar{t}_2 < \dots < \bar{t}_{2^{n-1}}$. Take an integer r larger than $R/2\pi$, then set $t_j = \bar{t}_j + 2\pi r(j-1)$, in such a way that $t_k - t_{k-1} > R$ for every $k = 2, \dots, 2^{n-1}$.

Now consider the arithmetic mean of the l-th (complex) coordinates of $\varphi(t_1), \dots, \varphi(t_{2^{n-1}})$. We show that this quantity is zero for $l = 2, \dots, \tilde{n}$. Indeed, from the definition of t_j, we have

$$\sum_{j=1}^{2^{n-1}} e^{it_j\gamma_l} = \prod_{k=1}^{n-1} \left(1 + e^{i\pi\gamma_l/|\gamma_{k+1}|}\right)\,,$$

which is zero since so is the k-th factor when $|\gamma_l| = |\gamma_{k+1}|$.

On the other hand, the arithmetic mean of the first coordinate is uniformly bounded away from zero. Indeed

$$\begin{aligned} \left|\frac{1}{2^{n-1}} \sum_{j=1}^{2^{n-1}} e^{i\gamma_1 t_j}\right| &= \prod_{k=1}^{n-1} \left|\frac{1 + e^{i\pi\gamma_1/|\gamma_{k+1}|}}{2}\right| \\ &= \prod_{k=2}^{n} \cos\left(\frac{\pi}{2|\gamma_k|}\right) \\ &= \exp\left(\sum_{k=2}^{n} \log\left(\cos\left(\frac{\pi}{2|\gamma_k|}\right)\right)\right) \\ &\ge \exp\left(\sum_{k=2}^{\infty} \log\left(\cos\left(\frac{\pi}{2k}\right)\right)\right) = \nu\,. \end{aligned} \tag{7}$$

Since $\log\left(\cos\left(\frac{\pi}{2k}\right)\right) \sim -\frac{\pi^2}{8k^2}$ as k tends to infinity, then the sum $\sum_{k\ge2} \log\left(\cos\left(\frac{\pi}{2k}\right)\right)$ converges to a (negative) finite value l.

Therefore we have found a sequence of numbers t_j such that the arithmetic mean of the first coordinate of $\varphi(t_1), \dots, \varphi(t_{2^{n-1}})$ is uniformly bounded away from zero and the arithmetic means of following $\tilde{n}-1$ coordinates are zero. According to (6), the role of $t_1, \dots, t_{2^{n-1}}$ can equivalently be played, for every $k \in \mathbf{N}$, by the 2^{n-1}-uple

$$t_j^k = t_j + 2\pi m k\,, \quad j = 1, \dots, 2^{n-1},$$

where the integer m is larger than $r + t_{2^{n-1}}/2\pi$. Now, let $l \in \{\tilde{n}+1, \dots, K\}$, so that $\gamma_l \notin \mathbf{Z}$. For every $h \in \mathbf{N}$, the arithmetic mean of the l-th coordinate of the points $\varphi(t_j^k)$ $(k = 0, \dots, h,\ j = 1, \dots, 2^{n-1})$ is

$$\frac{1}{2^{n-1}(h+1)} \sum_{j=1}^{2^{n-1}} \sum_{k=0}^{h} e^{it_j^k \gamma_l} =$$

$$= \frac{1}{2^{n-1}(h+1)} \sum_{j=1}^{2^{n-1}} e^{it_j\gamma_l} \sum_{k=0}^{h} e^{i2\pi m k \gamma_l}$$

$$= \left(\frac{1}{2^{n-1}} \sum_{j=1}^{2^{n-1}} e^{it_j\gamma_l} \right) \frac{1}{(h+1)} \frac{1 - e^{i2\pi m(h+1)\gamma_l}}{1 - e^{i2\pi m \gamma_l}} \overset{h\to\infty}{\longrightarrow} 0\,.$$

Therefore, we found a sequence of points in the convex hull of $\varphi([0,\infty))$ converging to $(2^{1-n}\sum_{j=1}^{2^{n-1}} e^{i\gamma_1 t_j}, 0, \dots, 0)$. The lemma follows from (7) and by rotation invariance (see (6)).

Remark 8. The constant ν can be easily computed numerically. One finds $\nu \approx 0.430 > \frac{2}{5}$.

3.3 An auxiliary system

Let (A, B, Φ) satisfy $(\mathfrak{A})$. With every $n \in \mathbf{N}$ we associate the control system on $\mathbf{C}^n$

$$\dot{x} = \nu |b_{jk}| \left(e^{i\theta} e_{jk}^{(n)} - e^{-i\theta} e_{kj}^{(n)} \right) x, \qquad (\Sigma_n)$$

where $\theta = \theta(t) \in \mathbb{S}^1$ and $(j,k) = (j(t), k(t)) \in \mathbf{N}^2 \setminus \{(p,p) : p \in \mathbf{N}\}$ are piecewise constant controls. Set $e_{jk}^{(n)}$ as the $n \times n$ matrix whose entries are all zero but the one of index (j,k) which is equal to 1.

The control system (Σ_n) is linear in x. For every θ in $\mathbb{S}^1$ and every $1 \le j, k \le n, j \ne k$, the matrix $e^{i\theta} e_{jk}^{(n)} - e^{-i\theta} e_{kj}^{(n)}$ is skew-adjoint with zero trace. Hence the control system (Σ_n) leaves the unit sphere $\mathbf{S}^n$ of $\mathbf{C}^n$ invariant. In order to take advantage of the rich Lie group structure of group of matrices, it is also possible to lift this system in the group $SU(n)$, considering x as a matrix.

4. PROOF OF THEOREM 2.1

4.1 Controllability of (Σ_n)

Proposition 9. Let (A, B, Φ, δ) satisfy $(\mathfrak{A})$. Then, for every n in $\mathbf{N}$, the finite dimensional control system (Σ_n) is completely controllable both in $\mathbf{S}^n \subset \mathbf{C}^n$ and in $SU(n)$.

Proof. Note that

$$\left[e_{jk}^{(n)} - e_{kj}^{(n)}, i(e_{jk}^{(n)} + e_{kj}^{(n)}) \right] = 2i(e_{jj}^{(n)} - e_{kk}^{(n)})\,.$$

Hence the Lie algebra generated by the linear vector fields $x \mapsto \nu|b_{jk}|(e^{i\theta} e_{jk} - e^{-i\theta} e_{kj})x$ contains the whole tangent space $\mathfrak{su}(n)x$ of $SU(n)$ at x. The controllability in $\mathbf{S}^n$ follows by projection.

4.2 Tracking (Σ_n) *via* (Θ_N)

Next proposition states that, for every $N \ge n$ system (Θ_N) can track, in projection on the first n components, without time reparametrization, every trajectory of system (Σ_n). Call $\Pi_n^{(N)}$ the projection mapping a $N \times N$ complex matrix to the $n \times n$ matrix obtained by dropping the last $N-n$ columns and the last $N-n$ rows.

Proposition 10. For every $n, N \in \mathbf{N}$, $N \ge n$, $\varepsilon, \delta > 0$, and for every solution $X : [0,T] \to SU(n)$ of system (Σ_n) with initial condition $X(0) = I_n$ there exist a piecewise constant control $u : [0,T] \to [1/\delta, +\infty)$ such that the solution $Y : [0,T] \to SU(N)$ of system (Θ_N) with initial condition $Y(0) = I_N$ corresponding to $v(t) = \int_0^t u(s)ds$ satisfies

$$\|\Pi_n^{(N)} Y(t) - X(t)\| < \varepsilon, \qquad \text{for every } t \in [0,T].$$

Proof. Given a trajectory $X(t)$ of system (Σ_n) with initial condition $X(0) = I_n$, denote by $(j,k) = (j(t), k(t)) \in \mathbf{N}^2$, $j \ne k$ and $\theta = \theta(t) \in \mathbb{S}^1$ its corresponding control functions. Being these function piecewise constant, it is possible to write $[0,T]$ as union of q intervals $[0,T] = \bigcup_{p=0}^{q} [t_p, t_{p+1}]$ in such a way that j, k, and θ are constant on $[t_p, t_{p+1})$ for every $p = 0, \dots, q$.

We are going to construct the control u by applying recursively Lemma 7. Let $\bar{\delta} > 1/\delta$. Fix $p \in \{0, \dots, q\}$ and j, k, θ such that $(j(t), k(t)) = (j,k)$ and $\theta(t) = \theta$ on $[t_p, t_{p+1})$. Apply Lemma 7 with $\gamma_1 = \lambda_j - \lambda_k$, $\{\gamma_2, \dots, \gamma_K\} = \{\lambda_l - \lambda_m \mid l, m \in \{1, \dots, N\}, \{l,m\} \ne \{j,k\}, \text{ and } l \ne m\}$,

$$R = \max_{1 \le l < m \le n} \frac{|b_{ll} - b_{mm}|}{|\lambda_l - \lambda_m|} T + \bar{\delta} T,$$

and $t_0 = t_0(p)$ to be fixed later depending on p. Then, for every $\eta > 0$, there exist $h = h(p) > 1/\eta$ and a sequence $(w_\alpha^p)_{\alpha=1}^h$ such that $w_1^p \ge t_0$, $w_\alpha^p - w_{\alpha-1}^p > R$, and such that

$$\left| \frac{1}{h} \sum_{\alpha=1}^{h} e^{i(\lambda_k - \lambda_j) w_\alpha^p} - \nu \frac{\bar{b}_{jk}}{|b_{jk}|} e^{i\theta} \right| < \eta,$$

and

$$\left| \frac{1}{h} \sum_{\alpha=1}^{h} e^{i(\lambda_l - \lambda_m) w_\alpha^p} \right| < \eta,$$

for every $l, m \in \{1, \dots, N\}$ such that $\{l,m\} \ne \{j,k\}$, and $l \ne m$.

Set $\tau_\alpha^p = t_p + (t_{p+1} - t_p)\alpha/h$, $\alpha = 0, \dots, h$, and define the piecewise constant function

$$v_\eta(t) = \sum_{p=0}^{q} \sum_{\alpha=1}^{h} \left(w_\alpha^p + i\tau_\alpha^p \frac{b_{jj} - b_{kk}}{\lambda_j - \lambda_k} \right) \chi_{[\tau_{\alpha-1}^p, \tau_\alpha^p)}(t)\,. \quad (8)$$

Note that by choosing $t_0(p) = w_{h(p-1)}^{p-1} + R$ for $p = 1, \dots, q$ and $t_0(0) = R$ we have that $v_\eta(t)$ is non-decreasing.

Set $M(t) = \nu |b_{j(t)k(t)}| \left(e^{i\theta(t)} e_{j(t)k(t)}^{(N)} - e^{-i\theta(t)} e_{k(t)j(t)}^{(N)} \right)$. The definition of v_η is such that

$$\int_0^t \vartheta_N(s, v_\eta(s))ds \overset{\eta\to 0}{\longrightarrow} \int_0^t M(s)ds, \qquad (9)$$

uniformly with respect to $t \in [0,T]$. This convergence guarantees (see for example [Agrachev and Sachkov, 2004,

Lemma 8.2]) that the projection $\Pi_n^{(N)}$ of the solution $Y_\eta(t)$ of system (Θ_N) with control v_η and initial condition I_N converges uniformly in $[0,T]$ to $X(t)$ as η tends to 0. In particular it is possible to choose η sufficiently small in such a way that

$$\|\Pi_n^{(N)} Y_\eta(t) - X(t)\| < \varepsilon/2 \quad \text{for every } t \in [0,T].$$

If $v_\eta(t)$ were of the type $\int_0^t u(s)ds$ for $u : [0,T] \to [1/\delta, \infty)$ piecewise constant then we would be done. Although v_η is piecewise constant, it is possible to construct a continuous piecewise linear approximation v'_η of v_η with slope not smaller than $\bar\delta$ and such that

$$\int_0^t \left(\vartheta_N(\tau, v_\eta(\tau)) - \vartheta_N(\tau, v'_\eta(\tau))\right) d\tau \overset{\eta\to 0}{\longrightarrow} 0,$$

uniformly with respect to $t \in [0,T]$. This is possible thanks to the choice of R and since the size of every interval $[\tau_k^p, \tau_{k+1}^p]$ is smaller that $1/h < \eta$.

In particular there exists η sufficiently small in such a way that, if Y'_η is the solution of system (Θ_N) with control v'_η and initial condition I_N, then

$$\|Y_\eta(t) - Y'_\eta(t)\| < \varepsilon/2 \quad \text{for every } t \in [0,T].$$

Finally, construct the control u as as the derivative of v'_η.

4.3 Tracking (Σ_n) in modulus via the original system

Next proposition extend the result of tracking for the system (Θ_N) to the infinite dimensional system (5).

Proposition 11. For every $\varepsilon > 0$, for every integer n, and for every trajectory $x : [0,T] \to \mathbf{S}^n$ of system (Σ_n) there exists a piecewise constant control function $u : [0,T] \to [1/\delta, +\infty)$ and a corresponding trajectory $\psi(t)$ of (5) satisfying $\big||\langle\phi_j, \psi(t)\rangle| - |x_j(t)|\big| < \varepsilon$ for every t in $[0,T]$, $j = 1,\dots,n$ and $\big|\langle\phi_j, \psi(0)\rangle - x_j(0)\big| < \varepsilon$ for every $j = 1,\dots,n$.

Proof. Consider $\mu > 0$. For every $j \in \mathbf{N}$ the hypothesis that ϕ_j belongs to $D(B)$ implies that the sequence $(b_{jk})_{k\in\mathbf{N}}$ is in ℓ^2. It is therefore possible to choose $N \geq n$ such that $\sum_{k>N} |b_{jk}|^2 < \mu$ for every $j = 1,\dots,n$.

Let $X : [0,T] \to SU(n)$ be the solution of system (Σ_n) corresponding to the control associated with the trajectory x and with initial condition $X(0) = I_n$. By Proposition 10, for every $\eta > 0$ there exist a piecewise constant control $u^\eta : [0,T] \to [1/\delta, +\infty)$ such that the solution $Y : [0,T] \to SU(N)$ of system (Θ_N) with initial condition I_N corresponding to $v^\eta(t) = \int_0^t u^\eta(\tau)d\tau$ satisfies

$$\|\Pi_n^{(N)} Y(t) - X(t)\| < \eta\,,$$

for every $t \in [0,T]$. Notice that $Y(t) = R^\eta(t,0)$ is the resolvent of (Θ_N) associated with the control v^η.

Let $\psi(t)$ be the solution of (5) corresponding to u^η with initial condition $\psi(0) = x_1(0)\phi_1 + \dots + x_n(0)\phi_n$. Call

$$q_j^\eta(t) = e^{-i\lambda_j v^\eta(t) - t b_{jj}} \langle\phi_j, \psi(t)\rangle, \quad j \in \mathbf{N}\,.$$

For almost every $t \in [0,T]$ and for every $j \in \mathbf{N}$,

$$\dot q_j^\eta(t) = \sum_{k=1}^\infty b_{jk} e^{i(\lambda_k - \lambda_j)v^\eta(t) + t(b_{kk} - b_{jj})} q_k^\eta(t). \tag{10}$$

Therefore $Q_N^\eta(t) = (q_1^\eta(t), \dots, q_N^\eta(t))^T$ satisfies the time-dependent linear equation

$$\dot Q_N^\eta(t) = \vartheta_N(t, v^\eta(t)) Q_N^\eta(t) + P_N^\eta(t),$$

with $\|\Pi_n^{(N)} P_N^\eta(t)\| \leq \sqrt{\mu n}$. Hence

$$Q_N^\eta(t) = R^\eta(t,0) \begin{pmatrix} x(0) \\ 0 \end{pmatrix} + \int_0^t R^\eta(s,t) P_N^\eta(s) ds.$$

Consider the projection on the first n coordinates. By (9), $R^\eta(s,t)$ converges uniformly, as η tends to 0, to a time dependent operator from $\mathbf{C}^N$ into itself which preserves the norm of the first n components, then there exists η sufficiently small such that

$$\left\|\Pi_n^{(N)} \left(\int_0^t R^\eta(t,s) P_N^\eta(s) ds\right)\right\| < 2T\sqrt{\mu n}.$$

Hence

$$\|Q_n^\eta(t) - x(t)\| < \varepsilon,$$

for η and μ small enough. In particular, for $j = 1,\dots,n$

$$\big||\langle\phi_j, \psi(t)\rangle| - |x_j(t)|\big| = \big||q_j^\eta(t)| - |x_j(t)|\big| < \varepsilon.$$

4.4 Approximate controllability

For $n \in \mathbf{N}$, let $\pi_n : \mathcal{H} \to \mathbf{C}^n$ be the map associating with $\psi \in \mathcal{H}$ the n-uple $(\langle\phi_1, \psi\rangle, \dots, \langle\phi_n, \psi\rangle)$.

Approximate controllability follows from Proposition 11 and Remark 6 applied both to (2) and to its time-reversed version.

Proof of Theorem 2.1. Fix $\psi_0, \psi_1 \in \mathbf{S}$, $\varepsilon > 0$, and n sufficiently large such that ψ_0 is ε-close to $\sum_{j=1}^n \langle\phi_j, \psi_0\rangle\phi_j$. Without loss of generality we can assume that $\lambda_1 \neq 0$.

Thanks to Proposition 9, there exists an admissible trajectory x of (Σ_n) steering $\pi_n(\psi_0)$ ε-close to $\pi_n(\phi_1) = (1, 0, \dots, 0)$. Applying Proposition 11 and Remark 6, there exists a piecewise constant control function $u : [0,T] \to [0,\delta]$ and a corresponding trajectory $\psi(t)$ of (2) satisfying $\big||\langle\phi_j, \psi(t)\rangle| - |x_j(t)|\big| < \varepsilon$ for every t in $[0,T]$, $j = 1,\dots,n$ and $\big|\langle\phi_j, \psi(0)\rangle - x_j(0)\big| < \varepsilon$ for every $j = 1,\dots,n$.

In particular, taking $t = 0, T$, we have that both $\|\psi(0) - \psi_0\| < 2\varepsilon$ and $\|\psi(T) - e^{i\theta}\phi_1\| < C\varepsilon$, for some $\theta \in \mathbb{S}^1$, with C independent of n, ε and ψ_0.

Hence, thanks to the unitarity of the evolution of (2), u steers ψ_0 to a $(C+2)\varepsilon$-neighborhood of $e^{i\theta}\phi_1$.

Note that if (A, B, Φ, δ) satisfies $(\mathfrak{A})$, then the same is true for $(-A, -B, \Phi, \delta)$. Notice, moreover, that, by unitarity of the evolution of (2), if $u : [0,T] \to [0,\delta]$ steers ψ_0 to a ε-neighborhood of ψ_1 for the time-reversed control system $\dot\psi(t) = -(A + u(t)B)\psi(t)$, then $u(T - \cdot) : [0,T] \to [0,\delta]$ steers ψ_1 ε-close to ψ_0 for the original system (2).

Hence there exists a control function $\tilde u : [0,\tilde T] \to [0,\delta]$ steering (2) from $e^{i\tilde\theta}\phi_1$ to a $(C+2)\varepsilon$-neighborhood of ψ_1, for some $\tilde\theta \in \mathbb{S}^1$.

Let $\tau > 0$ be such that

$$e^{\tau A}(e^{i\theta}\phi_1) = e^{i\tilde\theta}\phi_1.$$

Finally, the concatenation of u, of the control constantly equal to zero for a time τ, and of $\tilde u(\tilde T - \cdot)$ steers (2) from ψ_0 to a $2(C+2)\varepsilon$-neighborhood of ψ_1.

5. ESTIMATES OF THE L^1 NORM

We derive now estimates of the minimal L^1 norm of the control u steering ϕ_j to a ε-neighborhood of ϕ_k.

The strategy to get L^1 estimates is the following. Recall that, instead of considering the control system $\dot{x} = (A + uB)x$ driven by a piecewise constant function $u : [0, T_u] \to [0, \delta]$, we have defined the function $\mathcal{P}(u) : [0, \|u\|_{L^1}] \to [1/\delta, \infty)$ and considered the control system $\dot{x} = (\mathcal{P}(u)A + B)x$. To estimate the L^1 norm of u, it is enough to estimates the time needed to transfer the system $\dot{x} = (uA + B)x$ from a given source to a given target. We observe that, in view of Proposition 11, the time needed for the transfer of $\dot{x} = (uA + B)x$ from ϕ_j to an ε-neighborhood of ϕ_k is smaller than or equal to the time needed to transfer system (Σ_n) from $\pi_n(\phi_j)$ to an ε-neighborhood of $\pi_n(\phi_k)$ $(n \geq j, k)$.

Such time estimates have been given for general trajectories (i.e., trajectories that do not necessarily steer an eigenstate of A to another) in Agrachev and Chambrion [2006].

We proceed now to the proof of the Theorem 2.2.

Proof of Theorem 2.2. The solution $x : \mathbf{R} \to \mathbf{S}^n$ of the differential equation

$$\dot{x} = \nu |b_{jk}| \left(e_{jk}^{(n)} - e_{kj}^{(n)} \right) x$$

with initial condition $x(0) = \pi_n(\phi_j)$ is a trajectory of (Σ_n). For every t in $\mathbf{R}$, $x(t) = \exp\left(t\nu |b_{jk}| \left(e_{jk}^{(n)} - e_{kj}^{(n)} \right) \right) \pi_n(\phi_j)$. In particular, one checks that $x\left(\frac{\pi}{2\nu|b_{jk}|} \right) = -\pi_n(\phi_k)$.

In other words, the control system (Σ_n) can exchange (up to a phase factor) the eigenstates j and k of A, in time less than $\frac{\pi}{2\nu|b_{jk}|} < \frac{5\pi}{4|b_{jk}|}$.

Remark 12. Notice that the bound given in Theorem 2.2 does not depend on ε. However, it is possible that the time T_u needed to achieve the transfer of system (2) grows to infinity as ε tends to zero. When only some $|b_{jk}|$ are small, the bound given by Theorem 2.2 may be too conservative. It is possible to find other estimates by comparing the time needed to steer an eigenstate to another with ad-hoc metrics on $SU(n)$. The resulting expressions are somewhat intricate and can be found in [Agrachev and Chambrion, 2006, Section 5].

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